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Research Article

A Nonlinear Differential Equation Related to the Jacobi Elliptic Functions

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A nonlinear differential equation for the polar angle of a point of an ellipse is derived. The solution of this differential equation can be expressed in terms of the Jacobi elliptic function $\mathrm{dn}(u,k)$. If the polar angle is extended to the complex plane, the Jacobi imaginary transformation properties and the dependence on the real and complex quarter periods can be described. From the differential equation of the polar angle, exact solutions of the Poisson Boltzmann and the sinh-Poisson equations are found in terms of the Jacobi elliptic functions.

1. Introduction

Nonlinear differential equations with solutions expressed in terms of the Jacobi elliptic functions occur in many areas of physics such as the study of fields in wave guides, anharmonic oscillations, the period of the simple pendulum, the nonlinear Poisson Boltzmann equation, and the sinh-Poisson equation in connection with flows in fluids and plasmas [1–5]. The functions have been studied for many years starting back in the 18th century, and a detailed and strong basis of the theory has been developed starting from the Weierstrassian elliptic functions, the theta functions to the Jacobi elliptic functions [6, 7].

The aim of the present paper is briefly to present an alternate but rather simple way of describing aspects of the Jacobi elliptic functions, that at the same time result in new and interesting relationships. In particular this approach leads to exact solutions (even and odd) of the nonlinear Poisson Boltzmann equation and of the sinh-Poisson equation.

The first part of this paper contains a short derivation of the function describing the polar angle and gives expressions in terms of the Jacobi elliptic functions. The second part describes the consistency to the various existing relationships for these functions. The third part contains various transformations of the arguments. These transformations are used in the fourth part, where exact solutions (even and odd) of the Poisson Boltzmann and the sinh-Poisson equations are presented.

It is believed that valuable insight into the behavior of these functions can be gained even by students who are only familiar with differential calculus as taught in undergraduates courses in mathematics and physics and perhaps be a motivation for further studies.

2. Derivation of the Differential Equation for the Polar Angle

Consider an ellipse in Cartesian coordinates, $x^2/a^2 + y^2/b^2 = 1$, with semimajor axis, a, and semiminor axis, b, so that the eccentricity or the modulus k of the ellipse is given as, $k^2 = 1 - k'^2$, where the complementary modulus k' is, $k' = b/a$. Let us furthermore express a particular point of this ellipse in terms of a parameter u, so that its Cartesian coordinates are $x(u) = a \cdot \mathrm{cn}(u)$ and $y(u) = b \cdot \mathrm{sn}(u)$, where $\mathrm{sn}(u)$ and $\mathrm{cn}(u)$ are Jacobi elliptic functions [6, 7]. (The dependence on the modulus is implicit in the following except when explicit dependence is of importance). The polar radius of this point can be expressed as $r(u) = a \cdot b \cdot \mathrm{dn}(u)$. From these relationships, it follows directly that

$$\begin{aligned} x'(u) &= -\frac{1}{b^2} r(u) \cdot y(u), \\ y'(u) &= \frac{1}{a^2} r(u) \cdot x(u), \end{aligned} \tag{2.1}$$

and from the relationship $r^2(u) = x^2(u) + y^2(u)$, one obtains

$$r'(u) = \left(\frac{1}{a^2} - \frac{1}{b^2}\right) \cdot x(u) \cdot y(u) = \kappa \cdot x(u) \cdot y(u). \tag{2.2}$$

It should be emphasized, that comparison with expressions from the literature [6, 7], a value of $b = 1$, should always be applied, and this is assumed throughout this paper. However, for clarity the dependence on b is maintained in several expressions.

Now, let us introduce the polar angle $\theta(u)$ corresponding to the same particular point of the ellipse by the equation

$$\tan\theta(u) = \frac{y(u)}{x(u)}. \tag{2.3}$$

By differentiation of (2.3) and at the same time making use of the equation of the ellipse as well as (2.1), one obtains

$$\theta'(u) = \frac{1}{r(u)} = \frac{1}{a \cdot b \cdot \mathrm{dn}(u)}. \tag{2.4}$$

Differentiating one more time, one has

$$\theta''(u) = -\frac{1}{r^2(u)} \cdot r'(u) = -\frac{1}{r^2(u)} \cdot \kappa \cdot x(u) \cdot y(u) = -\kappa \cdot \cos\theta(u) \cdot \sin\theta(u) \tag{2.5}$$

or

$$\theta''(u) = -\frac{1}{2}\kappa \cdot \sin 2\theta(u) \tag{2.6}$$

with the solution

$$\theta(u) = \frac{1}{a \cdot b} \int_0^u \frac{1}{\mathrm{dn}(\overline{u})} d\overline{u}. \tag{2.7}$$

It should be noted that since $\mathrm{dn}(u)$ is a positive periodic function, the function $\theta(u)$ is an increasing function but not periodic, whereas $\theta'(u)$ is periodic. From elementary trigonometry, it follows that

$$\theta(u) = \arccos \frac{x(u)}{r(u)}. \tag{2.8}$$

So that

$$\int_0^u \frac{1}{\mathrm{dn}(\overline{u})} d\overline{u} = a \cdot b \cdot \arccos \frac{\mathrm{cn}(u)}{b \cdot \mathrm{dn}(u)}, \tag{2.9}$$

which is an expression that also can be found in [7] (Formula 16.24.6). One can equivalently express this integral as

$$\int_0^u \frac{1}{\mathrm{dn}(\overline{u})} d\overline{u} = a \cdot b \cdot \arcsin \frac{\mathrm{sn}(u)}{a \cdot \mathrm{dn}(u)} = a \cdot b \cdot \arctan \frac{b \cdot \mathrm{sn}(u)}{a \cdot \mathrm{cn}(u)}. \tag{2.10}$$

Equations (2.9) and (2.10) are easily proven to be consistent, and going back, one can easily verify that they satisfy (2.6). In terms of k and k', the important results of this section would be

$$\theta'(u) = \frac{k'}{\mathrm{dn}(u)}, \tag{5'}$$

$$\theta''(u) = \frac{1}{2}k^2 \cdot \sin 2\theta(u), \tag{6'}$$

$$\begin{gathered} \theta(u) = k' \int_0^u (1/\mathrm{dn}(\overline{u})) d\overline{u}, \\ \int_0^u \frac{1}{\mathrm{dn}(\overline{u})} d\overline{u} = \frac{1}{k'} \cdot \arccos \frac{\mathrm{cn}(u)}{\mathrm{dn}(u)} = \frac{1}{k'} \cdot \arcsin \frac{k' \cdot \mathrm{sn}(u)}{\mathrm{dn}(u)} = \frac{1}{k'} \cdot \arctan \frac{k' \cdot \mathrm{sn}(u)}{\mathrm{cn}(u)}, \end{gathered} \tag{7'}$$

where the Jacobi elliptic functions are implicitly dependent on k.

Since the derivative of the amplitude function [6, 7] is given as $\mathrm{am}'(u) = \mathrm{dn}(u)$, it follows from (5′) that the derivative of the polar angle is inversely proportional to the derivative of the amplitude function.

3. Complex Argument and the Relation to the Quarter Periods

In this section, the expression for the polar angle is extended to the complex plane by introducing the complex argument, $w = u + i \cdot v$, such that from (7′), one has

$$\theta(w) = \arctan \frac{k' \cdot \mathrm{sn}(w)}{\mathrm{cn}(w)}. \tag{3.1}$$

If in particular $w = K + i \cdot K'$, where K and K' are the real and complex quarter periods, respectively, and using $\mathrm{sn}(K + i \cdot K') = 1/k$ and $\mathrm{cn}(K + i \cdot K') = -i \cdot (k'/k)$ [6, 7], then

$$\theta(K + i \cdot K') = \frac{\pi}{2} + i \cdot \infty. \tag{3.2}$$

If, however, $w = i \cdot (K'/2)$, using $\mathrm{sn}(i \cdot (K'/2)) = i(1/\sqrt{k})$, $\mathrm{cn}(i \cdot (K'/2)) = \sqrt{(1+k)/k}$, and $\mathrm{dn}(i \cdot (K'/2)) = \sqrt{1+k}$ [6, 7], then one finds that

$$\theta\left(i \cdot \frac{K'}{2}\right) = \arctan i \cdot \frac{k'}{\sqrt{1+k}} = \frac{i}{2} \cdot \ln \frac{\sqrt{1+k} + k'}{\sqrt{1+k} - k'}, \tag{3.3}$$

and from (7′)

$$\begin{aligned} \sin\theta\left(i \cdot \frac{K'}{2}\right) &= i \cdot \frac{k'}{\sqrt{k}\sqrt{1+k}}, \\ \cos\theta\left(i \cdot \frac{K'}{2}\right) &= \frac{1}{\sqrt{k}}, \\ \tan\theta\left(i \cdot \frac{K'}{2}\right) &= i \cdot \frac{k'}{\sqrt{1+k}}. \end{aligned} \tag{3.4}$$

These expressions are consistent with the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$.

On the other hand, if instead $w = K/2$, using $\mathrm{sn}(K/2) = 1/\sqrt{(1+k')}$, $\mathrm{cn}(K/2) = \sqrt{k'/(1+k')}$, and $\mathrm{dn}(K/2) = \sqrt{k'}$ [6, 7], then one finds that

$$\theta\left(\frac{K}{2}\right) = \arctan \sqrt{k'}, \tag{3.5}$$

and thus from (7′),

$$\begin{aligned} \sin\theta\left(\frac{K}{2}\right) &= \sqrt{\frac{k'}{1+k'}}, \\ \cos\theta\left(\frac{K}{2}\right) &= \frac{1}{\sqrt{1+k'}}, \\ \tan\theta\left(\frac{K}{2}\right) &= \sqrt{k'}. \end{aligned} \tag{3.6}$$

Again these expressions are consistent with the trigonometric identity.

Various other relationships for the polar angle in terms of the real and complex quarter periods could have been derived, which as indicated would be consistent with existing theory.

4. Complex Argument and Transformation of Variables

The transformation of variables for the complex polar angle are dealt with in this section. Several expressions are derived, because they are used in the next section, where exact solutions of the nonlinear Poisson Boltzmann equation are presented. For clarity the Jacobi elliptic functions applied for the transformations are listed at the end of this section [6, 7].

If the complex argument of (3.1) is purely imaginary, $w = i \cdot v$, one finds that

$$\theta(i \cdot v, k) = \arctan\left(k' \cdot i \cdot \operatorname{sn}\left(v, k'\right)\right) = \frac{i}{2} \cdot \ln \frac{1 + k' \cdot \operatorname{sn}(v, k')}{1 - k' \cdot \operatorname{sn}(v, k')}, \tag{4.1}$$

for the imaginary transformation of the polar angle (principal value of the argument).

If, however, $w = i \cdot v + K$, then one finds that

$$\theta(i \cdot v + K) = \arctan \frac{i}{\operatorname{sn}(v, k')} = \frac{\pi}{2} + \frac{i}{2} \cdot \ln \frac{1 + \operatorname{sn}(v, k')}{1 - \operatorname{sn}(v, k')}. \tag{4.2}$$

In (4.2) the multiplicity of the real part on $n \cdot \pi$ has been skipped as the polar angle at the quarter period K corresponds to the value $\pi/2$, when considering the fundamental interval.

For $w = u + i \cdot K'$, one would find that

$$\theta\left(u + i \cdot K'\right) = \arctan \frac{i \cdot k'}{\operatorname{dn}(u, k)} = \frac{i}{2} \cdot \ln \frac{\operatorname{dn}(u, k) + k'}{\operatorname{dn}(u, k) - k'}, \tag{4.3}$$

where the denominator is positive and the expression is given in the fundamental interval.

The next three transformations are most easily obtained starting out from (4.3). First let us replace the argument of (4.3) by $w = u + K + i \cdot K'$, then

$$\theta\left(u + K + i \cdot K'\right) = \frac{i}{2} \cdot \ln \frac{\operatorname{dn}(u + K, k) + k'}{\operatorname{dn}(u + K, k) - k'} = \frac{i}{2} \cdot \ln \frac{1 + \operatorname{dn}(u, k)}{1 - \operatorname{dn}(u, k)}. \tag{4.4}$$

If, on the other hand, the argument of (4.3) is replaced by $w = i \cdot v + i \cdot K'$, then

$$\theta\left(i \cdot v + i \cdot K'\right) = \frac{i}{2} \cdot \ln \frac{\operatorname{dn}(v, k') + k' \cdot \operatorname{cn}(v, k')}{\operatorname{dn}(v, k') - k' \cdot \operatorname{cn}(v, k')}. \tag{4.5}$$

Again the denominator is positive.

Or the argument in (4.3) could be replaced by $w = i \cdot v + K + i \cdot K'$, then

$$\theta\left(i \cdot v + K + i \cdot K'\right) = \frac{i}{2} \cdot \ln \frac{1 + \operatorname{dn}(v, k')/cn(v, k')}{1 - \operatorname{dn}(v, k')/cn(v, k')} = \frac{\pi}{2} + \frac{i}{2} \cdot \ln \frac{\operatorname{dn}(v, k') + \operatorname{cn}(v, k')}{\operatorname{dn}(v, k') - \operatorname{cn}(v, k')} \tag{4.6}$$

such that the denominator is positive (principal value of the argument).

In Section 5, it is shown that these equations are related to the solution of the Poisson Boltzmann equation and the sinh-Poisson equation [4, 5].

The various transformations applied in this section are listed as follows [6, 7]:

$$\begin{gathered}
\operatorname{sn}(i\cdot v,k)=i\cdot\frac{\operatorname{sn}(v,k')}{\operatorname{cn}(v,k')},\qquad \operatorname{cn}(i\cdot v,k)=\frac{1}{\operatorname{cn}(v,k')},\qquad \operatorname{dn}(i\cdot v,k)=\frac{\operatorname{dn}(v,k')}{\operatorname{cn}(v,k')},\\
\operatorname{sn}(i\cdot v+K,k)=\frac{1}{\operatorname{dn}(v,k')},\qquad \operatorname{cn}(i\cdot v+K,k)=-i\cdot k'\cdot\frac{\operatorname{sn}(v,k')}{\operatorname{dn}(v,k')},\\
\operatorname{sn}\left(u+i\cdot K',k\right)=\frac{1}{k\cdot\operatorname{sn}(u,k)},\qquad \operatorname{cn}\left(u+i\cdot K',k\right)=\frac{-i}{k}\cdot\frac{\operatorname{dn}(u,k)}{\operatorname{sn}(u,k)},\\
\operatorname{dn}(u+K,k)=\frac{k'}{\operatorname{dn}(u,k)}.
\end{gathered}\tag{4.7}$$

5. Exact Solution of the Poisson Boltzmann Equation

Since $\theta(w)=\vartheta_1(w)+i\cdot\vartheta_2(w)$ is an analytic function the differential equations for the real and imaginary parts can be found from (6′)

$$\vartheta_1''+i\cdot\vartheta_2''=\frac{1}{2}k^2(\sin 2\vartheta_1\ \cosh 2\vartheta_2+i\cdot\cos 2\vartheta_1\ \sinh 2\vartheta_2).\tag{5.1}$$

From Cauchy-Riemann equations, one has

$$\frac{\partial^2\vartheta_1}{\partial u^2}+i\cdot\frac{\partial^2\vartheta_2}{\partial u^2}=-\frac{\partial^2\vartheta_1}{\partial v^2}-i\cdot\frac{\partial^2\vartheta_2}{\partial v^2}=\frac{1}{2}k^2(\sin 2\vartheta_1\ \cosh 2\vartheta_2+i\cdot\cos 2\vartheta_1\ \sinh 2\vartheta_2).\tag{5.2}$$

Thus, if $w=i\cdot v+K$, then from (4.2) the real part of the polar angle is $\vartheta_1=\pi/2$, and thus from (5.2), it follows that the imaginary part ϑ_2 will be a solution of the equation

$$\frac{\partial^2\vartheta_2}{\partial v^2}=\frac{1}{2}k^2\ \sinh 2\vartheta_2,\tag{5.3}$$

and substituting $\psi=2\vartheta_2$, one has

$$\psi''=k^2\ \sinh\psi,\tag{5.4}$$

which is the one dimensional nonlinear Poisson Boltzmann equation with the solution

$$\psi(v)=\ln\frac{1+\operatorname{sn}(v,k')}{1-\operatorname{sn}(v,k')}.\tag{5.5}$$

This solution is shown in Figure 1 for parameters of $k^2=0.75$ and $K'=1.686$.

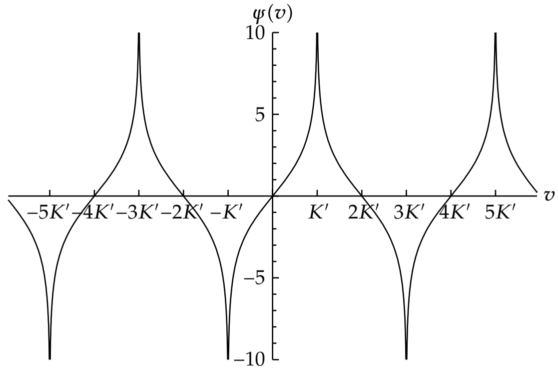

Figure 1: The solution $\psi(v) = \ln((1+\text{sn}(v,k'))/(1-\text{sn}(v,k')))$ for the Poisson Boltzmann equation. The parameters are $k^2 = 0.75$ and $K' = 1.686$.

As can be seen, this is an odd function with the period $4K'$. If instead $w = u + i \cdot K'$, then from (4.3) the real part of the polar angle is $\vartheta_1 = 0$, and thus from (5.2), it follows that the imaginary part ϑ_2 will be a solution of the equation

$$\frac{\partial^2 \vartheta_2}{\partial u^2} = \frac{1}{2} k^2 \sinh 2\vartheta_2, \tag{5.6}$$

and again substituting $\psi = 2\vartheta_2$, one has

$$\psi'' = k^2 \sinh \psi. \tag{5.7}$$

However, in this case differentiation is with respect to the variable u, and from (4.3), it follows that the solution is

$$\psi(u) = \ln \frac{\text{dn}(u,k) + k'}{\text{dn}(u,k) - k'}. \tag{5.8}$$

In this case, the solution is an even function with the period $2K$. In Figure 2, the solution given by (5.8) is shown.

The double prime in (5.4) and (5.7) indicates differentiation with respect to the variable v in case of the odd solution and u in case of the even solution. Of course to make consistency one could always replace these variables with some common variable.

As can be seen from (4.4) and (4.6), there exists in addition two more solutions (even) of the Poisson Boltzmann equation. The expressions are

$$\begin{aligned} \psi(u) &= \ln \frac{1+\text{dn}(u,k)}{1-\text{dn}(u,k)}, \\ \psi(v) &= \ln \frac{\text{dn}(v,k') + \text{cn}(v,k')}{\text{dn}(v,k') - \text{cn}(v,k')}. \end{aligned} \tag{5.9}$$

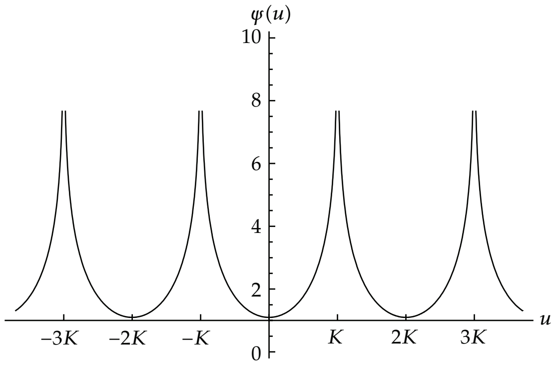

Figure 2: The solution $\psi(u) = \ln((\mathrm{dn}(u,k) + k')/(\mathrm{dn}(u,k) - k'))$ for the Poisson Boltzmann equation. The parameters are $k^2 = 0.75$ and $K = 2.156$.

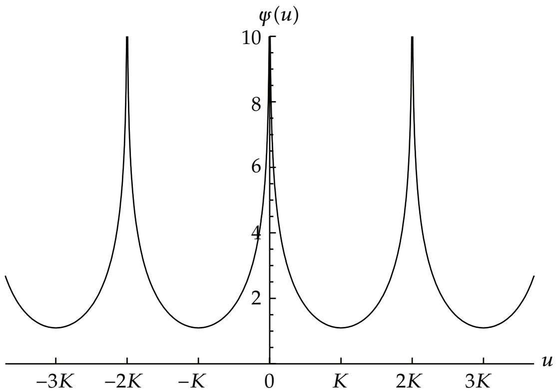

Figure 3: The solution $\psi(u) = \ln((1 + \mathrm{dn}(u,k))/(1 - \mathrm{dn}(u,k)))$ for the Poisson Boltzmann equation. The parameters are $k^2 = 0.75$ and $K = 2.156$.

Of course differentiations are with respect to the appropriate variable. These functions are illustrated in Figures 3 and 4.

In a similar way, one finds by use of (4.1) and (4.5) that the sinh-Poisson equation [5]

$$\psi'' = -k^2 \sinh \psi \tag{5.10}$$

has the odd solution

$$\psi(v) = \ln \frac{1 + k' \cdot \mathrm{sn}(v,k')}{1 - k' \cdot \mathrm{sn}(v,k')} \tag{5.11}$$

and the even solution

$$\psi(v) = \ln \frac{\mathrm{dn}(v,k') + k' \cdot \mathrm{cn}(v,k')}{\mathrm{dn}(v,k') - k' \cdot \mathrm{cn}(v,k')}. \tag{5.12}$$

These expressions are easily shown to be correct by direct substitution into (5.10).

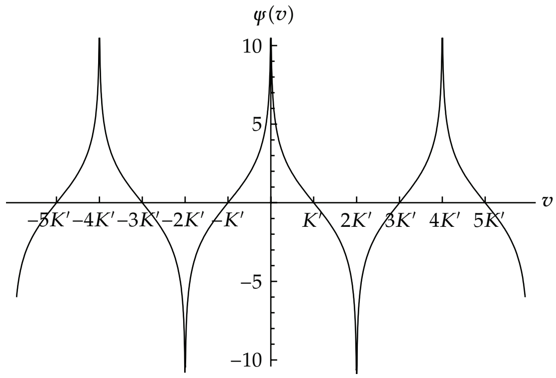

Figure 4: The solution $\psi(v) = \ln((\mathrm{dn}(v,k') + \mathrm{cn}(v,k'))/(\mathrm{dn}(v,k') - \mathrm{cn}(v,k')))$ for the Poisson Boltzmann equation. The parameters are $k^2 = 0.75$ and $K' = 1.686$.

6. Conclusions

Exact solutions of the nonlinear Poisson Boltzmann equation have been presented. In order to derive these solutions it was necessary to introduce a function related to the Jacobi elliptic functions, giving the polar angle of a particular point of an ellipse. This function was extended to the complex plane, and various relationships with the Jacobi elliptic functions were described and shown to be consistent.

A new nonlinear differential equation for the polar angle was derived, part of which could be shown to be associated with the nonlinear Poisson Boltzmann equation. Exact solutions were extracted for the nonlinear Poisson Boltzmann equation. In addition, exact solutions for the sinh-Poisson equation were also presented.

It is believed that valuable insight into the behavior of these functions can be gained even by students who are only familiar with differential calculus as taught in undergraduates courses in mathematics and physics and perhaps be a motivation for further studies.

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