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Research Article

Complex Dynamics of Mixed Triopoly Game with Quantity and Price Competition

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This article investigates the dynamics of a mixed triopoly game in which a state-owned public firm competes against two private firms. In this game, the public firm and private firms are considered to be boundedly rational and naive, respectively. Based on both quantity and price competition, the game's equilibrium points are calculated, and then the local stability of boundary points and the Nash equilibrium points is analyzed. Numerical simulations are presented to display the dynamic behaviors including bifurcation diagrams, maximal Lyapunov exponent, and sensitive dependence on initial conditions. The chaotic behavior of the two models has been stabilized on the Nash equilibrium point by using the delay feedback control method. The thresholds under price and quantity competition are also compared.

1. Introduction

The research on chaos theory and bifurcation theory based on different dynamical systems is the earliest chaotic dynamics branch with the most widespread application [1–5]. In recent years, mathematical models which describe the oligopoly dynamics of firms' competition have got much attention due to their important characteristics in analyzing competition patterns and market control. Literature includes many intensive studies on duopoly games (for example, see [6–9] for Cournot competition, [10, 11] for Bertrand competition, [12, 13] for Stackelberg competition, and [14–16] for Cournot–Bertrand case).

Compared with the duopoly game models, the analysis of oligopolies with more than two firms has been less addressed in the literature. In practice, such a game is close to the economic reality and it is widely deployed in oligopolies. However, analyzing the dynamics of such a game is a complex task. Therefore, various investigations were performed previously by considering Cournot triopoly game with three homogeneous players. For example, Puu [17] considered three naive players in the game model, in which an iso-elastic demand function and constant marginal cost were assumed. Agiza et al. [18] proposed Kopel triopoly Cournot game model; they assumed that the competitors did not immediately offer the optimal quantity computed on the basis of the (bounded rationality) profit maximization problem, but that they adjusted the previous quantity in the direction of the computed one. Tu and Wang [19] presented a dynamic master-slave Cournot triopoly game model with homogeneous bounded normal players. They analyzed the effects of changes in the adjustment speed parameters on the system dynamics. Alnowibet et al. [20] modeled a dynamic Cournot triopoly game that was composed of three homogeneous bounded rational players, and the demand function was characterized by log-concavity.

Additionally, Cournot triopoly games considering heterogeneous players were also investigated. Ma and Ji [21] considered a Cournot triopoly game with a square inverse demand. Ji [22] further investigated the game model based on heterogeneous expectations in electric power triopoly. Elabbasy et al. [23] studied the dynamical behavior of the triopoly game with heterogeneous players with linear cost function. The three players were considered to be boundedly rational, adaptive, and naive. Then, Elabbasy et al. [24] further generalized the triopoly game with heterogeneous players with nonlinear cost function. Based on a quantity competition, Askar and Alshamrani [25] studied four different models, namely, cooperative Cournot triopoly, rational Bertrand triopoly, rational Cournot triopoly, and Puu triopoly.

On the other hand, there are few studies on Bertrand triopoly game. For instance, in Sun and Ma [26], a Bertrand triopoly model was developed. They considered the bounded rational expectations, and local stability of the Nash equilibrium was studied. Furthermore, Sun and Ma [27] introduced a Bertrand triopoly model using nonlinear demand functions. Ma and Wu [28] studied the effect of delayed decision on the stability of a Bertrand triopoly model. Zhao [29] constructed a nonlinear dynamic Bertrand triopoly model based on the heterogeneity in expectations, namely, naive expectation and bounded rationality. The instability of the boundary equilibrium point and the stability condition of the internal equilibrium point were obtained.

As exceptions, in a differentiated triopoly model with heterogeneous firms, the local stability of the Nash equilibrium under both quantity and price competition was analyzed in [30, 31]. We can also cite Elsadany et al. [32], who considered an oligopoly game with four heterogeneous firms producing perfect substitute goods.

However, these results mentioned above depend on the assumption that all firms are private and profit-maximizers. Therefore, they may not apply to the increasingly important and popular mixed oligopolies, in which state-owned public firms compete with private firms. Analyses of mixed oligopolies date to Merrill and Schneider [33]. In most countries, there exist state-owned public firms that have substantial influence on their market competitors, such as the airline, steel, insurance, hospital, and banking industries. In the studies of mixed oligopolies, it is generally assumed that a public firm maximizes social surplus, while private firms maximize profits. For examples of mixed oligopolies and recent developments in research in this field, see [34, 35] and the references therein for more details.

This paper proposes a mixed triopoly game with heterogeneous players. Firm 0 is a state-owned public firm adopting bounded rationality, and its payoff is the social surplus. Firm $i\,(i = 1, 2)$ is a private firm, and its payoff is its own profit. We assume the private firms are both naive players. We analyze the local stability of the Nash equilibrium under both quantity and price competition. Numerical simulations are used to provide experimental evidence for the complex behavior of the evolution of the system, including bifurcation diagrams, maximal Lyapunov exponent, and sensitive dependence on initial conditions. A feedback control scheme is adopted to overcome the uncontrollable behavior of the dynamical system occurred due to chaos. To the best of our knowledge, this article is the first to use analytical and numerical tools to consider the dynamics of mixed triopoly game.

The remainder of the paper is organized as follows. The description of the mixed triopoly game with heterogeneous players is described in Section 2. In Section 3, we shall study the existence and local stability of fixed points of the three-dimensional dynamics system under Cournot competition. Dynamical behaviors under some change of control parameters of the game are investigated by numerical simulations, and chaos control of the system is also given. The dynamics under price competition are analyzed in Section 4. Finally, a conclusion is drawn in Section 5.

2. Model Specification

We adopt a standard differentiated oligopoly with a linear demand. The quasi-linear utility function of the representative consumer is

$$U(q_0, q_1, q_2) = \alpha \sum_{i=0}^{2} q_i - \beta \left(\sum_{i=0}^{2} q_i^2 + \delta \sum_{i=0}^{2} \sum_{i \neq j} q_i q_j \right) + y, \quad (1)$$

where q_0 is the quantity produced by the public firm, $q_i\,(i = 1, 2)$ is the quantity produced by the private firm, and y is the consumption of outside goods provided competitively (with a unitary price). Parameters α and β are positive constants and $\delta \in (0, 1)$ represents the degree of product differentiation; a smaller δ indicates a larger degree of product differentiation. For $i = 0, 1, 2$, the inverse demand function is given by

$$p_i = \alpha - \beta q_i - \beta\delta \sum_{i \neq j} q_j, \quad (2)$$

where p_i and q_i denote firm i's price and quantity, respectively. From (2), the corresponding direct demand function is given by

$$q_i = \frac{(1-\delta)\alpha - (1+\delta)p_i + \delta \sum_{i \neq j} p_j}{\beta(1-\delta)(1+2\delta)}. \quad (3)$$

The marginal production costs are constant. Let c_0 denote the state-owned public firm 0's marginal cost. To simplify the calculations, we assume that two private firms have the same marginal cost $c_1 = c_2$ with $\alpha > c_0 \geq c_1$. Furthermore, we assume that the equilibrium quantities of public and private firms are strictly positive under both Bertrand and Cournot competition. Let $a_i = \alpha - c_i\,(i = 0, 1)$; this assumption is satisfied if and only if $a_1 - \delta a_0 > 0$ and $(2+\delta)a_0 > 2\delta a_1$.

Since firm 0 is a public firm, its payoff is the social surplus, given by

$$\mathrm{SW} = \sum_{i=0}^{2} (p_i - c_i) q_i + \left(\alpha \sum_{i=0}^{2} q_i - \frac{\beta\left(\sum_{i=0}^{2} q_i^2 + \delta \sum_{i=0}^{2} \sum_{i \neq j} q_i q_j\right)}{2} - \sum_{i=0}^{2} p_i q_i \right). \quad (4)$$

Firm $i\,(i=1,2)$ is a private firm, and its payoff is its own profit: $\pi_i=(p_i-c_i)q_i$.

3. Cournot Competition

3.1. Local Stability of Equilibrium Points. The first-order conditions for public and private firms are, respectively,

$$\frac{\partial SW}{\partial q_0}=a_0-\beta q_0-\beta\delta(q_1+q_2), \tag{5}$$

$$\frac{\partial \pi_i}{\partial q_i}=a_1-2\beta q_i-\beta\delta(q_0+q_j), \quad (i,j=1,2,j\neq i). \tag{6}$$

From (5) and (6), we obtain the following reaction functions for public and private firms:

$$\begin{aligned} q_0&=\frac{a_0-\beta\delta(q_1+q_2)}{\beta},\\ q_1&=\frac{a_1-\beta\delta(q_0+q_2)}{2\beta},\\ q_2&=\frac{a_1-\beta\delta(q_0+q_1)}{2\beta}. \end{aligned} \tag{7}$$

We assume that the public firm 0 uses bounded rationality, hence does not have a complete knowledge of the demand function of the market, and builds its output decision on the basis of the expected marginal payoff $\partial SW/\partial q_0$. If the marginal payoff is positive (negative), it increases (decreases) its production q_0 at the next period output. Then, the dynamical equation of player 0 has the form

$$q_0^{t+1}=q_0^t+kq_0^t\frac{\partial SW}{\partial q_0}, \tag{8}$$

where k is a positive parameter which represents the relative speed adjustment. By using equation (5), the dynamical equation of the boundedly rational player is

$$q_0^{t+1}=q_0^t+kq_0^t\left[a_0-\beta q_0^t-\beta\delta\left(q_1^t+q_2^t\right)\right]. \tag{9}$$

The private firm $i\,(i=1,2)$ is a naive player; he computes his outputs using the reaction function in equation (7); the dynamical equation of player i has the form

$$q_i^{t+1}=\frac{a_1-\beta\delta\left(q_0^t+q_j^t\right)}{2\beta}, \quad (i,j=1,2,j\neq i). \tag{10}$$

Then, the dynamical mixed triopoly game with partial heterogeneous players is given by

$$\begin{cases} q_0^{t+1}=q_0^t+kq_0^t\left[a_0-\beta q_0^t-\beta\delta\left(q_1^t+q_2^t\right)\right],\\ q_1^{t+1}=\dfrac{a_1-\beta\delta\left(q_0^t+q_2^t\right)}{2\beta},\\ q_2^{t+1}=\dfrac{a_1-\beta\delta\left(q_0^t+q_1^t\right)}{2\beta}. \end{cases} \tag{11}$$

To study the qualitative behavior of the solutions of the discrete dynamical system (11), we define the equilibrium points of the mixed triopoly game as a nonnegative fixed point of the dynamical system (11). By setting $q_i^{t+1}=q_i^t\,(i=0,1,2)$ in (11), it is concluded that the dynamical system has two equilibrium points:

$$\begin{aligned} E_0&=\left(0,\frac{a_1}{(2+\delta)\beta},\frac{a_1}{(2+\delta)\beta}\right),\\ E_1&=\left(q_0^*,q_1^*,q_2^*\right), \end{aligned} \tag{12}$$

where

$$\begin{aligned} q_0^*&=\frac{(2+\delta)a_0-2\delta a_1}{\beta\left(2+\delta-2\delta^2\right)},\\ q_1^*&=q_2^*=\frac{a_1-\delta a_0}{\beta\left(2+\delta-2\delta^2\right)}. \end{aligned} \tag{13}$$

Note that E_0 is the boundary equilibrium point and E_1 is the Nash equilibrium point. The local stability of the equilibrium points depends on the eigenvalues of the Jacobian matrix given by

$$J(q_0,q_1,q_2)=\begin{pmatrix} 1+k\left[a_0-2\beta q_0-\beta\delta(q_1+q_2)\right] & -k\beta\delta q_0 & -k\beta\delta q_0\\ -\dfrac{\delta}{2} & 0 & -\dfrac{\delta}{2}\\ -\dfrac{\delta}{2} & -\dfrac{\delta}{2} & 0 \end{pmatrix}. \tag{14}$$

Theorem 1. *The boundary equilibrium point E_0 is a saddle point.*

Proof. At the boundary equilibrium point E_0, the Jacobian matrix takes the form

$$J(E_0)=\begin{pmatrix} 1+k\left(a_0-\dfrac{2\delta a_1}{2+\delta}\right) & 0 & 0 \\ -\dfrac{\delta}{2} & 0 & -\dfrac{\delta}{2} \\ -\dfrac{\delta}{2} & -\dfrac{\delta}{2} & 0 \end{pmatrix}. \quad (15)$$

It is easy to calculate that the three eigenvalues are $\lambda_1 = 1 + k(a_0 - (2\delta a_1/2 + \delta))$, $\lambda_2 = \delta/2$, and $\lambda_3 = -(\delta/2)$. By the condition $(2+\delta)a_0 > 2\delta a_1$, we conclude that $|\lambda_1| > 1$. On the other hand, $\delta \in (0, 1)$ implies that $|\lambda_{2,3}| < 1$. Then, E_0 is a saddle point of the dynamical system.

Next, we discuss the local stability of the Nash equilibrium point. The Jacobian matrix evaluated at E_1 takes the form

$$J(E_1)=\begin{pmatrix} 1-kd & -\delta kd & -\delta kd \\ -\dfrac{\delta}{2} & 0 & -\dfrac{\delta}{2} \\ -\dfrac{\delta}{2} & -\dfrac{\delta}{2} & 0 \end{pmatrix}, \quad (16)$$

where the constant d is determined by

$$d=\frac{(2+\delta)a_0-2\delta a_1}{2+\delta-2\delta^2}>0. \quad (17)$$

It is known that E_1 is asymptotically stable if and only if all the roots of the characteristic equation

$$\left|\lambda E - J(E_1)\right| = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \quad (18)$$

have magnitudes of eigenvalues less than one, in which

$$\begin{aligned} A_1 &= kd-1, \\ A_2 &= -\left(\frac{\delta^2}{4}+\delta^2 kd\right), \\ A_3 &= \frac{\delta^3 kd}{2}-\frac{\delta^2(kd-1)}{4}. \end{aligned} \quad (19)$$

From Schur–Cohn stability criterion, the coefficients of the characteristic polynomial need to satisfy the following conditions [29, 30]:

$$\begin{cases} 1+A_1+A_2+A_3>0, \\ 1-A_1+A_2-A_3>0, \\ 1-A_2+A_1A_3-A_3^2>0, \\ 3-A_2>0. \end{cases} \quad (20)$$

□

Theorem 2. *The Nash equilibrium E_1 is locally asymptotically stable provided that*

$$k<\frac{2\left(2-\delta^2\right)}{\left(2-\delta+2\delta^2\right)d}, \quad (21)$$

where $d>0$ is defined as in equation (17).

Proof. For $k, d, \delta > 0$, it can be calculated that

$$1+A_1+A_2+A_3 = kd\left(1-\frac{5}{4}\delta^2+\frac{\delta^3}{2}\right)>0. \quad (22)$$

Furthermore, the inequality $3 - A_2 > 0$ is obvious by noting that $A_2 < 0$. On the other hand, after some modifications, we obtain

$$1-A_1+A_2-A_3 = 2-\frac{\delta^2}{2}-kd\left(1+\frac{3}{4}\delta^2+\frac{\delta^3}{2}\right). \quad (23)$$

It concludes that the second inequality in (20) holds if

$$k<\frac{2\left(4-\delta^2\right)}{d\left(4+3\delta^2+2\delta^3\right)}=: k_1. \quad (24)$$

Finally, some tedious rearrangements lead to

$$\begin{aligned} &1-A_2+A_1A_3-A_3^2 \\ &= \frac{(2\delta-1)\delta^2}{16}\left(4+\delta^2-2\delta^3\right)d^2k^2+\frac{\delta^2}{8}\left(12-4\delta+\delta^2-2\delta^3\right)dk+1-\frac{\delta^4}{16}=: f(k). \end{aligned} \quad (25)$$

It is easy to see that $f(k)$ is a quadratic equation in k. When $\delta \in ((1/2), 1)$, $f(k) > 0$ for all $\alpha > 0$. In the case of $\delta \in (0, (1/2))$, the discriminant of $f(k)$ is positive; then, $f(k) = 0$ has two real solutions k_2 and k_3. We can easily obtain that one root $k_2 < 0$ and the other root $k_3 > (2(12 - 4\delta + \delta^2 - 2\delta^3)/(1 - 2\delta)(4 - 2\delta^3 + \delta^2)d)$. From these results, the threshold of k ensuring the local asymptotic stability of E_1 is given by $k < k_1$.

From Theorem 2, we can see that the local stability of Nash equilibrium point E_1 largely depends on the adjustment speed of the public firm 0's quantity decision. If the value of k is high (company is overresponsive), the Nash equilibrium will lose stability and dynamical behavior may occur. □

3.2. Numerical Simulations. In this section, the dynamical evolution of the nonlinear dynamical system (11) is numerically simulated, and the dynamical evolution process is visualized by simulation results such as stable region, bifurcation diagram, maximum Lyapunov exponent, strange attractor, and sensitive dependence on initial conditions.

We fixed the parameter values as $a_0 = 2$, $a_1 = 2.5$, and $\beta = 0.5$. The stability region of Nash equilibrium in the (δ, k)-plane is shown in Figure 1(a). It shows that the stability region increases (decreases) as δ increases when δ is smaller (larger) than 0.35. When the parameter k takes values out of the stability region, the equilibrium point loses its stability due to flip bifurcation, which is presented in Figure 1(b).

Figure 2(a) depicts the characteristic behavior of a period-doubling bifurcation, and one can get that the Nash equilibrium point is locally stable provided that $0 < k < 1.238$. Period-2 cycle occurs at the market when $k = 1.238$ as indicated by the two branches. After that, both branches split simultaneously, yielding a period-4 cycle. A cascade of further period doubling occurs when the first firm increases k, yielding period-8, period-16, and so on. The equilibrium undergoes a flip bifurcation at $k = 1.238$, and the system exhibits chaotic behavior while k approaches 1.737.

Figure 2(b) shows the corresponding maximum Lyapunov exponents of system (11), which is a good indicator for bifurcation and chaos. It is observed that the period-doubling bifurcation arises as k reaches the value of $k = 1.238$, and then the system turns into chaos as k increases.

Three-dimensional phase portraits for different values of k are shown in Figure 3, which can give a more detailed description for the orbits of system (11). It shows a flip bifurcation process, where chaos occurs in the end of periods $2, 4, \ldots$, and the strange attractors are shown in the fourth figure.

Figure 4 demonstrates the sensitivity of system (11) to initial conditions, which is one of the main characteristics of chaotic behavior. Figure 4 plots two orbits initially from the slightly deviated points $(p_0^1(0), p_1^1(0), p_2^1(0)) = (2.6924, 1.6346, 1.6346)$ and $(p_0^2(0), p_1^2(0), p_2^2(0)) = (p_0^1(0) + 10^{-4}, p_1^1(0), p_2^1(0))$, respectively. The blue curves represent outputs of the public firm and the red ones represent outputs of the private firm. It is seen clearly that even if the initial production of the public firm alters a little, great impact will emerge in all firms after a series of iterations.

3.3. Chaos Control. The occurrence of chaotic dynamics in the triopoly game is unacceptable and the firms are hoping to find some methods to control chaos to avoid the complexity. Many methods have been used to control chaos in oligopoly games. In the papers [24, 28, 29], it has been presented how the delay feedback control (DFC) method can be applied to control chaos in different economic models. We apply this technique to control the chaotic behavior for the present triopoly game.

We modify the first equation of system (11) by adding the control action $\mu(q_0^t - q_0^{t+1})$, where $\mu > 0$ is the control parameter. Then, the controlled system has the following form:

$$\begin{cases} q_0^{t+1} = q_0^t + kq_0^t\left[a_0 - \beta q_0^t - \beta\delta\left(q_1^t + q_2^t\right)\right] + \mu\left(q_0^t - q_0^{t+1}\right), \\ q_1^{t+1} = \dfrac{a_1 - \beta\delta\left(q_0^t + q_2^t\right)}{2\beta}, \\ q_2^{t+1} = \dfrac{a_1 - \beta\delta\left(q_0^t + q_1^t\right)}{2\beta}. \end{cases} \tag{26}$$

The Jacobian matrix of controlled system (26) is as follows:

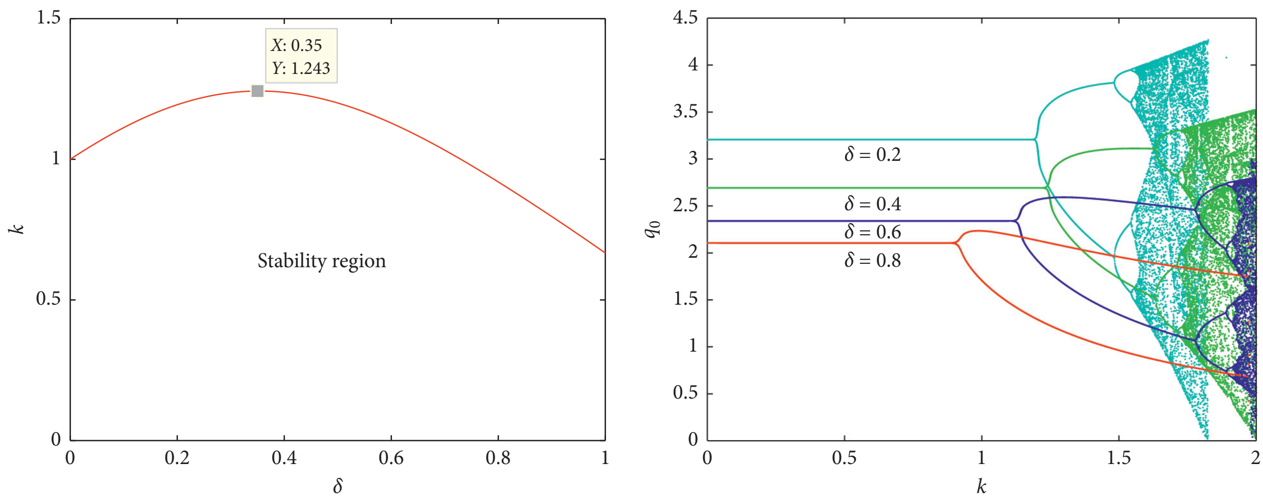
$$J(E_1) = \begin{pmatrix} 1 - \dfrac{kd}{1+\mu} & -\dfrac{\delta kd}{1+\mu} & -\dfrac{\delta kd}{1+\mu} \\ -\dfrac{\delta}{2} & 0 & -\dfrac{\delta}{2} \\ -\dfrac{\delta}{2} & -\dfrac{\delta}{2} & 0 \end{pmatrix}. \tag{27}$$

At the parameter values $(k, \delta, a_0, a_1, \beta) = (2, 0.4, 2, 2.5, 0.5)$, the Jacobian matrix (27) has the form

$$J(E_1) = \begin{pmatrix} \dfrac{1 + \mu - 2.6923}{1+\mu} & -\dfrac{1.0769}{1+\mu} & -\dfrac{1.0769}{1+\mu} \\ -0.2 & 0 & -0.2 \\ -0.2 & -0.2 & 0 \end{pmatrix}. \tag{28}$$

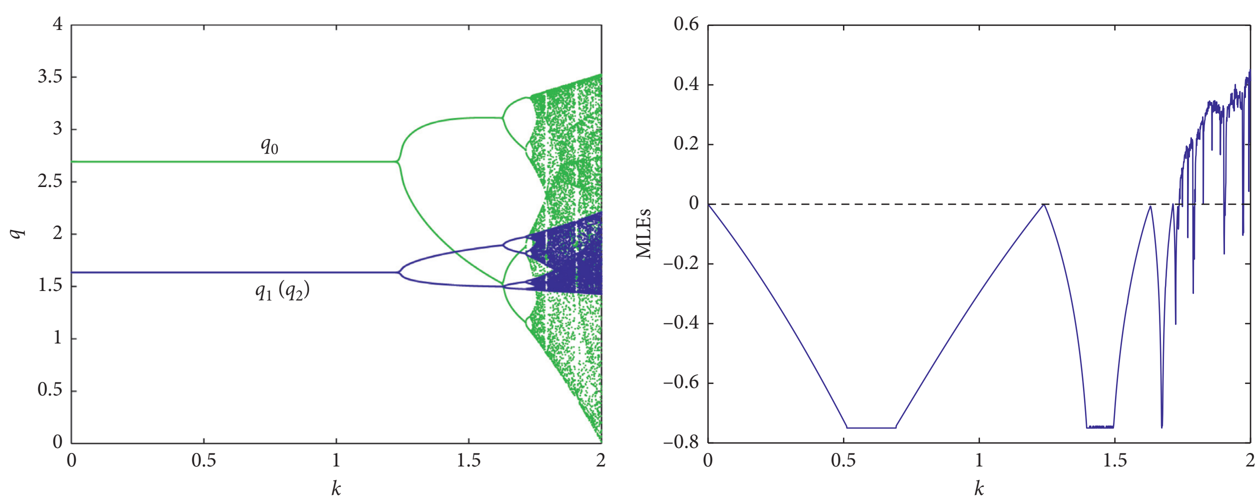
Applying Schur–Cohn conditions, matrix (28) has eigenvalues with an absolute less than one when $\mu > 0.6154$. It means that the system is stable around the Nash equilibrium point under the condition of $\mu > 0.6154$. From Figure 5, one can see that with the increase of the control parameter μ, system (26) gradually tends to be stable from chaos and quasi-period. When $\mu > 0.6154$, system (26) finally evolves to be stable.

Figure 6 depicts the behavior of controlled system (26) for $\mu = 0.1$ and $\mu = 0.8$, respectively, with initial values $(q_0(0), q_1(0), q_2(0)) = (2.7, 1.5, 1.5)$. From the two graphs, we can find that as the feedback strength value increases, the chaotic behavior can be quickly controlled to stable orbit. Therefore, the above control method is able to control chaos in the game when μ is large enough, and the market game can switch from chaotic trajectories to an equilibrium state.



(a) (b)

FIGURE 1: (a) The stability regions in the c-plane of Nash equilibrium point E_1 for system (11). (b) Bifurcation diagrams for system (11) with respect to the parameter k with the variable q_0 with various values of δ.



(a) (b)

FIGURE 2: Bifurcation diagram and MLEs for system (11) with respect to parameter k.

4. Bertrand Competition

4.1. Local Stability of Equilibrium Points. In this section, we discuss the Bertrand model in which all firms choose prices. From equation (3), the first-order conditions for public and private firms are, respectively,

$$\frac{\partial SW}{\partial p_0} = \frac{(1+\delta)(c_0 - p_0) + \delta(p_1 + p_2 - 2c_1)}{\beta(1-\delta)(1+2\delta)} = 0,$$
$$\frac{\partial \pi_i}{\partial p_i} = \frac{(1-\delta)\alpha + (1+\delta)(c_i - 2p_i) + \delta\sum_{i\neq j} p_j}{\beta(1-\delta)(1+2\delta)} = 0, \quad i \neq 0. \tag{29}$$

From the first-order conditions, we obtain the following reaction functions:

$$p_0 = \frac{(1+\delta)c_0 + \delta(p_1 + p_2 - 2c_1)}{1+\delta},$$
$$p_i = \frac{(1-\delta)\alpha + \delta\sum_{i\neq j} p_j + (1+\delta)c_1}{2(1+\delta)}, \quad i \neq 0. \tag{30}$$

Now, suppose that the decision rule of the public firm 0 is based on marginal profit, and the decision rule of the private firms follows the expected value of naive. Therefore, how to adjust the price of firm 0 mainly depends on whether the current marginal profit is positive or negative in the $t+1$-th period. Suppose that the public firm adjusts the decision mechanism over time as follows:

$$p_0^{t+1} = p_0^t + \rho p_0^t \frac{(1+\delta)(c_0 - p_0^t) + \delta(p_1 + p_2 - 2c_1)}{\beta(1-\delta)(1+2\delta)}, \tag{31}$$

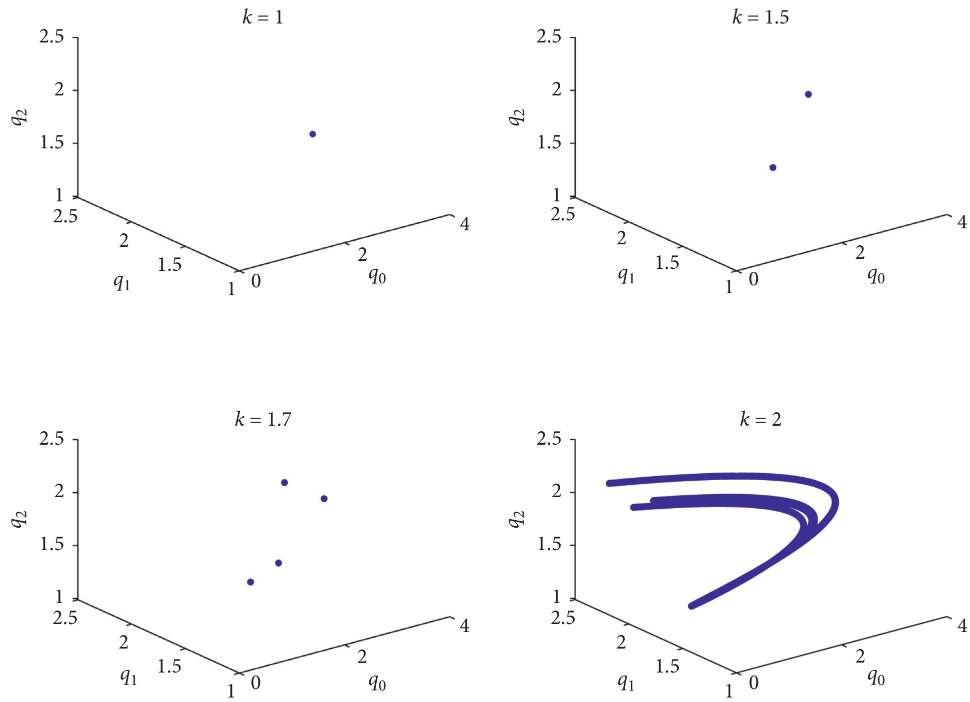


Figure 3: Phase portraits for Figure 2 with various values of k.

where $\rho > 0$ indicates the speed at which firm 0 adjusts the price based on marginal profit. Suppose that both private firms adopt the naive expected value decision rule to adjust the product price; it can be expressed in the following form:

$$p_i^{t+1} = \frac{(1-\delta)\alpha + \delta \sum_{i \neq j} p_j^t + (1+\delta)c_1}{2(1+\delta)}, \quad i \neq 0. \tag{32}$$

From equations (31) and (32), we can obtain the following three-dimensional dynamical system based on price competition:

$$\begin{cases} p_0^{t+1} = p_0^t + \rho p_0^t \dfrac{(1+\delta)\left(c_0 - p_0^t\right) + \delta\left(p_1^t + p_2^t - 2c_1\right)}{\beta(1-\delta)(1+2\delta)}, \\ p_1^{t+1} = \dfrac{(1-\delta)\alpha + \delta\left(p_0^t + p_2^t\right) + (1+\delta)c_1}{2(1+\delta)}, \\ p_2^{t+1} = \dfrac{(1-\delta)\alpha + \delta\left(p_0^t + p_1^t\right) + (1+\delta)c_1}{2(1+\delta)}. \end{cases} \tag{33}$$

By letting $p_i^{t+1} = p_i^t$ in (18) for $i = 0, 1, 2$, we have

$$\begin{cases} \rho p_0^t \dfrac{(1+\delta)(c_0 - p_0) + \delta\left(p_1^t + p_2^t - 2c_1\right)}{\beta(1-\delta)(1+2\delta)} = 0, \\ \dfrac{(1-\delta)\alpha + \delta\left(p_0^t + p_2^t\right) + (1+\delta)c_1}{2(1+\delta)} = 0, \\ \dfrac{(1-\delta)\alpha + \delta\left(p_0^t + p_1^t\right) + (1+\delta)c_1}{2(1+\delta)} = 0. \end{cases} \tag{34}$$

By solving (34), we get two equilibrium points for dynamical system (33) as follows:

$$\begin{aligned} F_0 &= \left(0, \frac{(1-\delta)a_1 + 2c_1}{2+\delta}, \frac{(1-\delta)a_1 + 2c_1}{2+\delta}\right), \\ F_1 &= \left(p_0^*, p_1^*, p_2^*\right), \end{aligned} \tag{35}$$

where $p_0^* = ((1+\delta)(2+\delta)c_0 + 2\delta(a_1 - \delta\alpha)/2 + 3\delta - \delta^2)$, $p_1^* = p_2^* = ((1 + 2\delta - \delta^2)c_1 + \delta(1+\delta)c_0 + (1-\delta^2)\alpha/2 + 3\delta - \delta^2)$.

Now, we shall discuss the dynamical behavior of system (33) around F_0 and F_1. As discussed in Section 3.1, we return to the three-dimensional map (33). Let $J(p_0, p_1, p_2)$ be the Jacobian matrix of system (33) corresponding to the state variables (p_0, p_1, p_2); then, we have

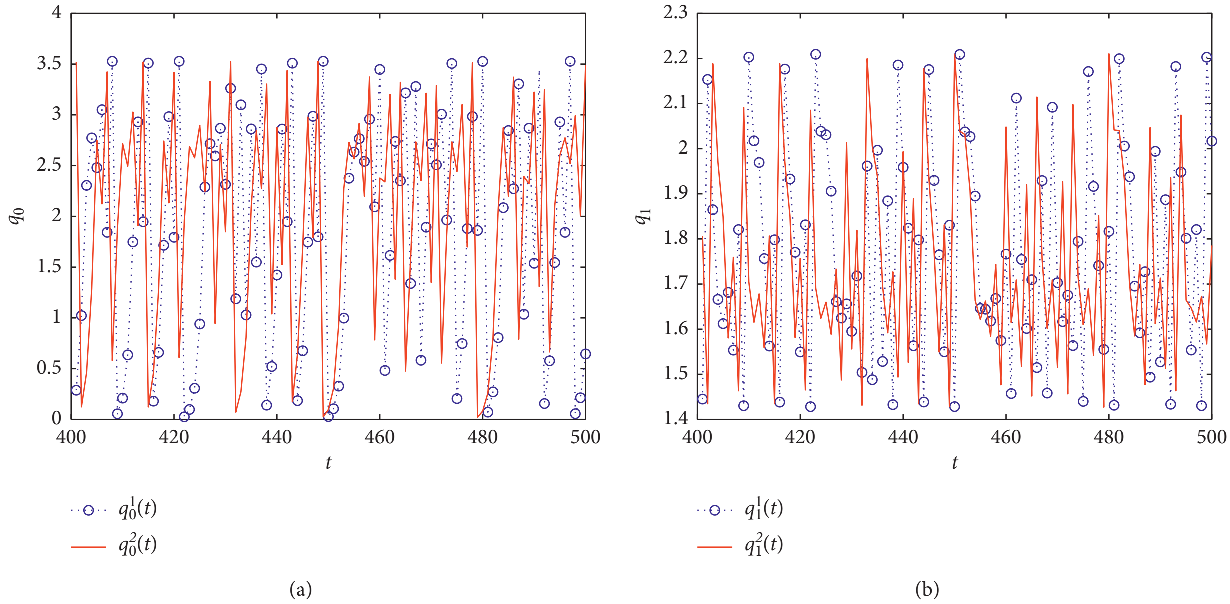


FIGURE 4: Sensitive dependence on initial conditions for system (11) in the time period [400, 500] when $k = 2$.

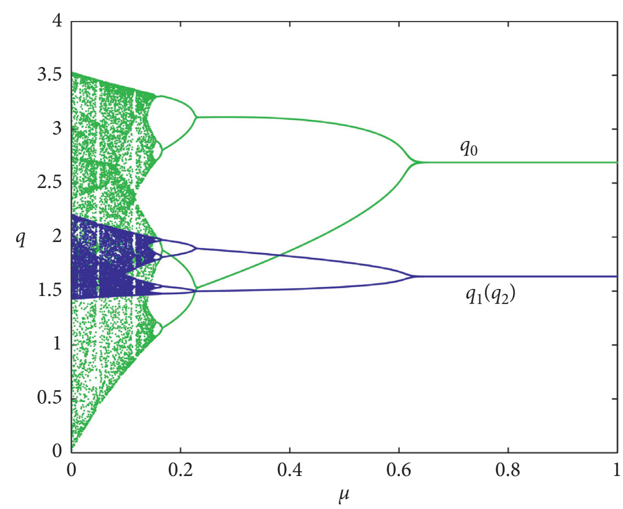


FIGURE 5: Bifurcation diagram for system (26) with control parameter μ.

$$J(p_0, p_1, p_2) = \begin{pmatrix} 1 + \rho \dfrac{(1+\delta)c_0 - 2(1+\delta)p_0 + \delta(p_1 + p_2) - 2\delta c_1}{\beta(1-\delta)(1+2\delta)} & \dfrac{\rho\delta p_0}{\beta(1-\delta)(1+2\delta)} & \dfrac{\rho\delta p_0}{\beta(1-\delta)(1+2\delta)} \\ \dfrac{\delta}{2(1+\delta)} & 0 & \dfrac{\delta}{2(1+\delta)} \\ \dfrac{\delta}{2(1+\delta)} & \dfrac{\delta}{2(1+\delta)} & 0 \end{pmatrix}. \quad (36)$$

Next, the stability of the boundary equilibrium point F_0 of the complex dynamical system (33) will be given in Theorem 3.

Theorem 3. *The boundary equilibrium point F_0 is a saddle point.*

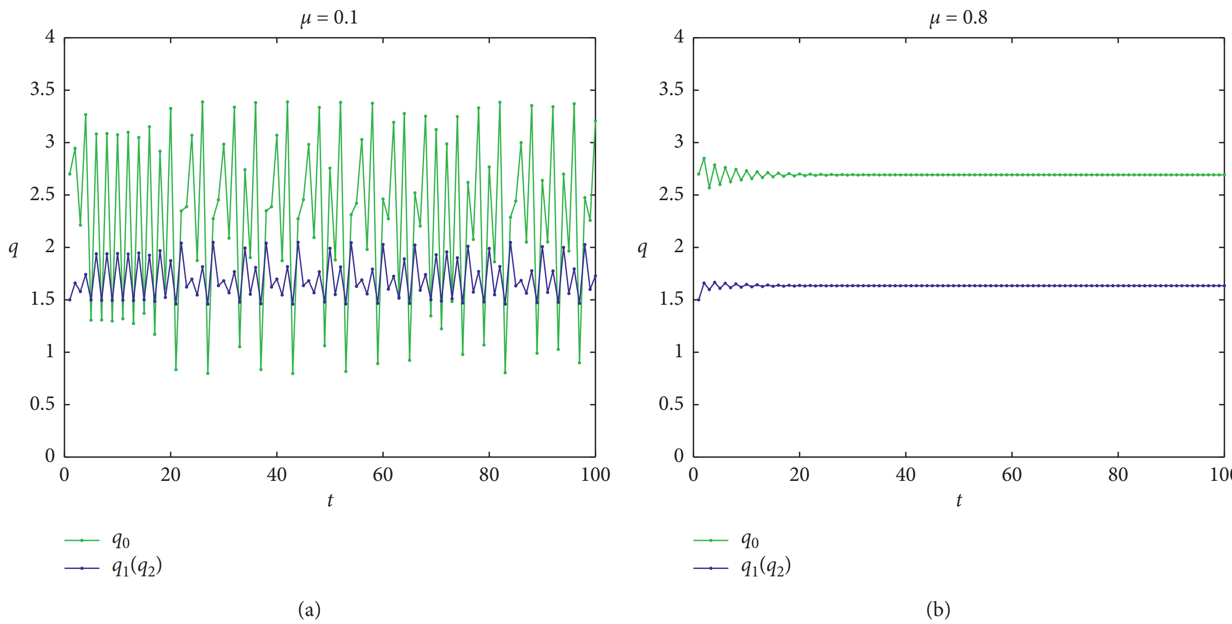


Figure 6: Revolution trajectories for system (26) when $\mu = 0.1$ and $\mu = 0.8$.

Proof. At the boundary equilibrium point F_0, the Jacobian matrix takes the form

$$J(F_0) = \begin{pmatrix} 1+\rho\dfrac{(1+\delta)(2+\delta)c_0 + 2\delta(1-\delta)a_1 - 2\delta^2 c_1}{\beta(1-\delta)(1+2\delta)} & 0 & 0 \\ \dfrac{\delta}{2(1+\delta)} & 0 & \dfrac{\delta}{2(1+\delta)} \\ \dfrac{\delta}{2(1+\delta)} & \dfrac{\delta}{2(1+\delta)} & 0 \end{pmatrix}. \quad (37)$$

The three eigenvalues can be calculated as $\lambda_1 = 1 + \rho((1+\delta)(2+\delta)c_0 + 2\delta(1-\delta)a_1 - 2\delta^2 c_1/\beta(1-\delta)(1+2\delta))$, $\lambda_2 = (\delta/2(1+\delta))$, and $\lambda_3 = -(\delta/2(1+\delta))$. By the condition $c_0 \geq c_1$, we conclude that $|\lambda_1| > 1$. On the other hand, $\delta \in (0,1)$ implies that $|\lambda_{2,3}| < 1$.

Now we consider the stability properties of F_1. The Jacobian matrix at F_1 takes the form

$$J(F_1) = \begin{pmatrix} 1-(1+\delta)b\rho & \delta b\rho & \delta b\rho \\ c & 0 & c \\ c & c & 0 \end{pmatrix}, \quad (38)$$

where $b = (p_0^*/\beta(1+2\delta)(1-\delta))$, $c = (\delta/2(1+\delta))$. The necessary and sufficient conditions for F_1 to be asymptotically stable are all eigenvalues of the Jacobian matrix $J(F_1)$ located in the unit circle of the complex plane. The characteristic equation of $J(F_1)$ is as follows:

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0, \quad (39)$$

where the coefficients are $B_1 = (1+\delta)b\rho - 1$, $B_2 = -(c^2 + 2\delta b\rho c)$, $B_3 = c^2[1-(1+3\delta)b\rho]$.

Again, according to Schur–Cohn stability criterion, we need to verify the four inequalities discussed as in (20). First, for $\delta \in (0,1)$ and $b, \rho > 0$, we obtain

$$1 + B_1 + B_2 + B_3 = \frac{4 + 12\delta + 7\delta^2 - 3\delta^3}{4(1+\delta)^2} b\rho > 0. \quad (40)$$

The inequality $3 - B_2 > 0$ is obvious. After some careful calculations, it is deduced that the inequality

$$1 - B_1 + B_2 - B_3 > 0 \quad (41)$$

holds if public firm 0's adjustment speed ρ satisfies

$$\rho < \frac{2(4 + 8\delta + 3\delta^2)}{b(4 + 12\delta + 15\delta^2 + 5\delta^3)} =: \rho_1. \quad (42)$$

Finally, we consider a quadratic equation in ρ defined by

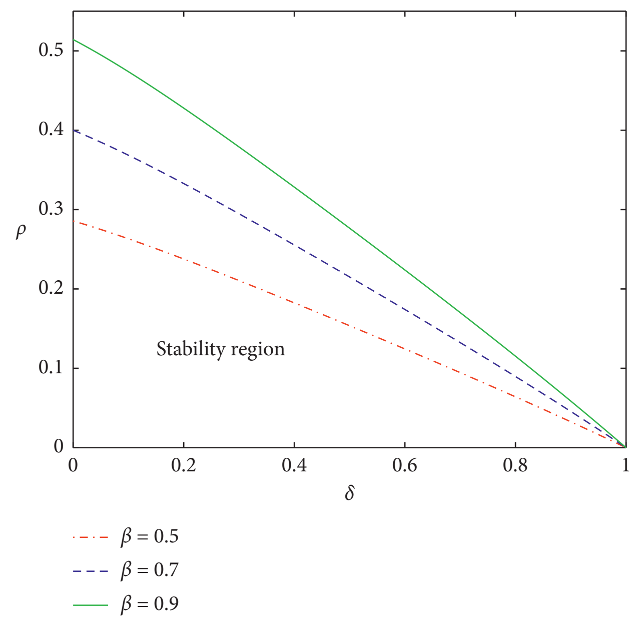


FIGURE 7: The stability region in the (δ, ρ)-plane of Nash equilibrium point E_1 for system (33).

$$\begin{aligned} &1 - B_2 + B_1 B_2 - B_3^2 \\ &= -\left[(1+\delta)(1+3\delta) + (1+3\delta)^2 c^2\right] b^2 c^2 \rho^2 + \left[2\delta + 2(1+2\delta)c + 2(1+3\delta)c^3\right] cb\rho + 1 - c^4 =: g(\rho). \end{aligned} \tag{43}$$

For $\delta \in (0, 1)$, it is not hard to see that the discriminant of $g(\rho)$ is positive and $g(\rho) = 0$ has two roots with $\rho_2 < 0$ and

$$\rho_3 > \frac{2\left(12 + 40\delta + 45\delta^2 + 19\delta^3\right)}{(1+3\delta)\left(4 + 12\delta + 13\delta^2 + 7\delta^3\right)b} =: \rho_4. \tag{44}$$

Comparing the thresholds ρ_1 and ρ_4 implies that $\rho_1 < \rho_4$. Therefore, we eventually obtain the following result. □

Theorem 4. *The Nash equilibrium F_1 is locally asymptotically stable provided that*

$$\rho < \frac{2\left(4 + 8\delta + 3\delta^2\right)}{b\left(4 + 12\delta + 15\delta^2 + 5\delta^3\right)}, \tag{45}$$

where the constant $b > 0$ is determined by

$$b = \frac{(1+\delta)(2+\delta)c_0 + 2\delta(a_1 - \delta\alpha)}{\beta(1+2\delta)(1-\delta)\left(2 + 3\delta - \delta^2\right)}. \tag{46}$$

As expected, the local stability of the Nash equilibrium in Bertrand game is determined by the adjustment speed of gradient player. If the value of ρ is high (company is overresponsive), the complex dynamical behavior may occur. Moreover, different from the quantity competition case, the threshold of the adjustment speed depends on an additional parameter β. The greater the value of β is, the higher the stability of the Nash equilibrium is (see Figure 7 for details).

4.2. Numerical Simulations. In this section, we provide some numerical evidence for the chaotic behavior of the dynamical game (33). To study the local stability properties of the Nash equilibrium point, it is convenient to consider the following set of parameters: $\alpha = 5.5$, $c_0 = 3.5$, and $c_1 = c_2 = 3$.

According to equation (42), we know that the stability region in the (δ, ρ) plane changes as δ varies. For fixed β, the stability region becomes small with δ increasing. Moreover, it is deduced that the greater the value of β is, the larger the stability of the Nash equilibrium is, regardless of the value of the parameter δ.

Figures 8(a) and 8(b) display the bifurcation diagram with respect to ρ and the corresponding maximum Lyapunov exponents of system (33), respectively. For values of ρ lower than 0.182, the Nash equilibrium point F_1 is locally stable. As ρ increases, system (33) becomes unstable and complex dynamical behavior occurs, including period bifurcations and chaos. This means that for large value of speed of adjustment of boundedly rational player, dynamical game (33) converges always to complex dynamics.

A more specific representation is shown in Figure 9. Figure 9 plots the price evolution with time when the system is in the stable region, 2,4-period bifurcations, and chaotic state, respectively. When $\rho = 0.15$, the price of all firms will be stable in equilibrium state gradually after a series of fluctuations. When $\rho = 0.27$, the complex behavior can be observed and the system is in chaos.

The three-dimensional chaotic attractors are given in Figure 10, which exhibit fractal structure. When all parameters are kept fixed and only the adjustment speed of public firm varies, one can see that the structure of the triopoly game becomes complicated through period-

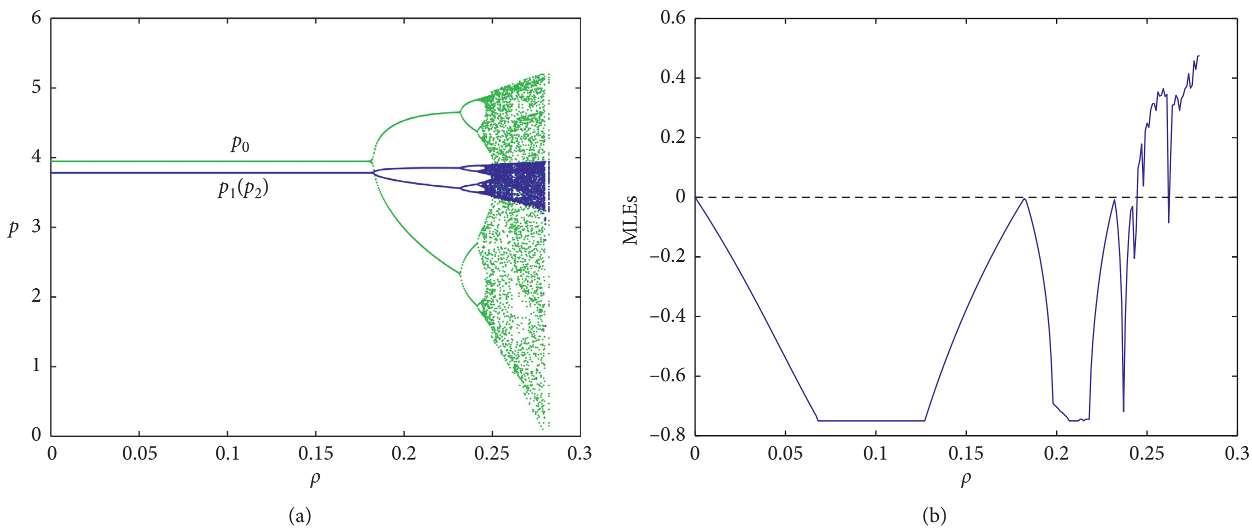


FIGURE 8: Bifurcation diagram and MLEs for system (33) with respect to parameter ρ.

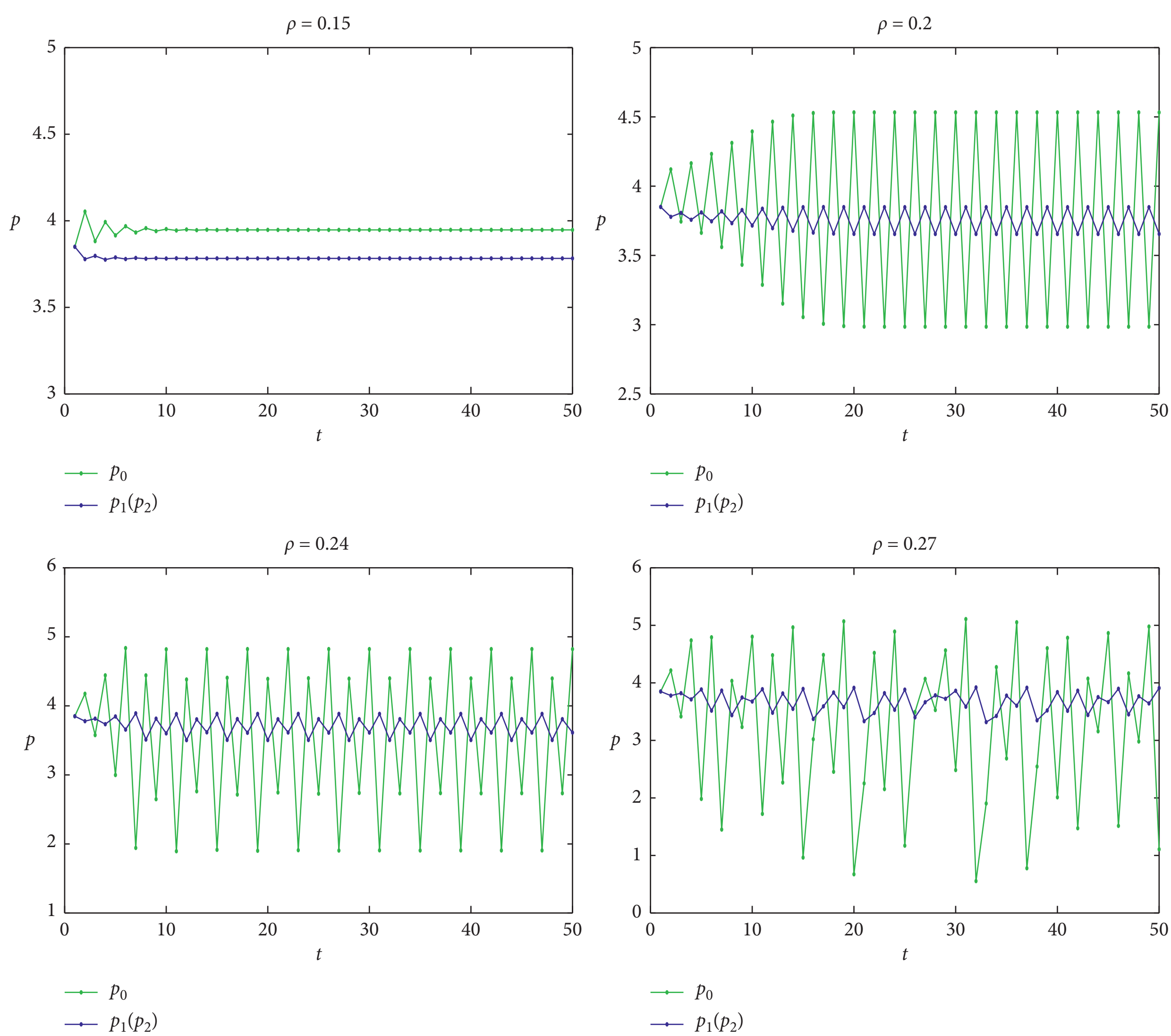


FIGURE 9: Evolutions for system (33) with various values of ρ.

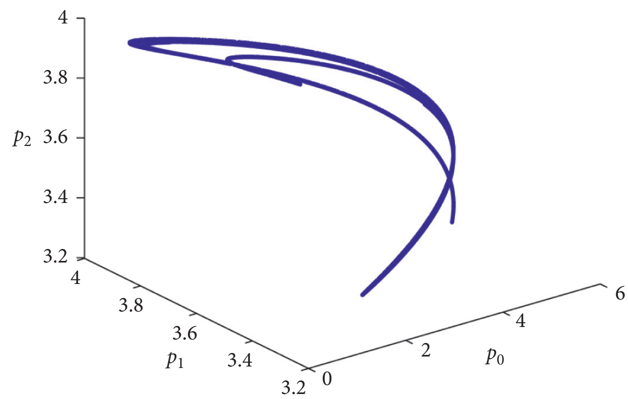


Figure 10: Strange attractor when $\rho = 0.27$.

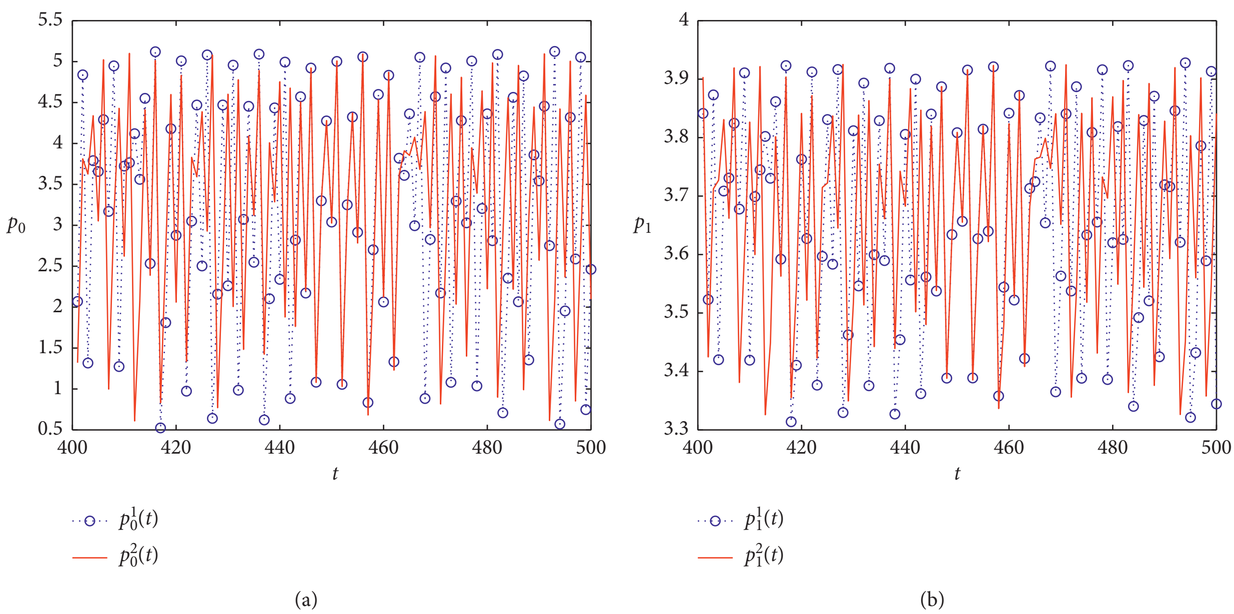


Figure 11: Sensitive dependence on initial conditions for system (33) in the time period [400, 500] when $\rho = 0.27$.

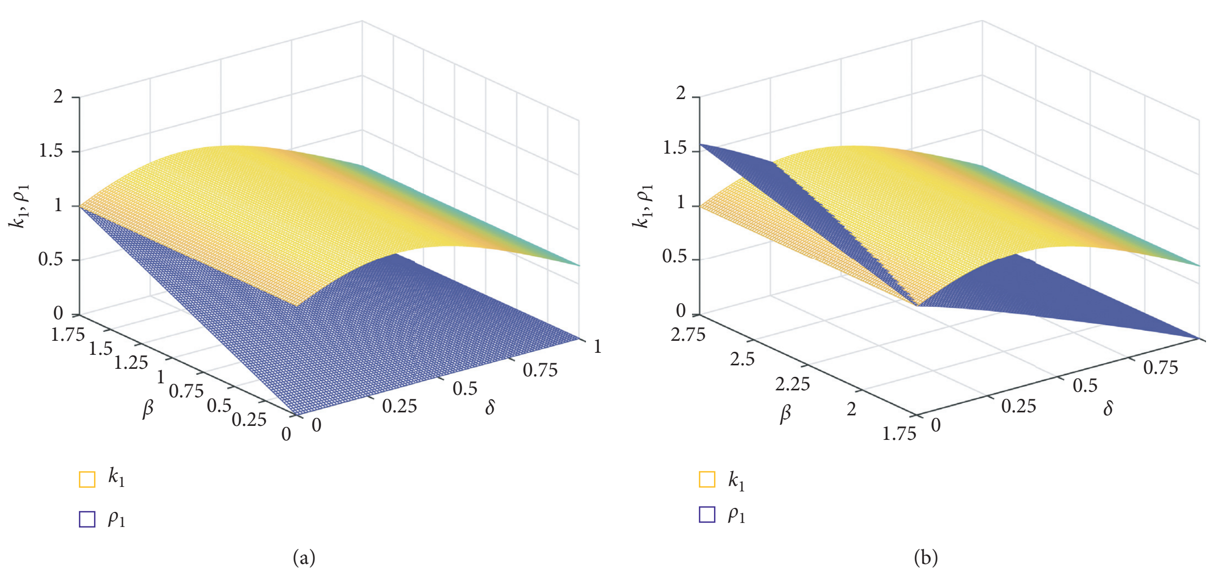
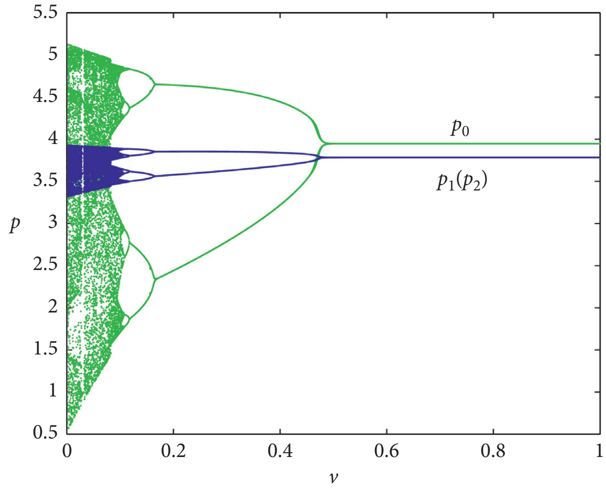
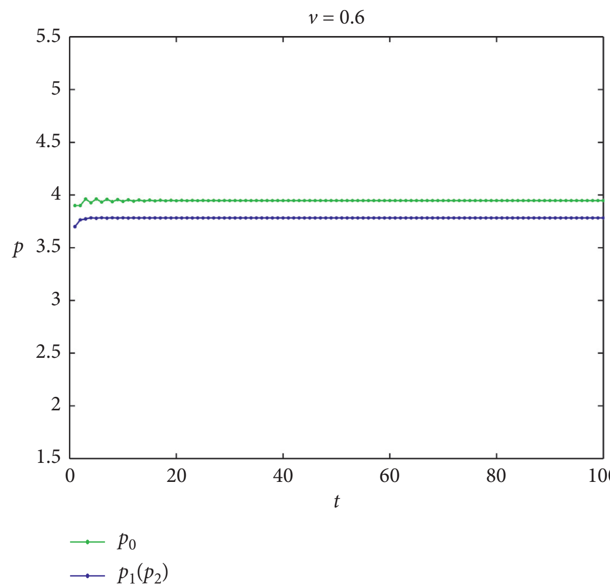


Figure 12: Comparison between thresholds k_1 and ρ_1. (a) For $\beta \in (0, 1.75)$. (b) For $\beta \in (1.75, 2.75)$.



(a)



(b)

FIGURE 13: (a) Bifurcation diagram for system (47) with control parameter ν. (b) Revolution trajectories for system (47) when $\nu = 0.6$.

doubling bifurcations. In addition, more complex bounded attractors are created around Nash equilibrium point, which are periodic cycles of high order or chaotic attractors.

Figures 11(a) and 11(b) show the two orbits of public firm's price and private firm's price, respectively, where the blue ones start from the initial point $(p_0^1(0), p_1^1(0), p_2^1(0)) = (3.9474, 3.7829, 3.7829)$ and the red ones start from the initial point $(p_0^2(0), p_1^2(0), p_2^2(0)) = (3.9475, 3.7829, 3.7829)$. From the two figures, it is clear that the time series are indistinguishable at the beginning. However, after a number of iterations, the difference between them builds up rapidly.

The comparisons between the thresholds k_1 and ρ_1 are depicted in Figure 12. It is shown that the value of the adjustment speed where the Nash equilibrium loses its stability is lower under Bertrand competition than under Cournot competition for small β. More precisely, $k_1 \leq \rho_1$ for all $\delta \in (0, 1)$ if $\beta \in (0, 1.75]$ (see Figure 12(a)). As stated in [30], it can be understood as the consequences of the stability of a fiercer competition under Bertrand competition than under Cournot competition regardless of the degree of product differentiation for smaller β. It is more complicated in the case of $\beta > 1.75$; Figure 12(b) shows us the relationship between the two critical values for $\beta \in (1.75, 2.75)$.

4.3. Chaos Control. As discussed in Section 3.3, by adding the control action $\nu(p_0^t - p_0^{t+1})$ to system (33), the controlled system has the following form:

$$\begin{cases} p_0^{t+1} = p_0^t + \rho p_0^t \dfrac{(1+\delta)\left(c_0 - p_0^t\right) + \delta\left(p_1^t + p_2^t - 2c_1\right)}{\beta(1-\delta)(1+2\delta)} + \nu\left(q_0^t - q_0^{t+1}\right), \\ p_1^{t+1} = \dfrac{(1-\delta)\alpha + \delta\left(p_0^t + p_2^t\right) + (1+\delta)c_1}{2(1+\delta)}, \\ p_2^{t+1} = \dfrac{(1-\delta)\alpha + \delta\left(p_0^t + p_1^t\right) + (1+\delta)c_1}{2(1+\delta)}. \end{cases} \tag{47}$$

The Jacobian matrix of the controlled system (47) is as follows:

$$J(F_1) = \begin{pmatrix} 1 - \frac{(1+\delta)b\rho}{1+\nu} & \frac{\delta b\rho}{1+\nu} & \frac{\delta b\rho}{1+\nu} \\ c & 0 & c \\ c & c & 0 \end{pmatrix}. \quad (48)$$

At the parameter values $(\rho, \delta, \alpha, c_0, c_1, \beta) = (0.27, 0.4, 5.5, 3.5, 3, 0.5)$, the Jacobian matrix (48) takes the form

$$J(F_1) = \begin{pmatrix} \frac{1+\nu-2.7632}{1+\nu} & \frac{0.7895}{1+\nu} & \frac{0.7895}{1+\nu} \\ 0.1429 & 0 & 0.1429 \\ 0.1429 & 0.1429 & 0 \end{pmatrix}. \quad (49)$$

Again, using Schur–Cohn conditions, the matrix (49) has eigenvalues with an absolute less than one when $\nu > 0.4803$. From Figure 13(a), one can see that the system is controlled from a chaotic state to a stable state when $\nu > 0.4803$. Figure 13(b) shows the behavior of the controlled system for $\nu = 0.6$, starting from initial values $(p_0(0), p_1(0), p_2(0)) = (3.9, 3.7, 3.7)$. One can conclude that a controlled behavior converges to the fixed point when ν is large enough.

5. Conclusion

In this paper, we study the dynamics in a triopoly game with product differentiation, in which a state-owned public firm competes with two private firms. We investigate the local stability of the equilibrium points under both quantity and price competition. Numerical simulations of the two kinds of games have been analyzed by means of period-doubling bifurcations, strange attractors, maximal Lyapunov exponent, and sensitive dependence on initial conditions. We have stabilized the chaotic behavior of the models to stable fixed points by the delay feedback control method. We find that the boundedly rational player accelerates the adjustment speed of the output quantity (price), and it leads to the instability of the system and makes the system go to a chaotic region. The comparison between Cournot and Bertrand competition implies that the Nash equilibrium is more stable in a quantity setting for smaller β.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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