Hindawi
Discrete Dynamics in Nature and Society
Volume 2021, Article ID 9512371, 15 pages
https://doi.org/10.1155/2021/9512371

Research Article

Mathematical Analysis of the TB Model with Treatment via Caputo-Type Fractional Derivative

Xiao-Hong Zhang,[1] Aatif Ali,[2] Muhammad Altaf Khan,[3] Mohammad Y. Alshahrani,[4] Taseer Muhammad,[5] and Saeed Islam[2]

[1]*College of Science, Hunan City University, Yiyang 413000, China*
[2]*Department of Mathematics, Abdul Wali Khan University, Mardan 23200, Khyber Pakhtunkhwa, Pakistan*
[3]*Institute for Ground Water Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, Bloemfontein, South Africa*
[4]*Department of Clinical Laboratory Sciences, College of Applied Medical Sciences, King Khalid University, P.O. Box 61413, Abha 9088, Saudi Arabia*
[5]*Department of Mathematics, College of Sciences, King Khalid University, Abha 61413, Saudi Arabia*

Correspondence should be addressed to Muhammad Altaf Khan; altafdir@gmail.com

Received 3 June 2021; Accepted 27 October 2021; Published 16 November 2021

Academic Editor: Juan Luis García Guirao

In this study, we formulate a noninteger-order mathematical model via the Caputo operator for the transmission dynamics of the bacterial disease tuberculosis (TB) in Khyber Pakhtunkhwa (KP), Pakistan. The number of confirmed cases from 2002 to 2017 is considered as incidence data for the estimation of parameters or to parameterize the model and analysis. The positivity and boundedness of the model solution are derived. For the dynamics of the tuberculosis model, we find the equilibrium points and the basic reproduction number. The proposed model is locally and globally stable at disease-free equilibrium, if the reproduction number $\mathscr{R}_0 < 1$. Furthermore, to examine the behavior of the various parameters and different values of fractional-order derivative graphically, the most common iterative scheme based on fundamental theorem and Lagrange interpolation polynomial is implemented. From the numerical result, it is observed that the contact rate and treatment rate have a great impact on curtailing the tuberculosis disease. Furthermore, proper treatment is a key factor in reducing the TB transmission and prevalence. Also, the results are more precise for lower fractional order. The results from various numerical plots show that the fractional model gives more insights into the disease dynamics and on how to curtail the disease spread.

1. Introduction

Tuberculosis (TB) usually caused by a bacterium called *Mycobacterium tuberculosis* bacterium (MTB) is a contagious infectious disease. This life-threatening disease is still imposing an alarming situation and health challenge across the globe, specifically for developing countries. TB is listed in the topmost death-causing diseases due to the high rate of mortality. Tuberculosis spread from MTB infects healthy people through the air, when they sneeze, spit, speak, or cough. Common symptoms of this disease are high fever with chills, chronic cough, night sweats, nail clubbing, weight loss, and fatigue [1]. The total number of people who died from TB in the year 2019 is 1.4 million, among which 208000 were reported HIV-positive. According to the WHO, around 10 million TB cases were estimated worldwide, in which over 95% has occurred in developing countries [2].

Several mathematical models for the transmission dynamics and on how to curtail the disease were developed. In the study of infectious diseases, the mathematical models play a key role and provide helpful information on how to control the spread and disease. There are a lot of models developed by researchers for TB dynamics. In 1962, a TB model was proposed by Waaler et al. [3], and Revelle et al.

formulated a TB disease compartmental model [4]. Liu and Zhang provided a TB model and discussed the effect of vaccination and treatment [5]. Liu et al. formulated another model and used statistical data of TB to check the seasonal effects [6]. The rabies transmission and control analysis were proposed in [7]. Wallis explored a TB model with reactivation, while Kim et al. presented a model to reduce the spread of TB with optimal control strategies in the Philippines [8, 9]. Recently, a TB model with slow and fast exposed classes using real data of Pakistan was proposed by Khan et al. [10].

All the abovementioned models are formulated via integer-order derivatives. These classical models do not provide information about the learning mechanism and memory effect and thus have some limitations, while fractional models provide more compatible and realistic results. Many fractional-order derivatives were proposed in [11, 12] and have a wide range of applications in the fields of epidemiology, physics, engineering, fluid dynamics, and many others [13–16]. The fractional TB mathematical model for children and adults is investigated in [17]. In recent years, fractional-order models gained much attention due to vast applications. In biological systems, fractional models are used for better understanding of the dynamics of various diseases. Recently, in the current situation of the pandemic, many fractional-order COVID-19 models and dengue transmission models were developed in [18–20]. The fractional calculus using the series approach of the type (p, q)-Mathieu-type series has been suggested in [21].

Tuberculosis disease is a main cause of mortality and morbidity and so is a massive health challenge in Pakistan. Pakistan is at the fifth position in the list of high-burden TB countries [22]. In Pakistan, the current incidence rate is more than 0.5 million and more than 50,000 die annually [23, 24]. TB is considered as a massive burden in the province Khyber Pakhtunkhwa (KP), Pakistan. A report issued by the National TB Control Programme showed that an estimated total of 462920 new cases were reported and treated in KP, Pakistan, from 2002 to 2017.

Motivated by the abovementioned work and the previous literature in view, we study the dynamical TB model with standard incidence rate explored in [10] by considering Caputo fractional derivatives for more insights into the disease. We also used the real data from year 2002 to year 2017 of Khyber Pakhtunkhwa (KP) for the parametrization of the model [25]. The remaining work is organized as follows: Section 2 contains preliminaries, and model description is given in Section 3. The analysis of the model and estimation of parameters are given in Section 4, and Section 5 contains the numerical simulation. Finally, the study is concluded in Section 6.

2. Fractional Basics Concepts

The basic definitions are presented related to fractional calculus.

Definition 1. The derivative for the function $w \in C^p$ having order α in the Caputo operator is defined as [26]

$$ {}^{C}D_t^{\alpha}(w(t)) = \frac{1}{\Gamma(p-\alpha)} \int_0^t w^p(\varepsilon)(t-\varepsilon)^{p-\alpha-1} \mathrm{d}\varepsilon. \tag{1} $$

Definition 2. For the given function w: $R^+ \longrightarrow R$ having fractional order $\alpha > 0$, the fractional integral is

$$ I_t^{\alpha}(w(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varepsilon)^{\alpha-1} w(\varepsilon) \mathrm{d}\varepsilon, \tag{2} $$

where $\Gamma(.)$ denotes the gamma function.

Definition 3. For the Caputo fractional dynamical system, let (p^*) be the equilibrium point. Then,

$$ {}^{C}D_t^{\alpha}(p(t)) = w(t, p(t)), \quad \alpha \in (0, 1), \tag{3} $$

iff $w(t, p^*) = 0$.

3. Description of the Model

To analyze the transmission dynamics of TB disease, we consider the tuberculosis model proposed in [10] for more insight into the disease dynamics studied via the Caputo fractional operator. The model is divided into six compartments: the susceptible compartment $S(t)$, slow and fast exposed compartments $E_1(t)$ and $E_2(t)$, and $I(t)$, $T(t)$, and $R(t)$ representing the infected, treated, and recovered compartments, respectively. When the susceptible person after getting infection by interacting with infected individual remains in the incubation period, then it is due to the nature of infection that the individual goes to fast or slow exposed class. We assumed that the slow exposed individuals before entering into the infected compartment must join the fast exposed individuals. Then, the sum of all compartments is $\mathbb{N}(t)$, that is,

$$ \mathbb{N}(t) = S(t) + E_1(t) + E_2(t) + I(t) + T(t) + R(t). \tag{4} $$

The nonlinear fractional differential system governed by these assumptions is described as follows [10]:

$$ \begin{cases} {}^{C}D_t^{\alpha} S = \Delta - \dfrac{\mathrm{BSI}}{N} - \nu S, \\ {}^{C}D_t^{\alpha} E_1 = \omega \dfrac{\mathrm{BSI}}{N} - (\nu + \varsigma_1) E_1 + (1-\rho)\delta T, \\ {}^{C}D_t^{\alpha} E_2 = (1-\omega) \dfrac{\mathrm{BSI}}{N} + \varsigma_1 E_1 - (\nu + \varsigma_2) E_2, \\ {}^{C}D_t^{\alpha} I = \varsigma_2 E_2 + \rho\delta T - (\nu + \gamma + \sigma_1) I, \\ {}^{C}D_t^{\alpha} T = \gamma I - (\nu + \delta + \sigma_2 + \phi) T, \\ {}^{C}D_t^{\alpha} R = \phi T - \nu R. \end{cases} \tag{5} $$

In model (5), after interaction between susceptible and infected persons, a ratio $\omega (0 < \omega < 1)$ of susceptible individuals $S(t)$ joins $E_1(t)$ and a fraction $(1-\omega)$ enters into fast exposed class E_2 directly. Δ and ν denote the birth and death

rates, respectively, while β represents the successful transmission coefficient. The compartments I- and T-induced rates due to disease are given by σ_1 and σ_2, respectively; the rate of progression from E_1 to E_2 is ς_1 and from E_2 to I is ς_2. For the infected individuals, the per capita treatment rate is represented by γ and δ is the rate, where individuals quit the $T(t)$ compartment due to incomplete treatment. A fraction of δ that reenters into the infected class I is denoted by $\rho\delta T$ and the remaining $\delta T(1-\rho)$ rejoins the slow exposed class depending on treatment state of the individuals. The evolution rate is ϕ, where treated T becomes recovered R. The parameter $\rho(0<\rho<0)$ in $(1-\rho)\delta$ represents the ratio of drug defiance people in compartment T; let

$$\begin{aligned} m_1 &= (\nu+\varsigma_1),\\ m_2 &= (\nu+\varsigma_2),\\ m_3 &= (\nu+\gamma+\sigma_1),\\ m_4 &= (\nu+\delta+\sigma_2+\phi). \end{aligned} \tag{6}$$

Model (5) can be written as

$$\begin{cases} {}^{C}D_t^{\alpha}S = \Delta - \dfrac{\mathrm{BSI}}{N} - \nu S,\\ {}^{C}D_t^{\alpha}E_1 = \omega\dfrac{\mathrm{BSI}}{N} - m_1E_1 + (1-\rho)\delta T,\\ {}^{C}D_t^{\alpha}E_2 = (1-\omega)\dfrac{\mathrm{BSI}}{N} + \varsigma_1E_1 - m_2E_2,\\ {}^{C}D_t^{\alpha}I = \varsigma_2E_2 + \rho\delta T - m_3I,\\ {}^{C}D_t^{\alpha}T = \gamma I - m_4T,\\ {}^{C}D_t^{\alpha}R = \phi T - \nu R, \end{cases} \tag{7}$$

subjected to appropriate nonnegative conditions

$$\begin{aligned} S(0)&\geq 0,\\ E_1(0)&\geq 0,\\ E_2(0)&\geq 0,\\ I(0)&\geq 0,\\ T(0)&\geq 0,\\ R(0)&\geq 0. \end{aligned} \tag{8}$$

In model (7), the fractional derivatives $\alpha\in(0,1]$ considered as in Caputo sense and with biological parameter values, both estimated and fitted, are displayed in Table 1.

4. Fractional Model Analysis

4.1. Invariant Region and Attractivity. Model (7) is explored in a feasible region $\Xi\subset\mathbb{R}_+^6$, such that

$$\Xi = \left\{\left(S(t),E(t),E_1(t),T(t),R(t)\right)\in\mathbb{R}_+^6 : \mathbb{N}(t)\leq\frac{\Delta}{\nu}\right\}. \tag{9}$$

Lemma 1. *$\Xi\subset\mathbb{R}_+^6$ is a positively invariant region with nonnegative initial conditions for model (7) in $\mathbb{R}_+^6$.*

Proof. The net population becomes

$${}^{C}D_t^{\alpha}\mathbb{N}(t) = {}^{C}D_t^{\alpha}S(t) + {}^{C}D_t^{\alpha}E_1(t) + {}^{C}D_t^{\alpha}I(t) + {}^{C}D_t^{\alpha}E_2(t) + {}^{C}D_t^{\alpha}T(t) + {}^{C}D_t^{\alpha}R(t), \tag{10}$$

and then, we have

$${}^{C}D_t^{\alpha}\mathbb{N}(t) + \nu\mathbb{N}(t)\leq\Delta. \tag{11}$$

Now by Laplace transform, we obtained

$$\begin{aligned} \mathbb{N}(t)&\leq\mathbb{N}(0)E_{\alpha,1}\left(\nu t^{\alpha}\right) + \Delta t^{\alpha}E_{\alpha,\alpha+1}\left(\nu t^{\alpha}\right),\\ \mathbb{N}(t)&\leq\frac{\Delta}{\nu}. \end{aligned} \tag{12}$$

Thus, the model solution with nonnegative conditions in Ξ remains in Ξ. So, Ξ is positive invariant and hence attracts all solutions in $\mathbb{R}_+^6$. □

Now, for the model solution positivity,

$$R_+^6 = \left\{u\in R^6 | z\geq 0\right\} \text{ and } u(t) = \left\{S(t),E(t),E_1(t),E_2(t),T(t),R(t)\right\}^T. \tag{13}$$

Corollary 1 (see [27]). *We assumed that $j(t)\in\mathscr{C}[c,d]$ and ${}^{C}D_t^{\alpha}j(t)\in(c,d]$, where $\alpha\in(0,1]$. If*

Table 1: Descriptions of parameters and their numerical values.

Parameter	Description	Value (years^{-1})	Reference
Δ	Birth rate	$\nu \times \mathbb{N}(0)$	Estimated
ν	Natural death rate	1/67.7	[23]
β	Rate of contact	0.6001	Fitted
γ	I treatment rate	0.1500	Fitted
ρ	Failure of treatment	0.2959	Fitted
σ_1	Disease-induced death rate in I	0.2738	Fitted
σ_2	Disease-induced death rate during treatment	0.1000	Fitted
ς_1	Rate of moving from E_1 to E_2	0.2738	Fitted
ς_2	Transfer rate from E_2 to I	0.1000	Fitted
δ	Leaving rate of T reentering to I or E	0.0649	Fitted
ω	Fraction of S becoming I	0.5259	Fitted
ϕ	Recovery rate	0.0100	Fitted

$${}^{C}D_t^{\alpha} j(t) \geq 0,\ \forall u \in (c,d),\ \textit{then } j(t) \textit{is nondecreasing}$$
$${}^{C}D_t^{\alpha} j(t) \leq 0,\ \forall u \in (c,d),\ \textit{then } j(t) \textit{ is nonincreasing}$$

4.2. Positivity and Boundedness

Proposition 1. *Model (7) solution is nonnegative and bounded by* $\forall\ (S(0), E_1(0), E_2(0), I(0), T(0), R(0)) \in \mathbb{R}_+^6$, *for* $t > 0$.

Proof. Using the result given in [28], we show the nonnegativity of the proposed model R_+^6. System (7) gives

$$\begin{cases} {}^{C}D_t^{\alpha} S|_{S=0} = \Delta > 0, \\ {}^{C}D_t^{\alpha} E_1|_{E_1=0} = \omega \dfrac{\beta SI}{S+E_2+I+T+R} + (1-\rho)\delta T \geq 0,\ \delta > \rho\delta, \\ {}^{C}D_t^{\alpha} E_2|_{E_2=0} = (1-\omega)\dfrac{\beta SI}{S+E_1+I+T+R} + \varsigma_1 E_1 \geq 0, \\ {}^{C}D_t^{\alpha} I|_{I=0} = \varsigma_2 E_2 + \rho\delta T \geq 0, \\ {}^{C}D_t^{\alpha} T|_{T=0} = \gamma I \geq 0, \\ {}^{C}D_t^{\alpha} R|_{R=0} = \phi T \geq 0. \end{cases} \tag{14}$$

So, Corollary 1 gives the required result, and we can say that the solution is in $\mathbb{R}_+^6$.

$$\Xi = \left\{(S, E_1, E_2, I, T, R) \in R_+^6 :\ (S, E_1, E_2, I, T, R) \geq 0\right\}. \tag{15}$$

The sum of all terms is positive; thus, the solution of model (7) is bounded. □

4.3. Parameter Estimation.

In this section, the method of least-square curve fitting is used to estimate the parameters from the confirmed cases of tuberculosis in Khyber Pakhtunkhwa, Pakistan, since 2002 − 2017. The birth rate Δ and natural death rate ν are estimated from the literature. The other biological parameters are fitted from incidence data. Figure 1 illustrates the model's best fitted curve, and the values of parameter with description are tabulated in Table 1. The value of control basic reproductive number is $\mathscr{R}_0 \approx 1.38$, estimated via fitted parameters. The curve fitting is summarized in a few steps for model (7) as follows:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \digamma(t, w, \Xi), \quad w(t_0) = w_0. \tag{16}$$

Let Ξ denote the unknown parameters and w be the vector-dependent variables. The objective function is taken for better possible fit and is given as

$$\widehat{\Xi} = \sum_{k=1}^{n} \left(w_{t_k} - \widetilde{w}_{t_k}\right)^2, \tag{17}$$

where w_{t_k} and $\widetilde{w}_{t_k}$ are considered as the model solution and actual data points at time t_k. To obtain model parameters by minimizing the objective function and for better agreement, we follow the optimization algorithm as well [29].

4.4. Model Equilibria and Reproduction Number.

To get the equilibria for fractional-order TB model (5), we have

$${}^{C}D_t^{\alpha} S = {}^{C}D_t^{\alpha} E_1 = {}^{C}D_t^{\alpha} E_2 = {}^{C}D_t^{\alpha} I = {}^{C}D_t^{\alpha} T = {}^{C}D_t^{\alpha} R = 0. \tag{18}$$

Model (7) has two equilibrium points:

(1) The risk-free or DFE $\mathscr{E}_0$ is

$$\mathscr{E}_0 = \left(\frac{\Delta}{\nu}, 0, 0, 0, 0, 0\right). \tag{19}$$

By using the next generation technique given in [30], we have

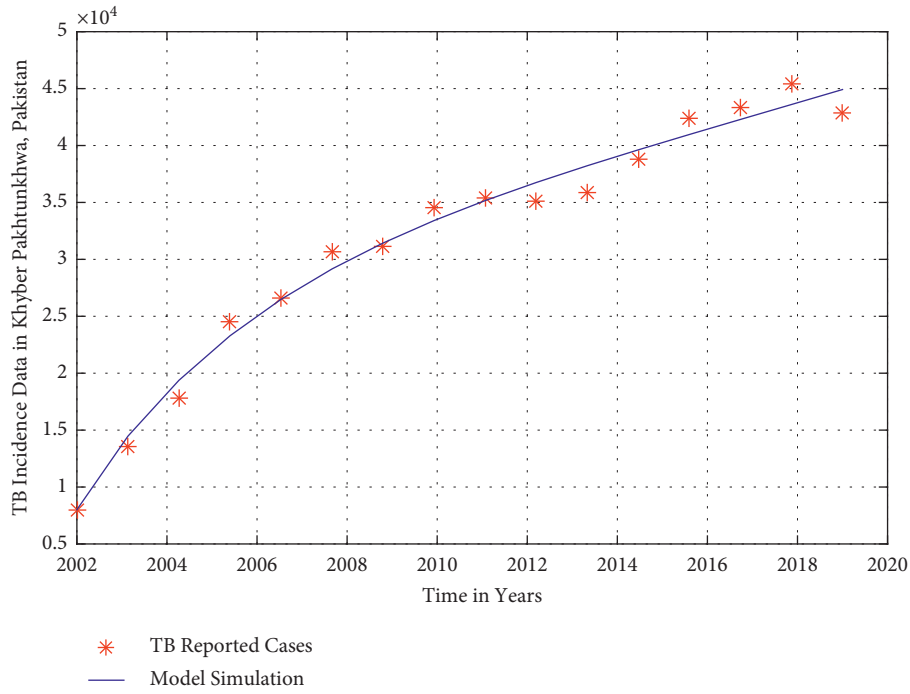

Figure 1: Model fitting (solid blue line) to the TB reported cases (red) from 2002 to 2017.

$$F = \begin{pmatrix} 0 & 0 & \omega\beta & 0 \\ 0 & 0 & (1-\omega)\beta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$V = \begin{pmatrix} m_1 & 0 & 0 & -\delta(1-\rho) \\ -\varsigma_1 & m_2 & 0 & 0 \\ 0 & -\varsigma_2 & m_3 & -\rho\delta \\ 0 & 0 & -\gamma & m_4 \end{pmatrix}. \tag{20}$$

Thus, the reproductive number $\mathscr{R}_0$ is given by

$$\mathscr{R}_0 = \rho\left(FV^{-1}\right) = \frac{\beta\varsigma_1\varsigma_2\omega}{m_1m_2m_3} + \frac{\beta\varsigma_2(1-\omega)}{m_2m_3} + \frac{\gamma\delta\varsigma_1\varsigma_2(1-\rho)}{m_1m_2m_3m_4} + \frac{\gamma\delta\rho}{m_3m_4}. \tag{21}$$

(2) Endemic equilibrium (EE) $\mathscr{E}_1$ is given by $\mathscr{E}_1 = (S^*, E_1^*, E_2^*, I^*, T^*, R^*)$, where

$$\begin{cases} S^* = \dfrac{\mathbb{N}^*}{\mathscr{R}_0}, \\ E_1^* = \dfrac{1}{m_1m_4}\left(\dfrac{m_4\omega\beta}{\mathscr{R}_0} + (1-\rho)\gamma\delta\right)I^*, \\ E_2^* = \dfrac{1}{\varsigma_2m_4}\left(m_4m_3 - \rho\gamma\delta\right)I^*. \\ T^* = \dfrac{\gamma}{m_4}I^*, \\ R^* = \dfrac{\phi\gamma}{\nu m_4}I^*, \\ I^* = \dfrac{m_1m_4\nu\varsigma_2(\mathscr{R}_0-1)\mathbb{N}^*}{m_4\nu\varsigma_2\omega\beta + \left(\nu\left(m_1\left(m_3m_4 - \rho\gamma\delta\right) + \gamma\varsigma_2\left(m_1 + (1-\rho)\delta\right)\right) + m_1\varsigma_2\left(\phi\gamma + m_4\nu\right)\right)\mathscr{R}_0}. \end{cases} \tag{22}$$

It is clear from above that $\mathscr{E}_1$ exists only if $\mathscr{R}_0 > 1$.

4.5. Stability of the DFE

Theorem 1. *The DFE ($\mathscr{E}_0$) is LAS if the eigenvalues λ_j of model (7) satisfy $|\arg(\lambda_j)| > \alpha\pi/2$.*

Proof. The Jacobian measure of system (7) evaluated at $\mathscr{E}_0$ is given by

$$J(E_0) = \begin{pmatrix} -\nu & 0 & 0 & -\beta & 0 & 0 \\ 0 & -m_1 & 0 & \beta\omega & \delta(1-\rho) & 0 \\ 0 & \varsigma_1 & -m_2 & (1-\omega)\beta & 0 & 0 \\ 0 & 0 & \varsigma_2 & -m_3 & \rho\delta & 0 \\ 0 & 0 & 0 & \gamma & -m_4 & 0 \\ 0 & 0 & 0 & 0 & \phi & -\nu \end{pmatrix}. \tag{23}$$

We have $-\nu$ (twice), while for the others, the following is shown:

$$P(\lambda) = \lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4, \tag{24}$$

where

$$\begin{aligned} c_1 &= m_1 + m_2 + m_3 + m_4, \\ c_2 &= m_2m_4 + m_1(m_2 + m_3 + m_4) \\ &\quad + m_2m_3(1 - \mathscr{R}_2) + m_3m_4(1 - \mathscr{R}_4), \\ c_3 &= (m_1 + m_2)m_3m_4(1 - \mathscr{R}_4) \\ &\quad + m_1m_2m_3(1 - (\mathscr{R}_1 + \mathscr{R}_2)) \\ &\quad + m_4(m_1m_2 - \beta\varsigma_2(1-\omega)), \\ c_4 &= m_1m_2m_3m_4(1 - \mathscr{R}_0). \end{aligned} \tag{25}$$

It can be observed that the coefficients shown above are positive, i.e., $c_i > 0$, for $i = 1, \ldots, 4$; furthermore, the Routh–Hurwitz criteria can be satisfied easily, $c_1c_2c_3 - c_1^2c - 4 - c_3^2 > 0$. So, the TB model is locally asymptotic at the DFE when $\mathscr{R}_0 < 1$. Thus, $|\arg(\lambda)| > \alpha\pi/2$ for all $\alpha \in [0, 1)$, i.e., local asymptotic stability. □

4.6. Global Stability of DFE

Theorem 2. *The fractional tuberculosis model is GAS at risk-free equilibrium ($\mathscr{E}_0$) if $\mathscr{R}_0 < 1$.*

Proof. The appropriate Lyapunov function ($\mathscr{L}(t)$) for global stability of model (7) is defined as

$$\mathscr{L}(t) = l_1E_1 + l_2E_2 + l_3I + l_4T, \tag{26}$$

where $l_n > 0$, with $n = 1, \ldots, 4$, are positive constants and the time fractional derivative of $\mathscr{L}(t)$ is

$${}^{C}D_t^{\alpha}\mathscr{L}(t) = l_1{}^{C}D_t^{\alpha}E_1 + l_2{}^{C}D_t^{\alpha}E_2 + l_3{}^{C}D_t^{\alpha}I + l_4{}^{C}D_t^{\alpha}T. \tag{27}$$

Considering fractional system (7), we obtain

$$\begin{aligned} {}^{C}D_t^{\alpha}\mathscr{L}(E_1, E_2, I, T) &= l_1\left\{\omega\frac{\beta SI}{N} - m_1E_1 + (1-\rho)\delta T\right\} \\ &\quad + l_2\left\{(1-\omega)\frac{\beta SI}{N} + \varsigma_1E_1 - m_2E_2\right\} \\ &\quad + l_3\{\varsigma_2E_2 + \rho\delta T - m_3I\} + l_4\{\gamma I - m_4T\}, \\ &\le l_1\{\omega\beta I - m_1E_1 + (1-\rho)\delta T\} \\ &\quad + l_2\{(1-\omega)\beta I + \varsigma_1E_1 - m_2E_2\} \\ &\quad + l_3\{\varsigma_2E_2 + \rho\delta T - m_3I\} \\ &\quad + l_4\{\gamma I - m_4T\}, \quad S \le N. \\ &= \{l_1\omega\beta + l_2(1-\omega)\beta + l_4\gamma - l_3m_3\}I \\ &\quad + \{l_2\varsigma_1 - l_1m_1\}E_1 \\ &\quad + \{l_3\varsigma_2 - l_2m_2\}E_2 \\ &\quad + \{l_1(1-\rho)\delta + l_3\rho\delta - l_4m_4\}T, \\ &= l_3m_3\left\{\frac{l_1\omega\beta + l_2(1-\omega)\beta + l_4\gamma}{l_3m_3} - 1\right\}I \\ &\quad + \{l_2\varsigma_1 - l_1m_1\}E_1 \\ &\quad + \{l_3\varsigma_2 - l_2m_2\}E_2 \\ &\quad + \{l_1(1-\rho)\delta + l_3\rho\delta - l_4m_4\}T. \end{aligned} \tag{28}$$

Now, we choose

$$\begin{aligned} l_1 &= \varsigma_1, \\ l_2 &= m_1, \\ l_3 &= \frac{m_1m_2}{\varsigma_2}, \\ l_4 &= \frac{\varsigma_1\varsigma_2(1-\rho)\delta + m_1m_2\rho\delta}{\varsigma_2m_4}, \end{aligned} \tag{29}$$

$${}^{C}D_t^{\alpha}\mathscr{L}(t) \le l_3m_3(\mathscr{R}_0 - 1)I.$$

Thus, ${}^{C}D_t^{\alpha}\mathscr{L}(t) \le 0$ if $\mathscr{R}_0 \le 1$. Therefore, the variables and parameters are nonnegative with ${}^{C}D_t^{\alpha}\mathscr{L}(t) \le 0$ iff $E_1 = E_2 = I = T = 0$. Hence, $(E_1, E_2, I, T) \longrightarrow (0, 0, 0, 0)$ as $t \longrightarrow \infty$. We get $S \longrightarrow \Delta/\nu$ and $R \longrightarrow 0$ as $t \longrightarrow \infty$ from system (7). Thus, the solution of model (7) with nonnegative initial conditions as $t \longrightarrow \infty$ approaches $\mathscr{E}_0$ according to the fractional case developed in [31], in the feasible region. Hence, it complies that the disease-free equilibrium of system (7) is GAS. □

Lemma 2. *Model (7) is unstable at DFE if $\mathscr{R}_0 > 1$.*

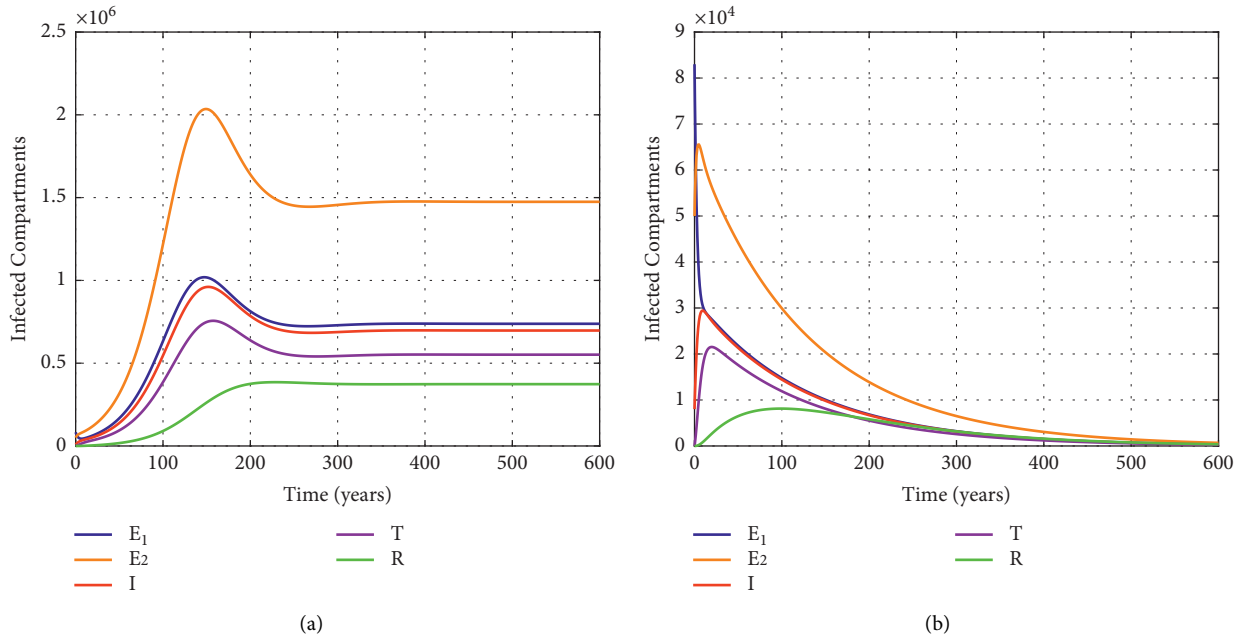

Figure 2: Numerical visualization of cumulative infected compartments of the TB model when $\mathscr{R}_0 > 1$ (a) and $\mathscr{R}_0 < 1$ (b).

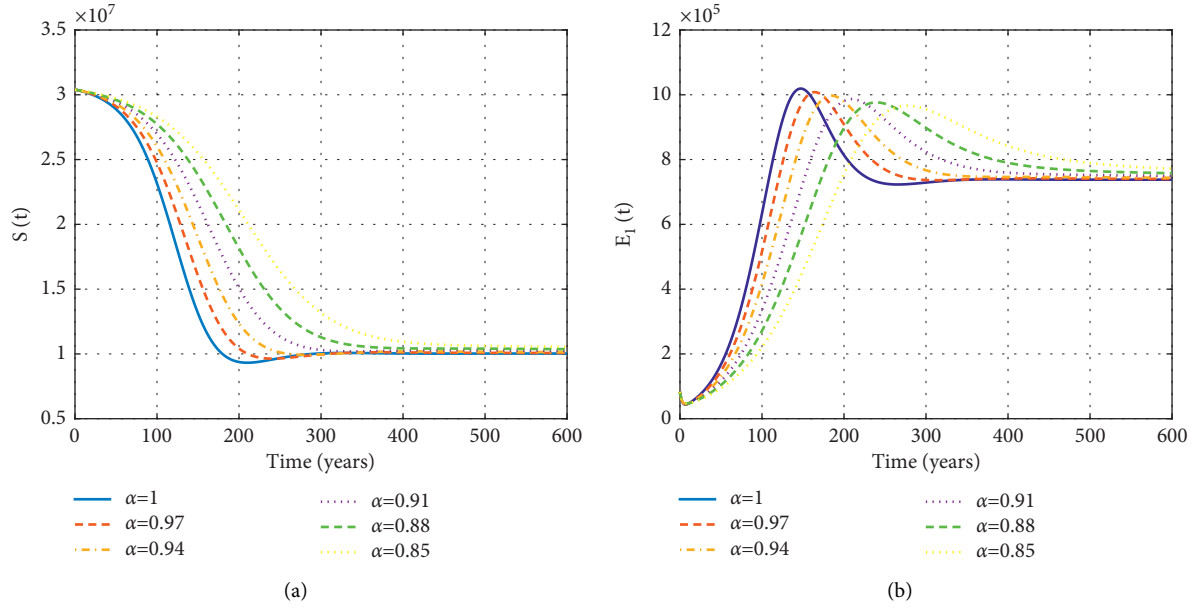

Figure 3: Continued.

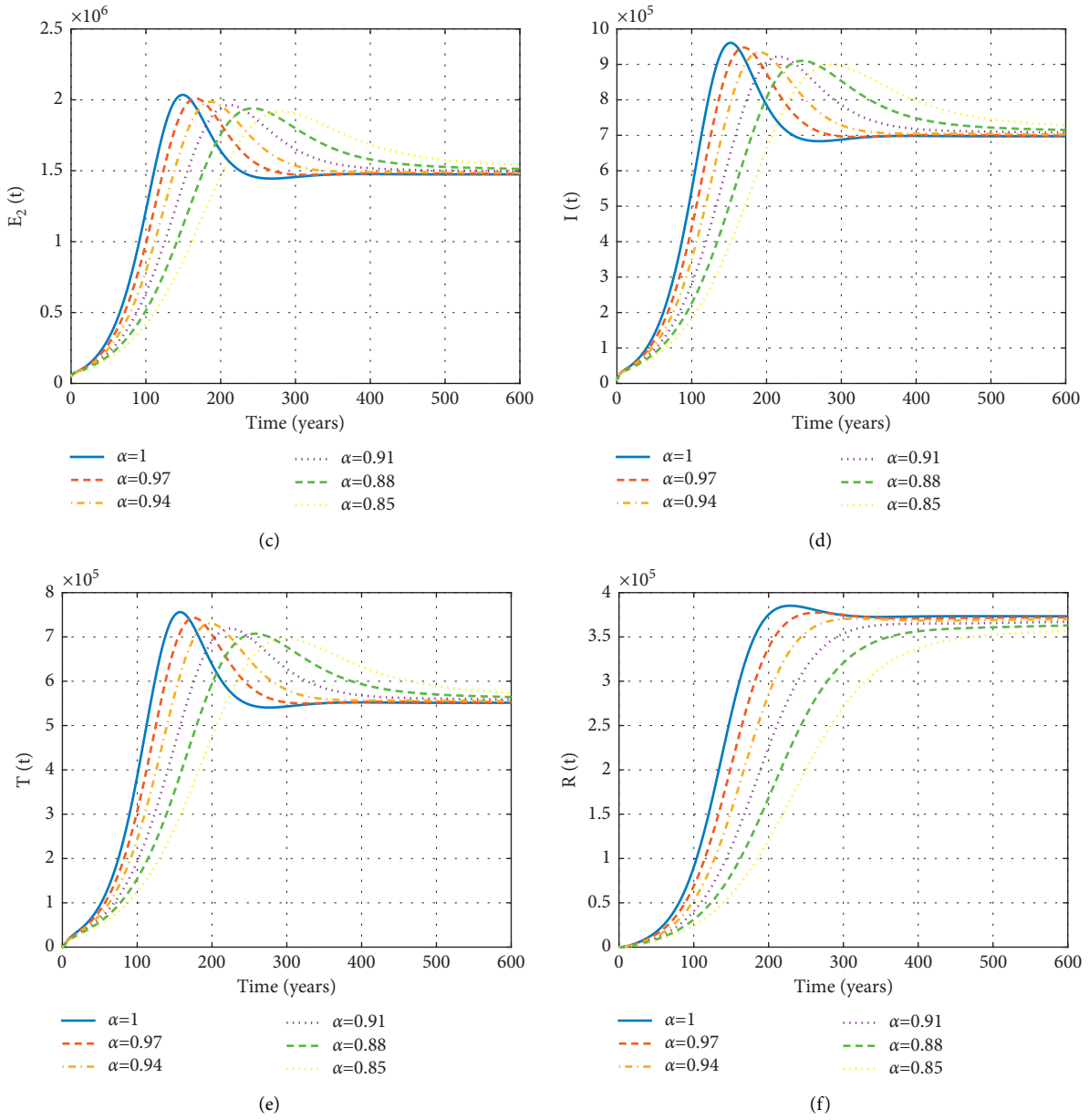

Figure 3: Numerical results of model (7) for different fractional orders α.

5. Numerical Scheme and Simulation

The numerical iterative method [32] is used for the numerical solution of model (7). A differential system is given as

$$\begin{cases} D_{t_0}^{\alpha}(w - w_0)(t) = \mathbb{F}(t, w(t)), \\ w(t_0) = w_0. \end{cases} \tag{30}$$

By applying the fractional integral operator, we obtained

$$w(t) - w(t_0) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \hbar)^{\alpha-1} \mathbb{F}(\hbar, w(\hbar)) \mathrm{d}\hbar. \tag{31}$$

By using the trapezoidal quadrature formula,

$$\int_{t_0}^{t_{n+1}} (t_{n+1} - \hbar)^{\alpha-1} j(\hbar) \mathrm{d}\hbar \approx \int_{t_0}^{t_{n+1}} (t_{n+1} - \hbar)^{\alpha-1} j_{n+1}(\hbar) \mathrm{d}\hbar. \tag{32}$$

The right-hand side of (32) yields

$$\int_{t_0}^{t_{n+1}} (t_{n+1} - \hbar)^{\alpha-1} j_{n+1}(\hbar) \mathrm{d}\hbar = \sum_{i=0}^{n+1} a_{i,n+1} j(t_i), \tag{33}$$

where

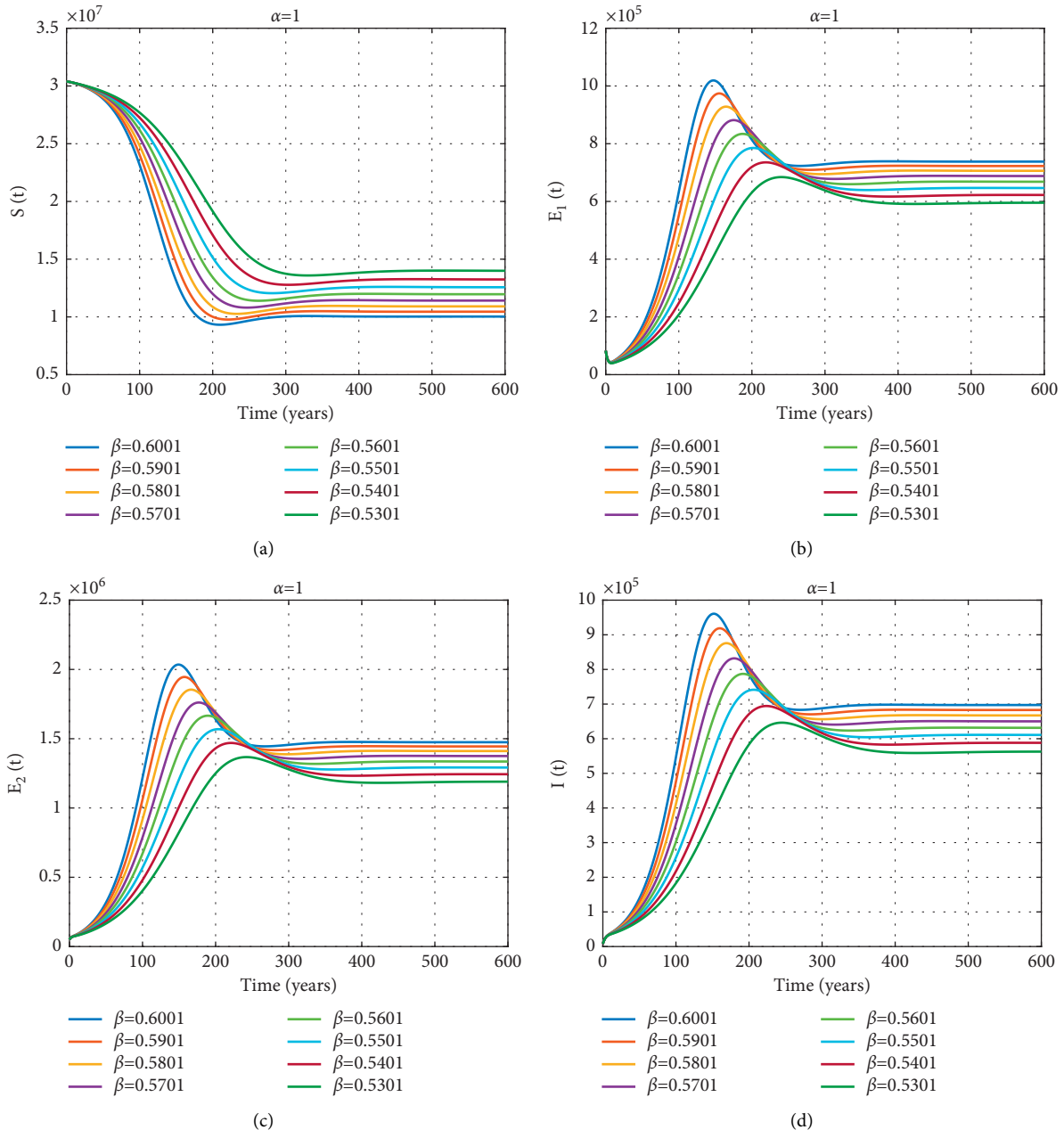

Figure 4: Continued.

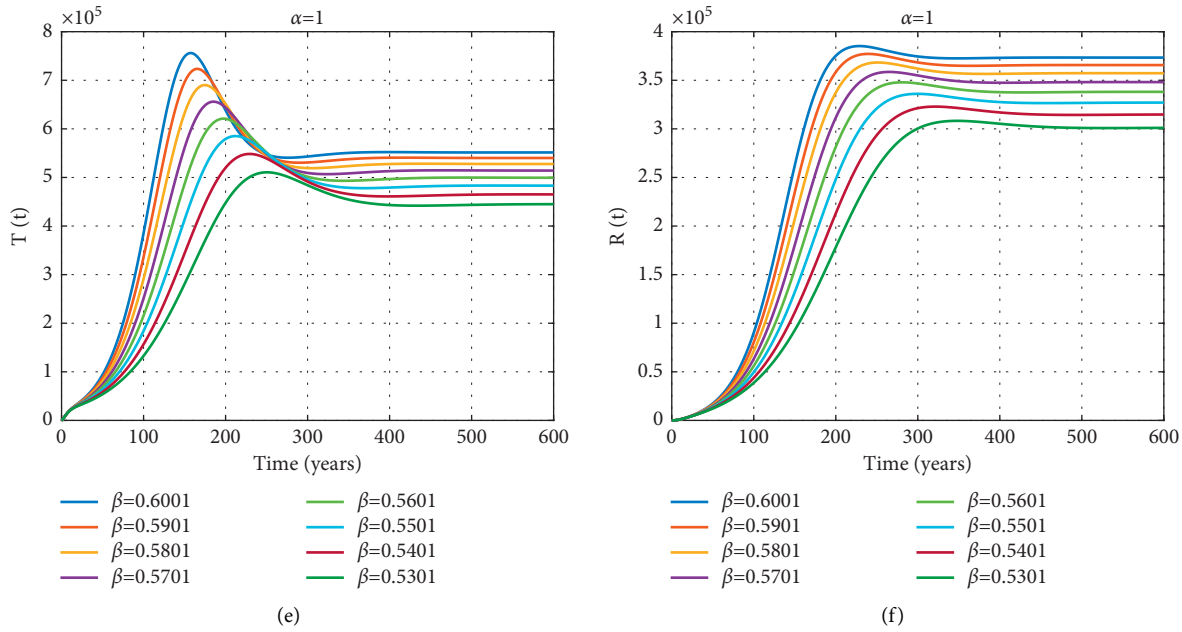

Figure 4: Behavior of the TB model with an impact of contact rate β at $\alpha = 1$.

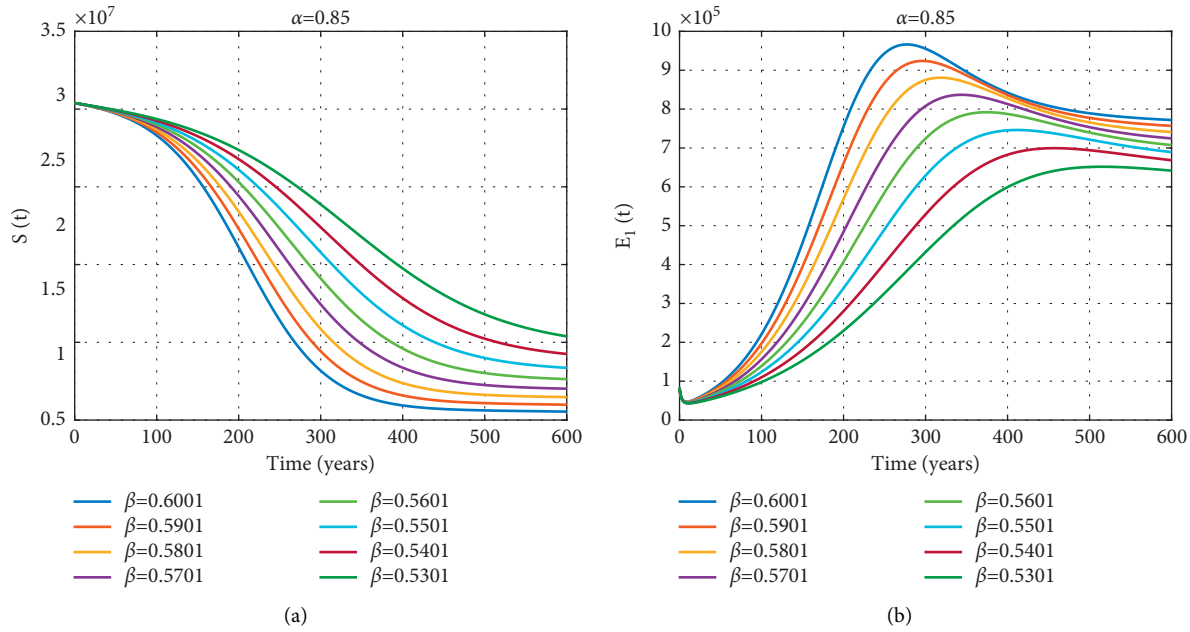

Figure 5: Continued.

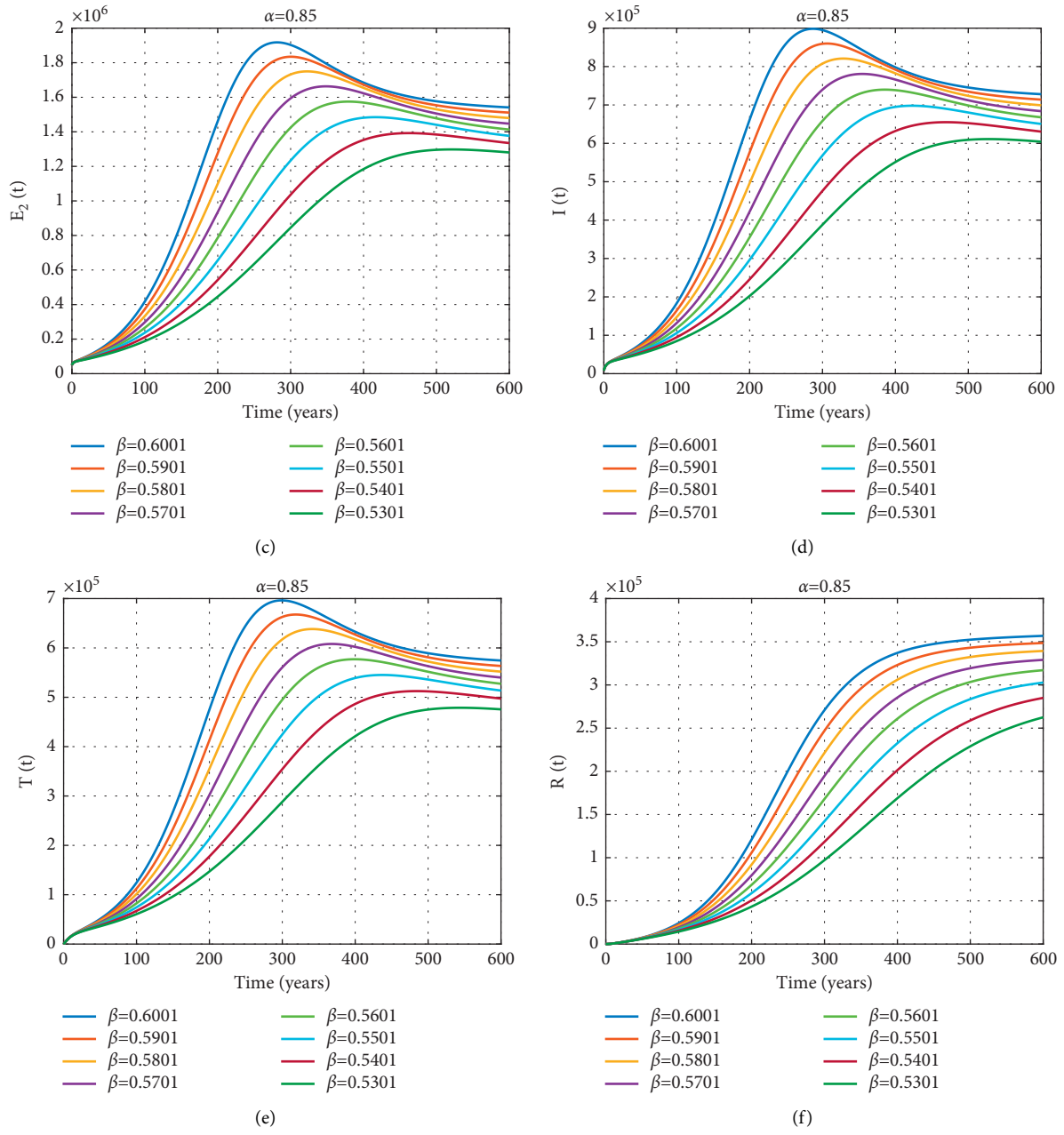

FIGURE 5: Behavior of the TB model with an impact of contact rate β at $\alpha = 0.85$.

$$a_{i,n+1} = \int_{t_0}^{t_{n+1}} \left(t_{n+1} - \hbar\right)^{\alpha-1} \phi_{j,n+1}(\hbar)\mathrm{d}\hbar \tag{34}$$

$$\chi_{i,n+1}(\hbar) = \begin{cases} \dfrac{(\hbar - t_{i-1})}{(t_i - t_{i-1})}, & \text{if } t_{i-1} < \hbar < t_i, \\ \dfrac{(t_{i+1} - \hbar)}{(t_{i+1} - t_i)}, & \text{if } t_i < \hbar < t_{i+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{35}$$

When nodes $t_i = t_0 + i_h$ are equispaced, then (34) reduces to

$$a_{i,n+1}(\hbar) = \begin{cases} \dfrac{g^\alpha}{\alpha(\alpha+1)}\left(n^{\alpha+1} - (n-\alpha)(n+1)^\alpha\right), & \text{if } i = 0, \\ \dfrac{g^\alpha}{\alpha(\alpha+1)}, & \text{if } i = n+1, \end{cases}$$

$$a_{i,n+1} = \frac{g^\alpha}{\alpha(\alpha+1)}\left((n-i+2)^{\alpha+1} - 2(n-i+1)^{\alpha+1} + (n-i)^{\alpha+1}\right). \tag{36}$$

The one-step fractional Adams–Moulton equation is

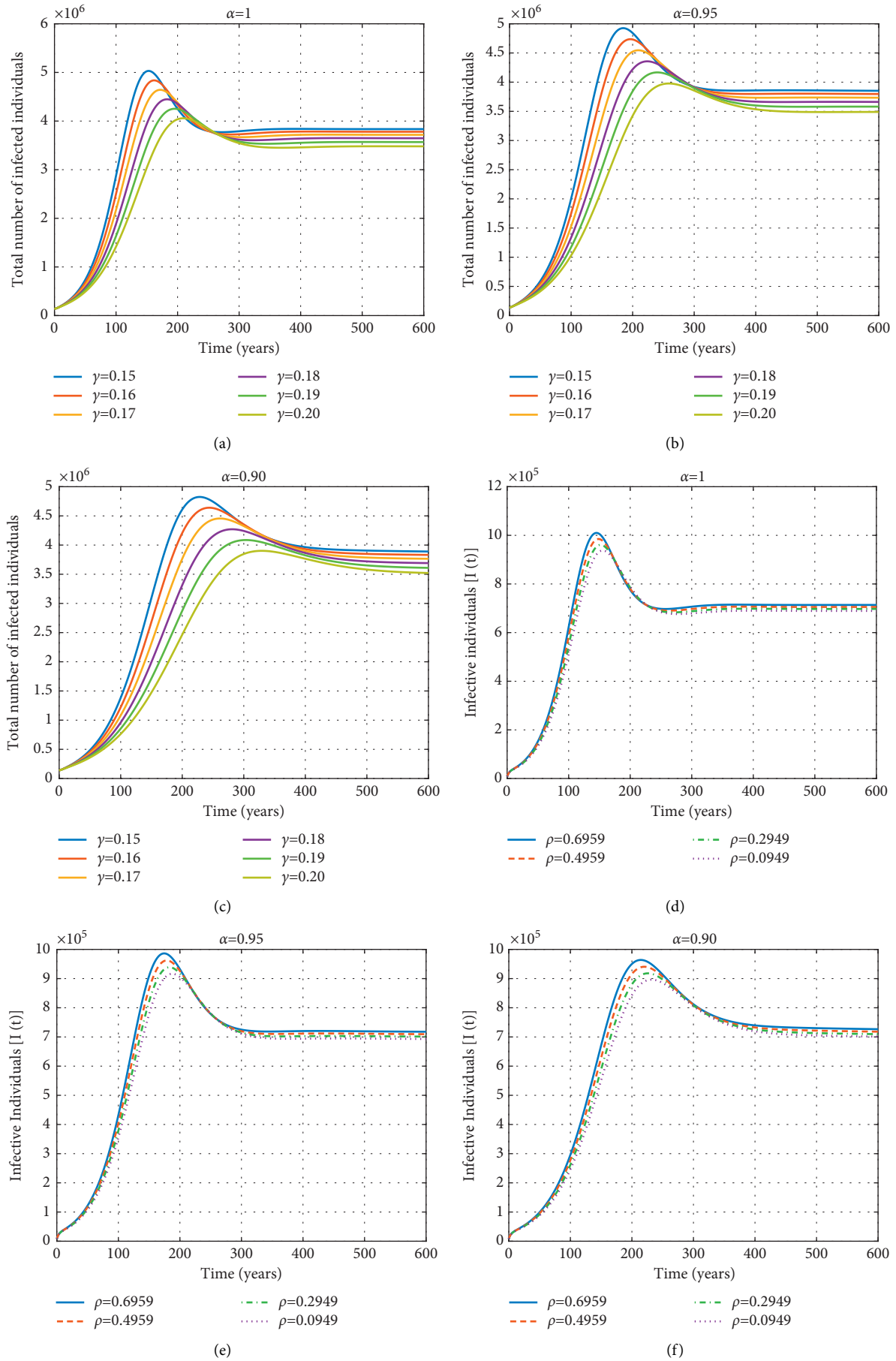

Figure 6: The treatment rate γ impact on cumulative infected persons (a–c) and variation of ρ on infectious individuals (d–f) for $\alpha = 1$, $\alpha = 0.95$, and $\alpha = 0.90$.

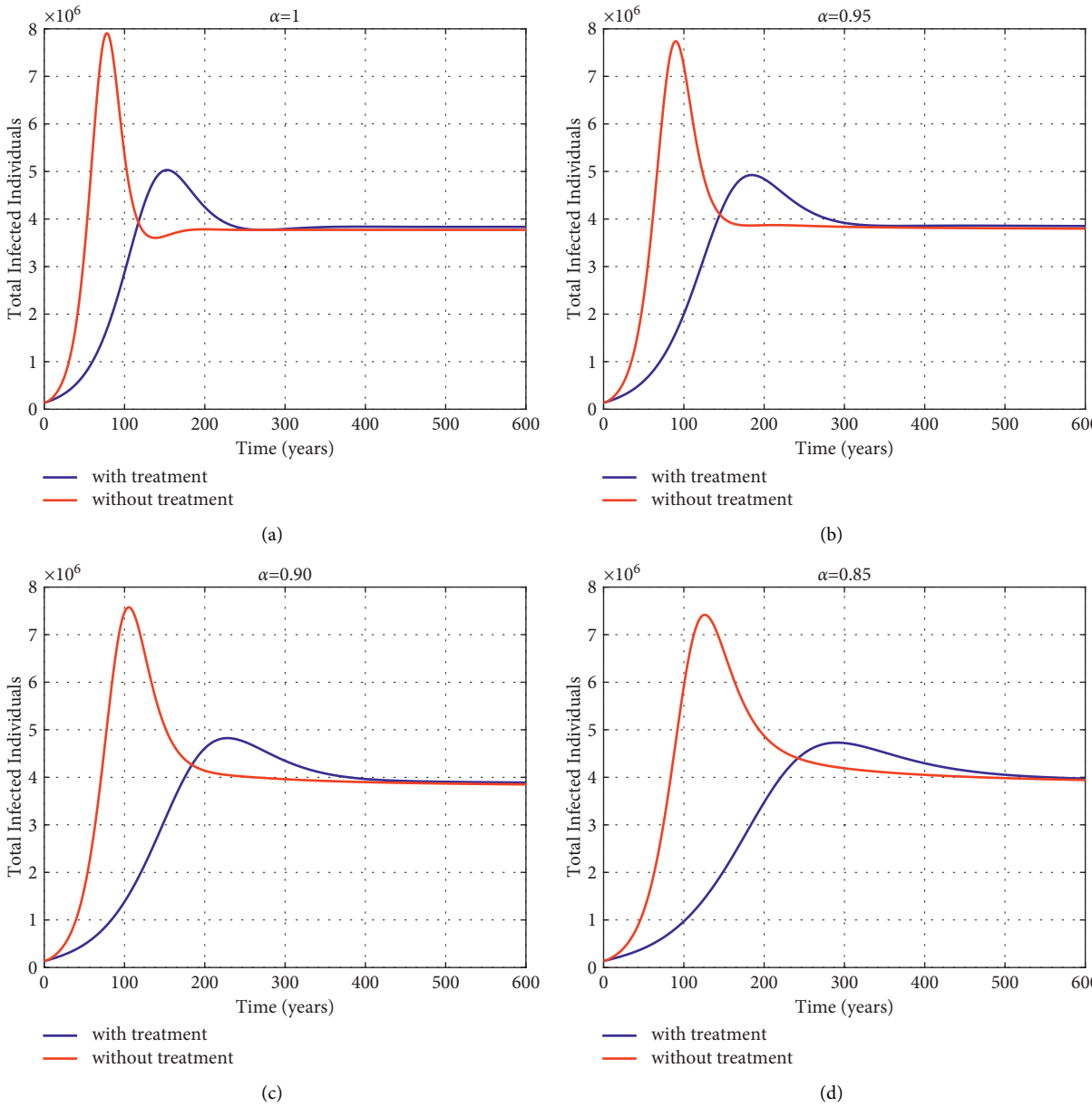

Figure 7: Behavior with and without treatment on total infected individuals for $\alpha = 1, 0.95, 0.90, 0.85$.

$$w_{n+1} = w_0 + \frac{1}{\Gamma(\alpha)}\left(\sum_{i=0}^{n} a_{i,n+1}\mathbb{F}(t_i, w_i) + a_{n+1,n+1}\mathbb{F}\left(t_{n+1}, w_{n+1}^{P}\right)\right). \tag{37}$$

(31) reduces to

$$\int_{t_0}^{t_{n+1}} (t_{n+1} - \hbar)^{\alpha-1} j(\hbar)\mathrm{d}\hbar \approx \sum_{i=0}^{n} b_{j,n+1} j(t_i), \tag{38}$$

where now

$$b_{i,n+1} = \int_{t_i}^{t_i+1} (t_{n+1} - \hbar)^{\alpha-1}\mathrm{d}\hbar = \frac{1}{\alpha}\left((t_{n+1} - t_i)^{\alpha} - (t_{n+1} - t_{i+1})^{\alpha}\right). \tag{39}$$

We have a simplified expression as follows:

$$b_{i,n+1} = \frac{g^{\alpha}}{\alpha}\left((n+1-i)^{\alpha} - (n-i)^{\alpha}\right). \tag{40}$$

Thus, the predictor $w_n^P + 1$ is determined by

$$w_{n+1}^{P} = w_0 + \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{n} b_{i,n+1}\mathbb{F}t_i, w(t_i, w_i). \tag{41}$$

We explore the dynamics of fractional TB model (7) numerically by the generalized predictor-corrector of Adams–Bashforth–Moulton method mentioned above. To control the disease up to some extent and for memory effect, we analyze the effect of different parameters by varying some sensitive parameters and fractional order α.

The infection must be reduced with the help of control strategies and treatment. To illustrate the effects of sensitive parameters, the population in 2017 of Khyber Pakhtunkhwa was 30, 523, 371 [33], which is to be considered as $\mathbb{N}(0)$. The state variables assumed as $E_1(0)$, $E_2(0)$, and $I(0)$ are 83000, 50000, and 8010, respectively. Moreover, no treated or recovered cases are considered initially, that is, $T(0) = R(0) = 0$. Then, the susceptible individuals can be obtained easily as $S(0) = 30$. The behaviors of cumulative infected people for $\mathscr{R}_0 = 1.38$ when $\beta = 0.6001$ and $\mathscr{R}_0 = 0.9300$ when $\beta = 0.4001$ are shown in Figure 2. The dynamical behavior of model (7) is depicted in Figure 3 for fractional orders $\alpha = 1, 0.97, 0.94, 0.91, 0.88, 0.85$. Figure 3 shows that only susceptible individuals increased, while the infected individuals decreased significantly by decreasing α. Figures 4 and 5 illustrate the impact of contact rate β on model (7) for $\alpha = 1$ and $\alpha = 0.85$, respectively. The decrease is observed in infected compartments with a decrease in β, and for smaller order, the impact of α is more pronounced. Also, the behaviors of cumulative infected people with an impact of treatment rate and infectious people with variation of failure of treatment ρ are plotted in Figure 6 for various fractional orders α. So, by increasing treatment rate γ, a decay in cumulative infected individuals is observed and vice versa for infective compartments in case of ρ. Finally, the behavior of total infected people with and without treatment is depicted in Figure 7, and results become more precise for smaller values of fractional order α. From numerical results, we can say that the tuberculosis infection can be controlled with proper treatment.

6. Conclusion

Tuberculosis has resulted in a lot of infected cases and deaths in Pakistan. The government of Pakistan and particularly the province KP have put many efforts for its minimization by treating the infected cases and also the relapse cases. To understand the TB infections in the KP province, we studied a mathematical model with slow and fast exposed cases and its effect on the model dynamics. We studied the essential mathematics involved in the modeling of the fractional-order model. Then, we investigated the stability of the model and proved the TB model to be locally as well as globally asymptotically stable. The stability results were obtained in the context of Caputo operator. Some discussion on the estimation procedures for the investigations of the model parameters is considered. We utilized the realistic parameters and obtained results graphically. We provided an efficient numerical scheme based on the Adams–Bashforth–Moulton method and obtained the graphical results. Numerical results were achieved by considering different parameters and fractional order values, and we discussed its impact on disease eradications. The effect of treatment rate and its related results was explored. Some results with treatment and without treatment are discussed and shown graphically, which show the disease can be eliminated by treating the infected people. While considering the effect of fractional-order parameter values, there were observed decreases in the infective compartments. From the analysis, the work shown suggests that the disease can be minimized more efficiently if the government takes serious actions by educating the people, making awareness, etc., and provides better treatment at doorsteps. Regarding the future of modeling of TB in Pakistan, we will explore and extend the results by considering the cases across the country with vaccinations and relapse model.

Data Availability

Data are available on the reasonable request to the corresponding author.

Conflicts of Interest

The authors declare that no conflicts of interest.

Acknowledgments

The authors extend their appreciation to King Khalid University, Saudi Arabia, for its support under number 78.

References

[1] M. Altaf Khan, S. Ullah, and M. Farooq, "A new fractional model for tuberculosis with relapse via Atangana-Baleanu derivative," *Chaos, Solitons & Fractals*, vol. 116, pp. 227–238, 2018.

[2] E. Harding, "Who global progress report on tuberculosis elimination," *The Lancet Respiratory Medicine*, vol. 8, no. 1, p. 19, 2020.

[3] H. Waaler, A. Geser, and S. Andersen, "The use of mathematical models in the study of the epidemiology of tuberculosis," *American Journal of Public Health and the Nation's Health*, vol. 52, no. 6, pp. 1002–1013, 1962.

[4] C. S. Revelle, W. R. Lynn, and F. Feldmann, "Mathematical models for the economic allocation of tuberculosis control activities in developing nations," *American Review of Respiratory Disease*, vol. 96, no. 5, pp. 893–909, 1967.

[5] J. Liu and T. Zhang, "Global stability for a tuberculosis model," *Mathematical and Computer Modelling*, vol. 54, no. 1-2, pp. 836–845, 2011.

[6] L. Liu, X.-Q. Zhao, and Y. Zhou, "A tuberculosis model with seasonality," *Bulletin of Mathematical Biology*, vol. 72, no. 4, pp. 931–952, 2010.

[7] J. K. K Asamoah, F. T. Oduro, E. Bonyah, and B. Seidu, "Modelling of rabies transmission dynamics using optimal control analysis," *Journal of Applied Mathematics*, vol. 2017, Article ID 2451237, 23 pages, 2017.

[8] R. S. Wallis, "Mathematical models of tuberculosis reactivation and relapse," *Frontiers in Microbiology*, vol. 7, p. 669, 2016.

[9] S. Kim, A. A. de los Reyes, and E. Jung, "Mathematical model and intervention strategies for mitigating tuberculosis in the Philippines," *Journal of Theoretical Biology*, vol. 443, pp. 100–112, 2018.

[10] M. A. Khan, M. Ahmad, S. Ullah, M. Farooq, and T. Gul, "Modeling the transmission dynamics of tuberculosis in khyber pakhtunkhwa Pakistan," *Advances in Mechanical Engineering*, vol. 11, no. 6, Article ID 1687814019854835, 2019.

[11] A. Atangana and B. Dumitru, "New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model," 2016, http://arxiv.org/abs/1602.03408.
[12] M. Caputo and M. Fabrizio, "A new definition of fractional derivative without singular kernel," *Progress in Fractional Differentiation and Applications*, vol. 1, no. 2, pp. 1–13, 2015.
[13] H. Yépez-Martínez and J. F. Gómez-Aguilar, "A new modified definition of Caputo-Fabrizio fractional-order derivative and their applications to the Multi Step Homotopy Analysis Method (MHAM)," *Journal of Computational and Applied Mathematics*, vol. 346, pp. 247–260, 2019.
[14] Y.-M. Chu, A. Ali, M. A. Khan, S. Islam, and S. Ullah, "Dynamics of fractional order covid-19 model with a case study of Saudi Arabia," *Results in Physics*, vol. 21, Article ID 103787, 2021.
[15] M. A. Khan, H. Putra Odinsyah, and F. Fatmawati, "Fractional model of HIV transmission with awareness effect," *Chaos, Solitons & Fractals*, vol. 138, Article ID 109967, 2020.
[16] Y. X. Li, M. G. Alshehri, E. A. Algehyne et al., "Fractional study of Huanglongbing model with singular and non-singular kernel," *Chaos, Solitons & Fractals*, vol. 148, Article ID 111037, 2021.
[17] M. A. Khan, F. Fatmawati, E. Bonyah, Z. Hammouch, and E. Mifta Shaiful, "A mathematical model of tuberculosis (tb) transmission with children and adults groups: a fractional model," *AIMS Mathematics*, vol. 5, no. 4, pp. 2813–2842, 2020.
[18] A. Ali, F. Salem Alshammari, S. Islam, M. A. Khan, and S. Ullah, "Modeling and analysis of the dynamics of novel coronavirus (covid-19) with caputo fractional derivative," *Results in Physics*, vol. 20, Article ID 103669, 2020.
[19] M. A. A. Oud, A. Ali, H. Alrabaiah, S. Ullah, M. A. Khan, and S. Islam, "A fractional order mathematical model for covid-19 dynamics with quarantine, isolation, and environmental viral load," *Advances in Difference Equations*, vol. 2021, no. 1, pp. 1–19, 2021.
[20] M. A. Khan, Windarto, and Fatmawati, "Parameter estimation and fractional derivatives of dengue transmission model," *AIMS Mathematics*, vol. 5, no. 3, pp. 2758–2779, 2020.
[21] D. Kaur, P. Agarwal, M. Rakshit, and M. Chand, "Fractional calculus involving (p, q)-mathieu type series," *Applied Mathematics and Nonlinear Sciences*, vol. 5, no. 2, pp. 15–34, 2020.
[22] World Health Organization, *Global Tuberculosis Report*, World Health Organization, Geneva, Switzerland, 2019, https://www.who.int/tb/country/data/profles/en.
[23] Pakistan Bureau of Statistics, *Pakistan's 6th Census: Population of Major Cities 583 Census*, http://www.pbscensus.gov.pk/, p. 584, Pakistan Bureau of Statistics, Islamabad, Pakistan, 2020, .
[24] National TB Control Program Pakistan (NTP). http://www.ntp.gov.pk/webdatabase.php.
[25] Provisional TB Control Program Pakistan (NTP). http://ptp.gkp.pk/,http://tb.kpdata.gov.pk/ptp/april_2017.
[26] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Elsevier, Amsterdam, Netherlands, 1998.
[27] Z. M. Odibat and N. T. Shawagfeh, "Generalized Taylor's formula," *Applied Mathematics and Computation*, vol. 186, no. 1, pp. 286–293, 2007.
[28] W. Lin, "Global existence theory and chaos control of fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 332, pp. 709–726, 2007.
[29] W. J. H. Stortelder, *Parameter Estimation in Nonlinear Dynamical Systems*, CWI, Amsterdam, The Netherlands, 1998.
[30] P. Van den Driessche and W. James, "Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission," *Mathematical Biosciences*, vol. 180, no. 1-2, pp. 29–48, 2002.
[31] C. Vargas-De-León, "Volterra-type lyapunov functions for fractional-order epidemic systems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 24, no. 1-3, pp. 75–85, 2015.
[32] K. Diethelm and A. D. Freed, "The fracpece subroutine for the numerical solution of differential equations of fractional order," in *Proceedings of the Forschung und wissenschaftliches Rechnen*, pp. 57–71, Göttingen, Germany, 1999.
[33] World Health Organization, *WHO Country Cooperation Strategic*, World Health Organization, Geneva, Switzerland, 2020, https://apps.who.int/iris/bitstream/10665/136607/1/ccsbrief_pak_en_2016.pdf.