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*Research Article*

# Asymptotic Distributions for Power Variations of the Solution to the Spatially Colored Stochastic Heat Equation

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Let $u_{\alpha,d} = \{u_{\alpha,d}(t,x), \quad t \in [0,T], x \in \mathbb{R}^d\}$ be the solution to the stochastic heat equations (SHEs) with spatially colored noise. We study the realized power variations for the process $u_{\alpha,d}$, in time, having infinite quadratic variation and dimension-dependent Gaussian asymptotic distributions. We use the underlying explicit kernels and spectral/harmonic analysis, yielding temporal central limit theorems for SHEs with spatially colored noise. This work builds on the recent works on delicate analysis of variations of general Gaussian processes and SHEs driven by space-time white noise.

## 1. Introduction

Throughout this work, we will consider the following $d$-dimensional stochastic heat equation (SHE):

$$\begin{aligned} &\frac{\partial}{\partial t}u_{\alpha,d}(t,x) = \frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}u_{\alpha,d}(t,x) + \sigma\big(u_{\alpha,d}(t,x)\big)\dot{W}_{\alpha,d}, \quad t\in_{+}, x \in \mathbb{R}^d, \\ &u_{\alpha,d}(0,x) = w(x), \quad x \in \mathbb{R}^d, \end{aligned} \tag{1}$$

with $\varepsilon > 0$ and Gaussian space-time colored noise $W_{\alpha,d}$. The noise $W_{\alpha,d}$ is assumed to have a particular covariance structure (see [1]):

$$\mathbb{E}\big[W_{\alpha,d}(t,A)W_{\alpha,d}(s,B)\big] = (t\wedge s)\int_A\int_B f_{\alpha,d}(x-y)\mathrm{d}x\mathrm{d}y, \quad t,s \in s_{+}, A, B \in \mathscr{B}_b\big(\mathbb{R}^d\big), \tag{2}$$

where

$$f_{\alpha,d}(x) = c_{\alpha,d}|x|^{-d+\alpha}, \quad 0 < \alpha < d, \tag{3}$$

with $c_{\alpha,d} = 2^{d-\alpha}\pi^{d/2}\Gamma((d-\alpha)/2)/\Gamma(\alpha/2)$. The initial condition, $w(x)$, is taken to be bounded and $\rho$-Hölder continuous. We will also assume $\sigma$ to be Lipschitz continuous, and there exists $c_0 \geq 0$ such that $|\sigma(x)-\sigma(y)| \leq c_0|x-y|$ and $|\sigma(x)| \leq c_0(1+|x|)$. Stochastic PDEs (SPDEs) such as (1) have been studied in [1–6] and others.

It is known (see [1, 7–10]) that (1) admits a unique mild solution if and only if $d < 2+\alpha$, and this mild solution is interpreted as the solution of the following integral equation:

$$u_{\alpha,d}(t,x) = \int_{\mathbb{R}^d} G(t,x-y)u_{\alpha,d}(0,y)\mathrm{d}y + \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)\sigma\big(u_{\alpha,d}(s,y)\big)W_{\alpha,d}(\mathrm{d}s,\mathrm{d}y), \tag{4}$$

for $t \in \mathbb{R}_+, x \in \mathbb{R}^d$, where the above integral is a Wiener integral with respect to the noise $W_{\alpha,d}$ (see, e.g., [2] for the definition) and $G$ is the Green kernel of the heat equation given by

$$G(t,x) = \begin{cases} (2\pi\varepsilon t)^{-1/2} e^{-|x|^2/(2\varepsilon t)}, & \text{if } t > 0, x \in \mathbb{R}^d, \\ 0, & \text{if } t \leq 0, x \in \mathbb{R}^d. \end{cases} \tag{5}$$

Bezdek [11] investigated weak convergence of probability measures corresponding to the solution of (1) in $d = 1$. He showed that probability measures corresponding to $u_{\alpha,1}$ weakly converge to those corresponding to the solution to the SHE with white noise when $\alpha\uparrow 1$, that is, the solution of (1) converges in the appropriate sense to the solution of the same equation, but with white noise $W$ instead of colored noise $W_{\alpha,1}$ as $\alpha\uparrow 1$. By that, we mean the solution to

$$\begin{aligned} &\frac{\partial}{\partial t}u(t,x) = \frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}u(t,x) + \sigma(u(t,x))\dot{W}, t \in \mathbb{R}_+, x \in \mathbb{R}, \\ &u(0,x) = w(x), \quad x \in \mathbb{R}, \end{aligned} \tag{6}$$

where $W$ denotes white noise. SPDEs such as (6) have been studied in [1, 2, 7, 10, 12, 13] and others.

Among others, Tudor and Xiao [14] investigated the exact uniform and local moduli of continuity and Chung-type laws of the iterated logarithm of the process $u_{\alpha,d}$ in time. In fact, they investigated these path properties for a more wide class, namely, the solution to the linear SHE driven by a fractional noise in time with correlated spatial structure. Swanson [13] showed that the solutions of the SHEs in (6) with $\varepsilon = \sigma = 1$, in time, have infinite quadratic variation and are not semimartingales and also investigated central limit theorems (CLTs) for modifications of the quadratic variations of the solutions of the SHEs with white noise. Pospíšil and Tribe [12] investigated the quartic variations of the solutions of the SHEs in (6) with $\varepsilon = \sigma = 1$, in time, having Gaussian asymptotic distributions. Inspired by Swanson [13] and Pospíšil and Tribe [12], in this work, we show that the realized power variations of the solutions of the SHEs in (1) with colored noise, in time, have infinite quadratic variation and Gaussian asymptotic distributions.

For $p > 0$, the $p$-power variation of a process $X$, with respect to a subdivision $\pi_n = \{0 = t_{n,0} < t_{n,1} < \ldots < t_{n,n} = 1\}$ of $[0,1]$, is defined to be the sum

$$\sum_{j=1}^{n} \left|X(t_{n,j}) - X(t_{n,j-1})\right|^p. \tag{7}$$

For simplicity, consider from now on the case where $t_{n,j} = j/n$, for $n \in \mathbb{N}$ and $j \in \{1,\ldots,n\}$. In this work, we wish to point out some interesting phenomena when $X$ is the solution to a SHE with colored noise. In fact, we will also drop the absolute value (when $p$ is odd). More precisely, we will consider

$$\sum_{j=1}^{n} \Delta X_j^p, \tag{8}$$

where $\Delta X_j = \Delta X(j/n)$ denotes the increment $X(j/n) - X((j-1)/n)$).

The analysis of the asymptotic behavior of quantities of type (8) is motivated, for instance, by the study of the exact rates of convergence of some approximation schemes of scalar stochastic differential equations driven by a Brownian motion (BM) $B$ (see, e.g., [15–17]), besides, of course, the traditional applications of quadratic variations to parameter estimation problems.

Now, let us recall some known results concerning the $p$-power variations (for $p \in \mathbb{N}_+$), which are today more or less classical. First, assume that $B$ is the standard BM. Let $\mu_p$ denote the $p$-moment of a standard Gaussian random variable following an $\mathcal{N}(0,1)$ law, that is, $\mu_{2p-1} = 0$ and $\mu_{2p} = (2p-1)!! = (2p)!/(p!2^p)$ for all $p \in \mathbb{N}_+$. By the scaling property of the BM and using the CLT, it is immediate that (see, e.g., [17]), as $n \longrightarrow \infty$:

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(n^{p/2}\Delta B_j^p - \mu_p\right) \xrightarrow{\mathscr{L}} \mathcal{N}\left(0, \mu_{2p} - \mu_p^2\right). \tag{9}$$

Assume that $H \neq 1/2$, that is, the case where the fractional Brownian motion (FBM) $B^H$ has no independent increments anymore. Then, (9) has been extended by Corcuera et al. [15], Nourdin [17], Dobrushin and Major [18], Taqqu [19], Breuer and Major [20], Giraitis and Surgailis [21], Wang [22], and Wang and Wang [23]. Swanson [13] extended (9) to modifications of the quadratic variation of the solutions of SHE driven by space-time white noise. Motivated by (9), in this work, we show that (9) with different mean and variance also holds for the solution to SHE with colored noise.

Our proofs are based on the method of Swanson [13]. We make use of the product moments of various orders of the normal correlation surface of two variates in Pearson and

Young [24] to establish exact convergence rates of variances of the realized power variation of the process $u$ with respect to time. This work builds on the recent works on delicate analysis of variations of general Gaussian processes and SHEs driven by space-time white noise.

## 2. Results

In order to state our results, we first introduce some notations. Let $X_{\alpha,d}(t) = u_{\alpha,d}(t,x)$, where $x \in \mathbb{R}$ is fixed. We consider discrete Riemann sums over a uniformly spaced time partition $t_j = j\Delta t$, where $\Delta t = n^{-1}$. Let $\Delta X_{\alpha,d;j} = X_{\alpha,d}(t_j) - X_{\alpha,d}(t_{j-1})$ and $\sigma^2_{\alpha,d;j} = \mathbb{E}[\Delta X^2_{\alpha,d;j}]$. For any $p \in \mathbb{N}_+$ and $n \in \mathbb{N}_+$, we define

$$V_p^n\left(X_{\alpha,d}\right)_t = \sum_{j=1}^{\lfloor nt \rfloor} \Delta X^p_{\alpha,d;j}. \tag{10}$$

Here and in the sequel, $\lfloor a \rfloor$ denotes an integer satisfying $a-1 < \lfloor a \rfloor \le a$ for $a \in \mathbb{R}_+$.

Let $\theta = \theta(\alpha,d) = (d-\alpha)/2$. For $j \in \mathbb{N}_+$, let $a_j = a(\alpha,d,j) = 2j^{1-\theta} - (j-1)^{1-\theta} - (j+1)^{1-\theta}$. For real number $r \ge 1$, define $b_r = b(\alpha,d,r) = \sum_{j=1}^{\infty} a_j^r$. It follows from (44) below that $b_r$ is a positive and finite constant depending on $\alpha, d$, and $r$. For any $p \in \mathbb{N}_+$, we put

$$\kappa_{\alpha,d,p} = \begin{cases} K^p_{\alpha,d}\left(\mu_{2p} - \mu_p^2 + \dfrac{p!p!}{2^{p-1}} \displaystyle\sum_{u=1}^{\lfloor p/2 \rfloor} \dfrac{2^{2u} b_{2u}}{(\lfloor p/2 \rfloor - u)!(\lfloor p/2 \rfloor - u)!(2u)!}\right), & \text{if } p \text{ is even}, \\ K^p_{\alpha,d}\left(\mu_{2p} - \dfrac{p!p!}{2^{p-2}} \displaystyle\sum_{u=0}^{\lfloor p/2 \rfloor} \dfrac{2^{2u} b_{2u+1}}{(\lfloor p/2 \rfloor - u)!(\lfloor p/2 \rfloor - u)!(2u+1)!}\right), & \text{if } p \text{ is odd}, \end{cases} \tag{11}$$

where

$$K_{\alpha,d} = K_{\alpha,d}(\theta,\varepsilon) = \frac{\Gamma(\theta)}{2^{(d+\alpha)/2}\pi^{d/2}\Gamma(d/2)(1-\theta)}\left(\frac{2}{\varepsilon}\right)^{d/4}, \tag{12}$$

where $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\mathrm{d}x$, $s>0$, is the gamma function.

We will first show the exact convergence rate of variance for the realized power variation of the process $X_{\alpha,d}$.

**Theorem 1.** *Fix $\varepsilon > 0$ and $x \in \mathbb{R}^d$ and assume $\alpha + 1 \le d < \alpha + 2$. Assume that $w = 0$ and $\sigma = 1$ in (1). Then, for each fixed $t > 0$ and any $p \in \mathbb{N}_+$,*

$$n^{-1+p(1-\theta)}\mathrm{Var}\left(V_p^n\left(X_{\alpha,d}\right)_t\right) \longrightarrow \kappa_{\alpha,d,p} t, \tag{13}$$

*as n tends to infinity.*

By (13), we have the following convergence in probability for the realized power variation of the process $X_{\alpha,d}$.

**Corollary 2.** *Fix $\varepsilon > 0$ and $x \in \mathbb{R}^d$ and assume $\alpha + 1 \le d < \alpha + 2$. Assume that $w = 0$ and $\sigma = 1$ in (1). Then, for each fixed $t > 0$ and any $p \in \mathbb{N}_+$,*

$$n^{-1+p(1-\theta)}V_{2p}^n\left(X_{\alpha,d}\right)_t \longrightarrow K^p_{\alpha,d}\mu_{2p}t, \tag{14}$$

*in $L^2$ and in probability as n tends to infinity.*

*Remark 3.* Since $V^n_{2p}(X_{\alpha,d})_t$ is monotone, (14) implies that $n^{-1+p(1-\theta)}V^n_{2p}(X_{\alpha,d})_t \longrightarrow K^p_{\alpha,d}\mu_{2p}t$ uniform convergence in probability in the time interval $[0,T]$ with some $T>0$. Moreover, (14) implies that the process $X_{\alpha,d}$ has infinite quadratic variation.

*Example 4.* If $\alpha \uparrow 1$ and $d=2$, the 4-th variation, namely, $p=2$ in (14), the corresponding constant of the right-hand side of (14) is equal to $3/(\varepsilon\pi)$.

The CLT for the realized power variation of the process $X_{\alpha,d}$ is as follows.

**Theorem 2.** *Fix $\varepsilon > 0$ and $x \in \mathbb{R}^d$ and assume $\alpha + 1 \le d < \alpha + 2$. Assume that $w = 0$ and $\sigma = 1$ in (1). Then, for any $p \in \mathbb{N}_+$,*

$$\left(X_{\alpha,d}(t), \frac{1}{\sqrt{n}}\sum_{j=1}^{\lfloor nt \rfloor}\left(n^{p(1-\theta)/2}\Delta X^p_{\alpha,d;j} - K^{p/2}_{\alpha,d}\mu_p\right)\right) \xrightarrow{\mathscr{L}} \left(X_{\alpha,d}(t), \kappa^{1/2}_{\alpha,d,p}B(t)\right), \tag{15}$$

*as n tends to infinity, where $B = \{B(t), t \in [0,T]\}$ is a BM independent of the process $X_{\alpha,d}$, and the convergence is in the space $D([0,T])^2$ equipped with the Skorokhod topology.*

*Remark 6.* Comparing (15) and (9), we have that the realized power variations of the process $X_{\alpha,d}$ for $\alpha+1 \le d < \alpha+2$ share similar Gaussian asymptotic properties with those of BM.

Throughout this paper, positive and finite constants are numbered as $c_{2,1}, c_{2,2}, \ldots$ or $c_{3,1}, c_{3,2}, \ldots$.

## 3. Proofs

*3.1. Preliminaries.* We need the following product moment of various orders of the normal correlation surface of two variates, which are equations (9) and (12) in Pearson and Young [24].

**Lemma 7.** *Suppose that $(\xi,\eta) \sim \mathcal{N}\left(0, \begin{pmatrix} \sigma_1^2 & \rho \\ \rho & \sigma_2^2 \end{pmatrix}\right)$, where $\rho = (\sigma_1\sigma_2)^{-1}\mathbb{E}[\xi\eta]$. Then,*

$$\mathbb{E}[\xi^p\eta^p] = \begin{cases} \dfrac{p!p!}{2^p}\sigma_1^p\sigma_2^p \displaystyle\sum_{j=1}^{p/2} \frac{(2\rho)^{2j}}{(p/2-j)!(p/2-j)!(2j)!}, & \text{if } p \text{ is even}, \\ \dfrac{\rho p!p!}{2^{p-1}}\sigma_1^p\sigma_2^p \displaystyle\sum_{j=0}^{\lfloor p/2\rfloor} \frac{(2\rho)^{2j}}{(\lfloor p/2\rfloor-j)!(\lfloor p/2\rfloor-j)!(2j+1)!}, & \text{if } p \text{ is odd}. \end{cases} \tag{16}$$

We also derive some needed estimates on the covariance function and the variance function of increments of $X_{\alpha,d}$.

**Lemma 8.** *Fix $\varepsilon > 0$ and $x \in \mathbb{R}^d$ and assume $\alpha+1 \le d < \alpha+2$. Assume that $w = 0$ and $\sigma = 1$ in (1). Then, for all $s,t \in [0,T]$,*

$$\mathbb{E}\left[X_{\alpha,d}(t)X_{\alpha,d}(s)\right] = K_{\alpha,d}\left((t+s)^{1-\theta} - (t-s)^{1-\theta}\right), \tag{17}$$

$$c_{2,1}|t-s|^{1-\theta} \le \mathbb{E}\left[\left(X_{\alpha,d}(t) - X_{\alpha,d}(s)\right)^2\right] \le c_{2,2}|t-s|^{1-\theta}, \tag{18}$$

*and*

$$\left|\mathbb{E}\left[\left(X_{\alpha,d}(t) - X_{\alpha,d}(s)\right)^2\right] - K_{\alpha,d}\left|t-s\right|^{1-\theta}\right| \le \frac{c_{2,3}}{s^{\theta+1}}(t-s)^2, \tag{19}$$

*where $K_{\alpha,d}$ is given in (12).*

*Proof.* By Proposition 2.3 of Tudor [10], one has that (17) holds with

$$K_{\alpha,d} = \frac{(2\pi)^{-d}}{1-\theta}\int_{\mathbb{R}^d}\frac{d\xi}{|\xi|^\alpha}e^{-|\xi|^2/2}. \tag{20}$$

By the following integral formula (see Corollary on page 23 in [25]):

$$\int_{\mathbb{R}^d} f\left(\sum_{i=1}^d x_i^2\right)dx_1\ldots dx_d = \frac{\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty y^{d/2-1}f(y)dy, \tag{21}$$

the constant $K_{\alpha,d}$ becomes

$$K_{\alpha,d} = \frac{\Gamma(\theta)}{2^{(d+\alpha)/2}\pi^{d/2}\Gamma(d/2)(1-\theta)}. \tag{22}$$

This is (12) and yields (17).

Equation (18) is cited from Theorem 2.2 in Tudor [10]. It remains to show (19). To show (19), we define the following pinned string process in time $\{U_{\alpha,d}(t), t \ge 0\}$ by

$$U_{\alpha,d}(t) = \int_0^t\int_{\mathbb{R}^d} G(t-u,x-y)W_{\alpha,d}(du,dy) + \int_{-\infty}^0\int_{\mathbb{R}^d}(G(t-u,x-y) - G(-u,x-y))W_{\alpha,d}(du,dy). \tag{23}$$

Note that $U_{\alpha,d}(0) = 0$ and $U_{\alpha,d}(t)$ can be expressed as

$$U_{\alpha,d}(t) = \int_{\mathbb{R}}\int_{\mathbb{R}^d}(G((t-u)_+,x-y) - G((-u)_+,x-y))W_{\alpha,d}(du,dy). \tag{24}$$

In the above, $a_+ = \max(a,0)$. Now for every $t \ge 0$, one has the following decomposition:

$$X_{\alpha,d}(t) = U_{\alpha,d}(t) - Y_{\alpha,d}(t), \tag{25}$$

where

$$Y_{\alpha,d}(t) = \int_R \int R^d (G(t-u, x-y))I_{0>u} - G((-u)_+, x-y)W_{\alpha,d}(\mathrm{d}u, \mathrm{d}y). \tag{26}$$

Following the same lines as the Proof of Theorem 1 of Tudor and Xiao [14], for any $0 \le s < t$,

$$\mathbb{E}\left[\left|U_{\alpha,d}(t) - U_{\alpha,d}(s)\right|^2\right] = K_{\alpha,d}(t-s)^{1-\theta}. \tag{27}$$

Denote by $\mu(d\xi) = |\xi|^{-\alpha} d\xi$ the tempered non-negative measure on $\mathbb{R}^d$. Let $\mathcal{F}\varphi$ denote the Fourier transform of the function $u \longmapsto \varphi(u)$ and $f$ be the Riesz kernel defined in (3). Then, for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$ (see, e.g., [10, 14]),

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x) f(x-y) \psi(y) \mathrm{d}x \mathrm{d}y = (2\pi)^{-d} \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi) \overline{\mathcal{F}\psi(\xi)} \mu(\mathrm{d}\xi). \tag{28}$$

It follows from (28) that for any $0 \le s < t$,

$$\begin{aligned}
\mathbb{E}\left[\left|Y_{\alpha,d}(t) - Y_{\alpha,d}(s)\right|^2\right] &= \mathbb{E}\left(\int_{\mathbb{R}} \int_{\mathbb{R}^d} (G(t-u, x-y)I_{0>u} - G(s-u, x-y)I_{0>u}) W_{\alpha,d}(\mathrm{d}u, \mathrm{d}y)\right)^2 \\
&= \int_{\mathbb{R}} \mathrm{d}u \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (G(t-u, x-y)I_{0>u} - G(s-u, x-y)I_{0>u}) \times (G(t-u, x-y')I_{0>u} \\
&\quad - G(s-u, x-y')I_{0>u}) f(y-y') \mathrm{d}y \mathrm{d}y' \\
&= (2\pi)^{-d} \int_R \mathrm{d}u \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi) \mathcal{F}(G(t-u, x-\cdot)I_{0>u} - G(s-u, x-\cdot)I_{0>u}) \\
&\quad (\xi) \times \overline{\mathcal{F}(G(t-u, x-\cdot)I_{0>u} - G(s-u, x-\cdot)I_{0>u})(\xi)} \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi) \int_{\mathbb{R}} \left(e^{-(t-u)\varepsilon|\xi|^2/2} I_{0>u} - e^{-(s-u)\varepsilon|\xi|^2/2} I_{0>u}\right)^2 \mathrm{d}u \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi) \int_{-\infty}^0 \left(e^{-(t-u)\varepsilon|\xi|^2/2} - e^{-(s-u)\varepsilon|\xi|^2/2}\right)^2 \mathrm{d}u.
\end{aligned} \tag{29}$$

Since $|1 - e^{-x}| \le 2x$ for all $x \ge 0$, one has for all $0 \le s < t$ and $\xi \in \mathbb{R}$,

$$\left|e^{-t\varepsilon|\xi|^2/2} - e^{-s\varepsilon|\xi|^2/2}\right| = e^{-s\varepsilon|\xi|^2/2}\left|1 - e^{-(t-s)\varepsilon|\xi|^2/2}\right| \le (t-s)\varepsilon|\xi|^2 e^{-s\varepsilon|\xi|^2/2}. \tag{30}$$

Thus, by (21), for any $0 < s < t$,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\mu(\mathrm{d}\xi)\int_{-\infty}^{0}\left(e^{-(t-u)\varepsilon|\xi|^2/2}-e^{-(s-u)\varepsilon|\xi|^2/2}\right)^2\mathrm{d}u\\
&=\int_{\mathbb{R}^d}\mu(\mathrm{d}\xi)\int_{-\infty}^{0}e^{u\varepsilon|\xi|^2}\left(e^{-t\varepsilon|\xi|^2/2}-e^{-s\varepsilon|\xi|^2/2}\right)^2\mathrm{d}u\\
&=\int_{\mathbb{R}^d}\frac{1}{|\xi|^{\alpha+2}}\left(e^{-t\varepsilon|\xi|^2/2}-e^{-s\varepsilon|\xi|^2/2}\right)^2\mathrm{d}\xi\le(t-s)^2\int_{\mathbb{R}^d}\frac{1}{|\xi|^{\alpha-2}}e^{-s\varepsilon|\xi|^2}\mathrm{d}\xi\\
&=\frac{\pi^{d/2}}{\Gamma(d/2)}\frac{(t-s)^2}{s^{\theta+1}}\int_0^\infty x^\theta e^{-x}\mathrm{d}x\le\frac{c_{2,4}}{s^{\theta+1}}(t-s)^2.
\end{aligned}\tag{31}$$

Combining (29) and (31), one has

$$\mathbb{E}\left[\left|Y_{\alpha,d}(t)-Y_{\alpha,d}(s)\right|^2\right]\le\frac{c_{2,5}}{(t-s)^2}s^{\theta+1}.\tag{32}$$

It follows from the argument of (29) that

$$\left|\mathbb{E}\left[\left|X_{\alpha,d}(t)-X_{\alpha,d}(s)\right|^2\right]-\mathbb{E}\left[\left|U_{\alpha,d}(t)-U_{\alpha,d}(s)\right|^2\right]\right|=\mathbb{E}\left[\left|Y_{\alpha,d}(t)-Y_{\alpha,d}(s)\right|^2\right].\tag{33}$$

This, together with (27) and (32), yields (19). The proof of Lemma 8 is completed. □

*3.2. Proof of Theorem 1*

*Proof* of Theorem 1. It is sufficient to prove (13) for the even $p$ case since the odd $p$ case can be proved similarly. For $1\le i<j\le\lfloor nt\rfloor$, define $\rho_{\alpha,d;ij}=(\sigma_{\alpha,d;i}\sigma_{\alpha,d;j})^{-1}\mathbb{E}[\Delta X_{\alpha,d;i}\Delta X_{\alpha,d;j}]$. Note that for a random variable $\xi$ following an $\mathcal{N}(0,\sigma^2)$ law,

$$\mathbb{E}[\xi^p]=\mu_p\sigma^p,\quad\forall p\in\mathbb{N}_+.\tag{34}$$

By (16) and (34), one has

$$\begin{aligned}
\operatorname{Var}\left(V_p^n\left(X_{\alpha,d}\right)_t\right)&=\mathbb{E}\left[\sum_{j=1}^{\lfloor nt\rfloor}\left(\Delta X^p_{\alpha,d;j}-\mu_p\sigma^p_{\alpha,d;j}\right)^2\right]\\
&=\sum_{j=1}^{\lfloor nt\rfloor}\mathbb{E}\left[\left(\Delta X^p_{\alpha,d;j}-\mu_p\sigma^p_{\alpha,d;j}\right)^2\right]+2\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\mathbb{E}\left[\left(\Delta X^p_{\alpha,d;i}-\mu_p\sigma^p_{\alpha,d;i}\right)\left(\Delta X^p_{\alpha,d;j}-\mu_p\sigma^p_{\alpha,d;j}\right)\right]\\
&=\sum_{j=1}^{\lfloor nt\rfloor}\left(\mathbb{E}\left[\Delta X^{2p}_{\alpha,d;j}\right]-\mu_p^2\sigma^{2p}_{\alpha,d;j}\right)+2\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\left(\mathbb{E}\left[\Delta X^p_{\alpha,d;i}\Delta X^p_{\alpha,d;j}\right]-\mu_p^2\sigma^p_{\alpha,d;i}\sigma^p_{\alpha,d;j}\right)\\
&=\left(\mu_{2p}-\mu_p^2\right)\sum_{j=1}^{\lfloor nt\rfloor}\sigma^{2p}_{\alpha,d;j}+\frac{p!p!}{2^{p-1}}\sum_{u=1}^{p/2}\frac{2^{2u}}{(p/2-u)!(p/2-u)!(2u)!}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\sigma^p_{\alpha,d;i}\sigma^p_{\alpha,d;j}\rho^{2u}_{\alpha,d;ij}.
\end{aligned}\tag{35}$$

It follows from (18) that

$$\sigma^2_{\alpha,d;j}\le c_{2,6}n^{-1+\theta}\ \text{for all}\ 1\le j\le\lfloor nt\rfloor.\tag{36}$$

By (19), (36), and Lagrange mean value theorem, it holds that for any real number $r>0$ and $1<j\le\lfloor nt\rfloor$,

$$\left|\sigma^r_{\alpha,d;j} - \left(K_{\alpha,d} n^{-1+\theta}\right)^{r/2}\right| \le c_{2,7}\left(\sigma^{r-2}_{\alpha,d;j} + \left(K_{\alpha,d} n^{-1+\theta}\right)^{(r-2)/2}\right)\left|\sigma^2_{\alpha,d;j} - K_{\alpha,d} n^{-1+\theta}\right| \le c_{2,8} n^{-2+(-1+\theta)(r-2)/2} t_{j-1}^{-(\theta+1)}. \tag{37}$$

Note that since $\alpha + 1 \le d < \alpha + 2$, one has $1/2 \le \theta < 1$. Thus,

$$\frac{1}{n}\sum_{j=2}^{\lfloor nt\rfloor} t_{j-1}^{-(\theta+1)/2} \longrightarrow \int_0^t x^{-(\theta+1)/2} \mathrm{d}x = \frac{2}{1-\theta} t^{-(\theta+1)/2}. \tag{38}$$

It follows from (37) (with $r = 2p$) and (38) that

$$n^{-1+p(1-\theta)} \sum_{j=1}^{\lfloor nt\rfloor} \left|\sigma^{2p}_{\alpha,d;j} - \left(K_{\alpha,d} n^{-1+\theta}\right)^p\right| \longrightarrow 0. \tag{39}$$

Hence,

$$n^{-1+p(1-\theta)} \sum_{j=1}^{\lfloor nt\rfloor} \sigma^{2p}_{\alpha,d;j} = n^{-1+p(1-\theta)} \sum_{j=1}^{\lfloor nt\rfloor}\left(\sigma^{2p}_{\alpha,d;j} - \left(K_{\alpha,d} n^{-1+\theta}\right)^p\right) + n^{-1+p(1-\theta)}\left(K_{\alpha,d} n^{-1+\theta}\right)^p \lfloor nt\rfloor \longrightarrow K^p_{\alpha,d} t. \tag{40}$$

It follows from (17) that

$$\begin{aligned}\mathbb{E}\left[\Delta X_{\alpha,d;i}\Delta X_{\alpha,d;j}\right] = {}& K_{\alpha,d} n - 1 + \theta (j+i)1-\theta - (j-i)1-\theta - (j+i-1)1-\theta + (j-i+1)1-\theta - (j+i-1)^{1-\theta} \\ & + (j-i-1)^{1-\theta} + (j+i-2)^{1-\theta} - (j-i)^{1-\theta},\end{aligned} \tag{41}$$

which simplifies to

$$\mathbb{E}\left[\Delta X_{\alpha,d;i}\Delta X_{\alpha,d;j}\right] = -K_{\alpha,d}\left(n^{-1+\theta} a_{j+i-1} + n^{-1+\theta} a_{j-i}\right), \tag{42}$$

where $a_j = 2j^{1-\theta} - (j-1)^{1-\theta} - (j+1)^{1-\theta}$. Thus, by binomial expansion, for every $1 \le u \le p/2$ and $1 \le i < j \le \lfloor nt\rfloor$,

$$\begin{aligned}\sigma^p_{\alpha,d;i}\sigma^p_{\alpha,d;j}\rho^{2u}_{\alpha,d;ij} &= \sigma^{p-2u}_{\alpha,d;i}\sigma^{p-2u}_{\alpha,d;j}\left(\mathbb{E}\left[\Delta X_{\alpha,d;i}\Delta X_{\alpha,d;j}\right]\right)^{2u} \\ &= K^{2u}_{\alpha,d}\sigma^{p-2u}_{\alpha,d;i}\sigma^{p-2u}_{\alpha,d;j}\left(n^{-1+\theta}a_{j+i-1} + n^{-1+\theta}a_{j-i}\right)^{2u} \\ &= K^{2u}_{\alpha,d}\sum_{v=0}^{2u}\binom{2u}{v}\sigma^{p-2u}_{\alpha,d;i}\sigma^{p-2u}_{\alpha,d;j}\left(n^{-1+\theta}a_{j+i-1}\right)^v\left(n^{-1+\theta}a_{j-i}\right)^{2u-v}.\end{aligned} \tag{43}$$

If we write $a_k = f(k-1) - f(k)$, where $f(x) = (x+1)^{1-\theta} - x^{1-\theta}$, then for each $k \ge 2$, the Lagrange mean value theorem gives $a_k = |f'(k-\zeta_1)| = \theta(1-\theta)(k-\zeta_1+\zeta_2)^{-\theta-1}$ for some $\zeta_1, \zeta_2 \in [0,1]$. This yields that for all $k \in \mathbb{N}_+$,

$$0 < a_k \le \frac{c_{2,9}}{k^{\theta+1}}, \tag{44}$$

and hence for any $r \ge 1$,

$$\sum_{k=1}^{M} a^r_k \longrightarrow b_r, \tag{45}$$

with some $b_r = b(r) > 0$ as $M \longrightarrow \infty$.

Note that since $j + i - 1 \ge (j+i)/2$, one has

$$n^{-1+\theta} a_{j+i-1} \le \frac{c_{2,10}}{n^2}\frac{1}{\left(t_i + t_j\right)^{\theta+1}}. \tag{46}$$

Note that (44) gives $n^{-1+\theta}a_{j-i} \le c_{2,11} n^{-1+\theta}$ and $n^{-1+\theta}a_{j+i-1} \le c_{2,12} n^{-1+\theta}$ for all $1 \le i < j \le \lfloor nt\rfloor$. Thus, by (36) and (46), for every $1 \le u \le p/2$ and $1 \le v \le 2u$,

$$n^{-1+p(1-\theta)}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\sigma^{p-2u}_{\alpha,d;i}\sigma^{p-2u}_{\alpha,d;j}\left(n^{-1+\theta}a_{j+i-1}\right)^v\left(n^{-1+\theta}a_{j-i}\right)^{2u-v} \le c_{2,13} n^{-\theta}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\left(n^{-1+\theta}a_{j+i-1}\right) \le c_{2,14} n^{-2-\theta}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\frac{1}{\left(t_i+t_j\right)^{\theta+1}}, \tag{47}$$

which tends to zero as $n \longrightarrow \infty$ since $\int_0^t\int_0^t (x+y)^{-(\theta+1)}\mathrm{d}x\mathrm{d}y < \infty$.

We now consider the term $v = 0$ in (43). Let $B^H = \{B^H(t), t \in \mathbb{R}_+\}$ be a FBM with index $H \in (0,1)$, which is a centered Gaussian process with $\mathbb{E}[(B^H(t) - B^H(s))^2] = |s-t|^{2H}$ for $s,t \in \mathbb{R}_+$. Then, for $H_0 = (1-\theta)/2$,

$$\mathbb{E}\left[\left(B^{H_0}\left(\frac{j+1}{n}\right) - B^{H_0}\left(\frac{j}{n}\right)\right)\left(B^{H_0}\left(\frac{i+1}{n}\right) - B^{H_0}\left(\frac{i}{n}\right)\right)\right] = -\frac{1}{2}\left[2\left(\frac{j-i}{n}\right)^{1-\theta} - \left(\frac{j-i-1}{n}\right)^{1-\theta} - \left(\frac{j-i+1}{n}\right)^{1-\theta}\right] = -\frac{1}{2}n^{-1+\theta}a_{j-i}. \tag{48}$$

Thus,

$$\begin{aligned} n^{-1+\theta}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor} a_{j-i} &= n^{-1+\theta}\sum_{i=1}^{\lfloor nt\rfloor-1}\sum_{j=i+1}^{\lfloor nt\rfloor} a_{j-i} \\ &= -2\sum_{i=1}^{\lfloor nt\rfloor-1}\sum_{j=i+1}^{\lfloor nt\rfloor}\mathbb{E}\left[\left(B^{H_0}\left(\frac{j+1}{n}\right) - B^{H_0}\left(\frac{j}{n}\right)\right)\left(B^{H_0}\left(\frac{i+1}{n}\right) - B^{H_0}\left(\frac{i}{n}\right)\right)\right] \\ &= -2\sum_{i=1}^{\lfloor nt\rfloor-1}\mathbb{E}\left[\left(B^{H_0}\left(\frac{\lfloor nt\rfloor+1}{n}\right) - B^{H_0}\left(\frac{i+1}{n}\right)\right)\left(B^{H_0}\left(\frac{i+1}{n}\right) - B^{H_0}\left(\frac{i}{n}\right)\right)\right] \\ &= -\sum_{i=1}^{\lfloor nt\rfloor-1}\left[-\left(\frac{\lfloor nt\rfloor - i}{n}\right)^{1-\theta} + \left(\frac{\lfloor nt\rfloor+1-i}{n}\right)^{1-\theta} - \left(\frac{1}{n}\right)^{1-\theta}\right] \\ &= -\left(\frac{\lfloor nt\rfloor}{n}\right)^{1-\theta} + \left(\frac{1}{n}\right)^{1-\theta} + \lfloor nt\rfloor n^{-1+\theta}. \end{aligned} \tag{49}$$

This yields

$$n^{-\theta}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\left(n^{-1+\theta}a_{j-i}\right) \longrightarrow t. \tag{50}$$

By (36) and (44), for every $1 \le u \le p/2$ and any $M > 0$,

$$\begin{aligned} n^{-1+p(1-\theta)}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+M+1}^{\lfloor nt\rfloor}\sigma_{\alpha,d;i}^{p-2u}\sigma_{\alpha,d;j}^{p-2u}\left(n^{-1+\theta}a_{j-i}\right)^{2u} &\le c_{2,15}M^{-(\theta+1)(2u-1)}n^{-\theta}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+M+1}^{\lfloor nt\rfloor}\left(n^{-1+\theta}a_{j-i}\right) \\ &\le c_{2,16}M^{-(\theta+1)(2u-1)}n^{-\theta}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\left(n^{-1+\theta}a_{j-i}\right). \end{aligned} \tag{51}$$

This, together with (45), yields

$$n^{-1+p(1-\theta)}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+M+1}^{\lfloor nt\rfloor}\sigma_{\alpha,d;i}^{p-2u}\sigma_{\alpha,d;j}^{p-2u}\left(n^{-1+\theta}a_{j-i}\right)^{2u} \le c_{2,17}M^{-(\theta+1)(2u-1)}t, \tag{52}$$

which tends to zero by letting $M \longrightarrow \infty$.

By (37) (with $r = p-2u$), (36), and (48), for every $1 \le u \le p/2$,

$$\begin{aligned}
&n^{-1+p(1-\theta)}\sum_{i=2}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\left|\sigma_{\alpha,d;i}^{p-2u}-\left(K_{\alpha,d}n^{-1+\theta}\right)^{(p-2u)/2}\right|\sigma_{\alpha,d;j}^{p-2u}\left(n^{-1+\theta}a_{j-i}\right)^{2u}\le c_{2,18}n^{-1-2\theta}\sum_{i=2}^{\lfloor nt\rfloor}\frac{1}{t_{i-1}^{\theta+1}}\sum_{j=i+1}^{\lfloor nt\rfloor}\left(n^{-1+\theta}a_{j-i}\right)\\
&=-2c_{2,18}n^{-1-2\theta}\sum_{i=2}^{\lfloor nt\rfloor}\frac{1}{t_{i-1}^{\theta+1}}\left[-\left(\frac{\lfloor nt\rfloor-i}{n}\right)^{1-\theta}+\left(\frac{\lfloor nt\rfloor+1-i}{n}\right)^{1-\theta}-\left(\frac{1}{n}\right)^{1-\theta}\right]\\
&\le c_{2,19}n^{-\theta}\sum_{i=2}^{\lfloor nt\rfloor}\left[-\left(\frac{\lfloor nt\rfloor-i}{n}\right)^{1-\theta}+\left(\frac{\lfloor nt\rfloor+1-i}{n}\right)^{1-\theta}\right]+c_{2,20}n^{-2-\theta}\sum_{i=2}^{\lfloor nt\rfloor}\frac{1}{t_{i-1}^{\theta+1}}\\
&\le c_{2,21}n^{-\theta}\left[\left(\frac{1}{n}\right)^{1-\theta}+\left(\frac{\lfloor nt\rfloor-1}{n}\right)^{1-\theta}\right]+c_{2,22}n^{-3/2-\theta/2}\sum_{i=2}^{\lfloor nt\rfloor}t_{i-1}^{-(\theta+1)/2},
\end{aligned}\tag{53}$$

which tends to zero as $n\longrightarrow\infty$ since $\int_0^t x^{-(\theta+1)/2}\mathrm{d}x<\infty$. Hence, for every $1\le u\le p/2$,

$$n^{-1+p(1-\theta)}\sum_{i=2}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\left(\sigma_{\alpha,d;i}^{p-2u}-\left(K_{\alpha,d}n^{-1+\theta}\right)^{(p-2u)/2}\right)\sigma_{\alpha,d;j}^{p-2u}\left(n^{-1+\theta}a_{j-i}\right)^{2u}\longrightarrow 0.\tag{54}$$

Similarly, for every $1\le u\le p/2$,

$$n^{-1+p(1-\theta)}\sum_{i=2}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\left(K_{\alpha,d}n^{-1+\theta}\right)^{(p-2u)/2}\times\left(\sigma_{\alpha,d;j}^{p-2u}-\left(K_{\alpha,d}n^{-1+\theta}\right)^{(p-2u)/2}\right)\left(n^{-1+\theta}a_{j-i}\right)^{2u}\longrightarrow 0.\tag{55}$$

For every $1\le u\le p/2$ and any $M>0$,

$$n^{-1+p(1-\theta)}\sum_{i=2}^{\lfloor nt\rfloor}\sum_{j=i+1}^{i+M}\left(K_{\alpha,d}n^{-1+\theta}\right)^{p-2u}\left(n^{-1+\theta}a_{j-i}\right)^{2u}=K_{\alpha,d}^{p-2u}\frac{\lfloor nt\rfloor-1}{n}\sum_{j=1}^{M}a_j^{2u}\longrightarrow K_{\alpha,d}^{p-2u}b_{2u}t,\tag{56}$$

as $n\longrightarrow\infty$ and $M\longrightarrow\infty$. Note that for every $1\le u\le p/2$ and $1\le i<j\le\lfloor nt\rfloor$,

$$\sigma_{\alpha,d;i}^{p-2u}\sigma_{\alpha,d;j}^{p-2u}=\left(\sigma_{\alpha,d;i}^{p-2u}-\left(K_{\alpha,d}n^{-1+\theta}\right)^{(p-2u)/2}\right)\sigma_{\alpha,d;j}^{p-2u}+\left(K_{\alpha,d}n^{-1+\theta}\right)^{(p-2u)/2}\left(\sigma_{\alpha,d;j}^{p-2u}-\left(K_{\alpha,d}n^{-1+\theta}\right)^{(p-2u)/2}\right)+\left(K_{\alpha,d}n^{-1+\theta}\right)^{p-2u}.\tag{57}$$

Hence, by (54)–(57), for every $1\le u\le p/2$,

$$n^{-1+p(1-\theta)}\sum_{i=2}^{\lfloor nt\rfloor}\sum_{j=i+1}^{i+M}\sigma_{\alpha,d;i}^{p-2u}\sigma_{\alpha,d;j}^{p-2u}\left(n^{-1+\theta}a_{j-i}\right)^{2u}\longrightarrow K_{\alpha,d}^{p-2u}b_{2u}t,\tag{58}$$

as $n\longrightarrow\infty$ and $M\longrightarrow\infty$. It follows from (36) that

$$n^{-1+p(1-\theta)}\sum_{j=2}^{1+M}\sigma_{\alpha,d;i}^{p-2u}\sigma_{\alpha,d;j}^{p-2u}\left(n^{-1+\theta}a_{j-1}\right)^{2u}\longrightarrow 0. \tag{59}$$

This, together with (43), (47), and (58), yields for every $1\le u\le p/2$,

$$n^{-1+p(1-\theta)}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\sigma_{\alpha,d;i}^{p}\sigma_{\alpha,d;j}^{p}\rho_{\alpha,d;ij}^{2u}\longrightarrow K_{\alpha,d}^{p}b_{2u}t. \tag{60}$$

Therefore, by (35), (40), and (60), one has

$$n^{-1+p(1-\theta)}\mathrm{Var}\left(V_p^n\left(X_{\alpha,d}\right)_t\right)\longrightarrow K_{\alpha,d}^{p}\left(\mu_{2p}-\mu_p^2+\frac{p!p!}{2^{p-1}}\sum_{u=1}^{p/2}\frac{2^{2u}b_{2u}}{(p/2-u)!(p/2-u)!(2u)!}\right)t=\kappa_{\alpha,d,p}t. \tag{61}$$

This proves (13). The Proof of Theorem 1 is completed. □

*Proof.* of Corollary 2. Write

$$\begin{aligned}n^{-1+p(1-\theta)/2}V_p^n\left(X_{\alpha,d}\right)_t-K_{\alpha,d}^{p/2}\mu_p t&=n^{-1+p(1-\theta)/2}\left(V_p^n\left(X_{\alpha,d}\right)_t-\mathbb{E}\left[V_p^n\left(X_{\alpha,d}\right)_t\right]\right)+\mu_p n^{-1+p(1-\theta)/2}\\&\sum_{j=1}^{\lfloor nt\rfloor}\left(\sigma_{\alpha,d;j}^{p}-\left(K_{\alpha,d}n^{-1+\theta}\right)^{p/2}\right)+K_{\alpha,d}^{p/2}\mu_p\left(\frac{\lfloor nt\rfloor}{n}-t\right).\end{aligned} \tag{62}$$

Obviously, the third term of (62) tends to zero as $n\longrightarrow\infty$. It follows from (37) (with $r=p$) and (38) that the second term of (62) tends to zero as $n\longrightarrow\infty$. Thus, by (13),

$$\mathbb{E}\left[\left|n^{-1+p(1-\theta)/2}V_p^n\left(X_{\alpha,d}\right)_t-K_{\alpha,d}^{p/2}\mu_p t\right|^2\right]\longrightarrow 0. \tag{63}$$

This proves (14). □

*3.3. Proof of Theorem 2.* The following lemma is needed to prove Theorem 2.

**Lemma 9.** *Let $F_1,\ldots,F_4$ be normal random variables with mean zero, $\mathbb{E}[F_j^2]=1$ and $\rho_{ij}=\mathbb{E}[F_iF_j]$. Put $\xi_j=F_j^p-E[F_j^p]$. Then, for any $p\in\mathbb{N}_+$,*

$$\left|\mathbb{E}\left[\prod_{j=1}^{4}\xi_j\right]\right|\le c_{3,1}\left(\left|\rho_{12}\rho_{34}\right|+\frac{1}{\sqrt{1-\rho_{12}^2}}\max_{i\le 2<j}\left|\rho_{ij}\right|\right), \tag{64}$$

*whenever $|\rho_{12}|<1$. Moreover,*

$$\left|\mathbb{E}\left[\prod_{j=1}^{4}\xi_j\right]\right|\le c_{3,2}\max_{2\le j\le 4}\left|\rho_{1j}\right|. \tag{65}$$

Furthermore, there exists $\varepsilon>0$ such that

$$\left|\mathbb{E}\left[\prod_{j=1}^{4}\xi_j\right]\right|\le c_{3,3}\max_{1\le i\ne j\le 4}\rho_{ij}^2, \tag{66}$$

whenever $|\rho_{ij}|<\varepsilon$ for all $1\le i\ne j\le 4$.

*Proof.* Following the same lines as the proof of Lemma 3.3 in Swanson [13] with $h_j(F_j)=\xi_j$, $1\le j\le 4$, we get Lemma 9 immediately. □

**Proposition 10.** *Fix $\varepsilon>0$ and $x\in\mathbb{R}^d$ and assume $\alpha+1\le d<\alpha+2$. Assume that $w=0$ and $\sigma=1$ in (1). Fix $r\in\mathbb{N}_+$. Put*

$$W_r^n\left(X_{\alpha,d}\right)_t=n^{-1/2+r(1-\theta)/2}\sum_{i=1}^{\lfloor nt\rfloor}\left(\Delta X_{\alpha,d;i}^{r}-\mu_r\sigma_{\alpha,d;i}^{r}\right). \tag{67}$$

Then, for all $0\le s<t$ and all $n\in\mathbb{N}_+$,

$$\mathbb{E}\left[\left|W_r^n\left(X_{\alpha,d}\right)_t-W_r^n\left(X_{\alpha,d}\right)_s\right|^4\right]\le c_{3,4}\left(\frac{\lfloor nt\rfloor-\lfloor ns\rfloor}{n}\right)^2. \tag{68}$$

The sequence $\{W_r^n(X_{\alpha,d})\}$ is therefore relatively compact in the Skorokhod space $D_{\mathbb{R}}[0,\infty)$.

*Proof.* We follow the method of Proposition 3.5 in Swanson [13] to prove (68). Let $\mathcal{S}=\{j\in\mathbb{N}_+^4\colon \lfloor ns\rfloor+1\le j_1\le\ldots\le j_4\le\lfloor nt\rfloor\}$. For $j\in\mathcal{S}$ and $k\in\{1,2,3\}$, define $h_k=j_{k+1}-j_k$ and let $\mathcal{S}_k=\{j\in\mathcal{S}\colon h_k=\max\{h_1,h_2,h_3\}\}$. Define $N=\lfloor nt\rfloor-(\lfloor ns\rfloor+1)$, and for $i\in\{0,1,\ldots,N\}$, let $\mathcal{S}_k^i=\{j\in\mathcal{S}_k\colon \max\{h_1,h_2,h_3\}=i\}$. Further define $\mathcal{T}_k^{\ell}=\mathcal{T}_k^{i,\ell}=\{j\in\mathcal{S}_k^i\colon \min\{h_1,h_2,h_3\}=\ell\}$ and $\mathcal{V}_k^{v}=\mathcal{V}_k^{i,\ell,v}=\{j\in\mathcal{T}_k^{\ell}\colon \mathrm{med}\{h_1,h_2,h_3\}=v\}$, where "med" denotes the median function. For $j\in\mathcal{S}$, define

$$U_{\alpha,d;j}=\prod_{k=1}^{4}\left(\Delta X_{\alpha,d;j_k}^{r}-\mu_r\sigma_{\alpha,d;j_k}^{r}\right). \tag{69}$$

Observe that

$$\mathbb{E}\left[\left|W_r^n(X_{\alpha,d})_t - W_r^n(X_{\alpha,d})_s\right|^4\right] = n^{-2+2r(1-\theta)}\mathbb{E}\left[\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\left(\Delta X_{\alpha,d;i}^r - \mu_r\sigma_{\alpha,d;i}^r\right)^4\right]$$
$$\leq 4!n^{-2+2r(1-\theta)}\sum_{j\in\mathcal{S}}\left|\mathbb{E}\left[U_{\alpha,d;j}\right]\right| \leq 4!n^{-2+2r(1-\theta)}\sum_{k=1}^{3}\sum_{j\in\mathcal{S}_k}\left|\mathbb{E}\left[U_{\alpha,d;j}\right]\right|, \quad (70)$$

and that

$$\sum_{j\in\mathcal{S}_k}\left|\mathbb{E}\left[U_{\alpha,d;j}\right]\right| = \sum_{i=0}^{N}\sum_{j\in\mathcal{S}_k^i}\left|\mathbb{E}\left[U_{\alpha,d;j}\right]\right|$$
$$= \sum_{i=0}^{N}\sum_{\ell=0}^{\lfloor i^\theta\rfloor}\sum_{j\in\mathcal{T}_k^\ell}\left|\mathbb{E}\left[U_{\alpha,d;j}\right]\right| + \sum_{i=0}^{N}\sum_{\ell=\lfloor i^\theta\rfloor+1}^{i}\sum_{j\in\mathcal{T}_k^\ell}\left|\mathbb{E}\left[U_{\alpha,d;j}\right]\right| \sum_{j\in\mathcal{V}_k^\nu}\left|\mathbb{E}\left[U_{\alpha,d;j}\right]\right|. \quad (71)$$
$$= \sum_{i=0}^{N}\sum_{\ell=0}^{\lfloor i^\theta\rfloor}\sum_{\nu=\ell}^{i}\sum_{j\in\mathcal{V}_k^\nu}\left|\mathbb{E}\left[U_{\alpha,d;j}\right]\right| + \sum_{i=0}^{N}\sum_{\ell=\lfloor i^\theta\rfloor+1}^{i}\sum_{\nu=\ell}^{i}$$

Let $F_{\alpha,d;k} = \sigma_{\alpha,d;j_k}^{-1}\Delta X_{\alpha,d;j_k}$ and

$$\xi_{\alpha,d;k} = F_{\alpha,d;k}^r - \mathbb{E}\left[F_{\alpha,d;k}^r\right] = \sigma_{\alpha,d;j_k}^{-r}\left(\Delta X_{\alpha,d;j_k}^r - \mu_r\sigma_{\alpha,d;j_k}^r\right). \quad (72)$$

Then,

$$\left|\mathbb{E}\left[U_{\alpha,d;j}\right]\right| = \left(\prod_{k=1}^{4}\sigma_{\alpha,d;j_k}^r\right)\left|\mathbb{E}\left[\prod_{k=1}^{4}\xi_{\alpha,d;k}\right]\right|. \quad (73)$$

By (42) and (44), for all $k \neq l \in \mathbb{N}_+$,

$$\left|\mathbb{E}\left[\Delta X_{\alpha,d;k}\Delta X_{\alpha,d;l}\right]\right| \leq \frac{c_{3,5}n^{-1+\theta}}{|k-l|^{\theta+1}}. \quad (74)$$

It follows from (36) and (74) that

$$\left|\rho_{\alpha,d;kl}\right| = \left|\mathbb{E}F_{\alpha,d;k}F_{\alpha,d,l}\right| = \sigma_{\alpha,d;j_k}^{-1}\sigma_{\alpha,d;j_l}^{-1}\left|\mathbb{E}\left[\Delta X_{\alpha,d;j_k}\Delta X_{\alpha,d;j_l}\right]\right| \leq \frac{c_{3,6}}{\left|j_k - j_l\right|^{\theta+1}}. \quad (75)$$

Suppose $0 \leq \ell \leq \lfloor i^\theta\rfloor$. Fix $\nu$ and let $j \in \mathcal{V}_k^\nu$ be arbitrary. If $k = 1$, then $i = \max\{h_1, h_2, h_3\} = h_1 = j_2 - j_1$. If $k = 3$, then $i = \max\{h_1, h_2, h_3\} = h_3 = j_4 - j_3$. In either case, by (65), (36), (73), and (75), one has

$$\left|\mathbb{E}\left[U_{\alpha,d;j}\right]\right| \leq \frac{c_{3,7}n^{-2r(1-\theta)}}{i^{\theta+1}} \leq c_{3,7}\left(\frac{1}{(\ell\nu)^{\theta+1}} + \frac{1}{i^{\theta+1}}\right)n^{-2r(1-\theta)}. \quad (76)$$

If $k = 2$, then $i = \max\{h_1, h_2, h_3\} = h_2 = j_3 - j_2$ and $\ell\nu = h_3h_1 = (j_4 - j_3)(j_2 - j_1)$. Hence, by (64), (36), (73), and (75),

$$\left|\mathbb{E}\left[U_{\alpha,d;j}\right]\right| \leq c_{3,8}\left(\frac{1}{(\ell\nu)^{\theta+1}} + \frac{1}{i^{\theta+1}}\right)n^{-2r(1-\theta)}. \quad (77)$$

Now choose $k' \neq k$ such that $h_{k'} = \ell$. With $k'$ given, $j$ is determined by $j_k$. Since there are two possibilities for $k'$ and $N + 1$ possibilities for $j_k$, $|\mathcal{V}_k^\nu| \leq 2(N + 1)$. Therefore,

$$\sum_{\ell=0}^{\lfloor i^\theta\rfloor}\sum_{\nu=\ell}^{i}\sum_{j\in\mathcal{V}_k^\nu}\left|\mathbb{E}\left[U_{\alpha,d;j}\right]\right| \leq c_{3,9}(N+1)\sum_{\ell=0}^{\lfloor i^\theta\rfloor}\sum_{\nu=\ell}^{i}\left(\frac{1}{(\ell\nu)^{\theta+1}} + \frac{1}{i^{\theta+1}}\right)n^{-2r(1-\theta)} \leq c_{3,10}(N+1)\sum_{\ell=0}^{\lfloor i^\theta\rfloor}\left(\frac{1}{\ell^{\theta+1}} + \frac{1}{i^{\theta}}\right)n^{-2r(1-\theta)} \leq c_{3,11}(N+1)n^{-2r(1-\theta)}. \quad (78)$$

For the second summation, suppose $\lfloor i^\theta\rfloor + 1 \leq \ell \leq i$. In this case, if $j \in \mathcal{T}_k^\ell$, then $\ell = \min\{h_1, h_2, h_3\}$, so that by (66), (36), (73), and (75),

$$\left|\mathbb{E}\left[U_{\alpha,d;j}\right]\right| \leq \frac{c_{3,12}n^{-2r(1-\theta)}}{\ell^{2(\theta+1)}}. \quad (79)$$

Since $\sum_{\nu=\ell}^{i}|\mathcal{V}_k^\nu| \leq 2(N+1)i$ and $1/2 \leq \theta < 1$, one has

$$\sum_{\ell=\lfloor i^\theta\rfloor+1}^{i}\sum_{v=\ell}^{i}\sum_{j\in\mathcal{V}_k^v}\left|\mathbb{E}\left[U_{\alpha,d;j}\right]\right|\le c_{3,13}(N+1)i\sum_{\lfloor i^\theta\rfloor+1}^{i}\frac{n^{-2r(1-\theta)}}{\ell^{2(\theta+1)}}\le c_{3,14}(N+1)i\left(\int_{\lfloor i^\theta\rfloor}^{\infty}\frac{1}{x^{2(\theta+1)}}\mathrm{d}x\right)n^{-2r(1-\theta)}\le c_{3,15}(N+1)n^{-2r(1-\theta)}. \tag{80}$$

Thus, using (70), (71), (78), and (80), one has

$$n^{-2+2r(1-\theta)}\mathbb{E}\left[\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\left(\Delta X^r_{\alpha,d;j}-\mu_r\sigma^r_{\alpha,d;j}\right)^4\right]\le c_{3,16}\sum_{i=0}^{N}(N+1)n^{-2}=c_{3,16}\left(\frac{\lfloor nt\rfloor-\lfloor ns\rfloor}{n}\right)^2, \tag{81}$$

which is (68).

To show that a sequence of càdlàg processes $\{X_n\}$ is relatively compact, it suffices to show that for each $T>1$, there exist constants $\beta>0$, $C>0$, and $q>1$ such that

$$\begin{aligned}R_{X_n}(t,h)&=\mathbb{E}\left[\left|X_n(t+h)-X_n(t)\right|^\beta\left|X_n(t)-X_n(t-h)\right|^\beta\right]\\&\le Ch^q,\end{aligned} \tag{82}$$

for all $n\in\mathbb{N}$, all $t\in[0,T]$, and all $h\in[0,t]$ (see, e.g., Theorem 3.8.8 in [26]). Taking $\beta=2$ and using (68) together with Hölder inequality gives

$$R_{W_r^n(X_{\alpha,d})}(t,h)\le c_{3,17}\left(\frac{\lfloor nt+nh\rfloor-\lfloor nt\rfloor}{n}\right)\left(\frac{\lfloor nt\rfloor-\lfloor nt-nh\rfloor}{n}\right). \tag{83}$$

If $nh<1/2$, then the right-hand side of this inequality is zero. Assume $nh\ge 1/2$. Then,

$$\frac{\lfloor nt+nh\rfloor-\lfloor nt\rfloor}{n}\le\frac{nh+1}{n}\le 3h. \tag{84}$$

The other factor is similarly bounded, so that $R_{W_r^n(X_{\alpha,d})}(t,h)\le c_{3,18}h^2$. □

**Proposition 11.** *Fix $\varepsilon>0$ and $x\in\mathbb{R}^d$ and assume $\alpha+1\le d<\alpha+2$. Assume that $w=0$ and $\sigma=1$ in (1). Then, for any $0\le s<t$ and $r\in\mathbb{N}_+$,*

$$W_r^n\left(X_{\alpha,d}\right)_t-W_r^n\left(X_{\alpha,d}\right)_s\xrightarrow{\mathscr{L}}\kappa_{\alpha,d,r}^{1/2}|t-s|^{1/2}\mathcal{N}, \tag{85}$$

*as $n\longrightarrow\infty$, where $\mathcal{N}$ is a standard normal random variable.*

*Proof.* Let $\{n(j)\}_{j=1}^{\infty}$ be any sequence of natural numbers. We will prove that there exists a subsequence $\{n(j_m)\}$ such that $W_r^{n(j_m)}(X_{\alpha,d})_t-W_r^{n(j_m)}(X_{\alpha,d})_s$ converges in law to the given random variable.

For each $m\in\mathbb{N}_+$, choose $n(j_m)\in\{n(j)\}$ such that $n(j_m)>n(j_{m-1})$ and $n(j_m)\ge m^{2/\theta}(t-s)^{-1}$. Let $b=b(m)=n(j_m)(t-s)/m$. For $0\le k\le m$, define $u_k=n(j_m)s+kb$, so that

$$\begin{aligned}W_r^{n(j_m)}\left(X_{\alpha,d}\right)_t-W_r^{n(j_m)}\left(X_{\alpha,d}\right)_s&=n(j_m)^{-1/2+r(1-\theta)/2}\sum_{i=\lfloor n(j_m)s\rfloor+1}^{\lfloor n(j_m)t\rfloor}\left(\Delta X^r_{\alpha,d;i}-\mu_r\sigma^r_{\alpha,d;i}\right)\\&=n(j_m)^{-1/2+r(1-\theta)/2}\sum_{k=1}^{m}\sum_{i=u_{k-1}+1}^{u_k}\left(\Delta X^r_{\alpha,d;i}-\mu_r\sigma^r_{\alpha,d;i}\right).\end{aligned} \tag{86}$$

Let us now introduce the filtration

$$\mathscr{F}_t=\sigma\left\{W_{\alpha,d}(A)\colon A\subset[0,t]\times\mathbb{R}^d,\lambda(A)<\infty\right\}, \tag{87}$$

where $\lambda$ denotes Lebesgue measure on $\mathbb{R}^{d+1}$. Let $\tau_k=n(j_m)^{-1}u_{k-1}$. For each pair $(i,k)$ such that $u_{k-1}<i\le u_k$, define

$$\xi_{\alpha,d;ik}=\Delta X_{\alpha,d;i}-\mathbb{E}\left[\Delta X_{\alpha,d;i}|\mathscr{F}_{\tau_k}\right]. \tag{88}$$

Note that $\xi_{\alpha,d;ik}$ is $\mathscr{F}_{\tau_{k+1}}$-measurable and independent of $\mathscr{F}_{\tau_k}$. Recall that

$$X_{\alpha,d}(t)=\int_0^t\int_{\mathbb{R}^d}G(t-x,y)W_{\alpha,d}(\mathrm{d}x,\mathrm{d}y). \tag{89}$$

Also, given constants $0\le\tau\le s\le t$, one has

$$\mathbb{E}\left[X_{\alpha,d}(t)|\mathscr{F}_\tau\right]=\int_0^\tau\int_{\mathbb{R}^d}G(t-x,y)W_{\alpha,d}(\mathrm{d}x,\mathrm{d}y). \tag{90}$$

It follows from (89) and (90) that

$$X_{\alpha,d}(t+\tau_k) - \mathbb{E}\left[X_{\alpha,d}(t+\tau_k)|\mathcal{F}_{\tau_k}\right] = \int_{\tau_k}^{t+\tau_k}\int_{\mathbb{R}^d} G(t+\tau_k - x, y) W_{\alpha,d}(\mathrm{d}x, \mathrm{d}y). \tag{91}$$

This yields that $\{\xi_{\alpha,d;ik}\}$ has the same law as $\{\Delta X_{\alpha,d;i-u_{k-1}}\}$. Now define $\sigma^2_{\alpha,d;ik} = \mathbb{E}[\xi^2_{\alpha,d;ik}] = \sigma^2_{\alpha,d;i-u_{k-1}}$ and

$$\zeta_{\alpha,d;mk} = \sum_{i=u_{k-1}+1}^{u_k}\left(\xi^r_{\alpha,d;ik} - \mu_r\sigma^r_{\alpha,d;ik}\right), \tag{92}$$

so that $\zeta_{\alpha,d;mk}$, $1 \le k \le m$, are independent and

$$W_r^{n(j_m)}\left(X_{\alpha,d}\right)_t - W_r^{n(j_m)}\left(X_{\alpha,d}\right)_s = n(j_m)^{-1/2+r(1-\theta)/2}\sum_{k=1}^{m}\zeta_{\alpha,d;mk} + \varepsilon_{\alpha,d;m}, \tag{93}$$

where

$$\varepsilon_{\alpha,d;m} = n(j_m)^{-1/2+r(1-\theta)/2}\sum_{k=1}^{m}\sum_{i=u_{k-1}+1}^{u_k}\left(\left(\Delta X^r_{\alpha,d;i} - \mu_r\sigma^r_{\alpha,d;i}\right) - \left(\xi^r_{\alpha,d;ik} - \mu_r\sigma^r_{\alpha,d;ik}\right)\right). \tag{94}$$

Since $\xi_{\alpha,d;ik}$ and $\Delta X_{\alpha,d;i} - \xi_{\alpha,d;ik} = \mathbb{E}[\Delta X_{\alpha,d;i}|\mathcal{F}_{\tau_k}]$ are independent, one has

$$\begin{aligned}\sigma^2_{\alpha,d;i} &= \mathbb{E}\left[\Delta X^2_{\alpha,d;i}\right]\\ &= \mathbb{E}\left[\xi^2_{\alpha,d;ik}\right] + \mathbb{E}\left[\left|\Delta X_{\alpha,d;i} - \xi_{\alpha,d;ik}\right|^2\right]\\ &= \sigma^2_{\alpha,d;i-u_{k-1}} + \mathbb{E}\left[\left|\Delta X_{\alpha,d;i} - \xi_{\alpha,d;ik}\right|^2\right].\end{aligned} \tag{95}$$

This, together with (19), gives

$$\mathbb{E}\left[\left|\Delta X_{\alpha,d;i} - \xi_{\alpha,d;ik}\right|^2\right] = \sigma^2_{\alpha,d;i} - \sigma^2_{\alpha,d;i-u_{k-1}} \le \frac{c_{3,19}n(j_m)^{-1+\theta}}{(i-u_{k-1})^{\theta+1}}. \tag{96}$$

Thus, since $\Delta X_{\alpha,d;i} - \xi_{\alpha,d;ik}$ is Gaussian, by (34) and (96),

$$\mathbb{E}\left[\left|\Delta X_{\alpha,d;i} - \xi_{\alpha,d;ik}\right|^4\right] \le \frac{c_{3,20}n(j_m)^{-2+2\theta}}{(i-u_{k-1})^{2\theta+2}}. \tag{97}$$

Note that (34) and (36) give $\mathbb{E}[|\Delta X_{\alpha,d;i}|^{4r-4}] \le c_{3,21}\sigma^{4r-4}_{\alpha,d;i} \le c_{3,22}n(j_m)^{(-1+\theta)(2r-2)}$ and $\mathbb{E}[|\xi_{\alpha,d;ik}|^{4r-4}] \le c_{3,23}\sigma^{4r-4}_{\alpha,d;i-u_{k-1}} \le c_{3,24}n(j_m)^{(-1+\theta)(2r-2)}$. By Lagrange mean value theorem,

$$\left|\Delta X^r_{\alpha,d;i} - \xi^r_{\alpha,d;ik}\right| \le c_{3,25}\left(\left|\Delta X_{\alpha,d;i}\right|^{r-1} + \left|\xi_{\alpha,d;ik}\right|^{r-1}\right)\left|\Delta X_{\alpha,d;i} - \xi_{\alpha,d;ik}\right|. \tag{98}$$

Thus, by (97) and Hölder inequality,

$$\mathbb{E}\left[\left|\Delta X^r_{\alpha,d;i} - \xi^r_{\alpha,d;ik}\right|^2\right] \le c_{3,26}\left(\mathbb{E}\left[\left|\Delta X_{\alpha,d;i}\right|^{4r-4}\right] + \mathbb{E}\left[\left|\xi_{\alpha,d;ik}\right|^{4r-4}\right]\right)^{1/2}\left(\mathbb{E}\left[\left|\Delta X_{\alpha,d;i} - \xi_{\alpha,d;ik}\right|^4\right]\right)^{1/2} \le \frac{c_{3,27}n(j_m)^{-r(1-\theta)}}{(i-u_{k-1})^{\theta+1}}. \tag{99}$$

Similarly, by (96) and Lagrange mean value theorem,

$$\left|\sigma^r_{\alpha,d;i}-\sigma^r_{\alpha,d;ik}\right|\le c_{3,28}\left(\left|\sigma_{\alpha,d;i}\right|^{r-2}+\left|\sigma_{\alpha,d;ik}\right|^{r-2}\right)\left|\sigma^2_{\alpha,d;i}-\sigma^2_{\alpha,d;ik}\right|\le\frac{c_{3,29}n(j_m)^{-r(1-\theta)/2}}{(i-u_{k-1})^{\theta+1}}. \tag{100}$$

Therefore, by (99), (100), and Hölder inequality,

$$\begin{aligned}\mathbb{E}\left[\left|\varepsilon_{\alpha,d;m}\right|\right]&\le n(j_m)^{-1/2+r(1-\theta)/2}\sum_{k=1}^{m}\sum_{j=u_{k-1}+1}^{u_k}\left(\left(\mathbb{E}\left[\left|\Delta X^r_{\alpha,d;i}-\xi^r_{\alpha,d;ik}\right|^2\right]\right)^{1/2}+\mu_r\left|\sigma^r_{\alpha,d;j}-\sigma^r_{\alpha,d;jk}\right|\right)\\&\le c_{3,30}n(j_m)^{-1/2}\sum_{k=1}^{m}\sum_{i=u_{k-1}+1}^{u_k}(i-u_{k-1})^{-(\theta+1)/2}=c_{3,31}n(j_m)^{-1/2}\sum_{k=1}^{m}\sum_{i=1}^{u_k-u_{k-1}}i^{-(\theta+1)/2}.\end{aligned} \tag{101}$$

Since $u_k-u_{k-1}\le b$, this gives

$$\mathbb{E}\left[\left|\varepsilon_{\alpha,d;m}\right|\right]\le c_{3,32}n(j_m)^{-1/2}mb^{(1-\theta)/2}=c_{3,32}m^{(\theta+1)/2}n(j_m)^{-\theta/2}(t-s)^{(1-\theta)/2}. \tag{102}$$

But since $n(j_m)$ was chosen so that $n(j_m)\ge m^{2/\theta}(t-s)^{-1}$, one has $E[|\varepsilon_{\alpha,d;m}|]\le c_{3,33}m^{-(1-\theta)/2}|t-s|^{1/2}$ and $\epsilon_{\alpha,d;m}\longrightarrow 0$ in $L^1$ and in probability. Therefore, by (93), one needs only to show that

$$n(j_m)^{-1/2+r(1-\theta)/2}\sum_{k=1}^{m}\zeta_{\alpha,d;mk}\xrightarrow{\mathscr{L}}\kappa^{1/2}_{\alpha,d,r}|t-s|^{1/2}\mathscr{N} \tag{103}$$

in order to complete the proof.

For this, we will use the Lindeberg–Feller theorem (see, e.g., Theorem 2.4.5 in [27]), which states the following: for each $m$, let $\zeta_{\alpha,d;mk}$, $1\le k\le m$, be independent random variables with $\mathbb{E}[\zeta_{\alpha,d;mk}]=0$. Suppose

(a) $n(j_m)^{-1+r(1-\theta)}\sum_{k=1}^m\mathbb{E}[\zeta^2_{\alpha,d;mk}]\longrightarrow\nu^2$.

(b) For all $\varepsilon>0$, $\lim_{m\longrightarrow\infty}n(j_m)^{-1}+r(1-\theta)\sum_{k=1}^m\mathbb{E}[|\zeta^2_{\alpha,d;mk}|^2I^{-1/2}_{\{n(j_m)+r(1-\theta)/2\times|\zeta_{\alpha,d;mk}|>\varepsilon\}}]\longrightarrow 0$.

Then, $n(j_m)^{-1/2+r(1-\theta)/2}\sum_{k=1}^m\zeta_{\alpha,d;mk}\longrightarrow^{\mathscr{L}}\nu\mathscr{N}$ as $n\longrightarrow\infty$.

To verify these conditions, recall that $\{\xi_{\alpha,d;ik}\}$ and $\{\Delta X_{\alpha,d;i-u_{k-1}}\}$ have the same law, so that

$$\mathbb{E}\left[\left|\zeta_{\alpha,d;mk}\right|^4\right]=n(j_m)^{-2+2r(1-\theta)}\mathbb{E}\left[\left|\sum_{i=1}^{u_k-u_{k-1}}\left(\Delta X^r_{\alpha,d;i}-\mu_r\sigma^r_{\alpha,d;i}\right)\right|^4\right]. \tag{104}$$

Hence, by (68),

$$n(j_m)^{-2+2r(1-\theta)}\mathbb{E}\left[\left|\zeta_{\alpha,d;mk}\right|^4\right]\le c_{3,34}(u_k-u_{k-1})^2n(j_m)^{-2}. \tag{105}$$

Jensen inequality now gives $m^{-1+r(1-\theta)}\sum_{k=1}^m\mathbb{E}[|\zeta_{\alpha,d;mk}{}^2|]\le c_{3,35}mbn(j_m)^{-1}=c_{3,35}(t-s)$, so that by passing to a subsequence, one may assume that (a) holds for some $\nu\ge 0$.

For (b), let $\varepsilon>0$ be arbitrary. Then,

$$\begin{aligned}n(j_m)^{-1+r(1-\theta)}\sum_{k=1}^{m}\mathbb{E}\left[\left|\zeta_{\alpha,d;mk}\right|^2I_{\{n(j_m)^{-1/2+r(1-\theta)/2}|\zeta_{\alpha,d;mk}|>\varepsilon\}}\right]&\le\varepsilon^{-2}n(j_m)^{-2+2r(1-\theta)}\sum_{k=1}^{m}\mathbb{E}\left[\left|\zeta_{\alpha,d;mk}\right|^4\right]\\&\le c_{3,36}\varepsilon^{-2}mb^2n(j_m)^{-2}=c_{3,36}\varepsilon^{-2}m^{-1}(t-s)^2,\end{aligned} \tag{106}$$

which tends to zero as $m\longrightarrow\infty$.

It therefore follows that $n(j_m)^{-1/2+r(1-\theta)/2}\sum_{k=1}^m\zeta_{\alpha,d;mk}\longrightarrow^{\mathscr{L}}\nu\mathscr{N}$ as $n\longrightarrow\infty$ and it remains only to show that $\nu=\kappa^{1/2}_{\alpha,d,r}|t-s|^{1/2}$. For this, observe that the continuous mapping theorem implies that $|W^m_r(X_{\alpha,d})_t-W^m_r(X_{\alpha,d})_s|^2\longrightarrow^{\mathscr{L}}\nu^2\mathscr{N}^2$. By the Skorokhod representation theorem, one may assume that the convergence is a.s. By Proposition 10, the family $|W^m_r(X_{\alpha,d})_t-W^m_r(X_{\alpha,d})_s|^2$ is uniformly integrable. Hence, $|W^m_r(X_{\alpha,d})_t-W^m_r(X_{\alpha,d})_s|^2\longrightarrow\nu^2\mathscr{N}^2$ in $L^1$, which implies $\mathbb{E}[|W^m_r(X_{\alpha,d})_t-W^m_r(X_{\alpha,d})_s|^2]\longrightarrow\nu^2$. But by Theorem 1, $\mathbb{E}[|W^m_r(X_{\alpha,d})_t-W^m_r(X_{\alpha,d})_s|^2]\longrightarrow\kappa_{\alpha,d,r}|t-s|$, so $\nu=\kappa^{1/2}_{\alpha,d,r}|t-s|^{1/2}$ and the proof is complete. □

*Proof.* of Theorem 2It is sufficient to prove (15) for the even $p$ case since the odd $p$ case can be proved similarly. Let $\{n(j)\}_{j=1}^{\infty}$ be any sequence of natural numbers. By Proposition 10, the sequence $\{(X_{\alpha,d}, W_p^{n(j)}(X_{\alpha,d}))\}$ is relatively compact. Therefore, there exists a subsequence $\{n(j_k)\}$ and a càdlàg process $Y_{\alpha,d}$ such that $(X_{\alpha,d}, W_p^{n(j_k)}(X_{\alpha,d})) \xrightarrow{\mathscr{L}} (X_{\alpha,d}, Y_{\alpha,d})$. Fix $0 < s_1 < s_2 < \ldots < s_\ell < s < t$. With notation as in Proposition 11, let

$$\zeta_{\alpha,d;n(j_k)} = n(j_k)^{-1/2+p(1-\theta)/2} \sum_{i=\lfloor n(j_k)s\rfloor+2}^{\lfloor n(j_k)t\rfloor} \left(\xi_{\alpha,d;ik}^{p} - \mu_p \sigma_{\alpha,d;ik}^{p}\right), \tag{107}$$

and define

$$\eta_{\alpha,d;n(j_k)} = W_p^{n(j_k)}(X_{\alpha,d})_t - W_p^{n(j_k)}(X_{\alpha,d})_s - \zeta_{\alpha,d;n(j_k)}. \tag{108}$$

As in the proof of Proposition 11, $\eta_{\alpha,d;n(j_k)} \longrightarrow 0$ in probability. It therefore follows that

$$\left(W_p^{n(j_k)}(X_{\alpha,d})_{s_1}, \ldots, W_p^{n(j_k)}(X_{\alpha,d})_{s_\ell}, \zeta_{\alpha,d;n(j_k)}\right) \xrightarrow{\mathscr{L}} \left(Y_{\alpha,d}(s_1), \ldots, Y_{\alpha,d}(s_\ell), Y_{\alpha,d}(t) - Y_{\alpha,d}(s)\right). \tag{109}$$

Note that $\mathscr{F}_{(\lfloor n(j_k)s\rfloor+1)n(j_k)^{-1}}$ and $\zeta_{\alpha,d;n(j_k)}$ are independent. Hence, $(W_p^{n(j_k)}(X_{\alpha,d})_{s_1}, \ldots, W_p^{n(j_k)}(X_{\alpha,d})_{s_\ell})$, $\ldots$, $W_p^{n(j_k)}(X_{\alpha,d})_{s_\ell})$ and $\zeta_{\alpha,d;n(j_k)}$ are independent, which implies that $Y_{\alpha,d}(t) - Y_{\alpha,d}(s)$ and $(Y_{\alpha,d}(s_1), \ldots, Y_{\alpha,d}(s_\ell))$ are independent. This yields that the process $Y_{\alpha,d}$ has independent increments.

By Proposition 11, the increment $Y_{\alpha,d}(t) - Y_{\alpha,d}(s)$ is normally distributed with mean zero and variance $\kappa_{\alpha,d,p}|t-s|$. Also, $X_{\alpha,d}(0) = 0$ since $W_p^n(X_{\alpha,d})_0 = 0$ for all $n$. Hence, $Y_{\alpha,d}$ is equal in law to $\kappa_{\alpha,d,p}^{1/2} B$, where $B$ is a standard BM. It remains only to show that $X_{\alpha,d}$ and $B$ are independent.

Fix $0 < s_1 < s_2 < \ldots < s_\ell \le T$. Let $Z_{\alpha,d} = (X_{\alpha,d}(s_1), \ldots, X_{\alpha,d}(s_\ell))^T$ and $\Sigma_{\alpha,d} = \mathbb{E}[Z_{\alpha,d} Z_{\alpha,d}^T]$. It is easy to see that $\Sigma_{\alpha,d}$ is invertible. Hence, one may define the vectors $\nu_{\alpha,d;j} \in \mathbb{R}^\ell$ by $\nu_{\alpha,d;j} = \mathbb{E}[Z_{\alpha,d} \Delta X_{\alpha,d;j}]$, and $w_{\alpha,d;j} = \Sigma_{\alpha,d}^{-1} \nu_{\alpha,d;j}$. Let $\xi_{\alpha,d;j} = \Delta X_{\alpha,d;j} - w_{\alpha,d;j}^T Z_{\alpha,d}$, so that $\xi_{\alpha,d;j}$ and $Z_{\alpha,d}$ are independent.

Define

$$\widetilde{W}_p^n(X_{\alpha,d})_t = n^{-1/2+p(1-\theta)/2} \sum_{j=1}^{\lfloor nt\rfloor} \left(\xi_{\alpha,d;j}^{p} - \mu_p \sigma_{\alpha,d;j}^{p}\right). \tag{110}$$

Then,

$$\left|W_p^n(X_{\alpha,d})_t - \widetilde{W}_p^n(X_{\alpha,d})_t\right| \le n^{-1/2+p(1-\theta)/2} \left|\sum_{j=1}^{\lfloor nt\rfloor} \left(\Delta X_{\alpha,d;j}^{p} - \xi_{\alpha,d;j}^{p}\right)\right|. \tag{111}$$

By (34), binomial expansion, and Hölder inequality,

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le T}\left|W_p^n(X_{\alpha,d})_t - \widetilde{W}_p^n(X_{\alpha,d})_t\right|\right] &\le c_{3,37} n^{-1/2+p(1-\theta)/2} \sum_{\nu=1}^{p} \sum_{j=1}^{\lfloor nt\rfloor} \left(\mathbb{E}\left[\Delta X_{\alpha,d;j}^{2p-2\nu}\right]\right)^{1/2} \left(\mathbb{E}\left[\left(w_{\alpha,d;j}^T Z_{\alpha,d}\right)^{2\nu}\right]\right)^{1/2} \\
&\le c_{3,38} \sum_{\nu=1}^{p} n^{-1/2+\nu(1-\theta)/2} \sum_{j=1}^{\lfloor nt\rfloor} \left(\mathbb{E}\left[\left(w_{\alpha,d;j}^T Z_{\alpha,d}\right)^{2\nu}\right]\right)^{1/2} \le c_{3,39} \max_{1\le i\le \ell} \sum_{\nu=1}^{p} n^{-1/2+\nu(1-\theta)/2} \sum_{j=1}^{\lfloor nt\rfloor} \left|\mathbb{E}\left[X_{\alpha,d}(s_i)\Delta X_{\alpha,d;j}\right]\right|^{\nu}.
\end{aligned} \tag{112}$$

Note that by (36) and Hölder inequality, one has $|\mathbb{E}[X_{\alpha,d}(s_i)\Delta X_{\alpha,d;j}]| \le c_{3,40}\sigma_{\alpha,d;j} \le c_{3,41} n^{-(1-\theta)/2}$ for all $1 \le i \le \ell$ and $1 \le j \le \lfloor nt\rfloor$, and note that by (17) and Lagrange mean value theorem, for any $1 \le i \le \ell$ and $1 \le j \le \lfloor nt\rfloor$,

$$\begin{aligned}
\mathbb{E}\left[X_{\alpha,d}(s_i)\Delta X_{\alpha,d;j}\right] &= K_{\alpha,d}\left(\left(s_i + t_j\right)^{1-\theta} - \left(s_i + t_{j-1}\right)^{1-\theta} - \left(s_i - t_j\right)^{1-\theta} + \left(s_i - t_{j-1}\right)^{1-\theta}\right) \\
&= \frac{K_{\alpha,d}(1-\theta)}{n}\left(\left(s_i + \frac{(j-\zeta_1)}{n}\right)^{-\theta} + \left(s_i - \left(\frac{(j-\zeta_2)}{n}\right)^{-\theta}\right)\right) \le \frac{2K_{\alpha,d}(1-\theta)}{n}\left(s_i - \left(\frac{(j-\zeta_2)}{n}\right)^{-\theta}\right),
\end{aligned} \tag{113}$$

where $\zeta_1, \zeta_2 \in (0,1)$. Then, for any $1 \le i \le \ell$ and $1 \le \nu \le 2p$,

$$n^{-1/2+\nu(1-\theta)/2} \sum_{j=1}^{\lfloor nt \rfloor} \left| \mathbb{E}\left[ X_{\alpha,d}(s_i) \Delta X_{\alpha,d;j} \right] \right|^{\nu} \le c_{3,42} n^{1/2-\nu(1+\theta)/2} \frac{1}{n} \sum_{j=1}^{\lfloor nt \rfloor} \left( s_i - \frac{(j-\zeta_2)}{n} \right)^{-\theta}, \tag{114}$$

which tends to zero as $n \longrightarrow \infty$ since $\int_0^T (s_i - x)^{-\theta} dx < \infty$. Thus,

$$\left( Z_{\alpha,d}, \widetilde{W}_p^n \left( X_{\alpha,d} \right)_{s_1}, \ldots, \widetilde{W}_p^n \left( X_{\alpha,d} \right)_{s_d} \right) \xrightarrow{\mathscr{L}} \left( Z_{\alpha,d}, \kappa_{\alpha,d,p}^{1/2} B(s_1), \ldots, \kappa_{\alpha,d,p}^{1/2} B(s_\ell) \right). \tag{115}$$

Since $Z_{\alpha,d}$ and $\widetilde{W}_p^n(X_{\alpha,d})$ are independent, this gives that $X_{\alpha,d}$ and $B$ are independent.

We now can complete the proof. Note that by (37) and (38),

$$\max_{0 \le t \le T} \left| \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} \left( n^{p(1-\theta)/2} \Delta X_{\alpha,d;j}^p - K_{\alpha,d}^{p/2} \mu_p \right) - W_p^n \left( X_{\alpha,d} \right)_t \right| \le \mu_p n^{-1/2+p(1-\theta)/2} \sum_{j=1}^{\lfloor nt \rfloor} \left| \sigma_{\alpha,d;j}^p - \left( K_{\alpha,d} n^{-1+\theta} \right)^{p/2} \right| \longrightarrow 0. \tag{116}$$

This finishes the proof. □

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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