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Research Article

Behaviors of the Solutions via Their Closed-Form Formulas for Two Rational Third-Order Difference Equations

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In this work, we derive the solution formulas and study their behaviors for the difference equations $x_{n+1} = (\alpha x_n x_{n-3}/(-\beta x_{n-3} + cx_{n-2})), n \in \mathbb{N}_0$ and $x_{n+1} = (\alpha x_n x_{n-3}/(\beta x_{n-3} - cx_{n-2})), n \in \mathbb{N}_0$ with real initials and positive parameters. We show that there exist periodic solutions for the second equation under certain conditions when $\beta^2 < 4\alpha\gamma$. Finally, we give some illustrative examples.

1. Introduction

In [1–5], the first author ([1] together with Kamal) solved and studied the solutions for the difference equations

$$\begin{aligned}
x_{n+1} &= \frac{x_n x_{n-1}}{x_n - x_{n-2}},\\
x_{n+1} &= \frac{x_n x_{n-1}}{-x_n + x_{n-2}},\\
x_{n+1} &= \frac{a x_{n-3}}{b + c x_{n-1} x_{n-3}},\\
x_{n+1} &= \frac{a x_n x_{n-1}}{\pm b x_{n-1} + c x_{n-2}},\\
x_{n+1} &= \frac{a x_n x_{n-k}}{b x_n - c x_{n-k-1}},\\
x_{n+1} &= \frac{x_n x_{n-2}}{-a x_{n-1} + b x_{n-2}},
\end{aligned} \tag{1}$$

where $n \in \mathbb{N}_0$, with real initials and positive parameters.

In [6], the authors explored the dynamics of the difference equation

$$x_{n+1} = a x_n + \frac{b x_n x_{n-2}}{c x_{n-2} + d x_{n-3}}, \quad n \in \mathbb{N}_0, \tag{2}$$

where a, b, c, d and $x_{-3}, x_{-2}, x_{-1}, x_0$ are positive real numbers. They provided the solution of the mentioned equation when $a = 1$, $b = 1$, $c = 1$, and $d = 1$.

By virtue of the wide applications in the last few decades, difference equations turned into one of the major areas of research. There are many books dealing with difference equations through the qualitative behavior of nonlinear equations (see [7–11]).

Closed-form solutions for nonlinear difference are not available, except for some few equations (see, for example, [6, 7, 10, 12–26]). In [27], the authors studied the two recursive equations

$$x_{n+1} = \frac{\alpha x_n x_{n-3}}{-\beta x_{n-3} + \gamma x_{n-2}}, \quad n \in \mathbb{N}_0, \tag{3}$$

$$x_{n+1} = \frac{\alpha x_n x_{n-3}}{\beta x_{n-3} - \gamma x_{n-2}}, \quad n \in \mathbb{N}_0, \tag{4}$$

with real initials and positive parameters. They provided the solutions for the two mentioned equations when $\alpha = \beta = \gamma$.

Motivated by [27], we shall solve, find the forbidden set, and study ($\forall$ positive real values of α, β, γ) global behavior of the admissible solutions for equations (3) and (4) where

α, β, γ are positive real numbers and x_{-3}, x_{-2}, x_{-1}, x_0 are nonzero real numbers.

If we set

$$\frac{\beta}{\alpha} = a, \quad \frac{\gamma}{\alpha} = b, \tag{5}$$

then equations (3) and (4) are reduced to

$$x_{n+1} = \frac{x_n x_{n-3}}{-a x_{n-3} + b x_{n-2}}, \quad n \in \mathbb{N}_0, \tag{6}$$

$$x_{n+1} = \frac{x_n x_{n-3}}{a x_{n-3} - b x_{n-2}}, \quad n \in \mathbb{N}_0. \tag{7}$$

2. The Difference Equation $x_{n+1} = (x_n x_{n-3} / (-a x_{n-3} + b x_{n-2}))$

During this section, we suppose that

$$\theta_n = \frac{\lambda_+^n - \lambda_-^n}{\sqrt{a^2 + 4b}}, \tag{8}$$

where

$$\lambda_- = -\frac{a}{2} - \frac{\sqrt{a^2 + 4b}}{2}, \quad \lambda_+ = -\frac{a}{2} + \frac{\sqrt{a^2 + 4b}}{2}. \tag{9}$$

2.1. Solution of Equation (6). According with the transformation,

$$y_n = \frac{x_{n-1}}{x_n}, \quad \text{with } y_{-2} = \frac{x_{-3}}{x_{-2}}, y_{-1} = \frac{x_{-2}}{x_{-1}}, y_0 = \frac{x_{-1}}{x_0}. \tag{10}$$

Equation (6) becomes

$$y_{n+1} = -a + \frac{b}{y_{n-2}}, \quad n = 0, 1, \ldots. \tag{11}$$

By solving equation (11) and after some calculations, the solution of equation (6) can be obtained.

Theorem 1. *Suppose $\{x_n\}_{n=-3}^{\infty}$ is an admissible solution of equation (6). The solution of equation (6) is*

$$x_n = \begin{cases} \dfrac{\nu}{\mu_0((n-1)/3)\mu_{-1}((n-1)/3)\mu_{-2}((n+2)/3)}, & n = 1, 4, \ldots, \\ \dfrac{\nu}{\mu_0((n-2)/3)\mu_{-1}((n+1)/3)\mu_{-2}((n+1)/3)}, & n = 2, 5, \ldots, \\ \dfrac{\nu}{\mu_0(n/3)\mu_{-1}(n/3)\mu_{-2}(n/3)}, & n = 3, 6, \ldots, \end{cases} \tag{12}$$

where $\nu = x_0 x_{-1} x_{-2} x_{-3}$ and

$$\mu_{-3+i}(n) = b\theta_n x_{-3+i} + \theta_{n+1} x_{-4+i}, \quad \text{where } i = 1, 2, 3 \text{ and } n = 0, 1, \ldots. \tag{13}$$

Proof. We can write the solution formula (12) as

$$x_{3m+1} = \frac{\nu}{\mu_0(m)\mu_{-1}(m)\mu_{-2}(m+1)}, \quad x_{3m+2} = \frac{\nu}{\mu_0(m)\mu_{-1}(m+1)\mu_{-2}(m+1)}, \quad x_{3m+3} = \frac{\nu}{\mu_0(m+1)\mu_{-1}(m+1)\mu_{-2}(m+1)}. \tag{14}$$

The proof is by using the mathematical induction on m. When $m = 0$,

$$x_1 = \frac{\nu}{\mu_0(0)\mu_{-1}(0)\mu_{-2}(1)} = \frac{x_0 x_{-3}}{-a x_{-3} + b x_{-2}},$$

$$x_2 = \frac{\nu}{\mu_0(0)\mu_{-1}(0)\mu_{-2}(1)} = \frac{x_{-2}}{(-a x_{-2} + b x_{-1})} \frac{x_0 x_{-3}}{(-a x_{-3} + b x_{-2})} = \frac{x_1 x_{-2}}{-a x_{-2} + b x_{-1}},$$

$$x_3 = \frac{\nu}{\mu_0(0)\mu_{-1}(0)\mu_{-2}(1)}$$

$$= \frac{x_{-1}x_{-2}}{(-ax_{-1}+bx_0)(-ax_{-2}+bx_{-1})}\frac{x_0x_{-3}}{(-ax_{-3}+bx_{-2})}$$
$$= \frac{x_{-1}}{(-ax_{-1}+bx_0)}\frac{x_1x_{-2}}{(-ax_{-2}+bx_{-1})} \tag{15}$$

Suppose that the equalities (14) are true for $m_0 > 0$. Then,

$$\begin{aligned} x_{3m_0+1} &= \frac{x_{3m_0}x_{3m_0-3}}{-ax_{3m_0-3}+bx_{3m_0-2}} \\ &= \frac{(\nu/\mu_0(m_0)\mu_{-1}(m_0)\mu_{-2}(m_0))(\nu/\mu_0(m_0-1)\mu_{-1}(m_0-1)\mu_{-2}(m_0-1))}{-a(\nu/\mu_0(m_0-1)\mu_{-1}(m_0-1)\mu_{-2}(m_0-1))+b(\nu/\mu_0(m_0-1)\mu_{-1}(m_0-1)\mu_{-2}(m_0))} \\ &= \frac{\nu}{\mu_0(m_0)\mu_{-1}(m_0)(-a\mu_{-2}(m_0)+b\mu_{-2}(m_0-1))}. \end{aligned} \tag{16}$$

But, we can see that

$$-a\mu_{-2}(m_0)+b\mu_{-2}(m_0-1)=\mu_{-2}(m_0+1). \tag{17}$$

Then,

$$x_{3m_0+1} = \frac{\nu}{\mu_0(m_0)\mu_{-1}(m_0)\mu_{-2}(m_0+1)}. \tag{18}$$

Therefore,

$$x_{3m+1} = \frac{\nu}{\mu_0(m)\mu_{-1}(m)\mu_{-2}(m+1)}, \quad m=0,1,\ldots. \tag{19}$$

Similarly, we can obtain x_{3m+2} and x_{3m+3}. This completes the proof. □

For equation (6), it is clear that if $x_0 = 0$ and $x_{-1}x_{-2}x_{-3} \neq 0$, then x_4 is not defined. Let $x_{-1} = 0$ and $x_0x_{-2}x_{-3} \neq 0$, then x_7 is not defined. Also, if $x_{-2} = 0$ and $x_0x_{-1}x_{-3} \neq 0$, then x_6 is not defined. Now, if $x_{-3} = 0$ and $x_0x_{-1}x_{-2} \neq 0$, then x_5 is undefined.

This implies that the point $(x_0, x_{-1}, x_{-2}, x_{-3})$ with $\prod_{j=0}^{3} x_{-j} = 0$ belongs to the forbidden point of equation (6).

Theorem 2. *Equation (6) has the forbidden set*

$$\begin{aligned} F = &\bigcup_{i=0}^{3}\left\{(u_0,u_{-1},u_{-2},u_{-3})\in\mathbb{R}^4:\ u_{-i}=0\right\}\cup \\ &\bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},u_{-2},u_{-3})\in\mathbb{R}^4:\ u_0=-\frac{1}{b}\frac{\theta_{m+1}}{\theta_m}u_{-1}\right\}\cup \\ &\bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},u_{-2},u_{-3})\in\mathbb{R}^4:\ u_{-1}=-\frac{1}{b}\frac{\theta_{m+1}}{\theta_m}u_{-2}\right\}\cup \\ &\bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},u_{-2},u_{-3})\in\mathbb{R}^4:\ u_{-2}=-\frac{1}{b}\frac{\theta_{m+1}}{\theta_m}u_{-3}\right\}. \end{aligned} \tag{20}$$

2.2. Global Dynamics of Equation (6). Here, we illustrate the global behavior result and provide some examples.

Theorem 3. *Assume that $\{x_n\}_{n=-3}^{\infty}$ is an admissible solution of equation (6). Then, the following hold:*

(1) If $a+b>1$, then the solution $\{x_n\}_{n=-3}^{\infty}$ converges to zero.

(2) If $a+b=1$, then the solution $\{x_n\}_{n=-3}^{\infty}$ converges to period-2 solution.

(3) If $a+b<1$, then the solution $\{x_n\}_{n=-3}^{\infty}$ is unbounded.

Proof. We can write $\theta_m = \lambda_-^m(((\lambda_+/\lambda_-)^m - 1)/\sqrt{a^2+4b})$.

(1) If $a+b>1$, then $|\lambda_-|>1$. It follows that $|\theta_m| \longrightarrow \infty$ and $|\mu_{-3+i}(m)| \longrightarrow \infty$ as $m \longrightarrow \infty$.

Therefore, $\{x_n\}_{n=-3}^{\infty}$ converges to zero.

(2) Assume that $a+b=1$, then $\lambda_- = -1$.

This implies that

$$\begin{aligned} \theta_{2m} &\longrightarrow \frac{-1}{\sqrt{a^2+4b}}, \\ \theta_{2m+1} &\longrightarrow \frac{1}{\sqrt{a^2+4b}}, \quad \text{as}\, m \longrightarrow \infty. \end{aligned} \tag{21}$$

Also,

$$\mu_{-3+i}(2m) \longrightarrow \frac{-bx_{-3+i}+x_{-4+i}}{\sqrt{a^2+4b}} = L_{-3+i}, \quad \text{as}\, m \longrightarrow \infty, i=1,2,3. \tag{22}$$

Similarly,

$$\mu_{-3+i}(2m+1) \longrightarrow \frac{bx_{-3+i}-x_{-4+i}}{\sqrt{a^2+4b}} = -L_{-3+i}, \quad \text{as}\, m \longrightarrow \infty, i=1,2,3. \tag{23}$$

Using formula (14), we can write

$$x_{6m+1} = \frac{\nu}{\mu_0(2m)\mu_{-1}(2m)\mu_{-2}(2m+1)} \longrightarrow -\frac{\nu}{L_0L_{-1}L_{-2}} = -M, \tag{24}$$

where $M = (\nu/L_0L_{-1}L_{-2})$.

Similarly, we can get $\lim_{n\longrightarrow\infty} x_{6m+j}$ for $2 \le j \le 6$ in terms of M.

The result is obtained by noting

$$M = \frac{\nu}{L_0L_{-1}L_{-2}} = \frac{(a^2+4b)^{3/2}\nu}{\prod_{t=1}^{3}(-bx_{-3+t}+x_{-4+t})}, \tag{25}$$

and as $m \longrightarrow \infty$, we have that

$$x_{6m+j} \longrightarrow \begin{cases} -M, & n = 1,3,5, \\ M, & n = 2,4,6. \end{cases} \tag{26}$$

(3) If $a+b<1$, then $\lambda_- > -1$. That is, $\theta_m \longrightarrow 0$ as $m \longrightarrow \infty$. Also, $\{\mu_{-2}(m)\}_{m=0}^{\infty}$, $\{\mu_{-1}(m)\}_{m=0}^{\infty}$, and $\{\mu_0(m)\}_{m=0}^{\infty}$ are converging to zero.

This implies that $\{x_{3m+t}\}_{m=-1}^{\infty}$ are unbounded, $t = 0,1,2$. □

Example 1. Consider the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (6) such that $a = 0.5$, $b = 0.1$ $(a+b<1)$, with initial values $x_{-3} = 2.1$, $x_{-2} = -1$, $x_{-1} = 0.2$, and $x_0 = -1.8$. Figure 1 shows that solution $\{x_n\}_{n=-3}^{\infty}$ is unbounded.

Example 2. Consider $\{x_n\}_{n=-3}^{\infty}$ of equation (6) such that $a = 1, b = 0.1$ $(a+b>1)$, with $x_{-3} = 2.1$, $x_{-2} = -1$, $x_{-1} = 0.2$, and $x_0 = -1.8$. Figure 2 shows that $\{x_n\}_{n=-3}^{\infty}$ converges to zero.

Example 3. Consider $\{x_n\}_{n=-3}^{\infty}$ of equation (6) such that $a = 0.6$, $b = 0.4$ $(a+b=1)$, with $x_{-3} = 2.1$, $x_{-2} = -1$, $x_{-1} = 0.2$, and $x_0 = -1.8$. Figure 3 shows that $\{x_n\}_{n=-3}^{\infty}$ converges to the period-2 solution

$$\{\ldots, M, -M, M, -M, \ldots\}, \tag{27}$$

where

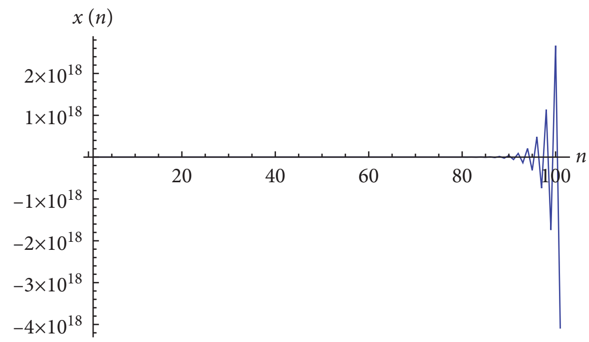

FIGURE 1: $x_{n+1} = (x_nx_{n-3}/(-0.5x_{n-3}+0.1x_{n-2}))$.

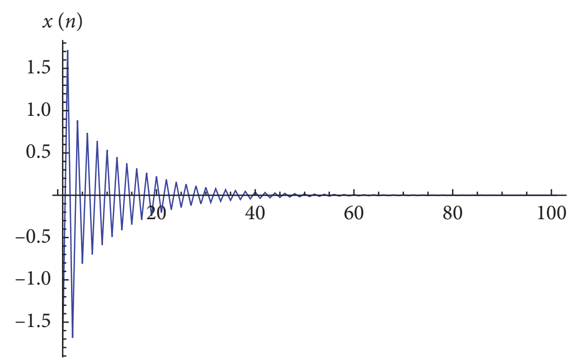

FIGURE 2: $x_{n+1} = (x_nx_{n-3}/(-x_{n-3}+0.1x_{n-2}))$.

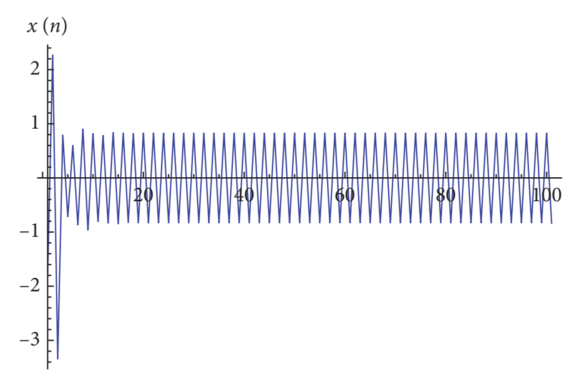

FIGURE 3: $x_{n+1} = (x_nx_{n-3}/(-0.6x_{n-3}+0.4x_{n-2}))$.

$$\begin{aligned} M &= \frac{(a^2+4b)^{3/2}\nu}{\prod_{t=1}^{3}(-bx_{-3+t}+x_{-4+t})} \\ &= \frac{((0.6)^2+4(0.4))^{3/2}(-1.8)(0.2)(-1)(2.1)}{(-0.4(-1.8)+0.2)(-0.4(0.2)-1)(-0.4(-1)+2.1)} \simeq -0.83513. \end{aligned} \tag{28}$$

Example 4. Consider $\{x_n\}_{n=-3}^{\infty}$ of equation (6) such that $a = 0.9$, $b = 0.1$ $(a + b = 1)$, with $x_{-3} = 2.1$, $x_{-2} = -1$, $x_{-1} = 0.2$, and $x_0 = -1.8$. Figure 4 shows that the solution $\{x_n\}_{n=-3}^{\infty}$ converges to the period-2 solution

$$\{\dots, M, -M, M, -M, \dots\}, \tag{29}$$

where

$$\begin{aligned} M &= \frac{\left(a^2 + 4b\right)^{3/2}\nu}{\prod_{t=1}^{3}\left(-bx_{-3+t} + x_{-4+t}\right)} \\ &= \frac{\left((0.9)^2 + 4(0.1)\right)^{3/2}(-1.8)(0.2)(-1)(2.1)}{(-0.1(-1.8) + 0.2)(-0.1(0.2) - 1)(-0.1(-1) + 2.1)} \approx -1.18003. \end{aligned} \tag{30}$$

3. The Difference Equation $x_{n+1} = (x_n x_{n-3}/(ax_{n-3} - bx_{n-2}))$

We discuss the behaviors of the solutions of equation (7). The transformation (10) reduces equation (7) into the recursive equation

$$y_{n+1} = a - \frac{b}{y_{n-2}}, \quad n = 0, 1, \dots. \tag{31}$$

During this section, we suppose that

$$\begin{aligned} \lambda_1 &= \frac{a}{2} - \frac{\sqrt{a^2 - 4b}}{2}, \\ \lambda_2 &= \frac{a}{2} + \frac{\sqrt{a^2 - 4b}}{2}. \end{aligned} \tag{32}$$

Clear that λ_1 and λ_2 are the roots of the equation

$$\lambda^2 - a\lambda + b = 0. \tag{33}$$

3.1. Case $a^2 > 4b$. In this subsection, we assume that $a^2 > 4b$. Clear that

$$0 < \lambda_1 < \frac{a}{2} < \lambda_2 < \frac{b}{a} < a. \tag{34}$$

During this subsection, we suppose that

$$\widehat{\theta}_n = \frac{\lambda_2^n - \lambda_1^n}{\sqrt{a^2 - 4b}}, \quad n = 0, 1, \dots. \tag{35}$$

Theorem 4. *Assume that $\{x_n\}_{n=-3}^{\infty}$ is an admissible solution of equation (7), then*

$$x_n = \begin{cases} \dfrac{\nu}{\widehat{\mu}_0((n-1)/3)\widehat{\mu}_{-1}((n-1)/3)\widehat{\mu}_{-2}((n+2)/3)}, & n = 1, 4, \dots, \\[2ex] \dfrac{\nu}{\widehat{\mu}_0((n-2)/3)\widehat{\mu}_{-1}((n+1)/3)\widehat{\mu}_{-2}((n+1)/3)}, & n = 2, 5, \dots, \\[2ex] \dfrac{\nu}{\widehat{\mu}_0(n/3)\widehat{\mu}_{-1}(n/3)\widehat{\mu}_{-2}(n/3)}, & n = 3, 6, \dots, \end{cases} \tag{36}$$

where $\nu = x_0 x_{-1} x_{-2} x_{-3}$ and

$$\widehat{\mu}_{-3+i}(n) = -b\widehat{\theta}_n x_{-3+i} + \widehat{\theta}_{n+1} x_{-4+i}, \quad \text{where } i = 1, 2, 3 \text{ and } n = 0, 1, \dots. \tag{37}$$

Proof. Its proof is same as the proof of Theorem 1 and is omitted. □

Theorem 5. *Let $\{x_n\}_{n=-3}^{\infty}$ be an admissible solution of recursive equation (7). Then, the following hold:*

(1) If $a < b + 1$, then one has the following:

(a) If $b < 1$, then the solution $\{x_n\}_{n=-3}^{\infty}$ is unbounded.
(b) If $b > 1$, then the solution $\{x_n\}_{n=-3}^{\infty}$ converges to zero.

(2) If $a = b + 1$, then one has the following:

(a) If $b < 1$, then the solution $\{x_n\}_{n=-3}^{\infty}$ converges to a finite limit.
(b) If $b > 1$, then the solution $\{x_n\}_{n=-3}^{\infty}$ converges to zero.

(3) If $a > b + 1$, then the solution $\{x_n\}_{n=-3}^{\infty}$ converges to zero.

Proof. We can write $\theta_m = \lambda_2^m((1 - (\lambda_1/\lambda_2)^m)/\sqrt{a^2 - 4b})$.

(1) If $a < b + 1$, then either $\lambda_1 > 1$ or $\lambda_2 < 1$.

(a) If $b < 1$, then $\lambda_2 < 1$. That is, $\widehat{\theta}_m \longrightarrow 0$ as $m \longrightarrow \infty$. This implies that $\{\widehat{\mu}_{-2}(m)\}_{m=0}^{\infty}$, $\{\widehat{\mu}_{-1}(m)\}_{m=0}^{\infty}$, and $\{\widehat{\mu}_0(m)\}_{m=0}^{\infty}$ are converging to zero. Therefore, $\{x_n\}_{n=-3}^{\infty}$ is unbounded.
(b) If $b > 1$, then $\lambda_1 > 1$. That is, $\widehat{\theta}_m \longrightarrow \infty$ as $m \longrightarrow \infty$. This implies that $\widehat{\mu}_{-2}(m)$, $\widehat{\mu}_{-1}(m)$, and $\widehat{\mu}_0(m)$ are unbounded. Therefore, $\{x_n\}_{n=-3}^{\infty}$ converges to zero.

(2) Suppose that $a = b + 1$, then either $\lambda_1 = 1$ or $\lambda_2 = 1$.

(a) If $b < 1$, then $\lambda_2 = 1$. It follows that $\widehat{\theta}_m \longrightarrow (2/\sqrt{a^2 - 4b})$ as $m \longrightarrow \infty$. This implies that

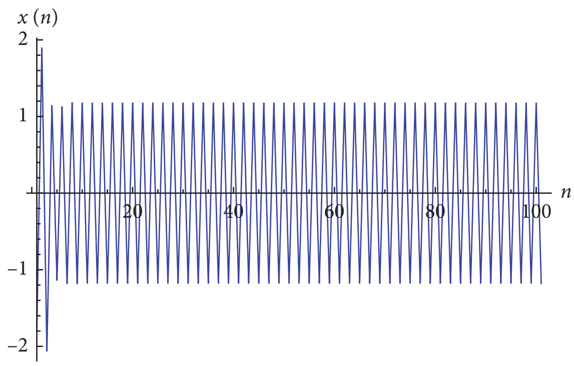

FIGURE 4: $x_{n+1} = (x_n x_{n-3}/(-0.9x_{n-3} + 0.1x_{n-2}))$.

$$\widehat{\mu}_{-3+i}(m) \longrightarrow \frac{-bx_{-3+i} + x_{-4+i}}{\sqrt{a^2 - 4b}} = \widehat{L}_{-3+i}, \quad \text{as } m \longrightarrow \infty, i = 1, 2, 3. \tag{38}$$

Then,

$$x_{3m+1} = \frac{\nu}{\widehat{\mu}_0(m)\widehat{\mu}_{-1}(m)\widehat{\mu}_{-2}(m+1)} \longrightarrow \frac{\nu}{\widehat{L}_0\widehat{L}_{-1}\widehat{L}_{-2}} = \widehat{M}, \tag{39}$$

where

$$\widehat{M} = \frac{\nu(a^2 - 4b)^{3/2}}{\prod_{t=1}^{3}(-bx_{-3+t} + x_{-4+t})}. \tag{40}$$

Similarly, we can show that $x_{3m+2} \longrightarrow \widehat{M}$ and $x_{3m+3} \longrightarrow \widehat{M}$ as $m \longrightarrow \infty$.

Therefore, $\{x_n\}_{n=-3}^{\infty}$ converges to M as $m \longrightarrow \infty$.

(b) If $b > 1$, then $\lambda_1 = 1$. That is, $\widehat{\theta}_m \longrightarrow \infty$ as $m \longrightarrow \infty$.

This implies that $\{\widehat{\mu}_{-2}(m)\}_{m=0}^{\infty}$, $\{\widehat{\mu}_{-1}(m)\}_{m=0}^{\infty}$, and $\{\widehat{\mu}_0(m)\}_{m=0}^{\infty}$ are unbounded.

Therefore, $\{x_n\}_{n=-3}^{\infty}$ converges to zero.

(3) If $a > b + 1$, then $\lambda_1 < 1 < \lambda_2$. That is, $\widehat{\theta}_m \longrightarrow \infty$ as $m \longrightarrow \infty$.

This implies that $\widehat{\mu}_{-2}(m)$, $\widehat{\mu}_{-1}(m)$, and $\widehat{\mu}_0(m)$ are unbounded.

Therefore, $\{x_n\}_{n=-3}^{\infty}$ converges to zero. □

Example 5. Consider $\{x_n\}_{n=-3}^{\infty}$ of recursive equation (7) where $a = 1$, $b = 0.2$ ($a < b + 1$ and $b < 1$), with $x_{-3} = 1$, $x_{-2} = -3$, $x_{-1} = -1$, and $x_0 = 1.8$. Figure 5 shows the unbounded solution $\{x_n\}_{n=-3}^{\infty}$.

Example 6. Consider $\{x_n\}_{n=-3}^{\infty}$ of recursive equation (7) such that $a = 1.2$, $b = 0.1$ ($a > b + 1$), with $x_{-3} = 1$, $x_{-2} = -3$, $x_{-1} = -1$, and $x_0 = 1$. Figure 6 shows that $\{x_n\}_{n=-3}^{\infty}$ converges to zero.

Example 7. Consider $\{x_n\}_{n=-3}^{\infty}$ of recursive equation (7) such that $a = 1.3$, $b = 0.3$ ($a = b + 1$ and $b < 1$), with $x_{-3} = -2$, $x_{-2} = -2$, $x_{-1} = 1$, and $x_0 = 2$. Figure 7 shows that $\{x_n\}_{n=-3}^{\infty}$ converges to $\widehat{M}$ where

$$\begin{aligned}\widehat{M} &= \frac{\nu(a^2 - 4b)^{3/2}}{\prod_{t=1}^{3}(-bx_{-3+t} + x_{-4+t})} \\ &= \frac{((1.3)^2 - 4(0.3))^{3/2}}{(-0.3(-2) - 2)(-0.3(1) - 2)(-0.3(2) + 1)} \simeq 2.1304.\end{aligned} \tag{41}$$

Example 8. Consider $\{x_n\}_{n=-3}^{\infty}$ of recursive equation (7) such that $a = 2.01$, $b = 1.01$ ($a = b + 1$ and $b > 1$), with $x_{-3} = -2$, $x_{-2} = -2$, $x_{-1} = 1$, and $x_0 = 2$. Figure 8 shows that the solution $\{x_n\}_{n=-3}^{\infty}$ converges to zero.

3.2. Case $a^2 = 4b$. In this subsection, we assume that $a^2 = 4b$. That is, $\lambda_1 = \lambda_2 = (a/2)$.

Theorem 6. *Assume that $\{x_n\}_{n=-3}^{\infty}$ is an admissible solution of recursive equation (7). Then,*

$$x_n = \begin{cases} \left(\dfrac{2}{a}\right)^n \dfrac{8\nu}{\delta_0((n-1)/3)\delta_{-1}((n-1)/3)\delta_{-2}((n+2)/3)}, & n = 1, 4, \ldots, \\ \left(\dfrac{2}{a}\right)^n \dfrac{8\nu}{\widehat{\mu}_0((n-2)/3)\widehat{\mu}_{-1}((n+1)/3)\delta_{-2}((n+1)/3)}, & n = 2, 5, \ldots, \\ \left(\dfrac{2}{a}\right)^n \dfrac{8\nu}{\delta_0(n/3)\delta_{-1}(n/3)\delta_{-2}(n/3)}, & n = 3, 6, \ldots, \end{cases} \tag{42}$$

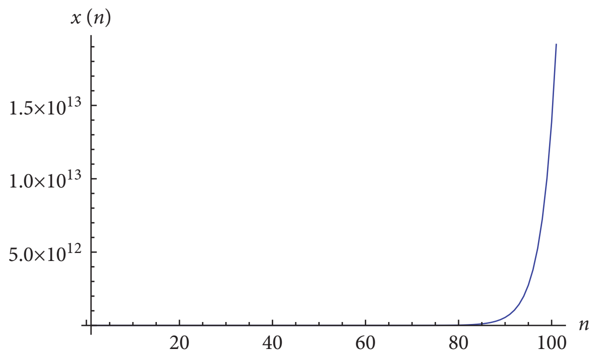

FIGURE 5: $x_{n+1} = (x_n x_{n-3}/(x_{n-3} - 0.2x_{n-2}))$.

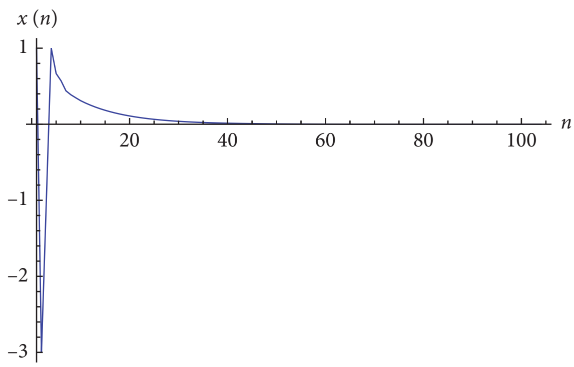

FIGURE 6: $x_{n+1} = (x_n x_{n-3}/(1.2x_{n-3} - 0.1x_{n-2}))$.

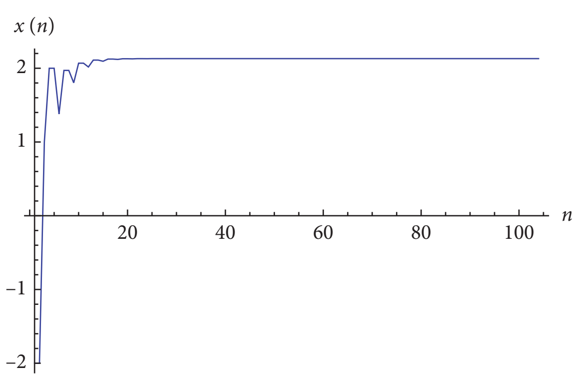

FIGURE 7: $x_{n+1} = (x_n x_{n-3}/(1.3x_{n-3} - 0.3x_{n-2}))$.

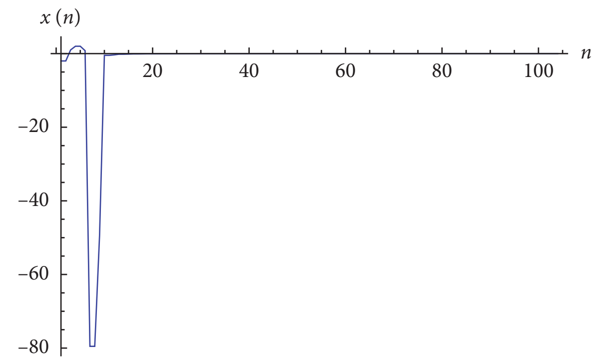

FIGURE 8: $x_{n+1} = (x_n x_{n-3}/(2.01x_{n-3} - 1.01x_{n-2}))$.

where $\nu = x_0 x_{-1} x_{-2} x_{-3}$ *and*

$$\delta_{-3+i}(n) = -anx_{-3+i} + 2(n+1)x_{-4+i}, \quad i = 1,2,3 \text{ and } n = 0,1,\ldots. \tag{43}$$

Proof. It is enough to see that

$$\delta_{-3+i}(n+1) = 2\delta_{-3+i}(n) - \delta_{-3+i}(n-1), \quad i = 1,2,3 \text{ and } n = 0,1,\ldots, \tag{44}$$

and its proof is same as of Theorem 1 and is omitted. □

Theorem 7. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be an admissible solution of recursive equation (7). Then, the following hold:*

(1) If $a \ge 2$*, then solution* $\{x_n\}_{n=-3}^{\infty}$ *converges to zero.*
(2) If $a < 2$*, then* $\{x_n\}_{n=-3}^{\infty}$ *is unbounded.*

Proof. Clear that $|\delta_{-3+i}(m)| \longrightarrow \infty$ as $m \longrightarrow \infty$, $i = 1,2,3$.

(1) If $a > 2$, then $(2/a)^n \longrightarrow 0$ as $n \longrightarrow \infty$. This implies that the solution $\{x_n\}_{n=-3}^{\infty}$ converges to zero.
Now, suppose that $a = 2$. Then,

$$x_{3m+1} = \frac{8\nu}{\delta_0(m)\delta_{-1}(m)\delta_{-2}(m+1)} \longrightarrow 0, \quad \text{as } m \longrightarrow \infty. \tag{45}$$

Similarly, for x_{3m+2} and x_{3m+3}.
Therefore, $\{x_n\}_{n=-3}^{\infty}$ converges to zero.

(2) If $a < 2$, then $(2/a)^m \longrightarrow \infty$ as $m \longrightarrow \infty$. Then,

$$\begin{aligned} |x_{3m+1}| &= \left| \left(\frac{2}{a}\right)^{3m+1} \frac{8\nu}{\delta_0(m)\delta_{-1}(m)\delta_{-2}(m+1)} \right| \\ &= \left| \left(\frac{2}{a}\right)^{3m+1} \frac{8\nu}{m^3 \prod_{t=0}^{1}(-ax_{-t} + 2((1/m)+1)x_{-t-1})(-a((1/m)+1)x_{-2} + 2((2/m)+1)x_{-3})} \right|. \end{aligned} \tag{46}$$

By applying L'Hospital's rule, we get

$$|x_{3m+1}| \longrightarrow \infty, \quad \text{as } m \longrightarrow \infty. \tag{47}$$

Similarly, for $|x_{3m+2}|$ and $|x_{3m+3}|$.

Hence, $\{x_n\}_{n=-3}^{\infty}$ is unbounded. □

Example 9. Consider $\{x_n\}_{n=-3}^{\infty}$ of recursive equation (7) such that $a = 1$, $b = 0.25$ ($a^2 = 4b$ and $a < 2$), with $x_{-3} = 0.2$, $x_{-2} = -2$, $x_{-1} = 2$, and $x_0 = -1.2$. Figure 9 shows that the unbounded solution $\{x_n\}_{n=-3}^{\infty}$.

Example 10. Consider $\{x_n\}_{n=-3}^{\infty}$ of recursive equation (7) such that $a = 2$, $b = 1$ ($a^2 = 4b$ and $a \geq 2$), with $x_{-3} = 0.2$, $x_{-2} = -2$, $x_{-1} = 2$, and $x_0 = -1.2$. Figure 10 shows that $\{x_n\}_{n=-3}^{\infty}$ converges to zero.

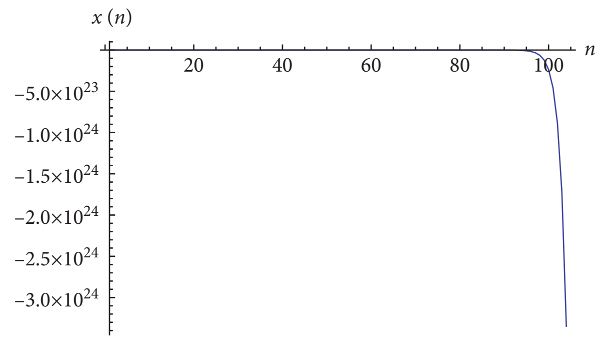

FIGURE 9: $x_{n+1} = (x_n x_{n-3}/(x_{n-3} - 0.25x_{n-2}))$.

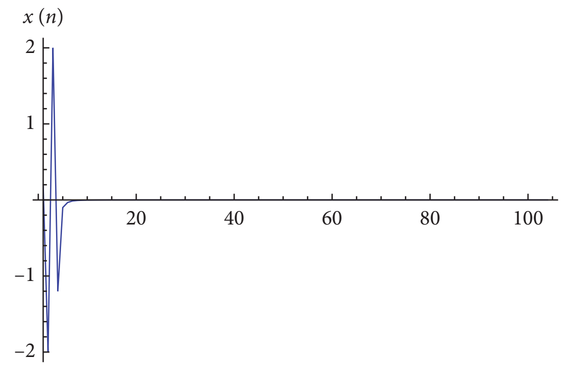

FIGURE 10: $x_{n+1} = (x_n x_{n-3}/(2x_{n-3} - x_{n-2}))$.

3.3. *Case $a^2 < 4b$.*

Hereafter, we study the final case when $a^2 < 4b$.

During this subsection, we suppose that

$$\lambda_1 = \frac{a}{2} - \frac{\sqrt{4b - a^2}}{2} i, \qquad \lambda_2 = \frac{a}{2} + \frac{\sqrt{4b - a^2}}{2} i. \tag{48}$$

That is,

$$|\lambda_1| = |\lambda_2| = \sqrt{b}, \qquad \sin\phi = \frac{\sqrt{4b - a^2}}{2\sqrt{b}}, \qquad \cos\phi = \frac{a}{2\sqrt{b}}. \tag{49}$$

Theorem 8. *Let $\{x_n\}_{n=-2}^{\infty}$ be an admissible solution of recursive equation (3). Then,*

$$x_n = \begin{cases} \dfrac{\sin^3 \alpha}{b^{(n/2)}} \dfrac{\nu}{\xi_0((n-1)/3)\xi_{-1}((n-1)/3)\xi_{-2}((n+2)/3)}, & n = 1, 4, \ldots, \\ \dfrac{\sin^3 \alpha}{b^{(n/2)}} \dfrac{\nu}{\xi_0((n-2)/3)\xi_{-1}((n+1)/3)\xi_{-2}((n+1)/3)}, & n = 2, 5, \ldots, \\ \dfrac{\sin^3 \alpha}{b^{(n/2)}} \dfrac{\nu}{\xi_0(n/3)\xi_{-1}(n/3)\xi_{-2}(n/3)}, & n = 3, 6, \ldots, \end{cases} \tag{50}$$

where $\nu = x_0 x_{-1} x_{-2}$, $\xi_{-3+i}(n) = -\sqrt{b} x_{-3+i} \sin n\phi + x_{-4+i} \sin(n+1)\phi$, $\phi = \tan^{-1}(\sqrt{4b - a^2}/b) \in \,]0, (\pi/2)[$.

Theorem 9. *Assume that $\{x_n\}_{n=-3}^{\infty}$ is an admissible solution of equation (7). Then, we have the following:*

(1) If $b > 1$, then the solution $\{x_n\}_{n=-3}^{\infty}$ converges to zero.

(2) If $b < 1$, then the solution $\{x_n\}_{n=-3}^{\infty}$ is unbounded.

Proof. Its proof is direct consequence and is omitted. □

Theorem 10. *Let $\{x_n\}_{n=-3}^{\infty}$ be an admissible solution of recursive equation (7) and let $b = 1$. If $\phi = (l/k)\pi$ (with $0 < l < (k/2)$), then $\{x_n\}_{n=-3}^{\infty}$ is periodic having prime period-6k.*

Proof. Using formula (50), we can write

$$x_{3m+3k+1} = \sin^3 \phi \frac{\nu}{\xi_0(m+k)\xi_{-1}(m+k)\xi_{-2}(m+1+k)}. \tag{51}$$

But, for each $i = 1, 2, 3$, we have

$$\begin{aligned}\xi_{-3+i}(m+2k) &= -x_{-3+i}\sin(m+2k)\phi + x_{-4+i}\sin(m+2k+1)\phi \\ &= -x_{-3+i}\sin(m\phi+2k\phi) \\ &\quad + x_{-4+i}\sin((m+1)\phi+2k\phi) \\ &= -x_{-3+i}\sin(m\phi+2l\pi) \\ &\quad + x_{-4+i}\sin((m+1)\phi+2l\pi) \\ &= -x_{-3+i}\sin m\phi + x_{-4+i}\sin(m+1)\phi \\ &= \xi_{-3+i}(m). \end{aligned} \tag{52}$$

Then, for $m \geq -1$, we have

$$\begin{aligned} x_{3m+6k+1} &= \sin^3\phi \frac{\nu}{\xi_0(m+2k)\xi_{-1}(m+2k)\xi_{-2}(m+1+2k)} \\ &= \sin^3\phi \frac{\nu}{\xi_0(m)\xi_{-1}(m)\xi_{-2}(m+1)} \end{aligned} \tag{53}$$

Similarly, we can show that for $m \geq -1$, we have

$$\begin{aligned} x_{3m+6k+2} &= x_{3m+2}, \\ x_{3m+6k+3} &= x_{3m+3}. \end{aligned} \tag{54}$$

This completes the proof. □

Example 11. Consider $\{x_n\}_{n=-3}^{\infty}$ of recursive equation (7) with $x_{-3} = -0.8$, $x_{-2} = 1.1$, $x_{-1} = -0.9$, and $x_0 = -1.2$. If $a = \sqrt{2+\sqrt{2}}, b = 1$ ($a^2 < 4b$ and $\phi = (\pi/8)$), then $\{x_n\}_{n=-3}^{\infty}$ is periodic with prime period $6k = 48$ (see Figure 11).

Example 12. Consider $\{x_n\}_{n=-3}^{\infty}$ of recursive equation (7) with $x_{-3} = -0.5$, $x_{-2} = 2$, $x_{-1} = 0.5$, and $x_0 = -1.1$. If $a = b = 1$ ($a^2 < 4b$ and $\phi = (\pi/3)$), then $\{x_n\}_{n=-3}^{\infty}$ is periodic having prime period-$6k = 18$ (see Figure 12).

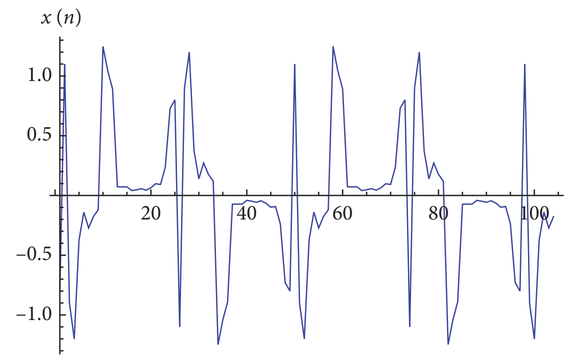

FIGURE 11: $x_{n+1} = (x_n x_{n-3}/(\sqrt{2+\sqrt{2}}\,x_{n-3} - x_{n-2}))$.

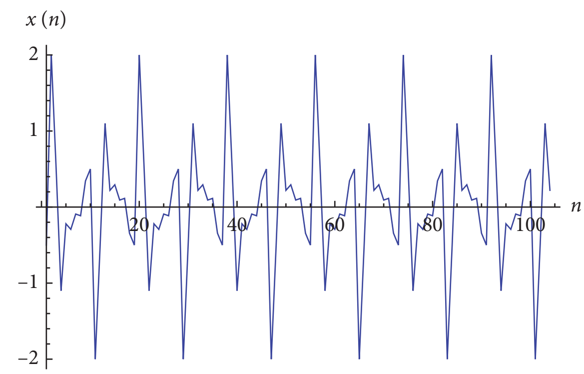

FIGURE 12: $x_{n+1} = (x_n x_{n-3}/(x_{n-3} - x_{n-2}))$.

3.4. Forbidden Sets. In this subsection, we give the forbidden set of recursive equation (7) when $a^2 > 4b$, $a^2 = 4b$, and $a^2 < 4b$.

Clear that, if $x_0 = 0$ and $x_{-1}x_{-2}x_{-3} \neq 0$, then x_4 is undefined. If $x_{-1} = 0$ and $x_0 x_{-2} x_{-3} \neq 0$, then x_7 is undefined. If $x_{-2} = 0$ and $x_0 x_{-1} x_{-3} \neq 0$, then x_6 is undefined. If $x_{-3} = 0$ and $x_0 x_{-1} x_{-2} \neq 0$, then x_5 is undefined

The following result gives the forbidden sets of equation (7) for all values of $a > 0$ and $b > 0$.

Theorem 11. *We have the following statements:*

(1) If $a^2 > 4b$, then equation (3) has the forbidden set

$$\begin{aligned} F_1 = &\bigcup_{i=0}^{3}\left\{(\nu_0,\nu_{-1},\nu_{-2},\nu_{-3}) \in \mathbb{R}^4 : \nu_{-i} = 0\right\} \cup \\ &\bigcup_{m=1}^{\infty}\left\{(\nu_0,\nu_{-1},\nu_{-2},\nu_{-3}) \in \mathbb{R}^4 : \nu_0 = \frac{1}{b}\frac{\widehat{\theta}_{m+1}}{\widehat{\theta}_m}\nu_{-1}\right\} \cup \\ &\bigcup_{m=1}^{\infty}\left\{(\nu_0,\nu_{-1},\nu_{-2},\nu_{-3}) \in \mathbb{R}^4 : \nu_{-1} = \frac{1}{b}\frac{\widehat{\theta}_{m+1}}{\widehat{\theta}_m}\nu_{-2}\right\} \cup \\ &\bigcup_{m=1}^{\infty}\left\{(\nu_0,\nu_{-1},\nu_{-2},\nu_{-3}) \in \mathbb{R}^4 : \nu_{-2} = \frac{1}{b}\frac{\widehat{\theta}_{m+1}}{\widehat{\theta}_m}\nu_{-3}\right\}. \end{aligned} \tag{55}$$

(2) If $a^2 = 4b$, then equation (3) has the forbidden set

$$\begin{aligned} F_2 = &\bigcup_{i=0}^{3}\left\{(\nu_0,\nu_{-1},\nu_{-2},\nu_{-3}) \in \mathbb{R}^4 : \nu_{-i} = 0\right\} \cup \\ &\bigcup_{m=1}^{\infty}\left\{(\nu_0,\nu_{-1},\nu_{-2},\nu_{-3}) \in \mathbb{R}^4 : \nu_0 = \frac{2}{b}\frac{1+m}{m}\nu_{-1}\right\} \cup \\ &\bigcup_{m=1}^{\infty}\left\{(\nu_0,\nu_{-1},\nu_{-2},\nu_{-3}) \in \mathbb{R}^4 : \nu_{-1} = \frac{2}{b}\frac{1+m}{m}\nu_{-2}\right\} \cup \\ &\bigcup_{m=1}^{\infty}\left\{(\nu_0,\nu_{-1},\nu_{-2},\nu_{-3}) \in \mathbb{R}^4 : \nu_{-2} = \frac{2}{b}\frac{1+m}{m}\nu_{-3}\right\}. \end{aligned} \tag{56}$$

(3) If $a^2 < 4b$, then equation (3) has the forbidden set

$$F_3 = \bigcup_{i=0}^{3} \left\{ \left(v_0, v_{-1}, v_{-2}, v_{-3} \right) \in \mathbb{R}^4 \colon v_{-i} = 0 \right\} \cup \bigcup_{m=1}^{\infty} \left\{ \left(v_0, v_{-1}, v_{-2}, v_{-3} \right) \in \mathbb{R}^4 \colon v_0 = \frac{1}{\sqrt{b}} \frac{\sin (m+1)\phi}{\sin m\phi} v_{-1} \right\} \cup \bigcup_{m=1}^{\infty} \left\{ \left(v_0, v_{-1}, v_{-2}, v_{-3} \right) \in \mathbb{R}^4 \colon v_{-1} = \frac{1}{\sqrt{b}} \frac{\sin (m+1)\phi}{\sin m\phi} v_{-2} \right\} \cup \bigcup_{m=1}^{\infty} \left\{ \left(v_0, v_{-1}, v_{-2}, v_{-3} \right) \in \mathbb{R}^4 \colon v_{-2} = \frac{1}{\sqrt{b}} \frac{\sin (m+1)\phi}{\sin m\phi} v_{-3} \right\}. \quad (57)$$

Data Availability

All the data utilized in this article have been included and the sources adopted were cited accordingly.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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