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Research Article

Dynamics and Stability Analysis of a Stackelberg Mixed Duopoly Game with Price Competition in Insurance Market

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This paper investigates the dynamical behaviors of a Stackelberg mixed duopoly game with price competition in the insurance market, involving one state-owned public insurance company and one private insurance company. We study and compare the stability conditions for the Nash equilibrium points of two sequential-move games, public leadership, and private leadership games. Numerical simulations present complicated dynamic behaviors. It is shown that the Nash equilibrium becomes unstable as the price adjustment speed increases, and the system eventually becomes chaotic via flip bifurcation. Moreover, the time-delayed feedback control is used to force the system back to stability.

1. Introduction

The insurance market in most countries has shown typical characteristics of an oligopoly market. Most of the existing literature on oligopoly games in the insurance market concentrates on static games, while less on dynamic games. The competition among oligarchs in the insurance market is mainly reflected in price competition. One of the most famous price game models is the Bertrand model. For a long time, the complex dynamics in Bertrand oligopoly games have been researched widely. For instance, Zhang and Ma [1] investigated a nonlinear Bertrand game of insurance market wherein one of the oligarchs made the decision only with bounded rationality without delay, while the other part made the delayed decision with one period and two periods. Xu and Ma [2] established a price game model with delay based on the insurance market and discussed the existence and stability of equilibrium points. Ahmed et al. [3] analyzed the dynamic behaviors of a differentiated Bertrand duopoly game, in which boundedly rational players apply a gradient adjustment mechanism to update their price in each period. Ma and Si [4] introduced a continuous Bertrand duopoly game model with a two-stage delay and investigated the influence of delay parameters on the dynamic characteristics of the system. Zhao [5] studied the dynamic properties of a Bertrand game model with three oligarchs in which enterprises have heterogeneous expectations. Askar and Al-khedhairi [6] introduced two different Bertrand duopoly models where the first one is the competition of price in which each player wants to maximize its relative profit, and the second model is the classic Bertrand competition in which the players want to maximize their profits. There is still some literature which studies the Bertrand game theory and the complexity of the dynamical system; see [7–12]. The above price game models assume that firms play simultaneous-move games; however, the competition among oligarchs in the real market is mostly a dynamic game or sequential game. It is well known that the most classical sequential game is the Stackelberg game. Shi et al. [13] proposed a price-Stackelberg duopoly game model with boundedly rational players and studied the complex dynamical behaviors. Shah et al. [14] applied the Stackelberg game with stochastic demand for the vendor-retailer system. Wang [15] investigated a manufacturer-Stackelberg game in a price competition supply chain under a fuzzy decision environment. Xiao et al. [16] analyzed a nonlinear two-dimensional duopoly Stackelberg game, including two types of heterogeneous players which are bounded rational players and adaptable players. Other interesting works that can be used to extend the applications of such economic games have been reported in [17–20].

Most countries have state-owned public firms that have a substantial influence on their market competitors. The competition in mixed oligopolies, in which a state-owned public firm competes against a private firm, is widespread in the real market. Such mixed oligopolies occur in various industries, such as automobile, postal services, hospitals, education, banking, and insurance [21]. Analysis of mixed oligopolies can be dated to 1966 when Merrill and Schneider first put forward the assumption that a public firm maximizes welfare (consumer surplus plus firm profits), while private firms maximize profits [22]. Some discussions of the mixed oligopolies were presented in [23–25]. In related works, there are few studies on Stackelberg mixed oligopolies. For instance, Wang and Mukherjee [26] showed welfare under different numbers of private firms under the assumption of a public firm as the Stackelberg leader and private firms as Stackelberg followers. Wang and Lee [27] examined the influence of the order of the firms' moves on the social efficiency with foreign ownership and free entry in a mixed oligopoly market. Tao et al. [28] studied and compared total welfare in Stackelberg mixed duopolies when either the public firm or the private firm acts as the leader. Gelves and Heywood [29] compared the merger between the public leader and the private follower with unilateral privatization of the public leader. Hirose and Matsumura [21] compared welfare and profit under price and quantity competition in Stackelberg mixed duopolies, wherein a state-owned public firm competes against a private firm. These studies mainly focus on quantity competition, less on price competition, and all of them discuss directly the decisions of competitors in Nash equilibrium, and their dynamics have not been studied. However, the dynamic adjustment process converging to Nash equilibrium, and the stability of Nash equilibrium are important in the real market.

In this paper, we pay attention to a Stackelberg mixed duopoly game of price competition between a state-owned public insurance company and a private insurance company in the insurance market and study how this duopoly game evolves to different Nash equilibriums in two sequential-move games, public leadership and private leadership games. Simulations of the complicated dynamic behaviors and chaos control are presented, and the welfare and profit in Nash equilibrium are also discussed.

The rest of this paper is organized as follows: in Section 2, the Stackelberg mixed duopoly game model with price competition is briefly described. In Section 3, the existence of equilibrium points, the instability of bounded equilibrium, and the local stability conditions of Nash equilibrium in two sequential-move cases are analyzed. In Section 4, numerical simulations are used to show the dynamic features of the game, including bifurcation diagram, maximum Lyapunov exponents, phase portrait, and sensitive dependence on initial conditions. The comparison of welfare and profit in Nash equilibrium under two sequential-move cases is also shown by the figures in this section. In Section 5, time-delayed feedback control is used to suppress the appearance of chaotic behavior for the proposed system. Finally, the paper is concluded in Section 6.

2. The Model

We consider the dynamical behaviors of a duopoly price game in the insurance market with two insurance companies (ICs), labeled by $i = 1, 2$. We assume IC_1 is a state-owned public insurance company, while IC_2 is a private insurance company, and the two ICs use different decisional mechanisms. Each IC_i chooses a nonnegative real number p_i, which is the price of its own product. The strategy profile $p = (p_1, p_2)$ results in a corresponding market quantity demanded $q = (q_1, q_2)$. We adopt a standard duopoly model with differentiated products and linear demand [30, 31]. The utility function of the representative consumer is given by

$$U(q_1, q_2) = \alpha(q_1 + q_2) - \frac{\beta}{2}\left(q_1^2 + 2\delta q_1 q_2 + q_2^2\right). \tag{1}$$

Parameters α and β are positive constants and $\delta \in (0, 1)$ measures the degree of horizontal differentiation, where a smaller δ indicates a larger degree of insurance product differentiation. Then, the inverse demand functions are

$$\begin{cases} p_1 = \alpha - \beta q_1 - \beta\delta q_2, \\ p_2 = \alpha - \beta q_2 - \beta\delta q_1. \end{cases} \tag{2}$$

According to equation (2), the direct demand functions can be given by

$$\begin{cases} q_1 = \dfrac{\alpha - \alpha\delta - p_1 + \delta p_2}{\beta\left(1 - \delta^2\right)}, \\[2ex] q_2 = \dfrac{\alpha - \alpha\delta - p_2 + \delta p_1}{\beta\left(1 - \delta^2\right)}. \end{cases} \tag{3}$$

We denote the marginal cost of IC_i with a positive constant c_i, assuming $\alpha > c_1 \geq c_2$. In addition, we assume that α is sufficiently large and that $c_1 - c_2$ is not too large to assure interior solutions in the following games. Since IC_1 is a public firm, its payoff is the domestic social surplus (welfare) [21, 28] and is given by

$$\pi_1 = (p_1 - c_1)q_1 + (p_2 - c_2)q_2 + \left[\alpha(q_1 + q_2) - \frac{\beta\left(q_1^2 + 2\delta q_1 q_2 + q_2^2\right)}{2} - p_1 q_1 - p_2 q_2\right]. \tag{4}$$

For the convenience of expression, equation (4) can be simplified to the following form:

$$\pi_1 = (\alpha - c_1)q_1 + (\alpha - c_2)q_2 - \frac{\beta\left(q_1^2 + 2\delta q_1 q_2 + q_2^2\right)}{2}, \tag{5}$$

and IC_2 is a private firm and its payoff is its own profit:

$$\pi_2 = (p_2 - c_2)q_2. \tag{6}$$

In this study, we consider two sequential-move games, public leadership and private leadership games. To construct and investigate the dynamic characteristics of the two games, we assume that both ICs are bounded rational, in which the players modify their price decisions dynamically according to the marginal payoff [32]. When IC_i (IC_j) is the leader (follower), $i \neq j$, the dynamic system for two ICs has the following form:

$$\begin{cases} p_i(t+1) = p_i(t) + k_i p_i(t)\dfrac{\partial \pi_i\left(p_i(t), p_j(t)\right)}{\partial p_i(t)}, \\ p_j(t+1) = p_j(t) + k_j p_j(t)\dfrac{\partial \pi_j\left(p_i(t+1), p_j(t)\right)}{\partial p_j(t)}, \end{cases} \tag{7}$$

where k_i and k_j are positive parameters known as the speed of price adjustment and $t = 0, 1, 2, \ldots$. At $t+1$, the leader IC_i takes the lead in determining the price $p_i(t+1)$. Then, the follower IC_j chooses $p_j(t+1)$ to maximize its own payoff after observing $p_i(t+1)$, which can be seen in the second equation of system (7).

3. Equilibrium Points and Local Stability

3.1. Public Leadership. In this case, we analyze a Stackelberg model in which IC_1 (IC_2) is the leader (follower). At $t+1$, the leader IC_1 takes the lead in determining the price $p_1(t+1)$ on the basis of marginal payoff. The marginal payoff of IC_1 is

$$\frac{\partial \pi_1\left(p_1(t), p_2(t)\right)}{\partial p_1(t)} = \frac{\delta\left[\alpha(1-\delta) - c_2\right] + 2c_1\left(2-\delta^2\right) + \left(3\delta^2 - 4\right)p_1(t)}{4\beta\left(1-\delta^2\right)}. \tag{8}$$

Then, we can get

$$p_1(t+1) = p_1(t) + \frac{k_1 p_1(t)}{4\beta\left(1-\delta^2\right)}\left\{\delta\left[\alpha(1-\delta) - c_2\right] + 2c_1\left(2-\delta^2\right) + \left(3\delta^2 - 4\right)p_1(t)\right\}. \tag{9}$$

The follower IC_2 has an advantage over the leader. IC_2 has known the current price of IC_1 when it chooses its price $p_2(t+1)$. Hence, the price of IC_2 at period $t+1$ is determined by its own price of period t and IC_1's price of period $t+1$. Then, the marginal payoff of IC_2 is given by

$$\frac{\partial \pi_2\left(p_1(t+1), p_2(t)\right)}{\partial p_2(t)} = \frac{1}{\beta\left(1-\delta^2\right)}\left[\alpha(1-\delta) + c_2 + \delta p_1(t) - 2p_2(t)\right] + \frac{\delta k_1 p_1(t)}{4\beta^2\left(1-\delta^2\right)^2}\left\{\delta\left[\alpha(1-\delta) - c_2\right] + 2c_1\left(2-\delta^2\right) + \left(3\delta^2 - 4\right)p_1(t)\right\}. \tag{10}$$

Then, we have

$$p_2(t+1) = p_2(t) + \frac{k_2 p_2(t)}{\beta\left(1-\delta^2\right)}\left[\alpha(1-\delta) + c_2 + \delta p_1(t) - 2p_2(t)\right] + \frac{\delta k_1 k_2 p_1(t) p_2(t)}{4\beta^2\left(1-\delta^2\right)^2}\left\{\delta\left[\alpha(1-\delta) - c_2\right] + 2c_1\left(2-\delta^2\right) + \left(3\delta^2 - 4\right)p_1(t)\right\}. \tag{11}$$

Thus, the duopoly game can be described by a discrete-time dynamic map as follows:

$$\begin{cases} p_1(t+1) = p_1(t) + \dfrac{k_1 p_1(t)}{4\beta\left(1-\delta^2\right)}\left\{\delta\left[\alpha(1-\delta) - c_2\right] + 2c_1\left(2-\delta^2\right) + \left(3\delta^2 - 4\right)p_1(t)\right\}, \\ p_2(t+1) = p_2(t) + \dfrac{k_2 p_2(t)}{\beta\left(1-\delta^2\right)}\left[\alpha(1-\delta) + c_2 + \delta p_1(t) - 2p_2(t)\right] + \dfrac{\delta k_1 k_2 p_1(t) p_2(t)}{4\beta^2\left(1-\delta^2\right)^2}\left\{\delta\left[\alpha(1-\delta) - c_2\right] + 2c_1\left(2-\delta^2\right) + \left(3\delta^2 - 4\right)p_1(t)\right\}. \end{cases} \tag{12}$$

When the market structure is stable enough at time t, $p_i(t+1)$ are approximately equal to $p_i(t)$. Setting $p_1(t+1) = p_1(t)$ and $p_2(t+1) = p_2(t)$ in equation (12), we can get the following equilibria:

$$\begin{aligned} E_1 &= (0,0),\\ E_2 &= \left(0, \frac{\alpha(1-\delta)+c_2}{2}\right),\\ E_3 &= \left(\frac{\delta[\alpha(1-\delta)-c_2]+2c_1(2-\delta^2)}{4-3\delta^2}, 0\right),\\ E_4 &= (p_1^*, p_2^*), \end{aligned} \tag{13}$$

where

$$\begin{aligned} p_1^* &= \frac{\delta[\alpha(1-\delta)-c_2]+2c_1(2-\delta^2)}{4-3\delta^2},\\ p_2^* &= \frac{(2-\delta^2)[\alpha(1-\delta)+c_1\delta]+2c_2(1-\delta^2)}{4-3\delta^2}. \end{aligned} \tag{14}$$

The points E_1, E_2, and E_3 are known as boundary equilibrium points while E_4 is the Nash equilibrium point. For the stake of economic significance, the equilibrium points should be nonnegative. It is easily concluded that E_2, E_3, and E_4 are all positive according to the conditions that α, β, c_1, and c_2 are positive parameters, $\delta \in (0,1)$, and $\alpha > c_1 \geq c_2$. E_1 represents that every IC has no price; E_2 and E_3 represent the monopolies IC_1 and IC_2, respectively; and E_4 represents both ICs competing in a duopoly game with equilibrium prices of p_1^* and p_2^*.

To analyze the local stability of the equilibrium points, we consider the Jacobian matrix of system (12), which can be given by

$$J(p_1,p_2) = \begin{pmatrix} 1+k_1\mu A + k_1(3\delta^2-4)\mu p_1 & 0\\ 4k_2\delta\mu p_2[1+k_1\mu A+k_1(3\delta^2-4)\mu p_1] & 1+4k_2\mu B - 8k_2\mu p_2 + 4k_1k_2\delta\mu^2 A p_1 \end{pmatrix}, \tag{15}$$

where

$$\begin{aligned} A &= \delta[\alpha(1-\delta)-c_2]+2c_1(2-\delta^2)+(3\delta^2-4)p_1,\\ B &= \alpha(1-\delta)+c_2+\delta p_1 - 2p_2,\\ \mu &= \frac{1}{4\beta(1-\delta^2)}. \end{aligned} \tag{16}$$

Then, the conditions for a stable equilibrium point can be obtained based on the following lemma [33, 34].

Lemma 1. *Suppose the Jacobian matrix (15) at any fixed point $E(\widetilde{p}_1,\widetilde{p}_2)$ has two eigenvalues λ_1 and λ_2, then*

(i) If $|\lambda_1|<1$ and $|\lambda_2|<1$, then $E(\widetilde{p}_1,\widetilde{p}_2)$ is locally asymptotically stable and $E(\widetilde{p}_1,\widetilde{p}_2)$ is called an attracting node

(ii) If $|\lambda_1|>1$ and $|\lambda_2|>1$, then $E(\widetilde{p}_1,\widetilde{p}_2)$ is an unstable repelling node

(iii) If $|\lambda_1|<1$ and $|\lambda_2|>1$ (or $|\lambda_1|>1$ and $|\lambda_2|<1$), then $E(\widetilde{p}_1,\widetilde{p}_2)$ is an unstable saddle point

(iv) If $|\lambda_1|=1$ and $|\lambda_2|\neq 1$ (or $|\lambda_1|\neq 1$ and $|\lambda_2|=1$), then $E(\widetilde{p}_1,\widetilde{p}_2)$ is a nonhyperbolic point

Theorem 1. *The boundary equilibrium E_1 is an unstable repelling node.*

Proof. At E_1, the Jacobian matrix takes the form:

$$J(E_1) = \begin{pmatrix} 1+\dfrac{k_1\{\delta[\alpha(1-\delta)-c_2]+2c_1(2-\delta^2)\}}{4\beta(1-\delta^2)} & 0\\ 0 & 1+\dfrac{k_2[\alpha(1-\delta)+c_2]}{\beta(1-\delta^2)} \end{pmatrix}. \tag{17}$$

The eigenvalues of $J(E_1)$ are given by

$$\lambda_1 = 1 + \frac{k_1\{\delta[\alpha(1-\delta) - c_2] + 2c_1(2-\delta^2)\}}{4\beta(1-\delta^2)},$$
$$\lambda_2 = 1 + \frac{k_2[\alpha(1-\delta) + c_2]}{\beta(1-\delta^2)}. \tag{18}$$

They are both greater than 1, so the point E_1 is an unstable repelling node. □

Theorem 2. *The boundary equilibrium E_2 is unstable. More precisely, we have the following:*

(i) *If $0 < k_2 < (\beta(1-\delta^2)/[\alpha(1-\delta) + c_2])$, then E_2 is a saddle point*

(ii) *If $k_2 = (\beta(1-\delta^2)/[\alpha(1-\delta) + c_2])$, then E_2 is a nonhyperbolic point*

(iii) *If $k_2 > (\beta(1-\delta^2)/[\alpha(1-\delta) + c_2])$, then E_2 is a repelling node*

Proof. At E_2, the Jacobian matrix becomes

$$J(E_2) = \begin{pmatrix} 1 + \dfrac{k_1\{\delta[\alpha(1-\delta) - c_2] + 2c_1(2-\delta^2)\}}{4\beta(1-\delta^2)} & 0 \\ C & 1 - \dfrac{2k_2[\alpha(1-\delta) + c_2]}{\beta(1-\delta^2)} \end{pmatrix}, \tag{19}$$

where

$$C = \frac{k_2\delta[\alpha(1-\delta) + c_2]}{2\beta(1-\delta^2)} + \frac{k_1k_2\delta[\alpha(1-\delta) + c_2]\{\delta[\alpha(1-\delta) - c_2] + 2c_1(2-\delta^2)\}}{8\beta^2(1-\delta^2)^2}. \tag{20}$$

The eigenvalues of $J(E_2)$ are given by

$$\lambda_1 = 1 + \frac{k_1\{\delta[\alpha(1-\delta) - c_2] + 2c_1(2-\delta^2)\}}{4\beta(1-\delta^2)},$$
$$\lambda_2 = 1 - \frac{2k_2[\alpha(1-\delta) + c_2]}{\beta(1-\delta^2)}, \tag{21}$$

$|\lambda_1| > 1$ is always satisfied, so E_2 is unstable. Simple calculations show that $|\lambda_2| < 1$ if $0 < k_2 < (\beta(1-\delta^2)/[\alpha(1-\delta) + c_2])$, $|\lambda_2| = 1$ if $k_2 = (\beta(1-\delta^2)/[\alpha(1-\delta) + c_2])$, and $|\lambda_2| > 1$ if $k_2 > (\beta(1-\delta^2)/[\alpha(1-\delta) + c_2])$. This concludes the proof. □

Theorem 3. *The boundary equilibrium E_3 is unstable. More precisely, we have the following:*

(i) *If $0 < k_1 < (8\beta(1-\delta^2)/\{\delta[\alpha(1-\delta) - c_2] + 2c_1(2-\delta^2)\})$, then E_3 is a saddle point*

(ii) *If $k_1 = (8\beta(1-\delta^2)/\{\delta[\alpha(1-\delta) - c_2] + 2c_1(2-\delta^2)\})$, then E_3 is a nonhyperbolic point*

(iii) *If $k_1 > (8\beta(1-\delta^2)/\{\delta[\alpha(1-\delta) - c_2] + 2c_1(2-\delta^2)\})$, then E_3 is a repelling node*

Proof. At E_3, the Jacobian matrix is

$$J(E_3) = \begin{pmatrix} 1 - \dfrac{k_1\{\delta[\alpha(1-\delta) - c_2] + 2c_1(2-\delta^2)\}}{4\beta(1-\delta^2)} & 0 \\ 0 & 1 + \dfrac{2k_2\{(2-\delta^2)[\alpha(1-\delta) + c_1] + 2c_1(1-\delta^2)\}}{\beta(1-\delta^2)(4-3\delta^2)} \end{pmatrix}. \tag{22}$$

The eigenvalues of $J(E_3)$ are given by

$$\lambda_1 = 1 - \frac{k_1\{\delta[\alpha(1-\delta)-c_2] + 2c_1(2-\delta^2)\}}{4\beta(1-\delta^2)},$$
$$\lambda_2 = 1 + \frac{2k_2\{(2-\delta^2)[\alpha(1-\delta)+c_1] + 2c_1(1-\delta^2)\}}{\beta(1-\delta^2)(4-3\delta^2)}, \tag{23}$$

$|\lambda_2| > 1$ is always satisfied, so E_3 is unstable. Simple calculations show that $|\lambda_1| < 1$ if $0 < k_1 < (8\beta(1-\delta^2)/\{\delta[\alpha(1-\delta)-c_2] + 2c_1(2-\delta^2)\})$, $|\lambda_1| = 1$ if $k_1 = (8\beta(1-\delta^2)/\{\delta[\alpha(1-\delta)-c_2] + 2c_1(2-\delta^2)\})$, and $|\lambda_1| > 1$ if $k_1 > (8\beta(1-\delta^2)/\{\delta[\alpha(1-\delta)-c_2] + 2c_1(2-\delta^2)\})$. This concludes the proof.

The boundary equilibrium points correspond to the situation of one or both ICs going bankrupt. That is, the duopoly market becomes a monopoly, or both of the ICs are out of the insurance market at the same time. However, from an economic point of view, we should pay more attention to the situation of the duopoly market. Hence, we are more interested in investigating the local stability properties of the Nash equilibrium point E_4. The Jacobian matrix at E_4 takes the form:

$$J(E_4) = \begin{pmatrix} 1 - \dfrac{k_1(4-3\delta^2)p_1^*}{4\beta(1-\delta^2)} & 0 \\ \dfrac{k_2\delta p_2^*}{\beta(1-\delta^2)} - \dfrac{k_1k_2\delta(4-3\delta^2)p_1^*p_2^*}{4\beta^2(1-\delta^2)^2} & 1 - \dfrac{2k_2p_2^*}{\beta(1-\delta^2)} \end{pmatrix}. \tag{24}$$

The characteristic equation is

$$\lambda^2 - \mathrm{Tr}(J(E_4))\lambda + \mathrm{Det}(J(E_4)) = 0, \tag{25}$$

where Tr is the trace and Det is the determinant, which are given by

$$\mathrm{Tr}(J(E_4)) = 2 - \frac{k_1(4-3\delta^2)p_1^*}{4\beta(1-\delta^2)} - \frac{2k_2p_2^*}{\beta(1-\delta^2)},$$
$$\mathrm{Det}(J(E_4)) = 1 - \frac{k_1(4-3\delta^2)p_1^*}{4\beta(1-\delta^2)} - \frac{2k_2p_2^*}{\beta(1-\delta^2)} + \frac{k_1k_2(4-3\delta^2)p_1^*p_2^*}{2\beta^2(1-\delta^2)^2}. \tag{26}$$

Then, we have

$$\mathrm{Tr}(J(E_4))^2 - 4\mathrm{Det}(J(E_4)) = \left(\frac{k_1(4-3\delta^2)p_1^*}{4\beta(1-\delta^2)} - \frac{2k_2p_2^*}{\beta(1-\delta^2)}\right)^2, \tag{27}$$

which indicates that the eigenvalues are real. According to Jury conditions [35], the necessary and sufficient conditions for the local stability of E_4 are as follows:

$$\begin{cases} 1 - \mathrm{Det}(J(E_4)) > 0, \\ 1 - \mathrm{Tr}(J(E_4)) + \mathrm{Det}(J(E_4)) > 0, \\ 1 + \mathrm{Tr}(J(E_4)) + \mathrm{Det}(J(E_4)) > 0. \end{cases} \tag{28}$$

The above conditions are, respectively, equivalent to

$$\text{(i)}\ \frac{k_1(4-3\delta^2)p_1^*}{4\beta(1-\delta^2)} + \frac{2k_2p_2^*}{\beta(1-\delta^2)} - \frac{k_1k_2(4-3\delta^2)p_1^*p_2^*}{2\beta^2(1-\delta^2)^2} > 0, \tag{29}$$

$$\text{(ii)}\ \frac{k_1k_2(4-3\delta^2)p_1^*p_2^*}{2\beta^2(1-\delta^2)^2} > 0, \tag{30}$$

$$\text{(iii)}\ 4 - \frac{k_1(4-3\delta^2)p_1^*}{2\beta(1-\delta^2)} - \frac{4k_2p_2^*}{\beta(1-\delta^2)} + \frac{k_1k_2(4-3\delta^2)p_1^*p_2^*}{2\beta^2(1-\delta^2)^2} > 0. \tag{31}$$

Clearly, the condition (ii) is always satisfied. Then, the following result can be obtained from the derivation of conditions (i) and (iii). □

Theorem 4. *The Nash equilibrium point E_4 is asymptotically locally stable if*

$$k_1 < \frac{8\beta(1-\delta^2)}{(4-3\delta^2)p_1^*},$$
$$k_2 < \frac{\beta(1-\delta^2)}{p_2^*}. \tag{32}$$

Proof. Inequality (29) can be rewritten as

$$\frac{8\beta(1-\delta^2)}{k_1(4-3\delta^2)p_1^*} + \frac{\beta(1-\delta^2)}{k_2p_2^*} > 2. \tag{33}$$

Inequality (31) can be modified as

$$4 - \frac{k_1(4-3\delta^2)p_1^*}{2\beta(1-\delta^2)} - \frac{4k_2p_2^*}{\beta(1-\delta^2)} + \frac{k_1k_2(4-3\delta^2)p_1^*p_2^*}{2\beta^2(1-\delta^2)^2} = 4\left(1 - \frac{k_1(4-3\delta^2)p_1^*}{8\beta(1-\delta^2)}\right)\left(1 - \frac{k_2p_2^*}{\beta(1-\delta^2)}\right) > 0. \tag{34}$$

Inequality (34) holds if and only if the following two conditions are satisfied:

$$k_1 < \frac{8\beta(1-\delta^2)}{(4-3\delta^2)p_1^*}, \quad k_2 < \frac{\beta(1-\delta^2)}{p_2^*}, \tag{35}$$

or

$$k_1 > \frac{8\beta(1-\delta^2)}{(4-3\delta^2)p_1^*}, \quad k_2 > \frac{\beta(1-\delta^2)}{p_2^*}. \tag{36}$$

It is obvious that the first condition implies inequality (33). On the other hand, inequality (33) is impossible in the (k_1, k_2)-plane determined by the second condition. This concludes the proof.

Condition (32) defines a stability region in the plane of the price adjustment speed (k_1, k_2) (see Figure 1(a)). The boundary intersects the axes k_1 and k_2 at points R_1 and R_2, respectively, whose coordinates are

$$R_1 = \left(\frac{8\beta(1-\delta^2)}{(4-3\delta^2)p_1^*}, 0\right), \quad R_2 = \left(0, \frac{\beta(1-\delta^2)}{p_2^*}\right). \tag{37}$$

Simple calculations show that if $k_1 = (8\beta(1-\delta^2)/(4-3\delta^2)p_1^*)$ or $k_2 = (\beta(1-\delta^2)/p_2^*)$, one of the absolute values of eigenvalues is equal to 1. Inequalities (29) and (31) define a bounded region of stability beyond which a flip bifurcation and a Neimark–Sacker bifurcation occur, respectively [33, 36]. According to Theorem 4, we can get that the Nash equilibrium point E_4 loses its stability only via flip bifurcation when one or both values of k_1 and k_2 are greater than the boundary values of the stability region.

According to the expressions of the coordinate value of the boundary points, we can clearly find out the effects of the changing values of parameters α, β, c_1, and c_2 on the stability region, respectively, but it is difficult to directly observe how parameter δ affects the stability regions from the expression. By computer work on the stability conditions for four cases $(\delta = 0.05, 0.3, 0.5, 0.7)$, the stability regions in the (k_1, k_2)-plane are numerically obtained and are plotted in Figure 1(a). We can see that increasing δ reduces the stability region, and the stability of system (12) is more sensitive to IC_2. If k_2 is relatively lower, even if k_1 is relatively higher, system (12) is stable. □

3.2. Private Leadership. When IC_2 (IC_1) is the leader (follower), we can write system (7) as

$$\begin{cases} p_1(t+1) = p_1(t) + \dfrac{k_1 p_1(t)}{\beta(1-\delta^2)}(c_1 - \delta c_2 - p_1(t) + \delta p_2(t)) \\ \qquad + \dfrac{\delta k_1 k_2 p_1(t) p_2(t)}{\beta^2(1-\delta^2)^2}\left[\alpha(1-\delta) + \delta c_1 + (1-2\delta^2)c_2 - 2(1-\delta^2)p_2(t)\right], \\ p_2(t+1) = p_2(t) + \dfrac{k_2 p_2(t)}{\beta(1-\delta^2)}\left[\alpha(1-\delta) + \delta c_1 + (1-2\delta^2)c_2 - 2(1-\delta^2)p_2(t)\right]. \end{cases} \tag{38}$$

Setting $p_1(t+1) = p_1(t)$ and $p_2(t+1) = p_2(t)$ in equation (38), we have the following equilibria:

$$\begin{aligned} E_5 &= (0,0), \\ E_6 &= \left(0, \frac{\alpha(1-\delta) + \delta c_1 + (1-2\delta^2)c_2}{2(1-\delta^2)}\right), \\ E_7 &= (c_1 - \delta c_2, 0), \\ E_8 &= (p_1^{**}, p_2^{**}), \end{aligned} \tag{39}$$

where

$$p_1^{**} = \frac{\delta[\alpha(1-\delta) - c_2] + (2-\delta^2)c_1}{2(1-\delta^2)}, \quad p_2^{**} = \frac{\alpha(1-\delta) + \delta c_1 + (1-2\delta^2)c_2}{2(1-\delta^2)}. \tag{40}$$

Points E_5, E_6, and E_7 are boundary equilibrium points and E_8 is the unique Nash equilibrium point. It is clear that E_6, E_7, and E_8 are all positive according to the conditions that α, β, c_1, and c_2 are positive parameters, $\delta \in (0,1)$, and $\alpha > c_1 \geq c_2$.

The Jacobian matrix of system (38) can be given by

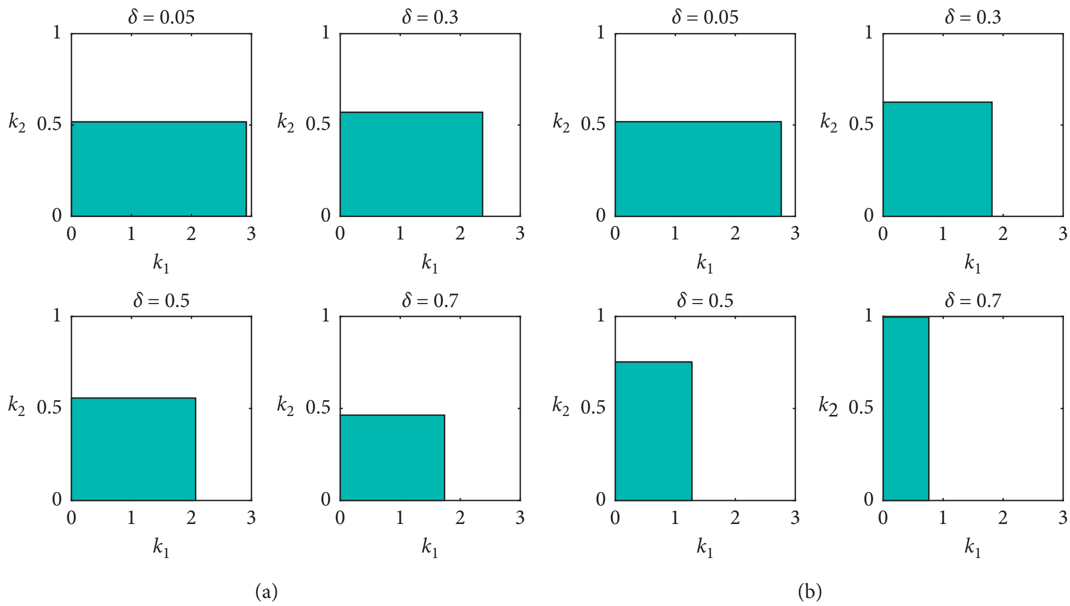


FIGURE 1: (a) The stability regions in the (k_1, k_2)-plane of Nash equilibrium point E_4 for system (12) with different levels of δ. (b) The stability regions in the (k_1, k_2)-plane of Nash equilibrium point E_8 for system (38) with different levels of δ. The other values of the parameters are $\alpha = 0.7$, $\beta = 0.2$, $c_1 = 0.13$, and $c_2 = 0.1$.

$$J(p_1, p_2) = \begin{pmatrix} 1 + k_1\mu' B' - k_1\mu' p_1 + k_1 k_2 \delta \mu'^2 A' p_2 & k_1 \delta \mu' p_1 \left[1 + k_2 \mu' A' - 2k_2\left(1-\delta^2\right)\mu' p_2\right] \\ 0 & 1 + k_2\mu' A' - 2k_2\left(1-\delta^2\right)\mu' p_2 \end{pmatrix}, \tag{41}$$

where

$$\begin{aligned} A' &= \alpha(1-\delta) + \delta c_1 + \left(1-2\delta^2\right)c_2 - 2\left(1-\delta^2\right)p_2, \\ B' &= c_1 - \delta c_2 - p_1 + \delta p_2, \\ \mu' &= \frac{1}{\beta\left(1-\delta^2\right)}. \end{aligned} \tag{42}$$

Theorem 5. *The boundary equilibrium E_5 is an unstable repelling node.*

Proof. The Jacobian matrix (41) at the point E_5 takes the form:

$$J(E_5) = \begin{pmatrix} 1 + \dfrac{k_1(c_1 - \delta c_2)}{\beta\left(1-\delta^2\right)} & 0 \\ 0 & 1 + \dfrac{k_2\left[\alpha(1-\delta) + \delta c_1 + \left(1-2\delta^2\right)c_2\right]}{\beta\left(1-\delta^2\right)} \end{pmatrix}. \tag{43}$$

The eigenvalues of $J(E_5)$ are given by

$$\begin{aligned} \lambda_1 &= 1 + \frac{k_1(c_1 - \delta c_2)}{\beta\left(1-\delta^2\right)}, \\ \lambda_2 &= 1 + \frac{k_2\left[\alpha(1-\delta) + \delta c_1 + \left(1-2\delta^2\right)c_2\right]}{\beta\left(1-\delta^2\right)}. \end{aligned} \tag{44}$$

It is clear that they are both greater than 1, so the point E_5 is an unstable repelling node. □

Theorem 6. *The boundary equilibrium E_6 is unstable. More precisely, we have the following:*

(i) *If* $0 < k_2 < (2\beta(1-\delta^2)/[\alpha(1-\delta) + \delta c_1 + (1-2\delta^2) c_2])$, *then E_6 is a saddle point* *(ii) If* $k_2 = (2\beta(1-\delta^2)/[\alpha(1-\delta)+\delta c_1+(1-2\delta^2)c_2])$, *then* E_6 *is a nonhyperbolic point*

(iii) If $k_2 > (2\beta(1-\delta^2)/[\alpha(1-\delta)+\delta c_1+(1-2\delta^2)c_2])$, *then* E_6 *is a repelling node*

Proof. At E_6, the Jacobian matrix becomes

$$J(E_6)=\begin{pmatrix} 1+\dfrac{k_1[2c_1-\delta c_2+\alpha\delta(1-\delta)]}{2\beta(1-\delta^2)^2} & 0 \\ 0 & 1-\dfrac{k_2[\alpha(1-\delta)+\delta c_1+(1-2\delta^2)c_2]}{\beta(1-\delta^2)} \end{pmatrix}. \quad (45)$$

The eigenvalues of $J(E_6)$ are given by

$$\lambda_1 = 1+\frac{k_1[2c_1-\delta c_2+\alpha\delta(1-\delta)]}{2\beta(1-\delta^2)^2}, \qquad \lambda_2 = 1-\frac{k_2[\alpha(1-\delta)+\delta c_1+(1-2\delta^2)c_2]}{\beta(1-\delta^2)}, \quad (46)$$

$|\lambda_1|>1$ is always satisfied, so E_6 is unstable. Simple calculations show that $|\lambda_2|<1$ if $0<k_2<(2\beta(1-\delta^2)/[\alpha(1-\delta)+\delta c_1+(1-2\delta^2)c_2])$, $|\lambda_2|=1$ if $k_2=(2\beta(1-\delta^2)/[\alpha(1-\delta)+\delta c_1+(1-2\delta^2)c_2])$, and $|\lambda_2|>1$ if $k_2>(2\beta(1-\delta^2)/[\alpha(1-\delta)+\delta c_1+(1-2\delta^2)c_2])$. This concludes the proof. □

Theorem 7. *The boundary equilibrium* E_7 *is unstable. More precisely, we have the following:*

(i) If $0<k_1<(2\beta(1-\delta^2)/(c_1-\delta c_2))$, *then* E_7 *is a saddle point*

(ii) If $k_1=(2\beta(1-\delta^2)/(c_1-\delta c_2))$, *then* E_7 *is a nonhyperbolic point*

(iii) If $k_1>(2\beta(1-\delta^2)/(c_1-\delta c_2))$, *then* E_7 *is a repelling node*

Proof. At E_7, the Jacobian matrix is

$$J(E_7)=\begin{pmatrix} 1-\dfrac{k_1(c_1-\delta c_2)}{\beta(1-\delta^2)} & \dfrac{k_1\delta(c_1-\delta c_2)}{\beta(1-\delta^2)}+\dfrac{k_1k_2\delta(c_1-\delta c_2)[\alpha(1-\delta)+\delta c_1+(1-2\delta^2)c_2]}{\beta^2(1-\delta^2)^2} \\ 0 & 1+\dfrac{k_2[\alpha(1-\delta)+\delta c_1+(1-2\delta^2)c_2]}{\beta(1-\delta^2)} \end{pmatrix}. \quad (47)$$

The eigenvalues of $J(E_7)$ are given by

$$\lambda_1 = 1-\frac{k_1(c_1-\delta c_2)}{\beta(1-\delta^2)}, \qquad \lambda_2 = 1+\frac{k_2[\alpha(1-\delta)+\delta c_1+(1-2\delta^2)c_2]}{\beta(1-\delta^2)}, \quad (48)$$

$|\lambda_2|>1$ is always satisfied, so E_7 is unstable. Simple calculations show that $|\lambda_1|<1$ if $0<k_1<(2\beta(1-\delta^2)/(c_1-\delta c_2))$, $|\lambda_1|=1$ if $k_1=(2\beta(1-\delta^2)/(c_1-\delta c_2))$, and $|\lambda_1|>1$ if $k_1>(2\beta(1-\delta^2)/(c_1-\delta c_2))$. This concludes the proof.

Next, we investigate the local stability properties of the Nash equilibrium point E_8. The Jacobian matrix at the point E_8 takes the form:

$$J(E_8)=\begin{pmatrix} 1-\dfrac{k_1p_1^{**}}{\beta(1-\delta^2)} & \dfrac{k_1\delta p_1^{**}}{\beta(1-\delta^2)}-\dfrac{2k_1k_2\delta p_1^{**}p_2^{**}}{\beta^2(1-\delta^2)} \\ 0 & 1-\dfrac{2k_2p_2^{**}}{\beta} \end{pmatrix}. \quad (49)$$

The trace and determinant of $J(E_8)$ are

$$\mathrm{Tr}(J(E_8)) = 2-\frac{k_1p_1^{**}}{\beta(1-\delta^2)}-\frac{2k_2p_2^{**}}{\beta}, \qquad \mathrm{Det}(J(E_8)) = 1-\frac{k_1p_1^{**}}{\beta(1-\delta^2)}-\frac{2k_2p_2^{**}}{\beta}+\frac{2k_1k_2p_1^{**}p_2^{**}}{\beta^2(1-\delta^2)}. \quad (50)$$

Then, we have

$$\mathrm{Tr}\left(J\left(E_8\right)\right)^2-4\mathrm{Det}\left(J\left(E_8\right)\right)=\left(\frac{k_1p_1^{**}}{\beta\left(1-\delta^2\right)}-\frac{2k_2p_2^{**}}{\beta}\right)^2. \quad (51)$$

It indicates that the eigenvalues are real. According to Jury conditions, the necessary and sufficient conditions for the local stability of E_8 can be given by

$$\text{(i) } \frac{k_1p_1^{**}}{\beta\left(1-\delta^2\right)}+\frac{2k_2p_2^{**}}{\beta}-\frac{2k_1k_2p_1^{**}p_2^{**}}{\beta^2\left(1-\delta^2\right)}>0, \quad (52)$$

$$\text{(ii) } \frac{2k_1k_2p_1^{**}p_2^{**}}{\beta^2\left(1-\delta^2\right)}>0, \quad (53)$$

$$\text{(iii) } 4-\frac{2k_1p_1^{**}}{\beta\left(1-\delta^2\right)}-\frac{4k_2p_2^{**}}{\beta}+\frac{2k_1k_2p_1^{**}p_2^{**}}{\beta^2\left(1-\delta^2\right)}>0. \quad (54)$$

Clearly, condition (ii) is always satisfied. According to conditions (i) and (iii), we have the following result. □

Theorem 8. *The Nash equilibrium point E_8 is asymptotically locally stable if*

$$k_1<\frac{2\beta\left(1-\delta^2\right)}{p_1^{**}},\qquad k_2<\frac{\beta}{p_2^{**}}. \quad (55)$$

Proof. Inequality (52) can be rewritten as

$$\frac{2\beta\left(1-\delta^2\right)}{k_1p_1^{**}}+\frac{\beta}{k_2p_2^{**}}>2. \quad (56)$$

Inequality (54) can be modified as

$$4-\frac{2k_1p_1^{**}}{\beta\left(1-\delta^2\right)}-\frac{4k_2p_2^{**}}{\beta}+\frac{2k_1k_2p_1^{**}p_2^{**}}{\beta^2\left(1-\delta^2\right)}=4\left(1-\frac{k_1p_1^{**}}{2\beta\left(1-\delta^2\right)}\right)\left(1-\frac{k_2p_2^{**}}{\beta}\right)>0. \quad (57)$$

We complete the proof by imitating the discussions in Theorem 4.

The stability region for the Nash equilibrium point E_8 is defined by the inequalities in condition (55). The boundary curve intersects the axes k_1 and k_2 at points R_3 and R_4, respectively, whose coordinates are

$$R_3=\left(\frac{2\beta\left(1-\delta^2\right)}{p_1^{**}},0\right),\qquad R_4=\left(0,\frac{\beta}{p_2^{**}}\right). \quad (58)$$

Simple calculations show that if $k_1=(2\beta(1-\delta^2)/p_1^{**})$ or $k_2=(\beta/p_2^{**})$, one of the absolute values of eigenvalues is equal to 1. According to Theorem 8, we can get that the Nash equilibrium point E_8 is stable inside the stability region, and loses its stability through flip bifurcation. By computer work on the stability conditions for four cases $(\delta=0.05,0.3,0.5,0.7)$, the stability regions of Nash equilibrium point E_8 in the (k_1,k_2)-plane are shown in Figure 1(b). Comparing Figures 1(a) and 1(b), we see that when the parameter δ is close to zero, the stability regions of Nash equilibrium points E_4 and E_8 are similar, but the difference between them becomes larger with the increase of the parameter δ. □

4. Numerical Simulation

In this section, we perform some numerical simulations for the complex dynamical behaviors of systems (12) and (38) and show how the systems evolve under different levels of parameters. Such simulations include a bifurcation diagram, maximum Lyapunov exponents, phase portrait, and sensitive dependence on initial conditions to further study the unpredictable behavior of the game. In all numerical simulations, parameters α, β, δ, c_1, and c_2 are commonly set as $\alpha=0.7$, $\beta=0.2$, $\delta=0.5$, $c_1=0.13$, and $c_2=0.1$. We perform numerical simulations for the following two situations, respectively.

4.1. Public Leadership. In a real insurance market, the demand function and their marginal costs are relatively certain, so the price adjustment speed is regarded as an important strategy for ICs to pursue profit maximization. In this case, we show by numerical simulations how system (12) evolves under different levels of the parameter k_1, the price adjustment speed of IC_1. We fix the parameter $k_2=0.5$, and the bifurcation diagram with respect to the fact that parameter k_1 is plotted in Figure 2(a). It shows that the equilibrium point begins stable; increasing the value of k_1 gives the appearance of a stable 2-cycle period through flip bifurcation, then increasing the value of k_1 further shows a sequence of period-doubling bifurcations followed by cycles with high periodicity; and chaotic scenario occurs in the end. The corresponding maximum Lyapunov exponents are plotted in Figure 2(b) to show bifurcation and chaos, where positive values show the chaotic behaviors.

The observations from Figure 2(a) tell that system (12) becomes unstable through the period-doubling bifurcation when the parameter takes suitable values. About the case in Figure 2, five two-dimension phase portraits for different values of k_1 are shown in Figure 3, which give a more detailed description of the orbits of system (12). The phase portraits show a flip bifurcation process, where chaos occurs when k_1 takes its value big enough, and strange attractors can be seen in the fifth phase portrait in Figure 3. The strange attractor reflects the complexity of ICs' dynamic price competition in chaos.

The sensitivity to initial conditions is also one of the important characteristics of chaos. Figure 4 reflects the case

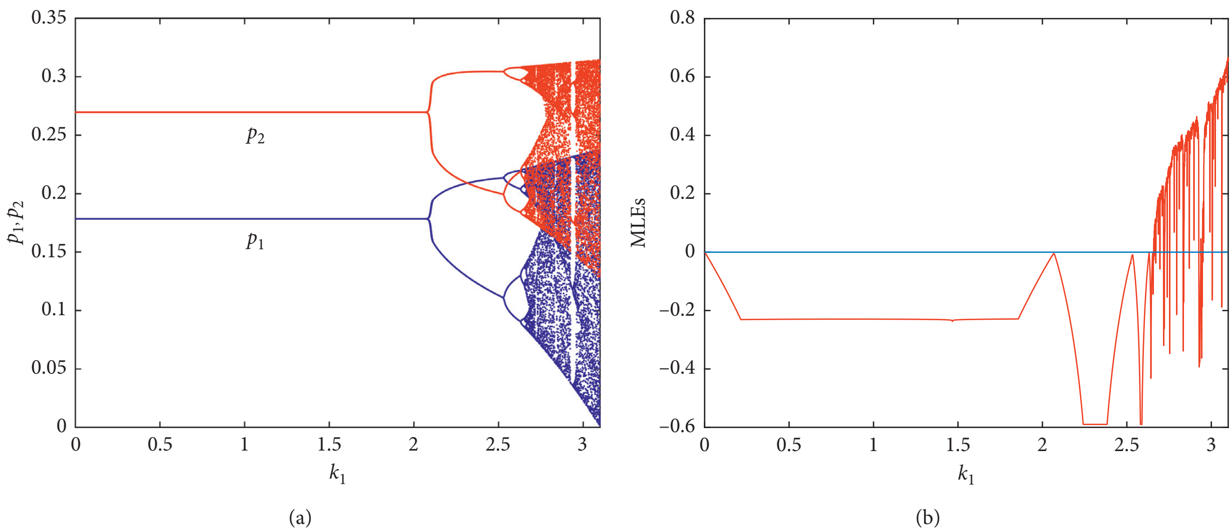


Figure 2: Bifurcation diagram and MLEs for system (12) with respect to parameter k_1 when $k_2 = 0.5$.

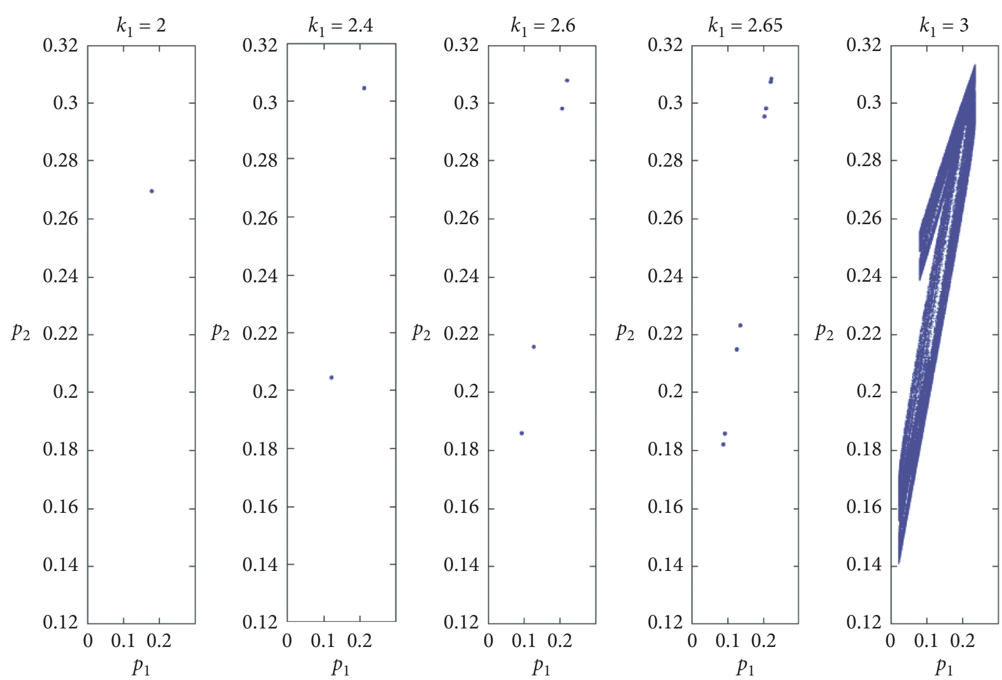


Figure 3: Phase portraits for Figure 2 with various values of k_1.

that the initial state p_1 ranges from 0.1785 to 0.1786 with other parameters keeping fixed. Figures 4(a) and 4(b) show the orbits of IC_1's price and IC_2's price, respectively, where the blue ones (labeled by superscript (1)) start from the initial point $(p_1^1(0), p_2^1(0)) = (0.1785, 0.2696)$ and the red ones (labeled by superscript (2)) start from the initial point $(p_1^2(0), p_2^2(0)) = (0.1786, 0.2696)$. After a series of iterations, great impacts will emerge in both ICs' prices, even though the initial price of IC_1 alters a little.

4.2. Private Leadership. In this case, the numerical simulations show the effect of the parameter k_2 and the price adjustment speed of IC_2, on the stability of system (38). Figures 5(a) and 5(b) show the bifurcation diagram with respect to k_2 and the corresponding maximum Lyapunov exponents of system (38), respectively. Figure 6 shows five situations of phase portraits with different k_2 of system (38). We can see system (38) loses its stability through flip bifurcation, and chaotic attractors occur after the accumulation of a period-doubling cascade. The results demonstrate that the insurance market is stable for relatively small values of k_2, and a faster adjustment speed is disadvantageous for system (38) to keep the stability. Figure 7 shows the sensitive dependence on the initial conditions of system (38) when $k_1 = 0.7$ and $k_2 = 0.95$. Figures 7(a) and 7(b) show the orbits of IC_1's price and IC_2's price, respectively, where the blue

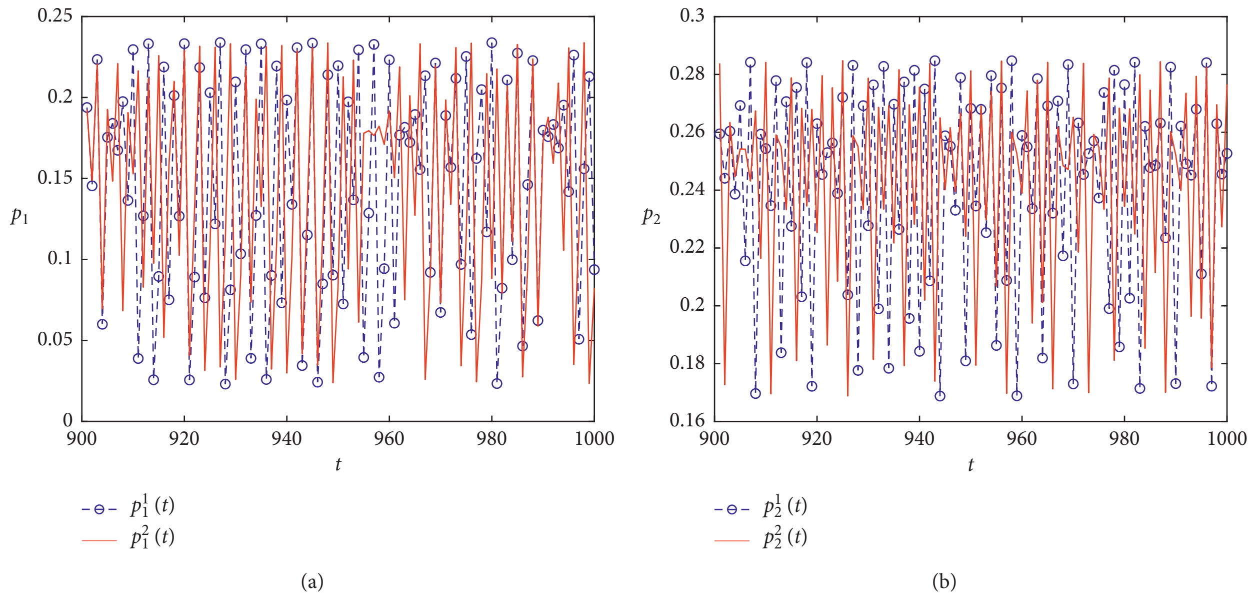


FIGURE 4: Sensitive dependence on initial conditions for system (12) in the time periods [900, 1000] when $k_1 = 3$ and $k_2 = 0.5$.

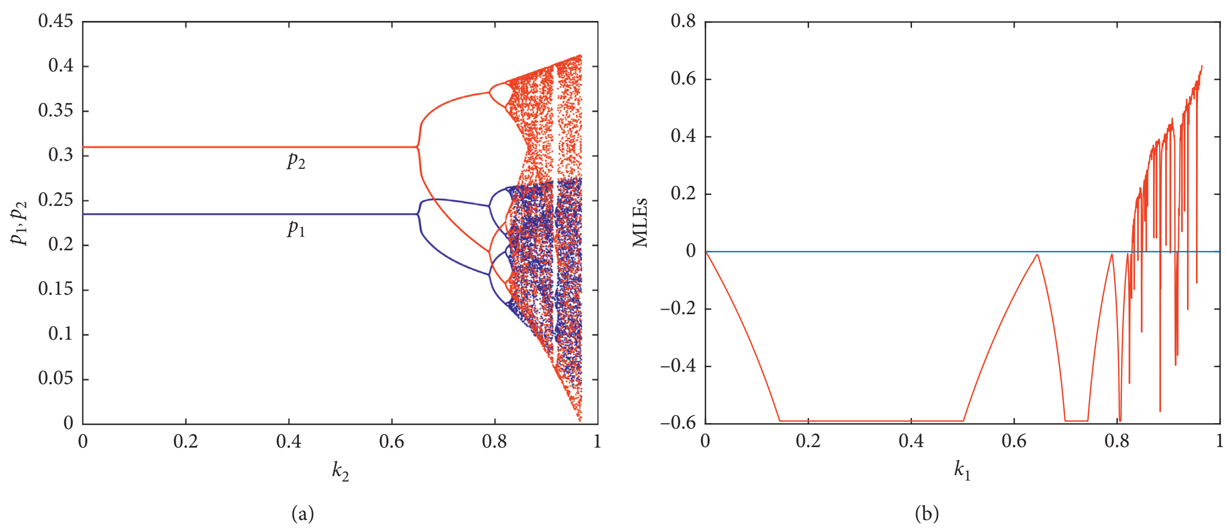


FIGURE 5: Bifurcation diagram and MLEs for system (38) with respect to parameter k_2 when $k_1 = 0.7$.

ones (labeled by superscript (1)) start from the initial point $(p_1^1(0), p_2^1(0)) = (0.2351, 0.3101)$ and the red ones (labeled by superscript (2)) start from the initial point $(p_1^2(0), p_2^2(0)) = (0.2351, 0.3102)$. We can see that the difference between the orbits with slightly deviated initial values builds up rapidly after a series of iterations, although their states are indistinguishable at the beginning.

4.3. The Comparison of Welfare and Profit for Two Games. We now compare the welfare and profit levels for the above two sequential-move games (public leadership and private leadership). Keeping fixed $\alpha = 0.7$, $\beta = 0.2$, $\delta = 0.5$, $c_1 = 0.13$, and $c_2 = 0.1$, Figure 8(a) shows the values of the welfare of IC_1, and Figure 8(b) shows the values of the profit of IC_2. The values of the welfare and profit in the Nash equilibrium state of both games are marked, where the red points refer to the welfare and profit at the Nash equilibrium point E_4 of the game (12), and the blue points refer to the welfare and profit at the Nash equilibrium point E_8 of the game (38). We can see that the values of the welfare and profit at the two Nash equilibrium points are positive, where $\pi_1(E_4)$ is greater than $\pi_1(E_8)$ and $\pi_2(E_4)$ is less than $\pi_2(E_8)$. The results show that an IC, whether public or private, is more profitable in the Nash equilibrium state when it is the leader in the price competition game.

5. Chaos Control

As can be seen from the above numerical simulations that price adjustment speeds have a great influence on the stability of systems (12) and (38), the dynamical behaviors of

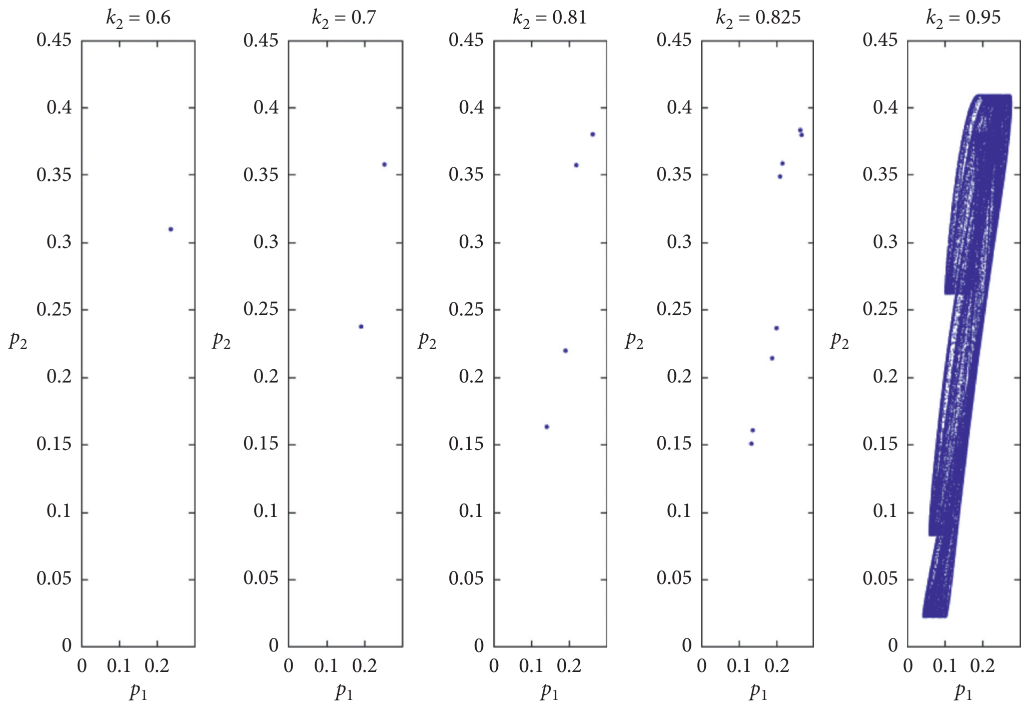


Figure 6: Phase portraits for Figure 5 with various values of k_2.

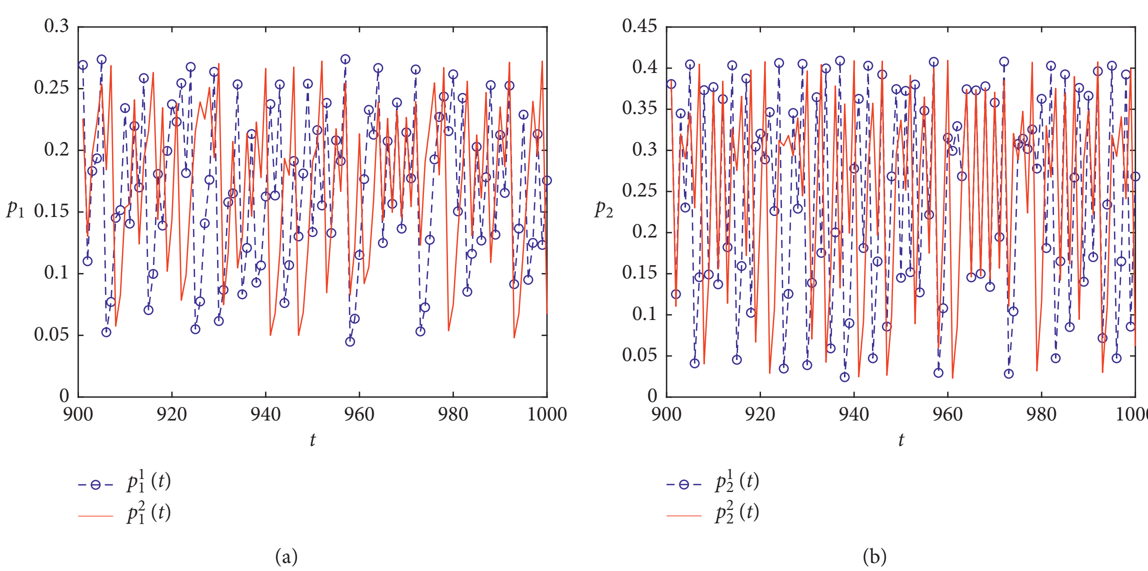


Figure 7: Sensitive dependence on initial conditions for system (38) in the time periods [900, 1000] when $k_1 = 0.7$ and $k_2 = 0.95$.

both two systems will be chaotic when the parameters fail to locate in the stable region. In practical application, the appearance of chaos is not expected, we hope that the occurrence of chaos can be avoided or controlled, and the insurance market can be controlled to a balance when it runs irregularly. In this section, we use the time-delayed feedback

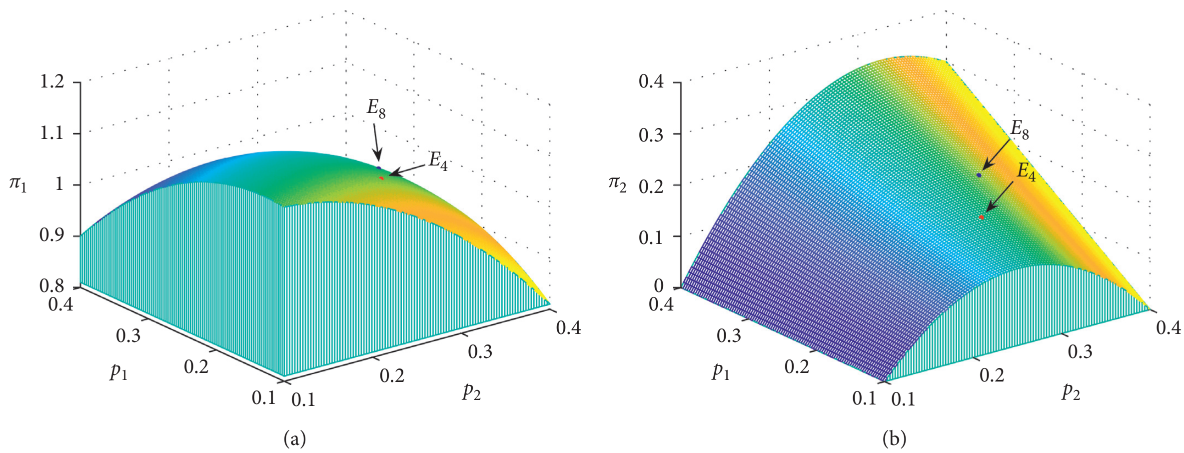


Figure 8: The welfare of IC_1 and the profit of IC_2.

control method to control the chaotic phenomenon [37–40]. Similar to the fourth section, we divide the following two cases for numerical simulations.

5.1. Public Leadership. We modify the equations of system (12) by inserting the control action $\phi(p_i(t) - p_i(t+1))$, where $\phi > 0$ is the controlling coefficient. Then, the controlled system can be given by

$$\begin{cases} p_1(t+1) = p_1(t) + \dfrac{k_1 p_1(t)}{4\beta(1-\delta^2)}\left\{\delta[\alpha(1-\delta) - c_2] + 2c_1(2-\delta^2) + (3\delta^2 - 4)p_1(t)\right\} + \phi(p_1(t) - p_1(t+1)), \\ p_2(t+1) = p_2(t) + \dfrac{k_2 p_2(t)}{\beta(1-\delta^2)}[\alpha(1-\delta) + c_2 + \delta p_1(t) - 2p_2(t)] \\ \qquad + \dfrac{\delta k_1 k_2 p_1(t) p_2(t)}{4\beta^2(1-\delta^2)^2}\left\{\delta[\alpha(1-\delta) - c_2] + 2c_1(2-\delta^2) + (3\delta^2 - 4)p_1(t)\right\} + \phi(p_2(t) - p_2(t+1)), \end{cases} \tag{59}$$

which can be rewritten as

$$\begin{cases} p_1(t+1) = p_1(t) + \dfrac{k_1 p_1(t)}{4\beta(1-\delta^2)(1+\phi)}\left\{\delta[\alpha(1-\delta) - c_2] + 2c_1(2-\delta^2) + (3\delta^2 - 4)p_1(t)\right\}, \\ p_2(t+1) = p_2(t) + \dfrac{k_2 p_2(t)}{\beta(1-\delta^2)(1+\phi)}[\alpha(1-\delta) + c_2 + \delta p_1(t) - 2p_2(t)] \\ \qquad + \dfrac{\delta k_1 k_2 p_1(t) p_2(t)}{4\beta^2(1-\delta^2)^2(1+\phi)}\left\{\delta[\alpha(1-\delta) - c_2] + 2c_1(2-\delta^2) + (3\delta^2 - 4)p_1(t)\right\}. \end{cases} \tag{60}$$

It is easy to see that the controlled system (60) has the same Nash equilibrium point E_4 as the original system (12). At E_4, the Jacobian matrix of system (60) takes the form

$$J(E_4)=\begin{pmatrix} 1-\dfrac{k_1\left(4-3\delta^2\right)p_1^*}{4\beta\left(1-\delta^2\right)(1+\phi)} & 0 \\ \dfrac{k_2\delta p_2^*}{\beta\left(1-\delta^2\right)(1+\phi)}-\dfrac{k_1k_2\delta\left(4-3\delta^2\right)p_1^*p_2^*}{4\beta^2\left(1-\delta^2\right)(1+\phi)} & 1-\dfrac{2k_2p_2^*}{\beta\left(1-\delta^2\right)(1+\phi)} \end{pmatrix}. \quad (61)$$

Figure 2(a) shows that chaotic behavior of the original system (12) occurs when parameter values are fixed as $(\alpha,\beta,\delta,c_1,c_2,k_1,k_2)=(0.7,0.2,0.5,0.13,0.1,3,0.5)$. By a similar approach in Section 3 to get the stability conditions (i)-(iii) for the original system (12), we can get that all the eigenvalues of the matrix (61) will lie within the unit circle provided that $\phi>0.45$, when the other parameters take above values. In other words, when $\phi>0.45$, the controlled system (60) will be asymptotically locally stable. This result can be numerically shown by Figure 9.

Figure 9(a) is the bifurcation diagram with respect to ϕ, where we see that, with the value of ϕ increasing, the system is gradually controlled from the chaotic state, 8, 4, 2-period bifurcation to a stable state. Figure 9(b) shows the stable behaviors of the orbits of the controlled system (60) beginning from the initial state $(p_1(0),p_2(0))=(0.19,0.27)$ for different levels of ϕ. We can see that the larger the feedback value is, the faster the system tends to be stable.

5.2. Private Leadership. Adding control action $\psi(p_i(t)-p_i(t+1))$ to system (38) and simplifying the system, we get the following form of the controlled system:

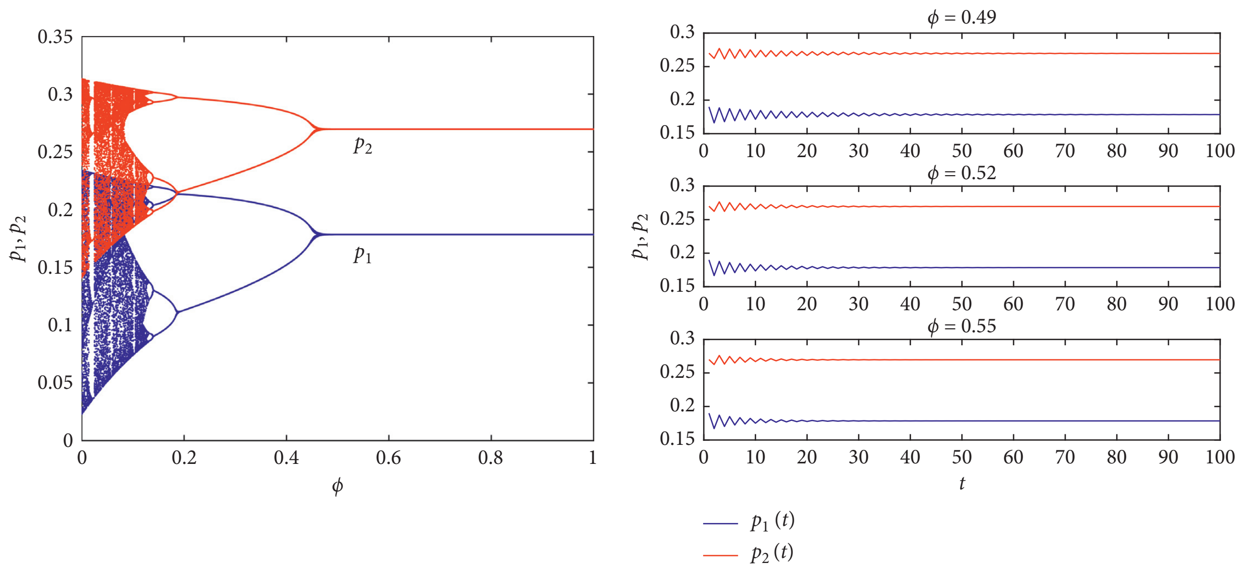
$$\begin{cases} p_1(t+1)=p_1(t)+\dfrac{k_1p_1(t)}{\beta\left(1-\delta^2\right)(1+\psi)}\left(c_1-\delta c_2-p_1(t)+\delta p_2(t)\right) \\ \qquad +\dfrac{\delta k_1k_2p_1(t)p_2(t)}{\beta^2\left(1-\delta^2\right)^2(1+\psi)}\left[\alpha(1-\delta)+\delta c_1+\left(1-2\delta^2\right)c_2-2\left(1-\delta^2\right)p_2(t)\right], \\ p_2(t+1)=p_2(t)+\dfrac{k_2p_2(t)}{\beta\left(1-\delta^2\right)(1+\psi)}\left[\alpha(1-\delta)+\delta c_1+\left(1-2\delta^2\right)c_2-2\left(1-\delta^2\right)p_2(t)\right]. \end{cases} \quad (62)$$

It is obvious that the controlled system (62) has the same Nash equilibrium point E_8 as the original system (38). At E_8, the Jacobian matrix of system (62) becomes

$$J(E_8)=\begin{pmatrix} 1-\dfrac{k_1p_1^{**}}{\beta\left(1-\delta^2\right)(1+\psi)} & \dfrac{k_1\delta p_1^{**}}{\beta\left(1-\delta^2\right)(1+\psi)}-\dfrac{2k_1k_2\delta p_1^{**}p_2^{**}}{\beta^2\left(1-\delta^2\right)(1+\psi)} \\ 0 & 1-\dfrac{2k_2p_2^{**}}{\beta(1+\psi)} \end{pmatrix}. \quad (63)$$

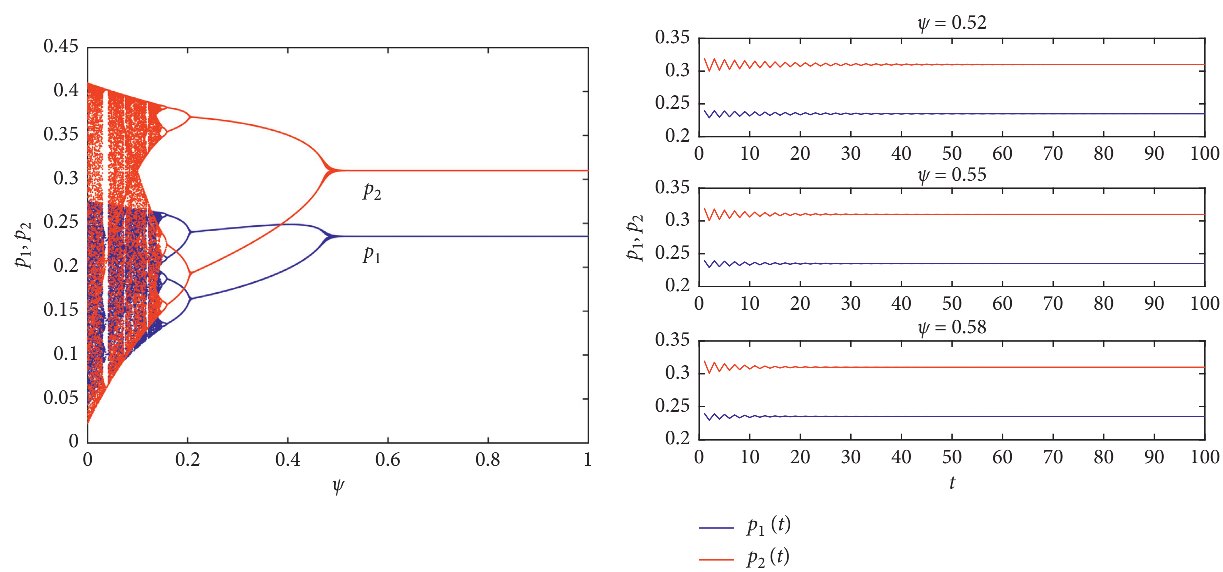
As has been shown in Figure 5(a), the chaotic behavior of the original system (38) occurs when parameter values are fixed as $(\alpha,\beta,\delta,c_1,c_2,k_1,k_2)=(0.7,0.2,0.5,0.13,0.1,0.7,0.95)$. By computer work on the Jury stability conditions, we can infer that when the other parameters take the above values, the controlled system (62) will be asymptotically locally stable provided that $\psi>0.4725$. As shown in Figure 10(a), with the value of ψ increasing, the system gradually gets out of chaos and achieves stability when $\psi>0.4725$. When $\psi=0.52$, 0.55, and 0.58, the stable behaviors of the orbits of the controlled system (62) beginning from the initial state $(p_1(0),p_2(0))=(0.24,0.32)$ are plotted in Figure 10(b). From Figure 10, we can see that the chaotic system is controlled at the fixed point when the controlling coefficient ψ is properly large.

In a real insurance market, we can consider the control action as the regulation on the price adjustment speed, and we can also consider the control action as the learning ability or adaptability of the market [40]. The time-delayed feedback control method can be used to make the system from a



(a) (b)

FIGURE 9: (a) Bifurcation diagram for system (60) with respect to the controlling factor ϕ. (b) Evolutions for system (60) with various values of ϕ.



(a) (b)

FIGURE 10: (a) Bifurcation diagram for system (62) with respect to the controlling factor ψ. (b) Evolutions for system (62) with various values of ψ.

chaotic state to stable state, which ensures that the insurance market develops in an orderly way.

6. Conclusion

In this paper, we have studied the complex dynamic behaviors of a Stackelberg mixed duopoly game with price competition in an insurance market, wherein a public insurance company competes against a private insurance company. We investigated this problem in two different sequential-move scenarios, public leadership and private leadership games. The equilibrium points including the Nash equilibrium point have been obtained as functions of the system parameters in two cases, and the conditions for the stability of equilibria have been found. We have made some numerical simulations for the system evolution, including stability region, bifurcation diagram, maximal Lyapunov exponents, phase portrait, and sensitive dependence on initial conditions. They show that the price adjustment speeds have a great influence on the system stability, and while varying the values of the price adjustment speeds, complex dynamic behaviors would occur, such as period bifurcations and chaos. Meanwhile, we have compared the welfare and profit levels for two sequential-move games. When an insurance company, whether public or private, is the Stackelberg leader in the price competition game, it yields greater welfare (or profit) than as the follower. In addition, we have also shown that the time-delayed feedback control method can be used to force the system back to its stable state from chaos.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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