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Research Article

An Algorithm to Compute the H-Bases for Ideals of Subalgebras

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The concept of H-bases, introduced long ago by Macauly, has become an important ingredient for the treatment of various problems in computational algebra. The concept of H-bases is for ideals in polynomial rings, which allows an investigation of multivariate polynomial spaces degree by degree. Similarly, we have the analogue of H-bases for subalgebras, termed as SH-bases. In this paper, we present an analogue of H-bases for finitely generated ideals in a given subalgebra of a polynomial ring, and we call them "HSG-bases." We present their connection to the SAGBI-Gröbner basis concept, characterize HSG-basis, and show how to construct them.

1. Introduction

The concept of H-bases, introduced long ago by Macaulay [1], is based solely on homogeneous terms of a polynomial. In [2], an extension of Buchberger's algorithm is presented to construct H-bases algorithmically. Some applications of H-bases are given in [3]; in addition, many of the problems in applications which can be solved by the Gröbner technique can also be treated successfully with H-bases. The concept of H-basis for ideals of a polynomial ring over a field K can be adopted in a natural way to K-subalgebras of a polynomial ring. In [4], SH-basis (Subalgebra Analogue to H-basis for Ideals) for the K-subalgebra of $K[x_1, \ldots, x_n]$ is defined. The properties of SH-bases are typically similar to H-basis results [3]. Like H-bases, the concept of SH-basis is also tied to homogeneous polynomials. In this paper, we will present an analogue to H-bases for ideals in a given subalgebra of a polynomial ring, and we call them "HSG-bases."

The paper is organized as follows. In Section 2, we briefly describe the underlying concept of grading which leads to SAGBI-Gröbner bases and HSG-basis. Then, we give the notion of *si*-reduction, which is one of the key ingredients for the characterization and construction of HSG-basis. After setting up the necessary notation, we present the *si*-reduction algorithm (see Algorithm 1). Also, here we present some properties characterizing HSG-basis (Theorem 1). In Section 3, we present a criterion through which we can check that the given system of polynomials is an HSG-basis of the subalgebra it generates (Theorem 2), and further on the basis of this theorem, we present an algorithm for the construction of HSG-basis (Algorithm 2).

2. HSG-Bases and SAGBI-Gröbner Bases

Here and in the following sections we consider polynomials in n variables $x_1, \ldots, x_n$ with coefficients from a field K. For short, we write

$$\mathrm{P} \coloneqq K[x_1, \ldots, x_n]. \tag{1}$$

If G is a subset of subalgebra $\mathscr{A}$ in $K[x_1, \ldots, x_n]$, then the set

$$I \coloneqq \left\{ \sum_{g \in G} h_g g \,|\, h_g \in \mathscr{A} \text{ and only finitely many } h_g \neq 0 \right\} \tag{2}$$

Input: a subalgebra $\mathscr{A}$, a finite subset $G \subset \mathscr{A}$, and a polynomial $f \in \mathscr{A}$.
Output: a polynomial h such that $f \longrightarrow_{G_A,*} h$.
(1) $h: = f$.
(2) While $(h \neq 0 \text{ and } G_h = \{\sum_i a_i g_i | M^{(H)}(\sum_i a_i g_i) = M^{(H)}(h)\} \neq \emptyset)$; where $a_i \in \mathscr{A}$ and $g_i \in G$.
(3) Choose $\sum_i a_i g_i \in G_h$.
(4) $h: = h - \sum_i a_i g_i$ and continue at 2.

Algorithm 1: Algorithm to compute si-reduction

Input: a subalgebra $\mathscr{A}$ and a finite subset $G \subset \mathscr{A}$.
Output: HSG-basis H for $\langle G \rangle_{\mathscr{A}}$.
(1) $H: = G$, $\text{Old}(H): = \emptyset$.
(2) $H = \{h_1, \ldots, h_s\}$.
(3)While $(H \neq \text{Old}(H))$ do
(4) Compute Q, an $M^{(H)}$–generating set for $syz(M^{(H)}(H))$.
(5) Compute $P: = \{\sum_{i=1}^{s} q_i h_i | (q_i)_{i=1}^{s} \in Q\}$.
(6) Compute $\text{red}(P): = \{\text{final } si\text{–reduction via } H \text{ of every element of } P\} - \{0\}$.
(7) $\text{Old}(H): = H \cup \text{red}(P)$.

Algorithm 2: Algorithm for the construction of HSG basis.

is the ideal of A in P generated by G and we write it shortly as $\langle G \rangle_{\mathscr{A}}$. In this section, we want to introduce HSG-bases and discuss some of their properties. This concept is very similar to the concept of SAGBI-Gröbner bases. Therefore, we will briefly explain the underlying common structure. Let Γ denote an ordered monoid, i.e., an abelian semigroup under an operation +, equipped with a total ordering $>$ such that, for all $\alpha, \beta, \gamma \in \Gamma$,

$$\alpha > \beta \Rightarrow \alpha + \gamma > \beta + \gamma. \tag{3}$$

A direct sum,

$$\mathrm{P} = \underset{\gamma \in \Gamma}{\oplus} \mathrm{P}_{\gamma}^{(\Gamma)}, \tag{4}$$

is called grading (induced by Γ) or briefly a Γ-grading if for all $\alpha, \beta \in \Gamma$,

$$f \in \mathrm{P}_{\alpha}^{(\Gamma)}, g \in \mathrm{P}_{\beta}^{(\Gamma)} \Rightarrow f \cdot g \in \mathrm{P}_{\alpha+\beta}^{(\Gamma)}. \tag{5}$$

Since the decomposition above is a direct sum, each polynomial $f \neq 0$ has a unique representation.

$$f = \sum_{i=1}^{s} f_{\gamma_i}, \quad 0 \neq f_{\gamma_i} \in \mathrm{P}_{\gamma_i}^{(\Gamma)}. \tag{6}$$

Assuming that $\gamma_1 > \gamma_2 > \cdots > \gamma_s$, the Γ-homogeneous term f_{γ_1} is called the maximal part of f, denoted by $M^{(\Gamma)}(f): = f_{\gamma_1}$, and $f - M^{(\Gamma)}(f)$ is called the d-reductum of f. For $G \subset \mathscr{A}$, $M^{(\Gamma)}(G): = \{M^{(\Gamma)}(g) | g \in G\}$.

There are two major examples of gradings. The first one is grading by degrees:

$$\mathrm{P}_{d}^{(\Gamma)} = \{p \in \mathrm{P} | p \text{ is homogeneous of degree } d\}, \quad \forall d \in \mathbb{N}. \tag{7}$$

Here, $\Gamma = \mathbb{N}$ with the natural total ordering. This grading is called the H-grading because of the homogeneous polynomials. Therefore, we also write H in place of this Γ. The space of all polynomials of degree at most d can now be written as

$$\mathrm{P}_d := \oplus_{k=0}^{d} \mathrm{P}_{k}^{(H)}. \tag{8}$$

The maximal part of a polynomial $f \neq 0$ is its homogeneous form of highest degree, $M^{(H)}(f)$. For simplicity, let $M^{(H)}(0): = 0$.

Definition 1. A subset $G = \{g_1, \ldots, g_s\} \subset \mathscr{A}$ (subalgebra) is called HSG-basis for the ideal $I_{\mathscr{A}} \subset \mathscr{A}$, if for all $0 \neq f \in I_{\mathscr{A}}$,

$$\exists h_1, \ldots, h_s \in \mathscr{A}: f = \sum_{i=1}^{s} h_i g_i, \deg(f) = \max_{i=1}^{s}\{\deg(h_i g_i)\} \Big(\text{Note that this condition is not obvious, } -x^3 y^3 + x^4 = \left(x^2\right)\left(x^3 y + x^2\right) + (-xy)\left(x^4 + x^2 y^2\right) \text{ see in } K\left[x^2, xy\right]\Big). \tag{9}$$

The representation for f in (9) is also called its HSG representation with respect to G.

Note that HSG-basis for ideal in a subalgebra is also a generating set of it. To obtain more insights into HSG-bases, we will give some equivalent definitions. First, we need a more technical notion.

Definition 2. For given $f, f_1, \ldots, f_m$, we say that f si–reduces to $\tilde{f}$ with respect to $F = \{f_1, \ldots, f_m\}$ in $\mathscr{A}$ if

$$\widetilde{f} = f - \sum_{i=1}^{m} a_i f_i, \deg(\widetilde{f}) < \deg(f) \tag{10}$$

holds with polynomials $a_i \in \mathscr{A}$ satisfying $\deg(a_i f_i) \le \deg(f)$. We write it as $f \longrightarrow_{F_{\mathscr{A}}} \widetilde{f}$. By $\longrightarrow_{F_{\mathscr{A}},*}$ we denote the transitive closure of the binary relation $\longrightarrow_{F_{\mathscr{A}}}$ 1.

The concept of *si*–reduction plays an important role in the characterization and construction of HSG-basis. For $f \in \mathscr{A}$ and $G \subset \mathscr{A}$, the following algorithm computes h such that $f \longrightarrow_{G_{\mathscr{A}},*} h$ (i.e., f reduces to h completely).

We note that such an element a_i in the subalgebra $\mathscr{A}$ can easily be determined as in the case of reduction in polynomial ring. We also note that $\deg(h - \sum_i a_i g_i)$ is strictly smaller than the $\deg(h)$ (by the choice of $\sum_i a_i g_i$). This shows that Algorithm 1 always terminates.

Theorem 1. *Let $G = \{g_1, \ldots, g_s\} \subset \mathscr{A}$ (subset of subalgebra $\mathscr{A}$) and $I_{\mathscr{A}}$ be an ideal of $\mathscr{A}$. Then, the following conditions are equivalent:*

(1) G is an HSG-basis for the ideal $I_{\mathscr{A}}$.

(2) $\langle M^{(H)}(g_1), \ldots, M^{(H)}(g_s) \rangle_{K[M(H)]} = \langle M^{(H)}(f) | f \in I_{\mathscr{A}} \rangle_{K[M(H)]}$.

(3) For all $f \in I$, $f \longrightarrow_{G_{\mathscr{A}},*} 0$.

Proof. (1)⇒(2). Let $M^{(H)}(p) \in \langle M^{(H)}(f) | f \in I_{\mathscr{A}} \rangle$ for some $p \in I_{\mathscr{A}}$. Since G is an HSG-basis, by (9), there are some $h_1, \ldots, h_s \in \mathscr{A}$ so that $p = \sum_{i=1}^{s} h_i g_i$ and $M^{(H)}(p) = M^{(H)}(\sum_{i=1}^{s} h_i g_i) = \sum_{i\in J} M^{(H)}(h_i) M^{(H)}(g_i) \in \langle M^{(H)}(g_1), \ldots, M^{(H)}(g_s) \rangle$, where $J = \{i | \deg(h_i g_i) = \deg(p)\}$.

(2)⇒(3). Let $0 \ne f \in I_{\mathscr{A}}$. By using Algorithm 1, we get $f \longrightarrow_{F_{\mathscr{A}}} h_1 \longrightarrow_{F_{\mathscr{A}}} h_2 \ldots \longrightarrow_{F_{\mathscr{A}}} h$, where h is *si*–reduced any further with respect to F. $M^{(H)}(f) \in \langle M^{(H)}(g_1), \ldots, M^{(H)}(g_s) \rangle$ implies $M^{(H)}(f) = \sum_{i\in J} M^{(H)}(h_i) M^{(H)}(g_i)$; then, $f \longrightarrow_{G_{\mathscr{A}}} \widetilde{f} = f - \sum h_i g_i \in I_{\mathscr{A}}$. If we follow the above process inductively, then $f \longrightarrow_{G_{\mathscr{A}},*} 0$.

(3)⇒(1). Let

$$f_0 \longrightarrow_{G_{\mathscr{A}}} f_1 \longrightarrow_{G_{\mathscr{A}}} \cdots \longrightarrow_{G_{\mathscr{A}}} f_d = 0, \tag{11}$$

where $M^{(H)}(f_{i-1}) = \sum_{j=1}^{s} M^{(H)}(h_{ij}) M^{(H)}(g_j)$, $i = 1, 2, \ldots, d$, $\deg(f_{i-1}) > \deg(f_i)$. Then,

$$f = \sum_{j=1}^{s} \sum_{i=1}^{d} M^{(H)}(h_{ij}) M^{(H)}(g_j). \tag{12}$$

Note that

$$\deg(f) = \deg(f_\circ) = \deg\left(\sum_{j=1}^{s} M^{(H)}(h_{1j}) M^{(H)}(g_j) \right), \tag{13}$$

and

$$\deg\left(\sum_{j=1}^{s} M^{(H)}(h_{ij}) M^{(H)}(g_j) \right) > \deg\left(\sum_{j=1}^{s} M^{(H)}(h_{i+1,j}) M^{(H)}(g_j) \right), i = 1, 2, \ldots, d. \tag{14}$$

Hence,

$$\deg(f) = \max_i \left(\deg\left(\sum_{j=1}^{s} M^{(H)}(h_{ij}) M^{(H)}(g_j) \right) \right). \tag{15}$$

(11) and (15) give the HSG representation. □

The second major example of gradings leads to the SAGBI-Gröbner basis concept. Here, $\Gamma = \mathbb{N}^n$ with component-wise addition equipped with a total ordering satisfying (11). In addition, $\gamma \ge 0, \forall \gamma \in \Gamma$. For arbitrary $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma$, the space $P_\gamma^{(\Gamma)}$ is a vector space of dimension 1, namely,

$$P_\gamma^{(\Gamma)} = \{c \cdot x^{\gamma_1} \ldots x^{\gamma_n} | c \in K\}. \tag{16}$$

The maximal part $M^{(\Gamma)}(f)$ of a polynomial f is a product of a leading coefficient $\mathrm{LC}(f)$ and a leading monomial $\mathrm{LM}(f)$, that is $M^{(\Gamma)}(f) = \mathrm{LC}(f) \cdot \mathrm{LM}(f)$, where $\mathrm{LC}(f) \in K$. The si-reduction $f \longrightarrow_{G_{\mathscr{A}}} \widetilde{f}$ is defined if there exists a polynomial $g \in G$ and $a \in \mathscr{A}$ such that $\mathrm{LM}(f) = \mathrm{LM}(g)\mathrm{LM}(a)$ and then we set $\widetilde{f} := (f - (M^{(\Gamma)}(f)) / (M^{(\Gamma)}(g) M^{(\Gamma)}(a)) ag)$. The relation $\longrightarrow_{G_{\mathscr{A}},*}$ is constructed as above.

A SAGBI-Gröbner basis G (with respect to a given monomial ordering and a given ideal $I_{\mathscr{A}}$ in a subalgebra $\mathscr{A}$) is a set of polynomials generating $I_{\mathscr{A}}$ and satisfying one of the following equivalent conditions:

(i) Every $f \in I_{\mathscr{A}}$ has a representation:

$$f = \sum_{i=1}^{s} h_i g_i, \quad \mathrm{LM}(f) = \max_{i=1}^{s} \{\mathrm{LM}(h_i) \mathrm{LM}(g_i)\}, \tag{17}$$

where $h_i \in \mathscr{A}$ and $g_i \in G$.

(ii) $\langle M^{(\Gamma)}(g) | g \in G \rangle = \langle M^{(\Gamma)}(f) | f \in I_{\mathscr{A}} \rangle$.

(iii) Every $f \in I_{\mathscr{A}}$ *si*-reduces to 0 with respect to G.

The proof of this equivalence and many other equivalent conditions can be found in [5]. If a monomial ordering is compatible with the semiordering by degrees,

$$\deg(x^\gamma) > \deg(x^\beta) \Rightarrow \gamma > \beta, \gamma, \beta \in \mathbb{N}^n, \tag{18}$$

then any SAGBI-Gröbner representation as given in (i) is an HSG representation; in other words, a SAGBI-Gröbner basis with respect to a degree compatible ordering is an HSG-basis as well. The converse is false, as the following example shows.

Example 1. Let $f_1 = x^4 + 2x^2y^2 + y^4 - 1$, $f_2 = x^2y^2 + y^4 - 2$, $f_3 = 2x^2 + y^2$. These polynomials belong to the subalgebra $\mathscr{A} = \mathbb{Q}[x^2, y^2]$. Then, we can see that f_1, f_2, and f_3 already constitute an HSG-basis for ideal $I_{\mathscr{A}} = \langle f_1, f_2, f_3\rangle$ in $\mathscr{A}$. If we order the monomials by degree lexicographical ordering, then $\langle M^{(H)}(f) | f \in I_{\mathscr{A}}\rangle_{\mathbb{Q}[M^{(H)}(\mathscr{A})]} = \langle x^4, x^2y^2, x^2\rangle_{\mathbb{Q}[M^{(H)}(\mathscr{A})]}$. Every SAGBI-Gröbner basis G with respect to this ordering contains at least four elements, for instance, $G = \{g_1, g_2, g_2, g_4\}$ with $g_1 = x^4 + 2x^2y^2 + y^4 - 1 = f_1$, $g_2 = x^2y^2 + y^4 - 2 = f_2$, $g_3 = 2x^2 + y^2 = f_3$, and $g_4 = y^4 - 4$. Obviously, this SAGBI-Gröbner basis is an HSG-basis as well.

3. Construction of HSG-Bases

In this section, we present an HSG-basis criterion, through which we can construct HSG-basis. For this purpose, we fix some notations which are necessary for this construction. Let $\mathscr{A}$ be a K−subalgebra of $K[x_1, \ldots, x_n]$.

(i) We denote $\mathscr{A}\oplus \ldots \oplus\mathscr{A}$ (s − times) by $\oplus_S \mathscr{A}$.

(ii) For a subset $G \subseteq \mathscr{A}$, we denote $\{M^{(H)}(g_i) | g_i \in G\}$ by $M^{(H)}(G)$.

Definition 3. For K−subalgebra $\mathscr{A}$ of $K[x_1, \ldots, x_n]$ and a subset $G = \{g_1, \ldots, g_s\} \subseteq \mathscr{A}$,

(1) $syz_{\mathscr{A}}(G) = \{\overrightarrow{a} = (a_i)_{i=1}^s \in \oplus_S\mathscr{A} | \sum_{i=1}^s a_i g_i = 0\}$. We call an element of $syz_{\mathscr{A}}(G)$ an $\mathscr{A}$−syzygy of G.

(2) For $\overrightarrow{a} = (a_i)_{i=1}^s \in \oplus_S\mathscr{A}$, let $M^{(H)}(\overrightarrow{a})$ represent the vector $(M^{(H)}(a_i)_{i=1}^s)$.

Definition 4. We call a subset $Q = \{\overrightarrow{q_1}, \overrightarrow{q_2}, \ldots, \overrightarrow{q_m}\}$ a $M^{(H)}$-generating set for $syz(M^{(H)}(G))$ if $\{M^{(H)}(\overrightarrow{q_i}) | 1 \le i \le m\}$ generates the $K[M^{(H)}(\mathscr{A})]$-module $syz[M^{(H)}(G)]$, i.e., for $\overrightarrow{a} \in syz[M^{(H)}(G)]$, there are some $h_1, h_2, \ldots h_m \in M^H(\mathscr{A})$ such that

$$M^{(H)}(a_i)_{i=1}^s = \sum_{j=1}^m M^{(H)}(h_j) M^{(H)}(q_{ij})_{i=1}^s. \tag{19}$$

In the case of SAGBI-Gröbner bases, there is an algorithm for computing SAGBI-Gröbner bases by means of syzygies (see [6]) where syzygies and their connection to SAGBI-Gröbner bases are studied in detail. The analogue for constructing HSG-bases by means of syzygies is connected to the following result [7].

Theorem 2 *(HSG-basis criterion). Let $G = \{g_1, \ldots, g_s\}$ be the subset of a subalgebra $\mathscr{A}$. Let Q be $M^{(H)}$−generating set for the $syz(M^{(H)}(G))$. Then, G is an HSG-basis for $\langle G\rangle_{\mathscr{A}}$ if and only if for every $\overrightarrow{q_j} = (q_{j,1}, \ldots, q_{j,s}) \in Q$, we have $\sum_{i=1}^s q_{j,i} g_i \longrightarrow_{G_{\mathscr{A},*}} 0$.*

Proof. ⇒: The statement is a direct result of Theorem 1.

⇐: Take $f \in \langle G\rangle_{\mathscr{A}}$. We need to show that $M^{(H)}(f) \in \langle M^{(H)}(G)\rangle_{K[M^{(H)}(\mathscr{A})]}$. For this, we write $f = \sum_{i=1}^m a_i g_i$ such that $p_0 = \max[M^{(H)}(a_i g_i)]$ (degree wise) is minimal among all such representations of f. We have $M^{(H)}(f) \le p_0$. Suppose that $M^{(H)}(f) < p_0$. Assume that $a_1 g_1, \ldots, a_{m_0} g_{m_0}$ are contributing to p_0, i.e., $M^{(H)}(a_i g_i) = p_0$ for all $1 \le i \le m_0$. If we set $\overrightarrow{a} = (a_1, \ldots, a_{m_0}, 0, \ldots, 0)$, we can see that $M^{(H)}(\overrightarrow{a}) \in syz(M^{(H)}(G))$. This implies that there are $b_1, \ldots, b_n \in \mathscr{A}$ and $\overrightarrow{Q_1}, \ldots, \overrightarrow{Q_n} \in \overrightarrow{Q}$ such that $M^{(H)}(\overrightarrow{a}) = \sum_{j=1}^n M^{(H)}(b_j) M^{(H)}(\overrightarrow{Q_j})$. We may assume that $M^{(H)}(b_j) M^{(H)}(q_{j,i}) M^{(H)}(g_i) = p_0$ for each j by homogeneity of the syzygies. Now,

$$\begin{aligned} f &= \sum_{i=1}^m a_i g_i - \sum_{i=1}^m \left(\sum_{j=1}^n b_j q_{j,i}\right) g_i + \sum_{j=1}^n b_j \left(\sum_{i=1}^m q_{j,i} g_i\right) \\ &= \sum_{i=1}^m \left(a_i - \sum_{j=1}^n b_j q_{j,i}\right) g_i + \sum_{j=1}^n b_j \left(\sum_{i=1}^m p_{j,i} g_i\right), \end{aligned} \tag{20}$$

where $\sum_{i=1}^m p_{j,i} g_i$ is an HSG representation for $\sum_{i=1}^m q_{j,i} g_i$ since $\sum_{i=1}^m q_{j,i} g_i \longrightarrow^G 0$. If we define $H_j = \max(M^{(H)}(p_{j,i} g_i))$, then

$$H_j = M^{(H)}\left(\sum q_{j,i} g_i\right) < \max\left(M^{(H)}\left(q_{j,i} g_i\right)\right), \quad \text{for all } j, \tag{21}$$

because $M^{(H)}(\overrightarrow{Q_j}) \in syz(M^{(H)}(G))$.

Consider the first sum of equation (20). For $i \le m_0$, we have $M^{(H)}(a_i) = M^{(H)}(\sum_{j=1}^n b_j q_{j,i})$, so by the cancellation of highest terms,

$$M^{(H)}\left[\left(a_i - \sum_{j=1}^n b_j q_{j,i}\right) g_i\right] < M^{(H)}(a_i g_i) = p_0. \tag{22}$$

For $i > m_0, M^{(H)}(a_i g_i) < p_0$ and $\sum_{j=1}^n M^{(H)}(b_j) M^{(H)}(q_{j,i}) = 0$ implies that

$$M^{(H)}\left(\sum_{j=1}^n b_j q_{j,i} g_i\right) < \max_j\left(M^{(H)}\left(b_j q_{j,i} g_i\right)\right) = p_0. \tag{23}$$

Since

$$M^{(H)}\left[\left(a_i - \sum_{j=1}^n b_j q_{j,i}\right) g_i\right] \le \max\left\{M^{(H)}(a_i g_i), M^{(H)}\left(\sum_{j=1}^n b_i q_{j,i} g_i\right)\right\} < p_0\ (\forall i). \tag{24}$$

So, first sum of equation (20) is less than p_0. For the second sum of equation (20), we have

$$M^{(H)}\left(\sum_{j=1}^{n} b_j \sum_{i=1}^{m} p_{j,i} g_i\right) \leq \max_{i,j} M^{(H)}\left(b_j p_{j,i} g_i\right) \leq \max_j \left[M^{(H)}\left(b_j\right) H_j\right] < \max_{i,j}\left(M^{(H)}\left(b_j q_{j,i} g_i\right)\right) = p_0. \quad (25)$$

Hence, equation (20) does provide a new representation for f such that $\max(M^{(H)}(a_i g_i)) < p_0$, a contradiction. Therefore, $M^{(H)}(f) = p_0$ and $M^{(H)}(f) = \sum_{i=1}^{m_0} M^{(H)}(a_i g_i) \in \langle M^{(H)}(G)\rangle$. □

On the basis of Theorem 2, now we present an algorithm which computes HSG-basis from a given set of generators. This algorithm is not necessarily terminating but does terminate, if and only if, the considered ideal in the subalgebra has a finite HSG-basis.

Now we present some examples which show the computation of HSG-basis through Algorithm 2.

Example 2. Let the subalgebra $\mathscr{A} = Q[x^2, xy]$ and $G = \{x^3y + x^2, xy + 2\} \subseteq \mathscr{A}$. Consider $H = G$; then, $M^{(H)}(H) = \{x^3y, xy\}$.

First pass through the while loop:

(i) $M^{(H)}(q_1)(x^3y) + M^{(H)}(q_2)(xy) = 0$ implies $Q = \{(-1, x^2)\}$. Then, $(-1)(x^3y + x^2) + (x^2)(xy + 2) = -x^3y - x^2 + x^3y + 2x^2 = x^2$ gives $P = \{x^2\}$.
(ii) As x^2 is *si*-reduced with respect to H, $\mathrm{red}(P) = \{x^2\}$.
(iii) Define: $\mathrm{Old}(H) = H \cup \{x^2\}$.
As $H \neq \mathrm{Old}(H)$, we repeat the whole process. Now we have $M^{(H)}(H) = \{x^3y, xy, x^2\}$.

Second pass through the while loop:

(i) $M^{(H)}(q_1)(x^3y) + M^{(H)}(q_2)(xy) + M^{(H)}(q_3)\ (x^2) = 0$ implies $(-1)(x^3y) + (0)(xy)\ + (xy)(x^2) = 0$. Therefore, $Q = \{(-1, x^2, 0),\ \ (-1, 0, xy)\}$. Then, $(-1)(x^3y + x^2) +\ (0)(xy + 2) + (xy)\ \ (x^2)\ = -x^3 y - x^2 + 0 + x^3\ y = -x^2$ gives $P = \{x^2, -x^2\}$.
(ii) Now, $\mathrm{red}(P) = \varnothing$.

Since $\mathrm{Old}(H) = H$, we stop here. The HSG-basis for $\langle G\rangle_{\mathscr{A}}$ is $\{x^3y + x^2, xy + 2, x^2\}$.

Example 3. Let $\mathscr{A} = Q[x^2, xy]$ and $G = \{x^3y + x^2y^2 + x^2, xy + 2\} \subseteq \mathscr{A}$. Consider $H = G$; then, $M^{(H)}(H) = \{x^3y + x^2y^2, xy\}$.

First pass through the while loop:

(i) $M^{(H)}(q_1)(x^3y + x^2y^2) + M^{(H)}(q_2)(xy) = 0$ gives $Q = \{(-1, x^2 + xy)\}$. Then, from $(-1)(x^3y + x^2y^2 + x^2) + (x^2 + xy)(xy + 2) = -x^3y - x^2y^2 - x^2 + x^3\ y + x^2\ y^2 + 2x^2 + 2xy = x^2 + 2xy$,
(ii) $\mathrm{red}(P) = \{x^2 - 4\}$.
(iii) Define: $\mathrm{Old}(H) = H \cup \{x^2 - 4\}$.

As $H \neq \mathrm{Old}(H)$, we repeat the whole process. Now we have $M^{(H)}(H) = \{x^3y + x^2y^2, xy, x^2\}$.

Second pass through the while loop:

(i) From the equation $M^{(H)}(q_1)\ (x^3y + x^2y^2) + M^{(H)}(q_2)(xy) + M^{(H)}(q_3)(x^2) = 0$, we have $Q = \{(-1, xy, xy), (-1, x^2 + xy, 0)\}$. We can compute P from $(-1)(x^3y + x^2y^2 + x^2) + (xy)(xy + 2) + (xy)(x^2 - 4) = -x^3y - x^2y^2 - x^2 + x^2y^2 + 2xy + x^3y - 4xy = -x^2 - 2xy$.
(ii) Now, $\mathrm{red}(P) = \varnothing$.

Since $\mathrm{Old}(H) = H$, we stop here. The HSG-basis for $\langle G\rangle_{\mathscr{A}}$ is $\{x^3y + x^2y^2 + x^2, xy + 2, x^2 - 4\}$.

4. Conclusion

In this paper, we presented the theory of HSG-bases, which are a good basis of an ideal in a subalgebra of a polynomial ring. We can further develop this theory for an arbitrary grading for which HSG-bases would be a special case for degree-based grading.

Data Availability

No data are required to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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