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Research Article

The Neimark–Sacker Bifurcation and Global Stability of Perturbation of Sigmoid Beverton–Holt Difference Equation

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We present the bifurcation results for the difference equation $x_{n+1} = x_n^2/(ax_n^2 + x_{n-1}^2 + f)$ where a and f are positive numbers and the initial conditions x_{-1} and x_0 are nonnegative numbers. This difference equation is one of the perturbations of the sigmoid Beverton–Holt difference equation, which is a major mathematical model in population dynamics. We will show that this difference equation exhibits transcritical and Neimark–Sacker bifurcations but not flip (period-doubling) bifurcation since this difference equation cannot have period-two solutions. Furthermore, we give the asymptotic approximation of the invariant manifolds, stable, unstable, and center manifolds of the equilibrium solutions. We give the necessary and sufficient conditions for global asymptotic stability of the zero equilibrium as well as sufficient conditions for global asymptotic stability of the positive equilibrium.

1. Introduction and Preliminaries

In this paper, we consider the difference equation

$$x_{n+1} = \frac{x_n^2}{ax_n^2 + x_{n-1}^2 + f}, \quad n = 0, 1, \ldots, \tag{1}$$

where the parameters a and f are positive numbers and the initial conditions x_{-1} and x_0 are nonnegative numbers.

Equation (1) can be considered as a nonlinear perturbation of the sigmoid Beverton–Holt difference equation

$$x_{n+1} = \frac{x_n^2}{x_n^2 + f}, \quad f > 0, x_0 \geq 0, n = 0, 1, \ldots, \tag{2}$$

which is a major mathematical model in population dynamics and is the simplest model that exhibits Allee's effect, see [1, 2]. A related difference equation of the form

$$x_{n+1} = \frac{\beta x_n^2}{x_{n-1}^2 + 1}, \quad \beta > 0, n = 0, 1, \ldots, \tag{3}$$

where the initial conditions x_{-1} and x_0 are nonnegative numbers considered in [3]. Equation (1) is a square version of the well-known Pielou difference equation

$$x_{n+1} = \frac{x_n}{x_{n-1} + f}, \quad f > 0, x_0 \geq 0, n = 0, 1, \ldots, \tag{4}$$

which is another major model in population dynamics [1, 4]. Equation (4) has the same global dynamics as more general difference equation

$$x_{n+1} = \frac{x_n}{ax_n + x_{n-1} + f}, \quad a, f > 0, x_0 \geq 0, n = 0, 1, \ldots. \tag{5}$$

Both equations (4) and (5) exhibit transcritical bifurcation, where the zero equilibrium is globally asymptotically stable up to some critical value where positive equilibrium appears and assumes global asymptotic stability.

A perturbation of original Beverton–Holt equation is the difference equation studied in [5]

$$x_{n+1} = \frac{x_n}{ax_n + x_{n-1}^2 + f}, \quad n = 0, 1, \ldots, \tag{6}$$

where the parameters and initial conditions x_{-1} and x_0 are nonnegative numbers exhibiting Naimark–Sacker bifurcation and chaos. Another perturbation of original delayed Beverton–Holt equation studied in [6]

$$x_{n+1} = \frac{x_{n-1}}{ax_n^2 + ex_{n-1} + f}, \quad n = 0, 1, \ldots, \tag{7}$$

where the parameters and initial conditions x_{-1} and x_0 are nonnegative numbers, exhibiting transcritical and period-doubling (flip) bifurcation.

A perturbation of delayed sigmoid Beverton–Holt equation (2) considered in [7] is

$$x_{n+1} = \frac{x_{n-1}^2}{ax_n^2 + cx_{n-1}^2 + f}, \quad n = 0, 1, \ldots, \tag{8}$$

where the parameters a, c and f are positive numbers and the initial conditions x_{-1} and x_0 are nonnegative numbers.

Equation (2) exhibits a global asymptotic stability of either zero or positive equilibrium solutions and exchange of stability or transcritical bifurcation. Equation (8), where all solutions converge to either an equilibrium or to one of the three period-two solutions, exhibits the transcritical and period-doubling bifurcations while equation (6) exhibits Neimark–Sacker bifurcation and possibly chaos, but not a period-doubling (flip) bifurcation. In view of Theorem 4.2 in [8], equation (1) cannot have period-two solution, so period-doubling (flip) bifurcation is impossible. Related models with similar dynamics were considered in [1, 2, 9]. If we search for a model that exhibits Allee's effect, transcritical bifurcation, and Neimark–Sacker bifurcation to a periodic solution, then equation (1) is probably the simplest such model. The dynamical difference between equations (1) and (6) is that equation (6) cannot exhibit Allee's effect and has at most two equilibrium solutions. The dynamical difference between equations (1) and (8) is that equation (1) exhibits transcritical and Neimark–Sacker bifurcation while equation (8) exhibits transcritical and period-doubling bifurcation, proving this difference is the main objective of this paper.

In this paper, we will show that local asymptotic stability of the zero equilibrium will also imply its global asymptotic stability. In the case of the positive equilibrium solution, we will show that global asymptotic stability holds in some subspaces of the parametric region of local asymptotic stability. The technique used in proving global asymptotic stability of the positive equilibrium solution is based on global attractivity results for maps with invariant boxes, see [10, 11]. Related rational difference equations which exhibit similar behavior were considered in [5]. Equation (1) is one of the possible perturbations of the sigmoid Beverton–Holt difference equation which can exhibit chaos and shows that models based on such perturbation of the sigmoid Beverton–Holt equation can be used to model any kind of dynamic behavior from Allee's effect, flip bifurcations to Naimark–Sacker bifurcation, and chaos. Consequently, all these dynamic scenarios can be fitted by appropriate perturbation of the sigmoid Beverton–Holt equation. For instance, equation (8) exhibits period-doubling bifurcation in addition to the transcritical bifurcation, so depending on the dynamics of the observed data, one can choose one of the models (1) and (6)–(8) to fit appropriate coefficients. See [12] and references therein for examples of fitting parameters of models. Some most recent applications of Neimark–Sacker bifurcation for differential equations can be found in [13] and for difference equations in [14]. Some global asymptotic results for second order difference equations related to the results from [10], which will be used in the present paper, can be found in [15].

Now, for the sake of completeness, we give the basic facts about the Neimark–Sacker bifurcation.

The Neimark–Sacker bifurcation is the discrete counterpart to the Hopf bifurcation for a system of ordinary differential equations in two or more dimensions, see [16–18]. It occurs for such a discrete system depending on a parameter, λ for instance, along with a fixed point, the Jacobian matrix of which has a pair of complex conjugate eigenvalues. These eigenvalues $\mu(\lambda)$ and $\overline{\mu}(\lambda)$ will cross the unit circle at $\lambda = \lambda_0$ transversally. When this occurs in the discrete setting a periodic solution, which will in general be of an unknown period, will appear and this solution will be locally stable. To represent this periodic solution, we use the Murakami computational approach, see [19], to identify an asymptotic formula for an invariant curve in the phase plane which is locally attracting.

The following result is referred to as the Neimark–Sacker bifurcation theorem, see [16–18].

Theorem 1 (Neimark–Sacker bifurcation). *Let*

$$\mathbf{F}: \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2; \quad (\lambda, x) \longrightarrow \mathbf{F}(\lambda, \mathbf{x}), \tag{9}$$

be a C^4 map depending on real parameter λ satisfying the following conditions:

(i) $\mathbf{F}(\lambda, \mathbf{0}) = 0$ for λ near some fixed λ_0;

(ii) $Jac_{\mathbf{F}}(\lambda, \mathbf{0})$ has two nonreal eigenvalues $\mu(\lambda)$ and $\overline{\mu}(\lambda)$ for λ near λ_0 with $|\mu(\lambda_0)| = 1$;

(iii) $d/d\lambda|\mu(\lambda)| = d(\lambda_0) < 0$ at $\lambda = \lambda_0$ (transversality condition);

(iv) $\mu^k(\lambda_0) \neq 1$ for $k = 1, 2, 3, 4$ (nonresonance condition).

Then, there is a smooth λ-dependent change of coordinate bringing $\mathbf{F}$ into the form

$$\mathbf{F}(\lambda, \mathbf{x}) = \mathcal{F}(\lambda, \mathbf{x}) + O\left(\|\mathbf{x}\|^5\right), \tag{10}$$

and there are smooth functions $\alpha(\lambda)$, $\beta(\lambda)$, and $\omega(\lambda)$ so that in polar coordinates, the function $\mathcal{F}(\lambda, x)$ is given by

$$\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} |\mu(\lambda)| r + \alpha(\lambda) r^3 \\ \theta + \omega(\lambda) + \beta(\lambda) r^2 \end{pmatrix}. \tag{11}$$

If $\alpha(\lambda_0) < 0$, then the Neimark–Sacker bifurcation at $\lambda = \lambda_0$ is supercritical and there exists a unique closed invariant curve $\Gamma(\lambda)$, which is attracting, and bifurcates from $\overline{x}$ for $\lambda < \lambda_0$.

Consider a general map $\mathbf{F}(\lambda_0, \mathbf{x})$ that has a fixed point at the origin with complex eigenvalues $\mu(\lambda_0) = \alpha(\lambda_0) + i\beta(\lambda_0)$ and $\overline{\mu}(\lambda_0) = \alpha(\lambda_0) - i\beta(\lambda_0)$ satisfying $\alpha(\lambda_0)^2 + \beta(\lambda_0)^2 = 1$ and $\beta(\lambda_0) \neq 0$. Assume that

$$\mathbf{F}(\lambda_0, \mathbf{x}) = \mathbf{A}(\lambda_0)\mathbf{x} + \mathbf{G}(\lambda_0, \mathbf{x}), \tag{12}$$

where $\mathbf{A}$ is the Jacobian matrix of $\mathbf{F}$ evaluated at the fixed point $(0,0)$, and

$$\mathbf{G}(\lambda_0, \mathbf{x}) := \begin{pmatrix} g_1(\lambda_0, x_1, x_2) \\ g_2(\lambda_0, x_1, x_2) \end{pmatrix}. \tag{13}$$

Here, we denote $\mu(\lambda_0) = \mu, \mathbf{A}(\lambda_0) = \mathbf{A}$ and $\mathbf{G}(\lambda_0, x) = \mathbf{G}(\mathbf{x})$. We let $\mathbf{p}$ and $\mathbf{q}$ be the eigenvectors of A associated with μ satisfying

$$\begin{aligned} \mathbf{A}\mathbf{q} &= \mu\mathbf{q}, \\ \mathbf{p}A &= \mu\mathbf{p}, \\ \mathbf{p}\mathbf{q} &= 1, \end{aligned} \tag{14}$$

and $\Phi = (\mathbf{q}, \overline{q})$. Assume that

$$\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \overline{z} \end{pmatrix}\right) = \frac{1}{2}\left(\mathbf{g}_{20}z^2 + 2\mathbf{g}_{11}z\overline{z} + \mathbf{g}_{02}\overline{z}^2\right) + O\left(|z|^3\right), \tag{15}$$

and

$$\begin{aligned} \mathbf{K}_{20} &= \left(\mu^2 I - A\right)^{-1}\mathbf{g}_{20}, \\ \mathbf{K}_{11} &= (I - A)^{-1}\mathbf{g}_{11}, \\ \mathbf{K}_{02} &= \left(\overline{\mu}^2 I - A\right)^{-1}\mathbf{g}_{02}. \end{aligned} \tag{16}$$

Let

$$\begin{aligned} &\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \overline{z} \end{pmatrix} + \frac{1}{2}\left(\mathbf{K}_{20}z^2 + 2\mathbf{K}_{11}z\overline{z} + \mathbf{K}_{02}\overline{z}^2\right)\right) \\ &= \frac{1}{2}\left(\mathbf{g}_{20}z^2 + 2\mathbf{g}_{11}z\overline{z} + \mathbf{g}_{02}\overline{z}^2\right) \\ &\quad + \frac{1}{6}\left(\mathbf{g}_{30}z^3 + 3\mathbf{g}_{21}z^2\overline{z} + 3\mathbf{g}_{12}z\overline{z}^2 + \mathbf{g}_{03}\overline{z}^3\right) + O\left(|z|^4\right), \end{aligned} \tag{17}$$

then

$$a(\lambda_0) = \frac{1}{2}\mathrm{Re}\left(\mathbf{p}\mathbf{g}_{21}\overline{\mu}\right). \tag{18}$$

The next result of Murakami [19] gives an approximate formula for the periodic solution.

Corollary 1. *Assume $a(\lambda_0) \neq 0$ and $\lambda = \lambda_0 + \eta$ where η is a sufficient small parameter. If $\overline{\mathbf{x}}$ is a fixed point of F, then the invariant curve $\Gamma(\lambda)$ from Theorem 1 can be approximated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \overline{x} + 2\rho_0 \mathrm{Re}\left(\mathbf{q}e^{i\theta}\right) + \rho_0^2\left(\mathrm{Re}\left(\mathbf{K}_{20}e^{2i\theta}\right) + \mathbf{K}_{11}\right), \tag{19}$$

where

$$\begin{aligned} d &= \frac{d}{d\eta}|\mu(\lambda)|\Big|_{\lambda=\lambda_0}, \\ \rho_0 &= \sqrt{-\frac{d}{a}\eta}, \quad \theta \in \mathbb{R}. \end{aligned} \tag{20}$$

Here, "Re" represents the real parts of these complex numbers. The calculation of $a(\lambda_0)$ is given by [20].

The rest of the paper is organized as follows; Section 2 gives local and global stability analysis of the zero equilibrium and positive equilibrium solutions in some regions of parameters; Section 3 presents the computation of Neimark–Sacker bifurcation; Section 4 presents the approximations of stable, unstable, and center manifolds of the equilibrium solutions of equation (1); finally, Section 5 establishes that the rate of convergence of the solutions that converge to the zero equilibrium is quadratic while the rate of convergence of the solutions that converge to any positive equilibrium solution is linear.

2. Local and Global Stability

Equation (1) has always the zero equilibrium $\overline{x}_0 = 0$. The positive equilibrium solutions of equation (1) are the positive solutions of the equation $(a+1)\overline{x}^2 - \overline{x} + f = 0$, that is,

$$\overline{x}_{\pm} = \frac{1 \pm \sqrt{1 - 4f(a+1)}}{2(a+1)}, \tag{21}$$

when

$$4f(a+1) < 1, \tag{22}$$

and

$$\overline{x} = \frac{1}{2(a+1)}, \tag{23}$$

when

$$4f(a+1) = 1. \tag{24}$$

The linearized equation associated with equation (1) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = pz_n + qz_{n-1}, \tag{25}$$

where

$$\begin{aligned} p &= f_u(\overline{x}, \overline{x}), \\ q &= f_v(\overline{x}, \overline{x}). \end{aligned} \tag{26}$$

Now, the following results hold.

Lemma 1. *For the equilibrium point $\overline{x}_0$ of equation (1), the equilibrium is always locally asymptotically stable.*

The proof of the lemma follows from the fact that linearized equation at $\overline{x}_0 = 0$ is $z_{n+1} = 0$.

Lemma 2. *Assume that (22) holds. The positive equilibrium* $\overline{x}_+ = 1 + \sqrt{1-4f(a+1)}/2(a+1)$ *of equation (1) satisfies the following:*

(i) If $f > (1-a)/4$, *the equilibrium point* $\overline{x}_+$ *is locally asymptotically stable.*

(ii) If $f < (1-a)/4$, *the equilibrium point* $\overline{x}_+$ *is a repeller.*

(iii) If $4f(a+1) = 1$, *the equilibrium point* $\overline{x} = 1/2(a+1)$ *is nonhyperbolic of stable type with eigenvalues 1 and* $4f < 1$.

Proof

(i) One can see that

$$p = f_u(\overline{x}_+, \overline{x}_+) = \frac{a+2-a\sqrt{1-4(a+1)f}}{a+1} = 2(1-a\overline{x}_+), \tag{27}$$

and

$$\begin{aligned} q &= f_v(\overline{x}_+, \overline{x}_+) = -\frac{\sqrt{1-4(a+1)f}+1}{a+1} = -2\overline{x}_+ < 0, \\ q-p-1 &= \frac{(a-1)\sqrt{1-4(a+1)f}-2(a+1)}{a+1}, \\ q+p-1 &= \frac{1}{2}\left(1-\sqrt{1-4(a+1)f}\right), \\ q+1 &= \frac{a-\sqrt{1-4(a+1)f}}{a+1}. \end{aligned} \tag{28}$$

The rest of the proof follows from Theorem 1.1 [10]. We notice that the linearized equation at any positive equilibrium is

$$y_{n+1} - 2(1-2a\overline{x})y_n + 2\overline{x}y_{n-1} = 0, \tag{29}$$

and the corresponding characteristic equation is

$$\lambda^2 - 2(1-2a\overline{x})\lambda + 2\overline{x} = 0. \tag{30}$$

(iii) In the case (iii), we have that the characteristic equation at the equilibrium point $\overline{x} = 1/2(a+1)$ is

$$\lambda^2 - 2(1-2af)\lambda + 4f = 0, \tag{31}$$

which solutions are 1 and $4f$. In view of the condition $4f(a+1) = 1$, we have $4f < 1$. □

Lemma 3. *Assume that (22) holds. The positive equilibrium* $\overline{x}_- = 1 - \sqrt{1-4f(a+1)}/2(a+1)$ *is always a saddle point.*

Proof. One can see that

$$p = f_u(\overline{x}_-, \overline{x}_-) = \frac{a+2+a\sqrt{1-4(a+1)f}}{a+1}, \tag{32}$$

and

$$q = f_v(\overline{x}_-, \overline{x}_-) = \frac{-1+\sqrt{1-4(a+1)f}}{a+1}, \tag{33}$$

which imply

$$\begin{aligned} q-p-1 &= \frac{(1-a)\sqrt{1-4(a+1)f}-2(a+2)}{a+1}, \\ q+p-1 &= \frac{-2+(a+1)\sqrt{1-4(a+1)f}}{a+1}, \\ q+1 &= \frac{a+\sqrt{1-4(a+1)f}}{a+1}. \end{aligned} \tag{34}$$

The rest of the proof follows from Theorem 1.1 [10].

First, we give the global asymptotic result for zero equilibrium. □

Theorem 2. *Assume that*

$$4f(a+1) > 1. \tag{35}$$

Then, the zero equilibrium of equation (1) is globally asymptotically stable.

Proof. Every solution $\{x_n\}$ of equation (1) satisfies

$$x_{n+1} = \frac{x_n^2}{ax_n^2 + x_{n-1}^2 + f} \le \frac{1}{a}, \quad n = 0, 1, \ldots, \tag{36}$$

and the function

$$f(u,v) = \frac{u^2}{au^2 + v^2 + f} \tag{37}$$

is increasing in u and decreasing in v, with property that it has an invariant and attracting interval $[0, 1/a]$. Now, we will employ Theorem 1.13 from [10]. Consider the system of equations

$$\begin{cases} f(M,m) = M, \\ f(m,M) = m, \end{cases} \tag{38}$$

and prove that $M = m$. This system becomes

$$\begin{cases} \dfrac{M^2}{aM^2 + m^2 + f} = M, \\[2ex] \dfrac{m^2}{am^2 + M^2 + f} = m, \end{cases} \tag{39}$$

which after eliminating M becomes

$$m\left(am^2+f-m\right)\left((a+1)m^2-m+f\right)\left(\left(a^3-a^2-a+1\right)m^2+\left(1-a^2\right)m+a^2f-2af+f+1\right)=0. \quad (40)$$

Now, the discriminant of first quadratic polynomial in (40) is $1-4af<0$ in view of (19) and the discriminant of second quadratic polynomial in (40) is $1-4f(a+1)<0$ in view of (35). Finally, the discriminant of the third polynomial in (20) is

$$-(a-1)^2(a+1)\left(4a^2f-8af-a+4f+3\right), \quad (41)$$

and for $a\neq 1$ is negative in view of (35). If $a=1$, the third polynomial simply becomes a constant 1. Thus, the only solution of (40) is $m=0$. The same holds for M in view of symmetry of the considered system. So, $m=M=0$, and by Theorem 1.13 in [10], the zero solution of equation (1) is globally asymptotically stable. □

As Figure 1 shows the boundary of the basins of attraction of two locally asymptotically stable equilibrium solutions $\overline{x}_0$ and $\overline{x}_+$ seem to be the global stable manifold of the smaller equilibrium solution $\overline{x}_-$, which is a saddle point for all values of parameters. In Section 3, we will derive the asymptotic formulas for both stable and unstable manifolds based on the functional equations that the two manifolds satisfy. We will visually compare these manifolds with the image of the basin of attraction.

Now, we give some results about the basins of attraction of the positive equilibrium solutions. We will show that local asymptotic stability of a positive equilibrium will also imply its global asymptotic stability in a substantial subregion of the parametric space and within the basins of attraction of locally stable equilibrium solutions.

Lemma 4. *Assume that (14) holds. If $\{x_n\}$ is nonzero solution of equation (1), then the following hold:*

(i) If $(x_{-1},x_0)\in R_1=\{(x,y)\colon 0\leq y\leq x\leq\overline{x}_-\}$, then $x_1\leq x_0$. The solution $\{x_n\}$ is a decreasing sequence and so

$$\lim_{n\longrightarrow\infty} x_n=0. \quad (42)$$

(ii) If $(x_{-1},x_0)\in R_2=\{(x,y)\colon \overline{x}_-\leq x\leq y\leq\overline{x}_+\}$, then $x_0\leq x_1$. The solution $\{x_n\}$ is an increasing sequence and so

$$\lim_{n\longrightarrow\infty} x_n=\overline{x}_+. \quad (43)$$

(iii) If $(x_{-1},x_0)\in R_3=\{(x,y)\colon \overline{x}_+\leq y\leq x\}$, then $x_1\leq x_0$. The solution $\{x_n\}$ is a decreasing sequence and so it converges to one of the equilibrium solutions.

Proof. Set

$$G(u)=(a+1)u^2-u+f. \quad (44)$$

Then, the positive equilibrium solutions $\overline{x}_\pm$ are solutions of the equilibrium equation $G(u)=0$ and $G(u)<0$ if and only if $u\in(\overline{x}_-,\overline{x}_+)$.

(i) Now, we have

$$\begin{aligned} x_1&=\frac{x_0^2}{ax_0^2+x_{-1}^2+f}\leq\frac{x_0^2}{ax_0^2+x_0^2+f}\\ &=\frac{x_0^2}{x_0+G(x_0)}\leq\frac{x_0^2}{x_0}=x_0. \end{aligned} \quad (45)$$

By using induction, we can prove that the solution $\{x_n\}$ is a decreasing sequence, and since $x_n<\overline{x}_-$, it can only converge to the zero equilibrium.

(ii) Now, we have

$$\begin{aligned} x_1&=\frac{x_0^2}{ax_0^2+x_{-1}^2+f}\geq\frac{x_0^2}{ax_0^2+x_0^2+f}\\ &=\frac{x_0^2}{x_0+G(x_0)}\geq\frac{x_0^2}{x_0}=x_0. \end{aligned} \quad (46)$$

By using induction, we can prove that the solution $\{x_n\}$ is an increasing and bounded sequence, and since $x_n>\overline{x}_-$, it can only converge to the larger positive equilibrium $\overline{x}_+$.

(iii) The proof is identical to the proof of part (i) and it will be omitted. □

Remark 1. An immediate consequence of Lemma 4 is that the set R_1 is a part of the basin of attraction of the zero equilibrium $\mathscr{B}(0)$ and R_2 is a part of the basin of attraction of the larger positive equilibrium $\mathscr{B}(\overline{x}_+)$, that is, $R_1\subset\mathscr{B}(0), R_2\subset\mathscr{B}(\overline{x}_+)$. Based on our simulations, we formulate the following conjecture.

Conjecture 1. $R_3=\{(x,y)\colon \overline{x}_-<x, y<\ \overline{x}_-\}=(\overline{x}\ _-,\infty)\times[0,\overline{x}_-)\subset\mathscr{B}(0)$.

3. The Neimark–Sacker Bifurcation

In this section, we bring the system that corresponds to equation (1) to the normal form which can be used for the computation of the relevant coefficients of the Neimark–Sacker bifurcation.

If we make a change of variable $y_n=x_n-\overline{x}_+$, then the transformed equation is given by

$$y_{n+1}=\frac{(\overline{x}_++y_n)^2}{a(\overline{x}_++y_n)^2+(\overline{x}_++y_{n-1})^2+f}-\overline{x}_+,\quad n=0,1,\ldots,. \quad (47)$$

Set

$$\begin{aligned} u_n&=y_{n-1},\\ v_n&=y_n,\quad \text{for } n=0,1,\ldots, \end{aligned} \quad (48)$$

and write equation (1) in the equivalent form:

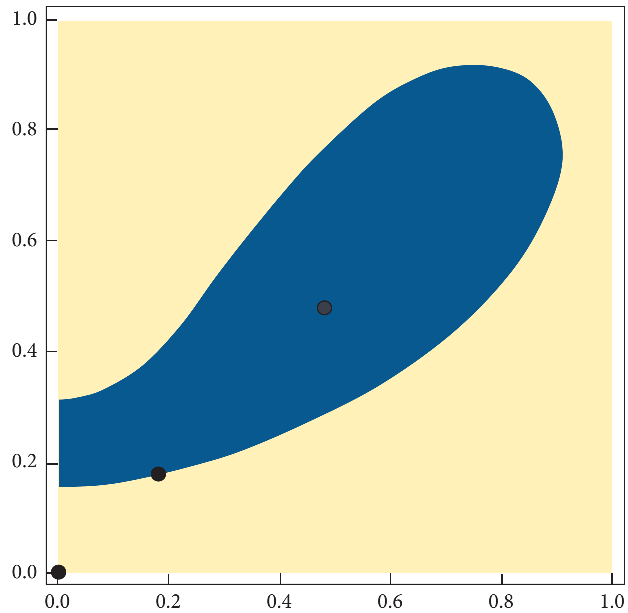

Figure 1: Basins of attraction for equation (1) for parameters $a = 0.49$ and $f = 10.125$. Picture is produced by Dynamica 5.

$$\begin{aligned} u_{n+1} &= v_n, \\ v_{n+1} &= \frac{(\overline{x}_+ + v_n)^2}{a(\overline{x}_+ + v_n)^2 + (\overline{x}_+ + u_n)^2 + f} - \overline{x}. \end{aligned} \tag{49}$$

Let $\mathbf{F}$ be the corresponding map defined by

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \dfrac{(\overline{x}_+ + v)^2}{a(\overline{x}_+ + v)^2 + (\overline{x}_+ + u)^2 + f} - \overline{x}_+ \end{pmatrix}. \tag{50}$$

Then, $\mathbf{F}$ has the unique fixed point $(0, 0)$ and the Jacobian matrix of $\mathbf{F}$ at $(0, 0)$ is given by

$$\mathrm{Jac}_{\mathbf{F}}(0,0) = \begin{pmatrix} 0 & 1 \\ -\dfrac{\sqrt{1-4(a+1)f}+1}{a+1} & \dfrac{a+2-a\sqrt{1-4(a+1)f}}{a+1} \end{pmatrix}. \tag{51}$$

The eigenvalues of $\mathrm{Jac}_{\mathbf{F}}(0,0)$ are $\mu(a)$ and $\overline{\mu(a)}$ where

$$\mu(a) = \frac{a+2-a\sqrt{1-4(a+1)f}+i\sqrt{2a^2(2(a+1)f-1)+2\left(a^2+4a+2\right)\sqrt{1-4(a+1)f}}}{2(a+1)}. \tag{52}$$

One can prove that for $a = a_0 = 1 - 4f$, we obtain $|\mu(a_0)| = 1$ and

$$\begin{aligned}
\mu(a_0) &= \frac{1}{2}\left(1+4f+i\sqrt{(1-4f)(4f+3)}\right),\\
\mu^2(a_0) &= 8f^2+4f-\frac{1}{2}+\frac{1}{2}i(4f+1)\sqrt{(1-4f)(4f+3)},\\
\mu^3(a_0) &= 32f^3+24f^2-1+4i(2f+1)\sqrt{(1-4f)(4f+3)}f,\\
\mu^4(a_0) &= 128f^4+128f^3+16f^2-8f-\frac{1}{2}+\frac{1}{2}i(4f+1)\left(16f^2+8f-1\right)\sqrt{(1-4f)(4f+3)}.
\end{aligned} \tag{53}$$

One can see that $\mu^k(a_0) \neq 1$ for $k = 1, 2, 3, 4$ and

$$|\mu(a)|^2 = \frac{\sqrt{1-4(a+1)f}+1}{a+1}. \tag{54}$$

Furthermore, we get

$$d(a_0) = \left.\frac{d|\mu(a)|}{da}\right|_{a=a_0} = -\frac{1}{4(1-4f)} < 0. \tag{55}$$

The eigenvectors corresponding to $\mu(a)$ and $\overline{\mu(a)}$ are $\mathbf{q}(a)$ and $\overline{\mathbf{q}(a)}$, where

$$\mathbf{q} = \mathbf{q}(a_0) = \left(\frac{1}{2}\left(1+4f-i\sqrt{(1-4f)(4f+3)}\right), 1\right)^T. \tag{56}$$

For $a = a_0$, we get

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{G}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{57}$$

where

$$\mathbf{A} = \mathrm{Jac}_{\mathbf{F}}(0,0)|_{a=a_0} = \begin{pmatrix} 0 & 1 \\ -1 & 4f+1 \end{pmatrix}, \tag{58}$$

and

$$\mathbf{G}\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} 0 \\ \dfrac{(v+1/2)^2}{(1-4f)(v+1/2)^2+f+(u+1/2)^2} - (4f+1)v + u - 1/2 \end{pmatrix}. \tag{59}$$

Hence, for $a = a_0$, system (49) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{A}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \mathbf{G}\begin{pmatrix} u_n \\ v_n \end{pmatrix}. \tag{60}$$

Define the basis of $\mathbb{R}^2$ by $\Phi = (\mathbf{q}, \overline{q})$.
Let

$$\begin{aligned}
\begin{pmatrix} u \\ v \end{pmatrix} &= \mathbf{\Phi}\begin{pmatrix} z \\ \overline{z} \end{pmatrix} = (\mathbf{q}z + \overline{qz}) = \\
&= \begin{pmatrix} \frac{1}{2}\left(4f - i\sqrt{(1-4f)(4f+3)}+1\right)z + \frac{1}{2}\left(4f + i\sqrt{(1-4f)(4f+3)}+1\right)\overline{z} \\ \overline{z}+z \end{pmatrix}.
\end{aligned} \tag{61}$$

By using this, one can see that

$$\mathbf{g}_{20} = \frac{\partial^2}{\partial z^2}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \overline{z} \end{pmatrix}\right)\Bigg|_{z=0} = \begin{pmatrix} 0 \\ 16f^2 + (12if - i)\sqrt{(1-4f)(4f+3)} - 3 \end{pmatrix},$$
$$\mathbf{g}_{11} = \frac{\partial^2}{\partial z\,\partial \overline{z}}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \overline{z} \end{pmatrix}\right)\Bigg|_{z=0} = \begin{pmatrix} 0 \\ -8f \end{pmatrix}, \tag{62}$$
$$\mathbf{g}_{02} = \frac{\partial^2}{\partial \overline{z}^2}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \overline{z} \end{pmatrix}\right)\Bigg|_{z=0} = \begin{pmatrix} 0 \\ 16f^2 + (i - 12if)\sqrt{(1-4f)(4f+3)} - 3 \end{pmatrix},$$

and

$$\mathbf{K}_{20} = \left(\mu^2 I - A\right)^{-1}\mathbf{g}_{20} = \begin{pmatrix} \dfrac{16f^2 + (12if - i)\sqrt{(1-4f)(4f+3)} - 3}{(2f+1)(4f-1)\left(16f^2 + 8f + i(4f+1)\sqrt{(1-4f)(4f+3)} - 1\right)} \\ \\ \dfrac{16f^2 + i\sqrt{(1-4f)(4f+3)}\,(12f-1) - 3}{4f(4f+1) - 2} \end{pmatrix},$$
$$\mathbf{K}_{11} = (I - A)^{-1}\mathbf{g}_{11} = \begin{pmatrix} \dfrac{8f}{4f-1} \\ \\ \dfrac{8f}{4f-1} \end{pmatrix}, \tag{63}$$
$$\mathbf{K}_{02} = \left(\overline{\mu}^2 I - A\right)^{-1}\mathbf{g}_{02} = \overline{\mathbf{K}_{20}}.$$

By using $\mathbf{K}_{20}, \mathbf{K}_{11}$, and $\mathbf{K}_{02}$, we have that

$$\mathbf{g}_{21} = \frac{\partial^3}{\partial z^2 \partial \overline{z}}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \overline{z} \end{pmatrix} + \frac{1}{2}\mathbf{K}_{20}z^2 + \mathbf{K}_{11}z\overline{z} + \frac{1}{2}\mathbf{K}_{02}\overline{z}^2\right)\Bigg|_{z=0}$$
$$= \begin{pmatrix} 0 \\ -2i\sqrt{\dfrac{4f+3}{1-4f}} + \dfrac{20}{3-12f} - \dfrac{74}{6f+3} + 18 \end{pmatrix}. \tag{64}$$

Next, we have that $\mathbf{pA} = \mu\mathbf{p}$ and $\mathbf{pq} = 1$, where

$$\mathbf{p} = \left(\frac{i}{\sqrt{(1-4f)(4f+3)}}, \frac{-4if + \sqrt{(1-4f)(4f+3)} - i}{2\sqrt{(1-4f)(4f+3)}}\right). \tag{65}$$

One can see that

$$\alpha(a_0) = \frac{1}{2}\mathrm{Re}(\mathbf{p}\mathbf{g}_{21}\overline{\mu}) = -\frac{1}{1-4f} < 0. \tag{66}$$

Theorem 3. *Let* $0 < f < 1/4(1+a)$ *and*

$$\overline{x}_+ = \frac{1+\sqrt{1-4(a+1)f}}{2(a+1)}. \tag{67}$$

Then, there is a neighborhood U of the equilibrium point $\overline{x}_+$ and some $\rho > 0$ such that for

$$|a - a_0| < \rho, \quad (a_0 = 1-4f), \tag{68}$$

and $x_0, x_{-1} \in U$, the ω-limit set of a solution of equation (1), with initial conditions x_0, x_{-1}, is the equilibrium point $\overline{x}_+$ if

$$a > a_0 = 1 - 4f, \tag{69}$$

and belongs to a closed invariant C^1 curve $\Gamma(a)$ encircling the equilibrium point $\overline{x}_+$ if

$$a < a_0 = 1-4f. \tag{70}$$

Furthermore, $\Gamma(a_0) = 0$ and the invariant curve $\Gamma(a)$ can be approximated by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \begin{pmatrix} \overline{x}_+ \\ \overline{x}_+ \end{pmatrix} + 2\rho_0 \mathrm{Re}\left(\mathbf{q}e^{i\theta}\right) + \rho_0^2\left(\mathrm{Re}\left(\mathbf{K}_{20}e^{2i\theta}\right) + \mathbf{K}_{11}\right), \tag{71}$$

where

$$\rho_0 = \frac{1}{2}\sqrt{(1-4f) - a}. \tag{72}$$

Proof. The proof follows from the above discussion and Theorem 1 and Corollary 1. See Figures 2 and 3 for a graphical illustration. □

4. The Invariant Manifolds

In this section, we derive the asymptotic formulas for the local stable and unstable manifolds for the equilibrium point $\overline{x}_-$ and provide some numerical examples where we compare visually the local approximations of stable and unstable manifolds and center manifold, obtained by using Mathematica, with the boundaries of the basins of attraction obtained by using the software package Dynamica.

From Lemma 3, it follows that $(\overline{x}_-, \overline{x}_-)$ is a saddle point if $4f(a+1) < 1$. In order to apply the theorem for the stable and unstable manifolds, we make a change of variable $y_n = x_n - \overline{x}_-$. Then, the transformed equation is given by

$$y_{n+1} = \frac{(\overline{x}_- + y_n)^2}{a(\overline{x}_- + y_n)^2 + (\overline{x}_- + y_{n-1})^2 + f} - \overline{x}_-, \quad n = 0, 1, \ldots. \tag{73}$$

Set

$$\begin{aligned} u_n &= y_{n-1} \\ v_n &= y_n, \quad \text{for } n = 0, 1, \ldots, \end{aligned} \tag{74}$$

and write equation (1) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n, \\ v_{n+1} &= \frac{(\overline{x}_- + v_n)^2}{a(\overline{x}_- + v_n)^2 + (\overline{x}_- + u_n)^2 + f} - \overline{x}_-. \end{aligned} \tag{75}$$

Let $\mathbf{G}$ be the corresponding map defined by

$$\mathbf{G}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f_1(u,v) \\ f_2(u,v) \end{pmatrix} = \begin{pmatrix} v \\ \dfrac{(\overline{x}_- + v)^2}{a(\overline{x}_+ + v)^2 + (\overline{x}_- + u)^2 + f} - \overline{x}_- \end{pmatrix}. \tag{76}$$

We expand $f_1(u,v)$ and $f_2(u,v)$ as a Taylor series about $(0,0)$ to write

$$\begin{aligned} f_1(u,v) &= v \\ f_2(u,v) &= -2u\overline{x}_- - 2(-1 + a\overline{x}_-)v + (-1 + 4\overline{x}_-)u^2 + 4(-1 + 2a\overline{x}_-)uv \\ &\quad + \frac{af(4a\overline{x}_- - 5) - (a+1)\overline{x}_- + 1}{f}v^2 + 4(1 - 2\overline{x}_-)u^3 \\ &\quad + \frac{2(\overline{x}_-(-12af + a + 1) + 2(a+2)f - 1)}{f}u^2 v \\ &\quad + \frac{2\overline{x}_-\left(-12a^2 f + a + 1\right) + 2(10af - 1)}{f}uv^2 \\ &\quad + \frac{4a\left(\overline{x}_-\left(-2a^2 f + a + 1\right) + 3af - 1\right)}{f}v^3 + O\left((|u| + |v|)^4\right). \end{aligned} \tag{77}$$

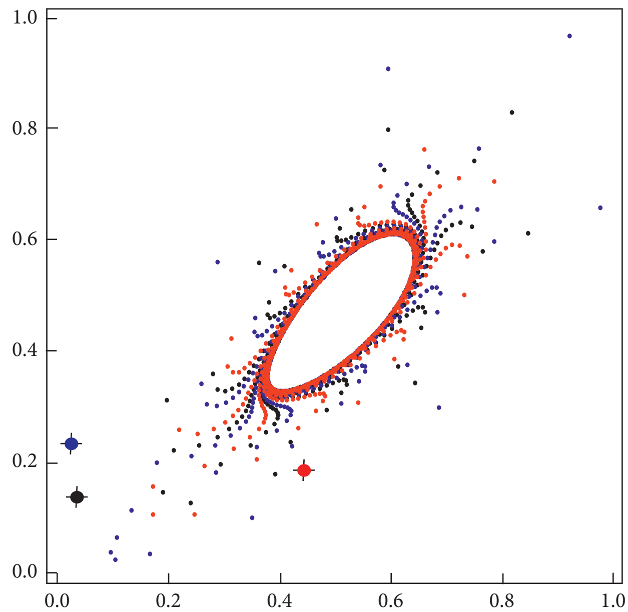

Figure 2: Trajectory of some solutions of equation (1) for parameters $a = 0.49$ and $f = 0.125$. Picture is produced by Dynamica 5.

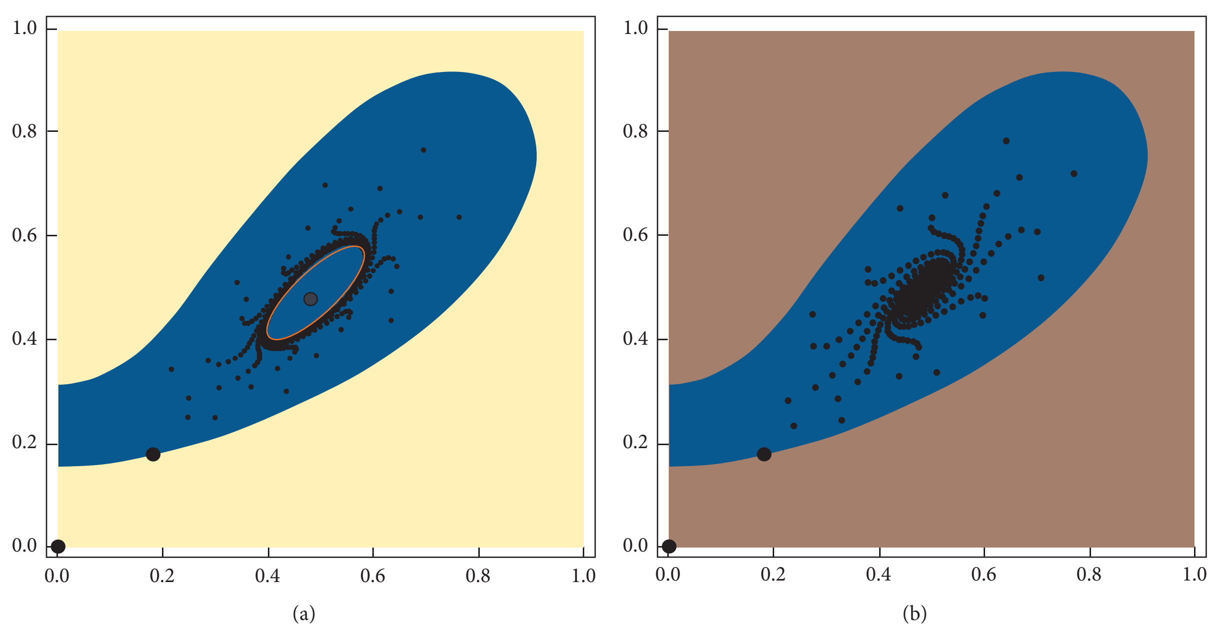

Figure 3: (a) Trajectory (black) of a solution of equation (1) and invariant curve (orange) for $a = 0.49$ and $f = 0.125$ and where $a_0 = 0.5$. (b) Trajectory (black) of a solution of equation (1) for $a = 0.51$ and $f = 0.125$ and where $a_0 = 0.5$.

Assume that the invariant manifold at $(0,0)$ is locally represented as the graph of a function $v = h(u)$ such that $h(u) = Au + Bu^2 + Cu^3 + O(|u|^4)$. Then, from $h(f_1(u, h(u))) - f_2(u, h(u)) = 0$, and by using package Mathematica, we obtain

$$A_{\pm} = 1 - a\overline{x}_{-} \pm \sqrt{\left(\overline{x}_{-}\left(a\left(a\overline{x}_{-} - 2\right) - 2\right) + 1\right)}, \quad (78)$$

and

$$B_{\pm} = \frac{(a+1)^2\left(\left(1 - A_{\pm}\right)\overline{x}_{-}^4\left(a\left(3A_{\pm} - 1\right) - A_{\pm} + 3\right) - f\overline{x}_{-}^2\left(A_{\pm}\left(\left(3a - 2\right)A_{\pm} + 4\right) + 1\right) + A^2 f^2\right)}{\overline{x}_{-}\left(-(a+1)f\left(2a\overline{x}_{-} + A^2 + A\right) + \overline{x}_{-}\left((a+1)A_{\pm}\left(A_{\pm} + 1\right) - 2\right) + 2f\right)}, \quad (79)$$

and

$$C_{\pm} = C_{\pm}\left(a, f, A_{\pm}, B_{\pm}\right). \tag{80}$$

Then, the dynamics restricted to the invariant manifold are given locally by the equation

$$\begin{aligned} u_{n+1} &= f_1\left(u_n, h\left(u_n\right)\right) = h\left(u_n\right) \\ &= A_{\pm}u_n + B_{\pm}u_n^2 + C_{\pm}u_n^3 + O\left(\left|u_n\right|^4\right). \end{aligned} \tag{81}$$

Note that the Jacobian matrix of $\mathbf{G}$ at $(0, 0)$ is given by

$$\mathrm{Jac}_{\mathbf{G}}(0,0) = \begin{pmatrix} 0 & 1 \\ \dfrac{\sqrt{1-4(a+1)f} - 1}{a+1} & \dfrac{a+2+a\sqrt{1-4(a+1)f}}{a+1} \end{pmatrix}, \tag{82}$$

and the eigenvalues of $\mathrm{Jac}_{\mathbf{G}}(0, 0)$ are $A_{\pm}$.

Theorem 4. *Assume that $4f(a+1) \leq 1$. Then, the equilibrium point $\overline{x}_-$ of equation (1) is a saddle point if $4f(a+1) < 1$. The stable manifold W^s and unstable manifold W^u at $(\overline{x}_-, \overline{x}_-)$ are given by*

$$W^u = \left\{(x,y)\colon\ y = \overline{x}_- + A_+\left(x-\overline{x}_-\right) + B_+\left(x-\overline{x}_-\right)^2 + C_+\left(x-\overline{x}_-\right)^3 + O\left(\left|x-\overline{x}_-\right|^4\right),\quad x>0, y>0\right\}, \tag{83}$$

and

$$W^s = \left\{(x,y)\colon\ y = \overline{x}_- + A_-\left(x-\overline{x}_-\right) + B_-\left(x-\overline{x}_-\right)^2 + C_-\left(x-\overline{x}_-\right)^3 + O\left(\left|x-\overline{x}_-\right|^4\right),\quad x>0, y>0\right\}. \tag{84}$$

The equilibrium point $(\overline{x}_-, \overline{x}_-)$ is nonhyperbolic if $4f(a+1) = 1$, and it is semiasymptotically stable from the right.

Proof. The proof of first part of the theorem follows from the above discussion. If $4f(a+1) = 1$, then $a = 1 - 4f/4f > 0$ and we obtain $A_- = 4f < 1, A_+ = 1$, and $B_+ = 1\ /4f(4f-1) < 0$. The rest of the proof follows from the fact that the dynamics of equation (1) are dynamics restricted to the center manifold which is given locally by equation (81). See Figures 4 and 5 for a graphical illustration.

Bifurcation diagram of equation (1) in certain range of parameters indicate chaos, see Figure 6. □

5. Rate of Convergence

In this section, we will shortly discuss the rate of convergence of solutions of equation (1) that converge to the equilibrium solutions. We will show that the convergence toward the zero equilibrium is quadratic and toward any positive equilibrium is linear.

Assume that $\lim_{n\longrightarrow\infty} x_n = 0$ for some solutions of equation (1). Then, equation (1) implies

$$\frac{x_{n+1}}{x_n^2} = \frac{1}{ax_n^2 + x_{n-1}^2 + f}, \tag{85}$$

and so

$$\lim_{n\longrightarrow\infty} \frac{x_{n+1}}{x_n^2} = \frac{1}{f}, \tag{86}$$

which shows that the convergence toward the zero equilibrium is quadratic.

Let $\lim_{n\longrightarrow\infty} x_n = \overline{x} > 0$ for some solutions of equation (1). Then, we have

$$\begin{aligned} x_{n+1} - \overline{x} &= \frac{x_n^2}{ax_n^2 + x_{n-1}^2 + f} - \frac{\overline{x}^2}{(a+1)\overline{x}^2 + f} = \frac{f\left(x_n^2 - \overline{x}^2\right) + \overline{x}^2\left(x_n^2 - x_{n-1}^2\right)}{\left(ax_n^2 + x_{n-1}^2 + f\right)\left((a+1)\overline{x}^2 + f\right)} \\ &= \frac{\left(f + \overline{x}^2\right)\left(x_n + \overline{x}\right)}{\left(ax_n^2 + x_{n-1}^2 + f\right)\left((a+1)\overline{x}^2 + f\right)}\left(x_n - \overline{x}\right) - \frac{\overline{x}^2\left(x_n + \overline{x}\right)}{\left(ax_n^2 + x_{n-1}^2 + f\right)\left((a+1)\overline{x}^2 + f\right)}\left(x_{n-1} - \overline{x}\right) \\ &= g_0\left(x_n - \overline{x}\right) + g_1\left(x_{n-1} - \overline{x}\right), \end{aligned} \tag{87}$$

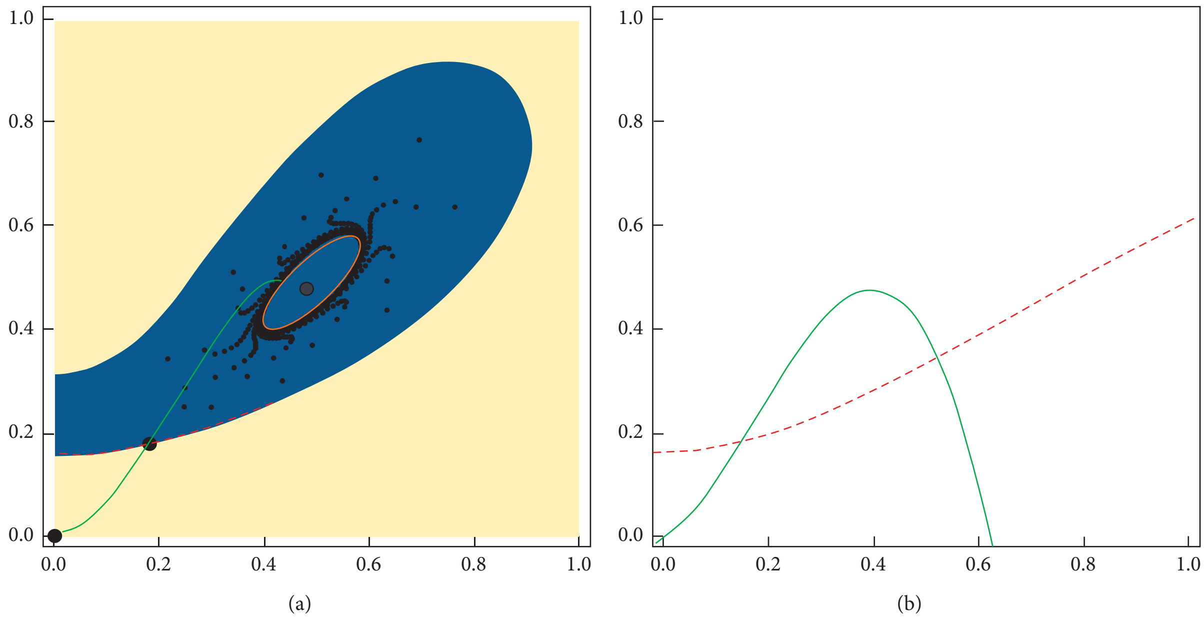

FIGURE 4: The local approximation of the stable (red) and unstable manifold (green) of equation (1) for $a = 0.49$ and $f = 0.125$

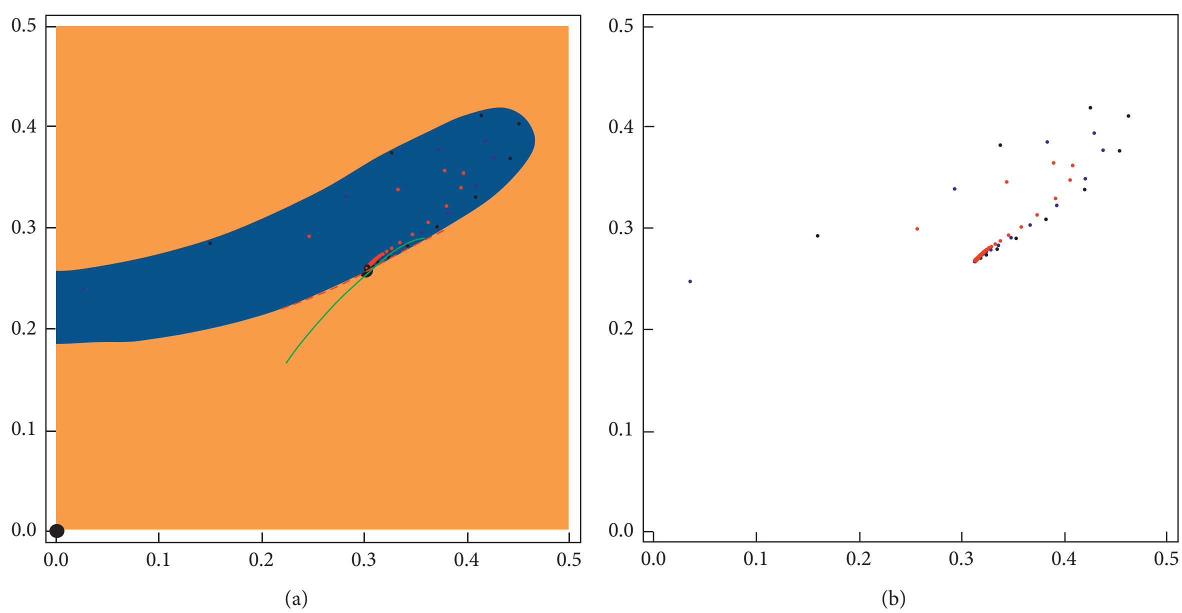

FIGURE 5: (a) The local approximation of the stable (red) and center manifold (green) of equation (1) in nonhyperbolic case, $a = 1$ and $f = 1/8 = 0.125$. (b) Trajectories of some solutions for $a = 1$ and $f = 0.125$.

where

$$\lim_{n\longrightarrow\infty} g_0 = \frac{\left(f+\overline{x}^2\right)2\overline{x}}{\left((a+1)\overline{x}^2+f\right)^2} = \frac{2\left(f+\overline{x}^2\right)}{\overline{x}} = 2\left(1-a\overline{x}\right), \tag{88}$$

and

$$\lim_{n\longrightarrow\infty} g_1 = -\frac{\overline{x}^2 2\overline{x}}{\left((a+1)\overline{x}^2+f\right)^2} = \frac{-2\overline{x}^3}{\overline{x}^2} = -2\overline{x}. \tag{89}$$

Setting $y_n = x_n - \overline{x}$, we see that the limiting equation is exactly the linearized equation (29). Now, in view of Poincaré-Perron theorem, we conclude that

$$\lim_{n\longrightarrow\infty} \frac{y_{n+1}}{y_n} = \lim_{n\longrightarrow\infty} \frac{x_{n+1}-\overline{x}}{x_n-\overline{x}} = \lambda_{1,2}, \tag{90}$$

where $\lambda_{1,2}$ are roots of the characteristic equation (30), and [6, 8, 21]. Thus, in this case, the convergence is linear since $\lambda_{1,2} \neq 0$. See Table 1 for numerical comparison of rates of convergence to zero equilibrium and positive equilibrium.

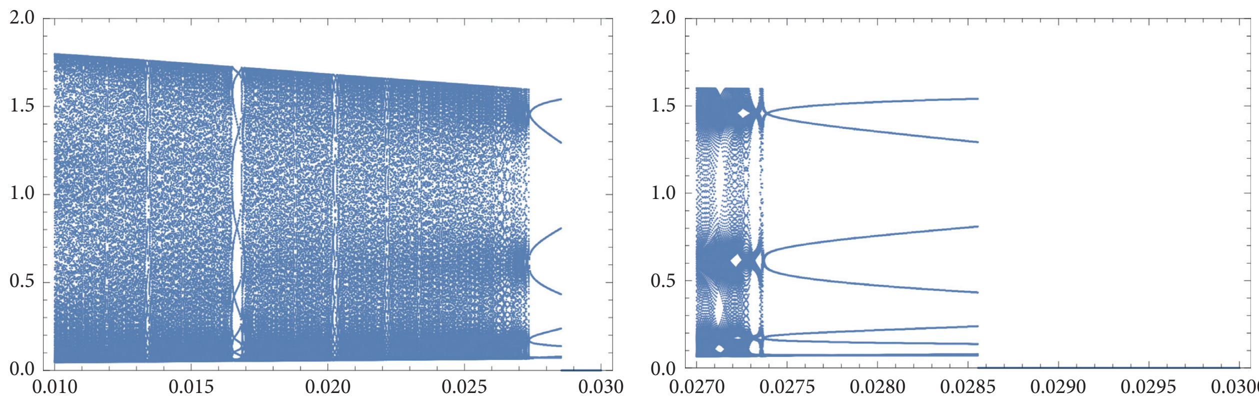

FIGURE 6: (a) Bifurcation diagram of equation (1) for $a = 0.52$ indicating occurrence of chaos and the convergence to the period 7 solution. (b) Bifurcation diagram of equation (1) for $a = 0.52$ indicating the global attractivity of period 7 solution and period 8 solutions following the chaos.

TABLE 1: The table with numerical comparison of rates of convergence when $a = 0.53$, $f = 0.140$, and $\overline{x}_+ = 0.450463$.

Iterations to $\overline{x}_0$	Iterations to $\overline{x}_+$
$x_0 = 0.2$	$x_0 = 0.36$
$x_1 = 0.1$	$x_1 = 0.40$
$x_2 = 0.0539665$	$\vdots$
$x_3 = 0.0192182$	$x_{75} = 0.451168$
$x_4 = 0.00258083$	$x_{76} = 0.45005$
$x_5 = 0.0000474498$	$x_{77} = 0.449201$
$x_6 = 1.60813 \times 10^{-8}$	$x_{78} = 0.448912$
$x_7 = 1.8472 \times 10^{-15}$	$x_{79} = 0.449237$
$x_8 = 2.43725 \times 10^{-29}$	$x_{80} = 0.449991$
$x_9 = 4.24298 \times 10^{-57}$	$x_{81} = 0.45085$
$x_{10} = 1.28592 \times 10^{-112}$	$x_{82} = 0.451477$
$x_{11} = 1.18114 \times 10^{-223}$	$x_{83} = 0.451659$
	$x_{84} = 0.451368$
	$x_{85} = 0.450763$
	$x_{105} = 0.450481$
	$x_{106} = 0.450262$
	$x_{107} = 0.450141$
	$x_{108} = 0.450153$
	$x_{109} = 0.450282$
	$x_{110} = 0.450467$

6. Conclusions

In this paper, we give the global dynamics of the difference equation $x_{n+1} = x_n^2/(ax_n^2 + x_{n-1}^2 + f)$ where a and f are positive numbers and the initial conditions x_{-1} and x_0 are nonnegative numbers in a part of parametric space. We show that this difference equation exhibits transcritical and Neimark–Sacker bifurcations but not flip (period-doubling) bifurcation since this difference equation cannot have period-two solutions. Moreover, we give the asymptotic approximation of the invariant manifolds, stable, unstable, and center manifolds of the saddle point and nonhyperbolic equilibrium solution. Finally, we give the rates of convergence toward all equilibrium solutions, proving that the convergence to the zero equilibrium solution is quadratic and convergence to the positive equilibrium solution is linear.

This difference equation is the simplest perturbation of sigmoid Beverton–Holt difference equation that exhibits Allee's effect, transcritical bifurcation, and Neimark–Sacker bifurcation, but not a flip bifurcation.

Data Availability

No data were used for this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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