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Research Article

Global Dynamics, Boundedness, and Semicycle Analysis of a Difference Equation

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In this paper, we explore local stability, attractor, periodicity character, and boundedness solutions of the second-order nonlinear difference equation. Finally, obtained results are verified numerically.

1. Introduction

For decades, the qualitative analysis of difference equations has been steadily increasing. This is due to the fact that difference equations appear as mathematical models in statistical problems, queuing theory, combinatorial analysis, electrical networks, genetics in biology, probability theory, economics, psychology, stochastic time series, sociology, geometry, number theory, etc. Precisely, there is an increasing interest in the qualitative analysis of difference equations. For instance, Devault et al. [1] have explored boundedness, existence of unbounded solutions, persistence, and global attractivity results for following nonautonomous difference equation:

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n}, \tag{1}$$

where p_n is a positive bounded sequence and initial conditions are positive. Amleh et al. [2] have explored global stability, periodic nature, and boundedness character of the following difference equation:

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, \tag{2}$$

where $\alpha \in (0, \infty)$ and x_{-1} and x_0 are positive constants. DeVault et al. [3] have explored boundedness, periodic character, and global stability of the following difference equation:

$$x_{n+1} = p + \frac{x_{n-k}}{x_n}, \tag{3}$$

where $k \in \{2, 3, \ldots,\}$, p is positive, and initial conditions are arbitrary positive numbers. Berenhaut and Stević [4] have explored the behaviour of the following difference equation:

$$x_{n+1} = A + \left(\frac{x_{n-2}}{x_{n-1}}\right)^p, \tag{4}$$

where p and $A \in (0, \infty)$, $p \neq 1$, and x_{-1} and $x_0 \in (0, \infty)$. Stević [5] has explored the behavior of the following difference equation:

$$x_{n+1} = a + \frac{x_{n-1}}{x_n}, \tag{5}$$

where α is a negative number. For more results on the behavior of the difference equation, we refer the reader to recent published articles [6–10] and books [11–13]. Motivated from aforementioned studies, we explore the behavior of the following difference equation:

$$x_{n+1} = a_n + \frac{x_n^p}{x_{n-1}^p}, \quad n = 0, 1, \ldots, \tag{6}$$

where p is a nonnegative real number. Moreover, initial conditions x_{-1} and x_0 are positive real numbers, and $\{a_n\}$ is a nonnegative periodic sequence with

$$a_n = \begin{cases} \alpha, & \text{if } n \text{ is even}, \\ \beta, & \text{if } n \text{ is odd}, \end{cases} \tag{7}$$

where α and $\beta \in (0, \infty)$.

2. Dynamics of Solutions of Equation (6)

In this study, we consider the following three cases of the function a_n.

2.1. Case 1: $a_n = a \in R^+$. In this case, (6) becomes

$$x_{n+1} = a + \frac{x_n^p}{x_{n-1}^p}, \quad n = 0, 1, \ldots. \tag{8}$$

It is easy to see that $\overline{x} = a + 1$ is the only positive fixed point of (8).

Now, the function $f\colon (0,\infty)^2 \longrightarrow (0,\infty)$ is defined by

$$f(x, y) = a + \frac{x^p}{y^p}. \tag{9}$$

Therefore,

$$\frac{\partial f(x,y)}{\partial x} = \frac{p x^{p-1}}{y^p}, \qquad \frac{\partial f(x,y)}{\partial y} = -\frac{p x^p}{y^{p+1}}. \tag{10}$$

Now,

$$\frac{\partial f(\overline{x},\overline{x})}{\partial x} = \frac{p}{a+1}, \qquad \frac{\partial f(\overline{x},\overline{x})}{\partial y} = -\frac{p}{a+1}. \tag{11}$$

So, the linearized equation of (8) about $\overline{x} = a + 1$ is

$$y_{n+1} - \frac{p}{a+1} y_n + \frac{p}{a+1} y_{n-1} = 0. \tag{12}$$

Theorem 1.

(i) *If $p < a + 1$, then $\overline{x} = a + 1$ of (8) is locally asymptotically stable, and so it is also called a sink*

(ii) *If $p > a + 1$, then $\overline{x} = a + 1$ of (8) is unstable and is called a repeller*

(iii) *If $p = a + 1$, then $\overline{x} = a + 1$ of (8) is unstable and is called a nonhyperbolic point*

Proof

(i) We set $p_1 = (p/(a+1))$ and $p_2 = -(p/(a+1))$. Now,

$$|p_1| < 1 - p_2 \Longleftrightarrow \frac{p}{a+1} < 1 + \frac{p}{a+1} \Longleftrightarrow 0 < 1, \tag{13}$$

and also,

$$1 - p_2 < 2 \Longleftrightarrow \frac{p}{a+1} < 1, \tag{14}$$

which is valid if

$$p < a + 1. \tag{15}$$

So, by Theorem 1.1.1 (a) and (c) of [13], one can obtain that $\overline{x} = a + 1$ is locally asymptotically stable when $p < a + 1$.

(ii) Again,

$$|p_2| - 1 = \frac{p}{a+1} - 1 > 0 \Longleftrightarrow \frac{p}{a+1} > 1, \qquad |p_1| - |1 - p_2| = \frac{p}{a+1} - 1 - \frac{p}{a+1} = -1 < 0, \tag{16}$$

and then, $|p_1| < |1 - p_2|$. Thus, by Theorem 1.1.1 (d) of [13], $\overline{x} = a + 1$ is unstable (repeller point) when $p > a + 1$.

(iii) Note that

$$p_2 = -1 \Longleftrightarrow -\frac{p}{a+1} = -1 \Longleftrightarrow -p = -(a+1) \Longleftrightarrow p = a + 1, \qquad |p_1| - 2 \le 0 \Longleftrightarrow \frac{p}{a+1} - 2 \le 0 \Longleftrightarrow p \le 2(a+1). \tag{17}$$

Thus, by Theorem 1.1.1 (e) of [13], $\overline{x} = a + 1$ is unstable (repeller point) when $p = a + 1$. □

Theorem 2. *Positive solution of (8) is bounded and persists if $0 < p < 1$.*

Proof. We obtain from (8) that

$$x_{n+1} > a, \quad \forall n \ge 0. \tag{18}$$

Hence, $\{x_n\}$ persists. Then, again, from (8), it follows that

$$x_{2n+1} \le a + \left(\frac{x_{2n}}{a}\right)^p, \quad n = 0, 1, \ldots,. \tag{19}$$

Now considering

$$y_{n+1} = a + \left(\frac{y_n}{a}\right)^p, \quad \forall n \ge 0. \tag{20}$$

If the solution of (20) with $y_0 = x_0$ is $\{y_n\}$, then

$$x_{2n+1} \le y_{n+1}, \quad (\text{Resp. } x_{2n+2} \le y_{n+1}). \tag{21}$$

Now, we have to show that $\{y_n\}$ is bounded. Let

$$f(x) = a + \frac{x^p}{a^p}, \tag{22}$$

and then,

$$f'(x) = \frac{1}{a^p} p x^{p-1} > 0,$$
$$f''(x) = \frac{1}{a^p} p(p-1) x^{p-2} < 0. \quad (23)$$

Therefore, f is nondecreasing and concave. Therefore, one gets y^* as the unique fixed point of $f(y) = y$. Moreover, f also satisfies

$$(f(y) - y)(y - y^*) < 0, \quad y \in (0, \infty). \quad (24)$$

By Theorem 2.6.2 of [14], y^* is a global attractor for all positive solutions of (20), and hence, it is bounded. So, from (8), $\{x_n\}$ is also bounded. □

Theorem 3. *Let $p \geq 4$, and then, (8) has unbounded solutions.*

Proof. It is noted that following holds:

$$x_{n+1} > \frac{x_n^p}{x_{n-1}^p}, \quad n \in N, \quad (25)$$

for every solution $\{x_n\}_{n=-1}^{\infty}$ of (8). Let $y_n = \ln x_n$. Then, it follows from (25) that

$$y_{n+1} > p y_n - p y_{n-1}. \quad (26)$$

Now, roots of

$$p(\lambda) = \lambda^2 - p\lambda + p, \quad (27)$$

are given by

$$\lambda_1, \lambda_2 = \frac{p \pm \sqrt{p^2 - 4p}}{2}. \quad (28)$$

Since $p \geq 4$, we have that $\lambda_1 > 1$ and

$$\lambda_2 = \frac{2p}{p + \sqrt{-4p + p^2}}. \quad (29)$$

Therefore, both roots of $p(\lambda)$ are positive if $p \geq 4$. Moreover, (26) can also written as

$$y_{n+1} - \lambda_1 y_n - \lambda_2 (y_n - \lambda_1 y_{n-1}) > 0. \quad (30)$$

Then, we see that

$$\frac{x_{n+1}}{x_n^{\lambda_1}} > \left(\frac{x_n}{x_{n-1}^{\lambda_1}}\right)^{\lambda_2}. \quad (31)$$

It follows that

$$\frac{x_n}{x_{n-1}^{\lambda_1}} > \left(\frac{x_{n-1}}{x_{n-2}^{\lambda_1}}\right)^{\lambda_2} > \cdots > \left(\frac{x_1}{x_0^{\lambda_1}}\right)^{\lambda_2} > \left(\frac{x_0}{x_{-1}^{\lambda_1}}\right)^{\lambda_2}. \quad (32)$$

Choose x_{-1} and x_0 so that

$$x_0 > 1,$$
$$x_0 = x_{-1}^{\lambda_1}. \quad (33)$$

It follows from this and (32) that

$$x_n > \left(\frac{x_0}{x_{-1}^{\lambda_1}}\right)^{\lambda_2} x_{n-1}^{\lambda_1} = x_{n-1}^{\lambda_1} > \cdots > x_0^{\lambda_1^n}, \quad (34)$$

and consequently,

$$x_n > x_0^{\lambda_1^n}, \quad n \in N. \quad (35)$$

It follows by letting $n \longrightarrow \infty$ in (35) that $x_n \longrightarrow \infty$ as $n \longrightarrow \infty$, and hence, it follows from this result. □

Theorem 4. *Let $a \geq 1$ and $0 < p < 1$; then, $\overline{x} = a + 1$ of (8) is globally asymptotically stable.*

Proof. By Theorem 1 (i), $\overline{x} = a + 1$ is a sink. Hence, it is enough to prove further that $\{x_n\}_{n=-1}^{\infty}$ of (8) tends to $\overline{x} = a + 1$. Recall that $\{x_n\}_{n=-1}^{\infty}$ of (8) is bounded by Theorem 2. Thus,

$$a \leq s = \liminf x_n,$$
$$S = \limsup x_n < \infty. \quad (36)$$

Then, from (8), we get

$$S \leq a + \frac{S^p}{s^p},$$
$$s \geq a + \frac{s^p}{S^p}. \quad (37)$$

Now, claiming that $S = s$, otherwise, $S > s$. From (37), we obtain

$$s^p S < s^p a + S^p,$$
$$s S^p > S^p a + s^p. \quad (38)$$

Since $0 < p < 1$ holds, then

$$s^{1-p} < S^{1-p}, \quad (39)$$

or equivalently

$$s S^p < S s^p. \quad (40)$$

It follows from (38) and (40) that

$$S^p a + s^p \leq s^p a + S^p. \quad (41)$$

Hence,

$$S^p (a-1) \leq s^p (a-1), \quad (42)$$

which is impossible for $a \geq 1$. This is contradiction, and hence, the result follows. □

Theorem 5. *Every positive solution of (8) oscillates about $a + 1 = \overline{x}$ with semicycles of length two or three, and extreme of every semicycle occurs at the first or the second term.*

Proof. Let the positive solution of (8) is $\{x_n\}_{n=-1}^{\infty}$. First, we prove that every positive semicycle except possibly the first term has two or three terms. Assuming $x_{\sigma-1} < \overline{x}$ and $x_\sigma \geq \overline{x}$ for some $\sigma \in \mathbb{N}$, we obtain from (8) that

$$x_{\sigma+1} = a + \frac{x_\sigma^p}{x_{\sigma-1}^p} > a + 1 = \overline{x}. \quad (43)$$

If $x_{\sigma+1} > x_\sigma$, then we have

$$x_{\sigma+2} = a + \frac{x_{\sigma+1}^p}{x_\sigma^p} > a + 1 = \overline{x}. \quad (44)$$

On the contrary, since $p \in (0,1]$, we see that

$$x_{\sigma+2} = a + \frac{x_{\sigma+1}^p}{x_\sigma^p} \leq a + \frac{x_{\sigma+1}^p}{\overline{x}^p} \leq a + \frac{x_{\sigma+1}^p}{a+1} \leq x_{\sigma+1}. \quad (45)$$

So, $\overline{x} < x_{\sigma+2} < x_{\sigma+1}$. Therefore,

$$x_{\sigma+3} = a + \frac{x_{\sigma+2}^p}{x_{\sigma+1}^p} < a + 1 = \overline{x}. \quad (46)$$

□

Theorem 6. *Equation (8) has no periodic solution having prime period two.*

Proof. Let

$$\ldots, \eta_1, \eta_2, \eta_1, \eta_2, \ldots, \quad (47)$$

be a periodic solution of period two of (8). It follows that

$$\eta_1 = a + \left(\frac{\eta_2}{\eta_1}\right)^p, \qquad \eta_2 = a + \left(\frac{\eta_1}{\eta_2}\right)^p, \quad (48)$$

which implies that

$$\eta_2 = a + \frac{1}{\eta_1 - a}. \quad (49)$$

Substituting from (49) into (48) and after some calculation, we get

$$(\eta_1 - a)^{p+1}\eta_1^p = (a(\eta_1 - a) + 1)^p. \quad (50)$$

From (50), one has

$$f(\eta_1) = (p+1)\ln(\eta_1 - a) + p \ln \eta_1 - p \ln[a(\eta_1 - a) + 1] = 0. \quad (51)$$

Obviously, $\eta_1 = a + 1$ is a solution of (51). But one has to prove that this is the unique solution of (51). Now,

$$f'(\eta_1) = \frac{(\eta_1 - a)(a\eta_1 + p(a(\eta_1 - a) + 1)) + (p+1)\eta_1}{\eta_1(\eta_1 - a)(a(\eta_1 - a) + 1)}. \quad (52)$$

Thus, $f'(\eta_1) > 0$ for $\eta_1 \in (a, \infty)$. This implies that, on (a, ∞), f is strictly nondecreasing. Hence, $\overline{x} = a + 1$ is the unique solution of (51), and consequently, $(a+1, a+1)$ is the unique solution of (48) completing the theorem's proof. □

2.2. Case 2: a_n be a Function of Period Two. We will explore dynamics of equation (6) when a_n is a periodic sequence having period two with α and $\beta \in (0,\infty)$ and $\alpha \neq \beta$. Consider $a_{2n} = \alpha$ and $a_{2n+1} = \beta$. Then, we have

$$x_{2n+1} = \alpha + \frac{x_{2n}^p}{x_{2n-1}^p}, \quad n = 0, 1, \ldots, \qquad x_{2n+2} = \beta + \frac{x_{2n+1}^p}{x_{2n}^p}, \quad n = 0, 1, \ldots,. \quad (53)$$

By separating the even-indexed and odd-indexed terms, equation (6) now becomes

$$\left.\begin{aligned} u_{n+1} &= \alpha + \frac{u_n^p}{v_n^p} \\ v_{n+1} &= \beta + \frac{v_n^p}{u_n^p} \end{aligned}\right\}, \quad n = 0, 1, \ldots, \quad (54)$$

where $(\overline{u}, \overline{v}) = (\alpha + 1, \beta + 1)$ is the unique fixed point of system (54).

Theorem 7. *If $p < (((\beta+1)(\alpha+1))/((\alpha+1)^p + (\beta+1)^p))$, then $E_{(\overline{u},\overline{v})} = (\alpha+1, \beta+1)$ of (54) is a sink.*

Proof. We consider the map T on $[0,\infty) \times [0,\infty))$, which is described as follows:

$$T(u,v) = \begin{bmatrix} T_1(u,v) \\ T_2(u,v) \end{bmatrix} = \begin{bmatrix} \alpha + \dfrac{u^p}{v^p} \\ \alpha + \dfrac{v^p}{u^p} \end{bmatrix}. \quad (55)$$

Then,

$$\frac{\partial T_1}{\partial u} = -\frac{pu^{p-1}v^p}{(u^p)^2}, \qquad \frac{\partial T_1}{\partial v} = \frac{pv^{p-1}}{u^p}, \qquad \frac{\partial T_2}{\partial u} = \frac{pu^{p-1}}{v^p}, \qquad \frac{\partial T_2}{\partial v} = -\frac{pv^{p-1}u^p}{(v^p)^2}. \quad (56)$$

Therefore, the Jacobian matrix of T evaluated at $E_{(\overline{u},\overline{v})} = (\alpha+1,\beta+1)$ is

$$J\left(E_{(\overline{u},\overline{v})}\right) = \begin{bmatrix} -\dfrac{pu^{p-1}v^p}{(u^p)^2} & \dfrac{pv^{p-1}}{u^p} \\ \\ \dfrac{pu^{p-1}}{v^p} & -\dfrac{pv^{p-1}u^p}{(v^p)^2} \end{bmatrix}, \tag{57}$$

and the auxiliary equation associated with $(\overline{u},\overline{v})$ is

$$\lambda^2 - \lambda p\left(\frac{(\beta+1)^{p-1}}{\alpha+1} + \frac{(\alpha+1)^{p-1}}{\beta+1}\right), \tag{58}$$

and then, we obtain

$$\begin{aligned} \lambda_1 &= 0, \\ \lambda_2 &= p\left(\frac{(\beta+1)^{p-1}}{\alpha+1} + \frac{(\alpha+1)^{p-1}}{\beta+1}\right). \end{aligned} \tag{59}$$

It follows by Corollary 1.3.1 of [14] that $(\overline{u},\overline{v}) = (\alpha+1,\beta+1)$ of (54) is locally stable if

$$p < \frac{(\beta+1)(\alpha+1)}{(\alpha+1)^p + (\beta+1)^p}. \tag{60}$$

Then, the proof is completed. □

2.3. Case 3: A Positive Bounded Sequence is a_n. We assume that $\{a_n\}$ is positive bounded with

$$\begin{aligned} \lim_{n\longrightarrow\infty} \inf a_n &= a \geq 0, \\ \lim_{n\longrightarrow\infty} \sup a_n &= b < \infty, \end{aligned} \tag{61}$$

for some real constants a and b.

Theorem 8. *$\{x_n\}_{n=-1}^{\infty}$ of (6) is bounded and persists if* $0 < p < 1$.

Proof. Its proof is same as proof of Theorem 2, and hence, it is omitted. □

Lemma 1. *Assume (61) is satisfied, and if*

$$\begin{aligned} \lambda &= \lim_{n\longrightarrow\infty} \inf x_n, \\ \eta &= \lim_{n\longrightarrow\infty} \sup x_n, \end{aligned} \tag{62}$$

then

$$\frac{ab-1}{b-1} \leq \lambda \leq \eta \leq \frac{ab-1}{a-1}. \tag{63}$$

Proof. Let $\varepsilon > 0$ for $n \geq N_0(\varepsilon)$, and we get

$$\begin{aligned} \lambda - \varepsilon &\leq x_n \leq \mu + \varepsilon, \\ a - \varepsilon &\leq a_n \leq b + \varepsilon. \end{aligned} \tag{64}$$

Therefore,

$$x_{n+1} \geq a - \varepsilon + \left(\frac{\lambda-\varepsilon}{\eta+\varepsilon}\right)^p. \tag{65}$$

Taking the $\lim_{n\longrightarrow\infty}\inf$ for (65), we obtain

$$\lambda \geq a - \varepsilon + \left(\frac{\lambda-\varepsilon}{\eta+\varepsilon}\right)^p. \tag{66}$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\lambda \geq a + \left(\frac{\lambda}{\eta}\right)^p. \tag{67}$$

Similarly,

$$\eta \leq b + \left(\frac{\eta}{\lambda}\right)^p. \tag{68}$$

We get from inequalities (67) and (68) that

$$\begin{aligned} \lambda\eta^p &\geq a\eta^p + \lambda^p, \\ \eta\lambda^p &\leq b\lambda^p + \eta^p. \end{aligned} \tag{69}$$

Since $0 < p < 1$ holds, we get

$$\lambda^{1-p} \leq \eta^{1-p}, \tag{70}$$

or equivalently

$$\lambda\eta^p \leq \eta\lambda^p. \tag{71}$$

It follows from equation (69) that

$$a\eta^p + \lambda^p \leq b\lambda^p + \eta^p. \tag{72}$$

So,

$$\eta^p(a-1) \leq \lambda^p(b-1), \tag{73}$$

and one has

$$\begin{aligned} \left(\frac{\eta}{\lambda}\right)^p &\leq \frac{b-1}{a-1}, \\ \left(\frac{\lambda}{\eta}\right)^p &\geq \frac{a-1}{b-1}. \end{aligned} \tag{74}$$

We have from (67), for all $n > N_0(\varepsilon)$,

$$\lambda \geq a + \left(\frac{\lambda}{\eta}\right)^p \geq a + \frac{a-1}{b-1} = \frac{ab-1}{b-1}. \tag{75}$$

Similarly, we obtain from (68) that

$$\eta \leq \frac{ab-1}{a-1}. \tag{76}$$

This completes the proof.

Now, we will explore attractively of solutions of equation (6).

Let $\{\overline{x}_n\}$ represent the arbitrary positive solution of (6). Now, one can find appropriate conditions such that $\{\overline{x}_n\}$ attracts all positive solutions of (6), that is,

$$\lim_{n\longrightarrow\infty}\frac{x_n}{\overline{x}_n}=1. \tag{77}$$

Now, define $\{y_n\}$:

$$y_n=\frac{x_n}{\overline{x}_n},\quad n=-1,0,1,\ldots, \tag{78}$$

and then, equation (6) becomes

$$y_{n+1}=\frac{a_n+(\overline{x}_n/\overline{x}_{n-1})^p(y_n/y_{n-1})^p}{a_n+(\overline{x}_n/\overline{x}_{n-1})^p}. \tag{79}$$

□

Lemma 2. *Let $\{\overline{x}_n\}$ be a positive solution of (6), and then,*

(i) $\overline{y}=1$ is the positive fixed point of (79).

(ii) If for some n and $y_{n-1}\geq y_n$, then $y_{n+1}<1$. Moreover, if for some n and $y_{n-1}<y_n$, then $y_{n+1}\geq 1$.

(iii) Every semicycle, except first one, of any oscillatory solution of (79) contains exactly one term.

Proof

(i) The proof of (i) is trivial.

(ii) If $y_{n-1}\geq y_n$, then $(y_n/y_{n-1})<1$ and

$$y_{n+1}=\frac{a_n+(\overline{x}_n/\overline{x}_{n-1})^p(y_n/y_{n-1})^p}{a_n+(\overline{x}_n/\overline{x}_{n-1})^p}<\frac{a_n+(\overline{x}_n/\overline{x}_{n-1})^p}{a_n+(\overline{x}_n/\overline{x}_{n-1})^p}=1. \tag{80}$$

In a similar way, the case is the same when $y_{n-1}<y_n$ is proven.

(iii) Let $\{y_n\}$ be an eventually oscillatory solution of (27) such that $y_{n-1}<1$ and $y_n\geq 1$. It follows from part (ii) that $y_{n+1}<1$. So, the positive semicycle has exactly one term. In similar way, one can prove for the negative semicycle. □

Lemma 3. *Every nonoscillatory solution of (79) converges to 1.*

Proof. Assuming $\{y_n\}$ be a nonoscillatory solution of (79). We may assume, without losing generality, that $y_n<1$, for $n\geq N_0$. Clearly, for $n\geq N_0$, one has $y_{n+1}<y_n$; otherwise, there exists $k>N_0$ such that $y_{k-1}\leq y_k$, and it follows by Lemma 1 (ii) that $y_{k+1}\geq 1$, that is, not possible. As $\{y_n\}$ is decreasing and $y_n<1$, it converges. Assume

$$\lim_{n\longrightarrow\infty}y_n=\tau, \tag{81}$$

where $0<\tau\leq 1$. We have to prove that $\tau=1$. Since

$$\lim_{n\longrightarrow\infty}\frac{y_n}{y_{n-1}}=1, \tag{82}$$

for $\varepsilon>0$ and n, the sufficiently large one has

$$\left|\left(\frac{y_n}{y_{n-1}}\right)^p-1\right|<\varepsilon. \tag{83}$$

Hence,

$$|y_{n+1}-1|=\left|\frac{(\overline{x}_n/\overline{x}_{n-1})^p}{a_n+(\overline{x}_n/\overline{x}_{n-1})^p}\right|\left|\left(\frac{y_n}{y_{n-1}}\right)^p-1\right|\leq\left|\left(\frac{y_n}{y_{n-1}}\right)^p-1\right|<\varepsilon. \tag{84}$$

So, we obtain $\lim_{n\longrightarrow\infty}y_n=1$. □

Lemma 4. *If $\{y_n\}$ is a positive solution of (79) and suppose that there exists $m\in\{1,2,\ldots,\}$, s.t.,*

$$\begin{aligned}y_{2m-1}&<1,\\ y_{2m}&\geq 1,\end{aligned} \tag{85}$$

then

$$\begin{aligned}y_{2n-1}&\geq 1,\\ y_n&<1,\quad n=m,m+1,\ldots,.\end{aligned} \tag{86}$$

Moreover, if

$$\begin{aligned}y_{2m-1}&\geq 1,\\ y_{2m}&<1,\end{aligned} \tag{87}$$

then

$$\begin{aligned}y_{2n-1}&<1,\\ y_n&\geq 1,\quad n=m,m+1,\ldots,.\end{aligned} \tag{88}$$

Proof. If $\{y_n\}$ be a solution of (79) such that (85) holds for $m\in\{1,2,\ldots,\}$, then we obtain

$$y_{2m-1}=\frac{a_n+(\overline{x}_n/\overline{x}_{n-1})^p(y_n/y_{n-1})^p}{a_n+(\overline{x}_n/\overline{x}_{n-1})^p}\geq\frac{a_n+(\overline{x}_n/\overline{x}_{n-1})^p}{a_n+(\overline{x}_n/\overline{x}_{n-1})^p}=1, \tag{89}$$

and by working inductively, one can prove that (86) is satisfied.

In similar way, one can prove that if (87) holds for $m\in\{1,2,\ldots,\}$, then (88) is satisfied. □

Theorem 9. *If $\{\overline{x}_n\}$ is a particular positive solution of (6) and $\{y_n\}$ is a positive solution of (79) and suppose that $0<p\leq(1/2)$ or $(1/2)<p<1$, $a>1$, and $a(a-1)>b-1$, then*

$$\lim_{n\longrightarrow\infty}y_n=1. \tag{90}$$

Proof. If $\{y_n\}$ is a solution of (79), then it is enough to prove that

$$\lim_{n\longrightarrow\infty}y_n=1. \tag{91}$$

Assuming there exists $m\in\{1,2,\ldots,\}$ such that (86) or (88) hold. We may also assume that (86) holds for $m\in\{1,2,\ldots,\}$, and $0<p\leq(1/2)$ holds. Let

$$\mu = \lim_{n\longrightarrow\infty} \inf y_n, \qquad \theta = \lim_{n\longrightarrow\infty} \sup y_n, \tag{92}$$

and also, let

$$k_1 = \lim_{n\longrightarrow\infty} \inf \overline{x}_n, \qquad k_2 = \lim_{n\longrightarrow\infty} \sup \overline{x}_n, \tag{93}$$

$$k = \frac{k_2}{k_1}. \tag{94}$$

Now, considering

$$F(x, y, z) = \frac{x + y^p z^p}{x + y^p}, \tag{95}$$

for $x, y,$ and $z > 0$, then one has

$$\frac{\partial F}{\partial x} = \frac{y^p (1 - z^p)}{(x + y^p)^2}, \qquad \frac{\partial F}{\partial y} = \frac{pxy^{p-1}(z^p - 1)}{(x + y^p)^2}. \tag{96}$$

Thus, it can be observed that

(i) F is nonincreasing in x and nondecreasing in y if $z > 1$

(ii) F is nonincreasing in y and nondecreasing in x if $z < 1$

Let $n \geq m$. Using (79), one has

$$y_{2n+1} = F\left(a_{2n}, \frac{\overline{x}_{2n}}{\overline{x}_{2n-1}}, \frac{y_{2n}}{y_{2n-1}}\right), \qquad y_{2n+2} = F\left(a_{2n+1}, \frac{\overline{x}_{2n+1}}{\overline{x}_{2n}}, \frac{y_{2n+1}}{y_{2n}}\right). \tag{97}$$

Since (85) holds, so by Lemma 4, one can obtain

$$\frac{y_{2n-1}}{y_{2n}} < 1, \qquad \frac{y_{2n}}{y_{2n-1}} \geq 1, \quad n \geq m. \tag{98}$$

Using (61) and (92)–(95) and the monotone properties of F, we get

$$\theta \leq F\left(a, k, \frac{\theta}{\mu}\right) = \frac{a + (\theta/\mu)^p k^p}{a + k^p}, \qquad \mu \geq F\left(a, k, \frac{\mu}{\theta}\right) = \frac{a + (\mu/\theta)^p k^p}{a + k^p}, \tag{99}$$

or

$$\theta\mu^p \leq \frac{a\mu^p + \theta^p k^p}{a + k^p}, \qquad \mu\theta^p \geq \frac{a\theta^p + \mu^p k^p}{a + k^p}. \tag{100}$$

Then,

$$a\theta^p \mu^{p-1} + \mu^{2p-1} k^p \leq \theta^p \mu^p \leq a\mu^p \theta^{p-1} + \theta^{2p-1} k^p. \tag{101}$$

Hence,

$$a\theta^p \mu^{p-1} + \mu^{2p-1} k^p \leq a\mu^p \theta^{p-1} + \theta^{2p-1} k^p. \tag{102}$$

So,

$$\theta^p \left(a\mu^{p-1} + \mu^{p-1}\left(\frac{\mu}{\theta}\right)^p k^p\right) \leq \mu^p \left(a\theta^{p-1} + \theta^{p-1}\left(\frac{\theta}{\mu}\right)^p k^p\right), \tag{103}$$

or

$$\left(\frac{\theta}{\mu}\right)^p \left(a\left(\frac{\mu}{\theta}\right)^{p-1} - k^p\right) \leq a - \left(\frac{\mu}{\theta}\right)^{p-1} k^p. \tag{104}$$

Thus,

$$a\frac{\theta}{\mu} - k^p \left(\frac{\theta}{\mu}\right)^p \leq a - \left(\frac{\theta}{\mu}\right)^{1-p} k^p, \qquad a\left(\frac{\theta}{\mu} - 1\right) \leq k^p \left(\left(\frac{\theta}{\mu}\right)^p - \left(\frac{\theta}{\mu}\right)^{1-p}\right). \tag{105}$$

But, from $0 < p \leq (1/2)$, one can obtain $p \leq 1 - p$. This implies that

$$a\left(\frac{\theta}{\mu} - 1\right) \leq 0, \tag{106}$$

or

$$\theta \leq \mu. \tag{107}$$

Thus, we get that $\theta = \mu$. The proof is completed.

Now, suppose that $(1/2) < p < 1$, $a > 1$, and $a(a - 1) > b - 1$ hold. Then, using relations (61) and (92)–(95) and $(\eta/\lambda)^p \leq ((b - 1)/(a - 1))$, $(\lambda/\eta)^p \geq ((a - 1)/(b - 1))$ holds; we obtain

$$\theta \leq F\left(a, \frac{\eta}{\lambda}, \frac{\theta}{\mu}\right) = \frac{a + (\eta/\lambda)^p (\theta/\mu)^p}{a + (\eta/\lambda)^p} \leq \frac{a + ((b - 1)/(a - 1))(\theta/\mu)^p}{a + ((b - 1)/(a - 1))},$$
$$\mu \geq F\left(a, \frac{\eta}{\lambda}, \frac{\mu}{\theta}\right) = \frac{a + (\eta/\lambda)^p (\mu/\theta)^p}{a + (\eta/\lambda)^p} \geq \frac{a + ((b - 1)/(a - 1))(\mu/\theta)^p}{a + ((b - 1)/(a - 1))}, \tag{108}$$

or

$$\mu^p\theta \le \frac{a\mu^p}{a+((b-1)/(a-1))} + \frac{((b-1)/(a-1))\theta^p}{a+((b-1)/(a-1))}, \quad (109)$$

$$\mu\theta^p \ge \frac{a\theta^p}{a+((b-1)/(a-1))} + \frac{((b-1)/(a-1))\mu^p}{a+((b-1)/(a-1))}. \quad (110)$$

Since $\mu \le \theta$, it follows that $\mu\theta^p \le \theta\mu^p$. Therefore, from equation (109), we get

$$\frac{a\theta^p}{a+((b-1)/(a-1))} + \frac{((b-1)/(a-1))\mu^p}{a+((b-1)/(a-1))} \le \frac{a\mu^p}{a+((b-1)/(a-1))} + \frac{((b-1)/(a-1))\theta^p}{a+((b-1)/(a-1))}, \quad (111)$$

or

$$\frac{a\theta^p}{a+((b-1)/(a-1))} - \frac{((b-1)/(a-1))\theta^p}{a+((b-1)/(a-1))} \le \frac{a\mu^p}{a+((b-1)/(a-1))} - \frac{((b-1)/(a-1))\mu^p}{a+((b-1)/(a-1))}. \quad (112)$$

Since $(1/2) < p < 1$, $a > 1$, and $a(a-1) > b-1$ hold, then from (112), we have $\theta \le \mu$; so, $\theta = \mu$. Then, the proof is completed. ☐

Theorem 10. *Assume that $0 < p < 1$, and $\{a_n\}$ is a periodic sequence, s.t., $a_{n+2} = a_n$ for all $n = 0, 1, \ldots$, and then, (6) has a periodic solution of the prime period two.*

Proof. For (6) possesses a periodic solution $\{x_n\}$ having prime period two, one can find positive numbers x_0 and x_{-1}, s.t.,

$$x_{-1} = x_1 = a_0 + \left(\frac{x_0}{x_{-1}}\right)^p, \qquad x_0 = x_2 = a_1 + \left(\frac{x_1}{x_0}\right)^p. \quad (113)$$

Or, equivalently,

$$x_{-1} = a_0 + \left(\frac{x_0}{x_{-1}}\right)^p, \qquad x_0 = a_1 + \left(\frac{x_{-1}}{x_0}\right)^p. \quad (114)$$

Now, one has to prove that system (114) is consistent. From (114), one gets

$$(x_{-1} - a_0)(x_0 - a_1) = 1, \quad (115)$$

and from this, it follows that

$$(x_{-1} - a_0)^{p+1} = \frac{(a_1(x_{-1} - a_0) + 1)^p}{x_{-1}^p}, \quad (116)$$

$$(x_0 - a_1)^{p+1} = \frac{(a_0(x_0 - a_1) + 1)^p}{x_0^p}. \quad (117)$$

Define

$$F(x) = (x - a_0)^{p+1} - \frac{(a_1(x - a_0) + 1)^p}{x^p}, \quad x > a_0. \quad (118)$$

Then,

$$F(a_0) = -\frac{1}{a_0} < 0, \qquad F(a_0 + 1) = -\frac{(a_1 + 1)^p}{(a_0 + 1)^p} + 1 > 0, \quad (119)$$

where $a_1 < a_0$. So, F has a zero, say $x_{-1} \in (a_0, a_0 + 1)$, and in view of equation (115) and (116), one can get that (6) has a two-periodic solution. Now, assuming that $a_1 > a_0$, one can define

$$H(x) = -\frac{(a_0(x - a_1) + 1)^p}{x^p} + (x - a_1)^{p+1}, \quad x > a_1. \quad (120)$$

Then,

$$H(a_1) = -\frac{1}{a_1} < 0, \qquad H(a_1 + 1) = -\frac{(a_0 + 1)^p}{(a_1 + 1)^p} + 1 > 0. \quad (121)$$

Thus, H has a zero, say $x_0 \in (a_1, a_1 + 1)$, and in view of equations (115) and (116), one can get that (6) has a two-periodic solution. ☐

3. Numerical Simulation

In this section, we will provide some simulation in order to verify obtained theoretical results, for these following cases are to be considered for the completeness of this section ():

Case 1: If $a = 2$ and $p = 1.4 < a + 1 = 3$, then Figure 1 implies that fixed point $\overline{x} = 3$ of equation (8) is a sink. This simulation agrees with the conclusion of Theorem 1.

Case 2: If $a = 2$ and $p = 2.1 < a + 1 = 3$, then Figure 2 implies that fixed point $\overline{x} = 3$ of equation (8) is a sink. This simulation again agrees with the conclusion of Theorem 1.

Case 3: If $\alpha = 0.6, \beta = 0.103$, and $p = 0.4$, then Figure 3 implies that fixed point $(\overline{u}, \overline{v}) = (1.6, 1.103)$ of system (54) is a sink. Additionally, if $\alpha = 0.6, \beta = 0.103$, and $p = 0.4$, then $p = 0.4 < (((\beta + 1)(\alpha + 1))/((\beta + 1)^p + (\alpha + 1)^p)) = 0.78$

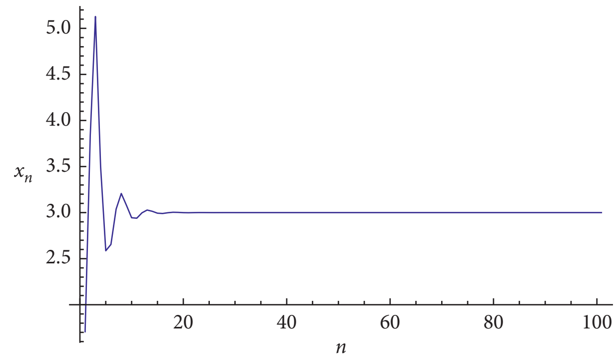


Figure 1: Phase portrait of equation (8) with $x_{-1} = 1.1$ and $x_0 = 1.7$.

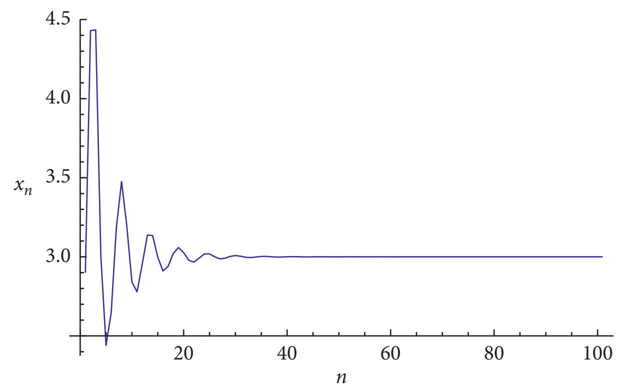
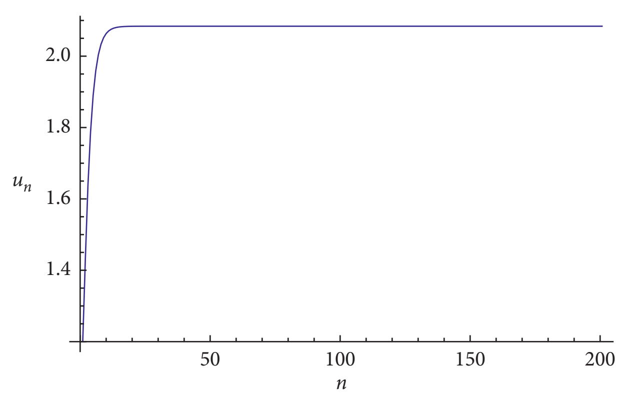
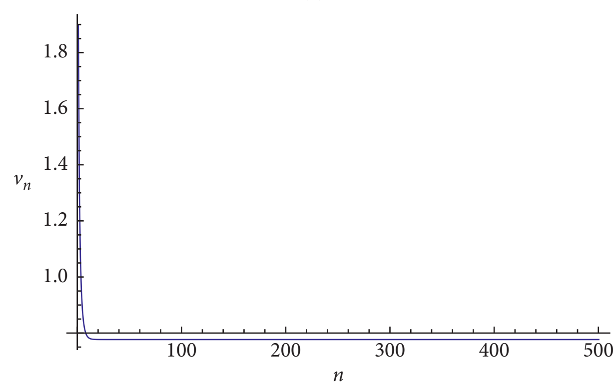


Figure 2: Phase portrait of equation (8) with $x_{-1} = 1.9$ and $x_0 = 2.9$.



(a)



(b)

Figure 3: Phase portrait of system (54) with $(u_0, v_0) = (1.2, 1.9)$.

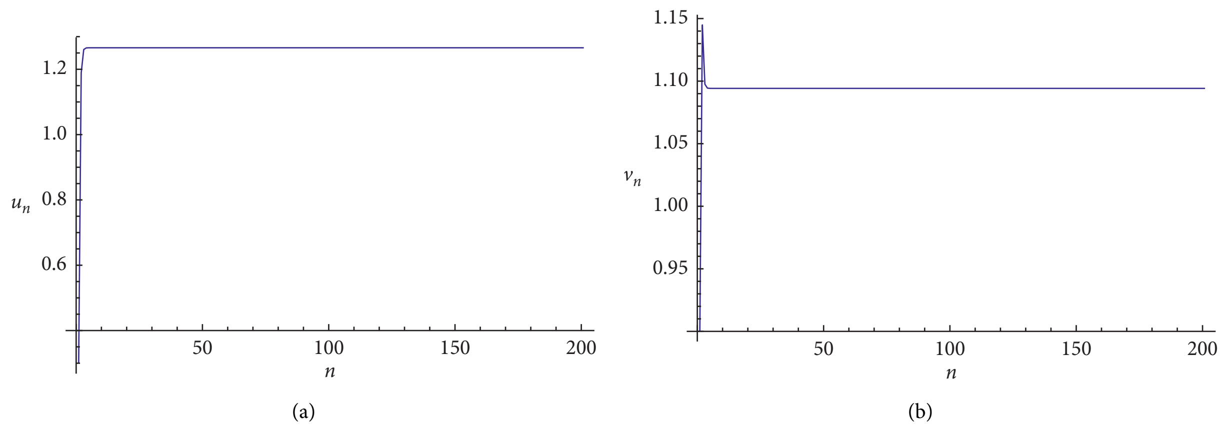


Figure 4: Phase portrait of system (54) with $(u_0, v_0) = (0.3, 1.9)$.

54629670125335, and hence, this simulation agrees with the conclusion of Theorem 7.

Case 4: If $\alpha = 0.26, \beta = 0.1$, and $p = 0.04$, then Figure 4 implies that fixed point $(\overline{u}, \overline{v}) = (1.26, 1.1)$ of system (54) is a sink. Additionally, if $\alpha = 0.26, \beta = 0.1$, and $p = 0.04$, then $p = 0.04 < (((\beta + 1)(\alpha + 1))/((\beta + 1)^p + (\alpha + 1)^p)) =$ 0.6884879889459452, and hence, this simulation again agrees with the conclusion of Theorem 7.

Data Availability

The data used to support the findings of the study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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