Hindawi
Discrete Dynamics in Nature and Society
Volume 2021, Article ID 1535920, 14 pages
https://doi.org/10.1155/2021/1535920

Research Article

Further Results on Bifurcation for a Fractional-Order Predator-Prey System concerning Mixed Time Delays

Zhenjiang Yao[1] and Bingnan Tang[2]

[1]*College of Economics and Management, Nanjing University of Aeronautics and Astronautics, Nanjing 211100, China*
[2]*Business School, Jiangsu University of Technology, Changzhou 213001, China*

Correspondence should be addressed to Bingnan Tang; regales1988@sina.com

Received 22 April 2021; Revised 11 May 2021; Accepted 10 June 2021; Published 18 June 2021

Academic Editor: Abdul Qadeer Khan

In the present work, we mainly focus on a new established fractional-order predator-prey system concerning both types of time delays. Exploiting an advisable change of variable, we set up an isovalent fractional-order predator-prey model concerning a single delay. Taking advantage of the stability criterion and bifurcation theory of fractional-order dynamical system and regarding time delay as bifurcation parameter, we establish a new delay-independent stability and bifurcation criterion for the involved fractional-order predator-prey system. The numerical simulation figures and bifurcation plots successfully support the correctness of the established key conclusions.

1. Introduction

Setting up mathematical models to describe the natural phenomena has become an important topic in real life. The interaction of predator population and prey population plays a significant role in maintaining ecological balance in nature. In order to grasp the change law of predator population and prey population, a large number of predator-prey models have been established and many fruits on dynamical properties of various predator-prey models have been reported. Usually, time delay often exists in biological systems due to the lag of the response of different predators and preys. In many situations, time delay will lead to the loss of stability, periodic oscillation, bifurcation, and chaotic behavior of predator-prey models. Thus, the study on the impact of time delay on dynamical nature of predator-prey models has attracted great interest of many scholars in the fields of biology and mathematics. For a long time in the past, lots of valuable works on predator-prey models have been published. For instance, Dubey et al. [1] investigated the stability behavior, Hopf bifurcation, and chaos of delayed predator-prey system. Ren and Shi [2] dealt with the global boundedness and stability of solutions of a predator-prey system with time delay. Li and Guo [3] introduced a new way to study the permanence and extinction for a stochastic prey-predator system involving functional response. Alsakaji et al. [4] made a detailed discussion on permanence, local and global stabilities, Hopf bifurcation, and a predator-prey model with time delay. For more publications about this topic, one can see [5–8].

Here we notice that the works of [1–8] are concerned with the integer-order predator-prey models. In recent years, fractional-order dynamical systems have found potential application in numerous areas such as all sorts of physical waves, neural network systems, biological technique, finance, automatic control, and so on [9–11]. A lot of researchers think that fractional-order dynamical system can more accurately describe the real phenomenon in realistic world than the classical integer-order ones due to its owned memory trait and hereditary nature [12]. Nowadays a great deal of valuable works on fractional-order dynamical systems have been published (see [13–22]). In particular, the study on fractional-order predator-prey systems is also continuously displayed. For example, Yousef and Chandan Maji [23] revealed the effect of fear for a fractional-order predator-prey model. Xie et al. [24] proved the non-negative and boundedness of a fractional-order predator-prey model and established some conditions to ensure the existence and stability of the positive equilibrium point of the fractional-order predator-prey model. In 2019, Zhou et al. [25] considered the bifurcation control issue for a fractional-order predator-prey system involving delays. For more details, one can see [26–28].

Hopf bifurcation caused by time delay is a vital dynamical phenomenon in predator-prey systems. Up to now, plenty of publications on Hopf bifurcation of integer-order predator-prey models have been available. The impact of time delay on Hopf bifurcation has been revealed. However, the investigation on Hopf bifurcation for fractional-order predator-prey models is comparatively few. Recently, some scholars are devoted to Hopf bifurcation of fractional-order predator-prey models and some valuable fruits have been derived. For instance, Alidousti [29] investigated the stability and Hopf bifurcation problem of a fractional predator-prey system. Yuan et al. [30] established a set of sufficient conditions to ensure the stability and the onset of Hopf bifurcation for a fractional-order predator-prey model. Wang et al. [31] discussed the stability and bifurcation for a generalized fractional-order predator-prey system involving time delay and interspecific competition. Huang et al. [32] applied a new technique to control Hopf bifurcation of a fractional predator-prey system involving delays. In 2019, Xu et al. [42] did a very valuable work on Hopf bifurcation for delayed neural networks. As to more works about this theme, we refer the readers to [33–36].

Up to now, the investigation on Hopf bifurcation of fractional-order delayed predator-prey systems merely involves discrete time delay. To reflect the time lag of response of predator population and prey population during the course of interaction of predator and prey in biological systems, it is very essential to introduce the distributed time delay into predator-prey models. Now there are only very few works on Hopf bifurcation of predator-prey system involving distributed time delay. Thus, a natural problem arises: what is the impact of distributed time delay on Hopf bifurcation of predator-prey system involving distributed time delay? This motives us to deal with the Hopf bifurcation for predator-prey system involving distributed time delay.

In 2020, Rahman et al. [37] investigated the following predator-prey system concerning both types of delays:

$$\begin{cases} \dfrac{\mathrm{d}w_1(t)}{\mathrm{d}t} = w_1(t)\left[a_1 - \alpha_{11}\displaystyle\int_{-\infty}^{t} U(t-\nu)w_1(\nu)\mathrm{d}\nu - \alpha_{12}w_2(t-\varsigma)\right], \\ \dfrac{\mathrm{d}w_2(t)}{\mathrm{d}t} = w_2(t)\left[-a_2 + \alpha_{21}w_1(t-\varsigma) - \alpha_{22}\displaystyle\int_{-\infty}^{t} U(t-\nu)w_2(\nu)\mathrm{d}\nu\right], \end{cases} \tag{1}$$

where $w_1(t)$ denotes the population density of prey at time t and $w_2(t)$ stands for the population density of predator at time t, $a_1 > 0$ stands for the growth rate of the prey populations without predators, $a_2 > 0$ stands for the death rate of the predator populations without prey, α_{11} represents the self-regulation rate for the prey, α_{12} represents the rate of predation of the prey by predators, α_{21} represents the conversion rate of predators and α_{22} represents the intraspecific competition among predators, $U(.)$ denotes the non-negative continuous delay kernel which is defined on $[0,\infty)$ and is integrable on $[0,\infty)$, and $\varsigma \geq 0$ denotes the feedback time delay between the predator and the prey. For details, see [37].

Usually, the kernel function owns the following two forms:

(i) $\int_{-\infty}^{t} U(\nu)\mathrm{d}\nu = 1$, $U(\nu) = \delta e^{-\delta(t-\nu)}$, $\delta > 0$.
(ii) $\int_{-\infty}^{t} U(t-\nu)\mathrm{d}\nu = 1$, $U(\nu) = \delta e^{-\delta\nu}$, $\delta > 0$.

Rahman et al. [37] chose kernel function as case (ii). By means of stability criterion and Hopf bifurcation theory of delayed differential equation, Rahman et al. [37] established a sufficient criterion ensuring the stability and the appearance of Hopf bifurcation of model (1). Meanwhile, the concrete formula determining bifurcation peculiarities is presented by virtue of center manifold theory and normal form theorem.

Inspired by the analysis above, we are to analyze the stability and Hopf bifurcation for fractional-order predator-prey model involving discrete time delay and distributed time delay. On the basis of the research of Rahman et al. [37], in this work, we revise model (1) as the fractional-order form:

$$\begin{cases} \dfrac{\mathrm{d}w_1^{\varrho}(t)}{\mathrm{d}t^{\varrho}} = w_1(t)\left[a_1 - \alpha_{11}\displaystyle\int_{-\infty}^{t} U(t-\nu)w_1(\nu)\mathrm{d}\nu - \alpha_{12}w_2(t-\varsigma)\right], \\ \dfrac{\mathrm{d}w_2^{\varrho}(t)}{\mathrm{d}t^{\varrho}} = w_2(t)\left[-a_2 + \alpha_{21}w_1(t-\varsigma) - \alpha_{22}\displaystyle\int_{-\infty}^{t} U(t-\nu)w_2(\nu)\mathrm{d}\nu\right], \end{cases} \tag{2}$$

where $0 < \rho < 1$ is a constant, $w_1(t)$ denotes the population density of prey at time t and $w_2(t)$ stands for the population density of predator at time t, $a_1 > 0$ stands for the growth rate of the prey populations without predators, $a_2 > 0$ stands for the death rate of the predator populations without prey, α_{11} represents the self-regulation rate for the prey, α_{12} represents the rate of predation of the prey by predators, α_{21} represents the conversion rate of predators and α_{22} represents the intraspecific competition among predators, $U(.)$ denotes the non-negative continuous delay kernel which is defined on $[0,\infty)$ and is integrable on $[0,\infty)$, and $\varsigma \geq 0$ denotes the feedback time delay between the predator and the prey. For more implication of the parameters in system (2), one can see [37]. In this research, we choose the kernel function $U(.)$ as (ii).

This article is organized as follows. Section 2 lists several necessary theories about fractional-order dynamical system. Section 3 gives the bifurcation condition for model (2) involving kernel function (ii). Section 4 presents simulation plots to support the validity of the obtained key conclusions. Section 5 ends this article.

2. Basic Principle on Fractional-Order Dynamical System

In this section, we present some indispensable basic knowledge about fractional-order dynamical system.

Definition 1 (see [38]). The Caputo-type fractional-order derivative is given by

$$\mathscr{D}^{\varrho}w(\tau)=\frac{1}{\Gamma(k-\varrho)}\int_{\tau_0}^{\tau}\frac{w^{(k)}(u)}{(\tau-u)^{\varrho-k+1}}\mathrm{d}u, \tag{3}$$

where $w(\tau)\in([\tau_0,\infty),R),\Gamma(u)=\int_0^{\infty}\tau^{u-1}e^{-\tau}\mathrm{d}\tau,\ \tau\geq\tau_0,\ k\in Z^+,k-1\leq\varrho<k$.

Lemma 1 (see [39, 40]). *For the fractional-order model*

$$\frac{\mathrm{d}^{\varrho}v(t)}{\mathrm{d}t^{\varrho}}=w(t,v(t)),\quad v(0)=v_0, \tag{4}$$

where $0<\varrho\leq 1$ and $w(t,v(t))$: $R^+\times R^n\longrightarrow R^n$. Let v_0 be the equilibrium point of (4). We say that v_0 is locally asymptotically stable provided that each eigenvalue λ of $(\partial w(t,v)/\partial v)|_{v=v_0}$ obeys $|\arg(\lambda)|>(\varrho\pi/2)$.

Lemma 2 (see [41]). *For the fractional-order model*

$$\begin{cases}\dfrac{\mathrm{d}^{\varrho_1}\mathscr{V}_1(t)}{\mathrm{d}t^{\varrho_1}}=f_{11}\mathscr{V}_1(t-\varsigma_{11})+f_{12}\mathscr{V}_2(t-\varsigma_{12})+\cdots+f_{1l}\mathscr{V}_l(t-\varsigma_{1l}),\\ \dfrac{\mathrm{d}^{\varrho_2}\mathscr{V}_2(t)}{\mathrm{d}t^{\varrho_2}}=f_{21}\mathscr{V}_1(t-\varsigma_{21})+f_{22}\mathscr{V}_2(t-\varsigma_{22})+\cdots+f_{2l}\mathscr{V}_l(t-\varsigma_{2l}),\\ \vdots\\ \dfrac{\mathrm{d}^{\varrho_l}\mathscr{V}_l(t)}{\mathrm{d}t^{\varrho_l}}=f_{l1}\mathscr{V}_1(t-\varsigma_{l1})+f_{l2}\mathscr{V}_2(t-\varsigma_{l2})+\cdots+f_{ll}\mathscr{V}_l(t-\varsigma_{ll}),\end{cases} \tag{5}$$

where $\varrho_j\in(0,1)$ $(j=1,2,\ldots,l)$, denote

$$\Delta(s)=\begin{bmatrix}s^{\varrho_1}-f_{11}e^{-s\varsigma_{11}} & -f_{12}e^{-s\varsigma_{12}} & \cdots & -f_{1l}e^{-s\varsigma_{1l}}\\ -f_{21}e^{-s\varsigma_{12}} & s^{>_2}-f_{22}e^{-s\varsigma_{22}} & \cdots & -f_{2l}e^{-s\varsigma_{2l}}\\ \vdots & \vdots & \ddots & \vdots\\ -f_{l1}e^{-s\varsigma_{l1}} & -f_{l2}e^{-s\varsigma_{l2}} & \cdots & s^{>_l}-f_{ll}e^{-s\varsigma_{ll}}\end{bmatrix}. \tag{6}$$

We say that the zero solution of model (5) is asymptotically stable provided that $\det(\Delta(s))=0$ possesses the roots with negative real parts.

For the fractional-order model

$$\begin{cases}\dfrac{\mathrm{d}^{\varrho_1}\mathscr{V}_1(t)}{\mathrm{d}t^{\varrho_1}}=f_{11}\mathscr{V}_1(t)+f_{12}\mathscr{V}_2(t)+\cdots+f_{1l}\mathscr{V}_l(t),\\ \dfrac{\mathrm{d}^{\varrho_2}\mathscr{V}_2(t)}{\mathrm{d}t^{\varrho_2}}=f_{21}\mathscr{V}_1(t)+f_{22}\mathscr{V}_2(t)+\cdots+f_{2l}\mathscr{V}_l(t),\\ \vdots\\ \dfrac{\mathrm{d}^{\varrho_l}\mathscr{V}_l(t)}{\mathrm{d}t^{\varrho_l}}=f_{l1}\mathscr{V}_1(t)+f_{l2}\mathscr{V}_2(t)+\cdots+f_{ll}\mathscr{V}_l(t),\end{cases} \tag{7}$$

where $0<\varrho_j\leq 1$ $(j=1,2,\ldots,l)$, the characteristic equation of model (7) owns the following expression:

$$\det\begin{bmatrix}s^{\varrho_1}-f_{11} & -f_{12} & \cdots & -f_{1l}\\ -f_{21} & s^{\varrho_2}-f_{22} & \cdots & -f_{2l}\\ \vdots & \vdots & \ddots & \vdots\\ -f_{l1} & -f_{l2} & \cdots & s^{\varrho_l}-f_{ll}\end{bmatrix}=0. \tag{8}$$

Assume that $\phi_h=(\epsilon_h/\varepsilon_h),\epsilon_h,\varepsilon_h\in Z^+$, $(\epsilon_h,\varepsilon_h)=1$ and let ε be the lowest common multiple of β_h of ψ_h, $h=1,2,\ldots,l$.

Lemma 3 (see [41]). *Assume that each root λs of the following equation:*

$$\det\begin{bmatrix}\lambda^{\varepsilon\phi_1}-f_{11} & -f_{12} & \cdots & -f_{1l}\\ -f_{21} & \lambda^{\varepsilon\phi_2}-f_{22} & \cdots & -q_{2l}\\ \vdots & \vdots & \ddots & \vdots\\ -f_{l1} & -f_{l2} & \cdots & \lambda^{\varepsilon\phi_l}-f_{ll}\end{bmatrix}=0, \tag{9}$$

conforms to $|\arg(\lambda)|>(\pi/2\varepsilon)$; then, the zero solution to model (7) is locally asymptotically stable.

3. Bifurcation Exploration for Predator-Prey Model (2)

In this section, we are to study the stability property and the appearance of Hopf bifurcation of predator-prey model (2). Set

$$\begin{cases} w_3(t) = \int_{-\infty}^{t} U(t-v) w_1(v) \mathrm{d}v \\ = \int_{-\infty}^{t} \delta e^{-\delta(t-v)} w_1(v) \mathrm{d}v, \\ w_4(t) = \int_{-\infty}^{t} U(t-s) w_2(s) \mathrm{d}s \\ = \int_{-\infty}^{t} \delta e^{-\delta(t-v)} w_2(v) \mathrm{d}v, \end{cases} \quad (10)$$

and then

$$\begin{aligned} \frac{\mathrm{d}w_3(t)}{\mathrm{d}t} &= \left[\int_{-\infty}^{t} \delta e^{-\delta(t-v)} w_1(v) \mathrm{d}v \right]' \\ &= -\delta w_3(t) + \delta w_1(t), \\ \frac{\mathrm{d}w_4(t)}{\mathrm{d}t} &= \left[\int_{-\infty}^{t} \delta e^{-\delta(t-v)} w_2(v) \mathrm{d}v \right]' \\ &= -\delta w_4(t) + \delta w_2(t). \end{aligned} \quad (11)$$

Thus, system (2) becomes the following equivalent form:

$$\begin{cases} \frac{\mathrm{d}w_1^{\varrho}(t)}{\mathrm{d}t^{\varrho}} = w_1(t)\left[a_1 - \alpha_{11} w_3(t) - \alpha_{12} w_2(t-\varsigma)\right], \\ \frac{\mathrm{d}w_2^{\varrho}(t)}{\mathrm{d}t^{\varrho}} = w_2(t)\left[-a_2 + \alpha_{21} w_1(t-\varsigma) - \alpha_{22} w_4(t)\right], \\ \frac{\mathrm{d}w_3(t)}{\mathrm{d}t} = -\delta w_3(t) + \delta w_1(t), \\ \frac{\mathrm{d}w_4(t)}{\mathrm{d}t} = -\delta w_4(t) + \delta w_2(t). \end{cases} \quad (12)$$

Assume that

$$(\mathscr{H}1) \quad a_1 \alpha_{21} > a_2 \alpha_{11}. \quad (13)$$

It is easy to obtain that system (12) owns the equilibrium points $\mathscr{E}_1^0(0,0,0,0)$, $\mathscr{E}_2^0((a_1/\alpha_{11}), 0, (a_1/\alpha_{11}), 0)$, and $\mathscr{E}_3^0(w_1^0, w_2^0, w_3^0, w_4^0)$, where

$$\begin{cases} w_1^* = \frac{a_1\alpha_{22} + a_2\alpha_{12}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}, \\ w_2^* = \frac{a_1\alpha_{21} - a_2\alpha_{11}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}, \\ w_3^* = \frac{a_1\alpha_{22} + a_2\alpha_{12}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}, \\ w_4^* = \frac{a_1\alpha_{21} - a_2\alpha_{11}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}. \end{cases} \quad (14)$$

If $(\mathscr{H}1)$ holds, then the equilibrium point $\mathscr{E}_3^0(w_1^0, w_2^0, w_3^0, w_4^0)$ is a positive equilibrium point. Considering the biological implication of predator-prey model (2), we only deal with the positive equilibrium point $\mathscr{E}_3^0(w_1^0, w_2^0, w_3^0, w_4^0)$. The linear system of equation (12) around $\mathscr{E}_3^0(w_1^0, w_2^0, w_3^0, w_4^0)$ is

$$\begin{cases} \frac{\mathrm{d}w_1^{\varrho}(t)}{\mathrm{d}t^{\varrho}} = b_1 w_1(t) + b_2 w_2(t-\varsigma) + b_3 w_3(t), \\ \frac{\mathrm{d}w_2^{\varrho}(t)}{\mathrm{d}t^{\varrho}} = c_1 w_1(t-\varsigma) + c_2 w_2(t) + c_3 w_4(t), \\ \frac{\mathrm{d}w_3(t)}{\mathrm{d}t} = -\delta w_3(t) + \delta w_1(t), \\ \frac{\mathrm{d}w_4(t)}{\mathrm{d}t} = -\delta w_4(t) + \delta w_2(t), \end{cases} \quad (15)$$

where

$$\begin{cases} b_1 = a_1 - \alpha_{11} w_3^0 - \alpha_{12} w_2^0, \\ b_2 = -\alpha_{12} w_1^0, \\ b_3 = -\alpha_{11} w_1^0, \\ c_1 = \alpha_{21} w_2^0, \\ c_2 = \alpha_{21} w_1^0 - a_2 - \alpha_{22} w_4^0, \\ c_3 = \alpha_{22} w_2^0. \end{cases} \quad (16)$$

The characteristic equation of (15) takes the following form:

$$\det\begin{bmatrix} s^{\varrho}-b_1 & -b_2e^{-s\varsigma} & -b_3 & 0 \\ -c_1e^{-s\varsigma} & s^{\varrho}-c_2 & 0 & -c_3 \\ -\delta & 0 & s+\delta & 0 \\ 0 & -\delta & 0 & s+\delta \end{bmatrix}=0. \quad (17)$$

Set $\varrho=\epsilon/\varepsilon$, where $\epsilon,\varepsilon\in Z^{+}$ and $(\epsilon,\varepsilon)=1$. Let $\lambda=s^{1/\varepsilon}$. When $\varsigma=0$, then equation (17) becomes

$$\det\begin{bmatrix} s^{\varrho}-b_1 & -b_2 & -b_3 & 0 \\ -c_1 & s^{\varrho}-c_2 & 0 & -c_3 \\ -\delta & 0 & s+\delta & 0 \\ 0 & -\delta & 0 & s+\delta \end{bmatrix}=0. \quad (18)$$

Lemma 4. *Assume that $\varsigma=0$ and all the roots λ of equation (18) obey $|\arg(\lambda)|>(\pi/2\varepsilon)$; then, the positive equilibrium point $\mathscr{E}_3^0(w_1^0,w_2^0,w_3^0,w_4^0)$ of model (12) is locally asymptotically stable.*

Proof. Clearly, when $\varsigma=0$, then characteristic equation (17) becomes equation (18). By virtue of Lemma 3, one can easily obtain that Lemma 4 holds.

By virtue of equation (17), one obtains

$$\begin{aligned} &s^{2\varrho+2}+\sigma_1s^{2\varrho+1}+\sigma_2s^{2\varrho}+\sigma_3s^{\varrho+2}+\sigma_4s^{\varrho+1}+\sigma_5s^{\varrho} \\ &+\sigma_6s^2+\sigma_7s+\sigma_8+\left(\sigma_9s^2+\sigma_{10}s+\sigma_{11}\right)e^{-2s\varsigma}=0, \end{aligned} \quad (19)$$

where

$$\begin{cases} \sigma_1=2\delta, \\ \sigma_2=\delta^2, \\ \sigma_3=-(b_1+c_2), \\ \sigma_4=-c_3\delta-2\delta(b_1+c_2)-b_3\delta, \\ \sigma_5=-c_3\delta^2-(b_1+c_2)\delta^2-b_3\delta^2, \\ \sigma_6=b_1c_2, \\ \sigma_7=\delta(b_1c_3+2b_1c_2+b_3c_2), \\ \sigma_8=\delta^2(b_3c_3+b_1c_3+b_3c_2+b_1c_2), \\ \sigma_9=-b_2c_1, \\ \sigma_{10}=-2\delta^2b_2c_1, \\ \sigma_{11}=-b_2c_1\delta^2. \end{cases} \quad (20)$$

Assume that $s=i\chi=\chi(\cos(\pi/2)+i\sin(\pi/2))$ is the root of equation (19); then, one gets

$$\begin{aligned} &\left[\chi^{2\varrho+2}\left(\cos\frac{(2\varrho+2)\pi}{2}+i\sin\frac{(2\varrho+2)\pi}{2}\right)+\sigma_1\chi^{2\varrho+1}\left(\cos\frac{(2\varrho+1)\pi}{2}+i\sin\frac{(2\varrho+1)\pi}{2}\right)\right. \\ &+\sigma_2\chi^{2\varrho}(\cos\varrho\pi+i\sin\varrho\pi)+\sigma_3\chi^{\varrho+2}\left(\cos\frac{(\varrho+2)\pi}{2}+i\sin\frac{(\varrho+2)\pi}{2}\right)+\sigma_4\chi^{\varrho+1}\left(\cos\frac{(\varrho+1)\pi}{2}+i\sin\frac{(\varrho+1)\pi}{2}\right) \\ &+\sigma_5\sigma^{\varrho}\left(\cos\frac{\varrho\pi}{2}+i\sin\frac{\varrho\pi}{2}\right)-\sigma_6\chi^2+i\sigma_7\chi+\sigma_8+\left(-\sigma_9\chi^2+i\sigma_{10}\chi+\sigma_{11}\right)(\cos2\chi\varsigma-i\sin2\chi\varsigma)=0. \end{aligned} \quad (21)$$

By means of equation (21), we get

$$\begin{cases} \mathscr{A}_1 \cos 2\,\chi\varsigma + \mathscr{A}_2 \sin 2\,\chi\varsigma = \mathscr{A}_3, \\ \mathscr{A}_2 \cos 2\,\chi\varsigma - \mathscr{A}_1 \sin 2\,\chi\varsigma = \mathscr{A}_4, \end{cases} \tag{22}$$

where

$$\begin{cases} \mathscr{A}_1 = \sigma_{11} - \sigma_9 \chi^2, \\ \mathscr{A}_2 = \sigma_{10}\chi, \\ \mathscr{A}_3 = -\chi^{2\varrho+2} \cos\dfrac{(2\varrho+2)\pi}{2} - \sigma_1 \chi^{2\varrho+1} \cos\dfrac{(2\varrho+1)\pi}{2} - \sigma_2 \chi^{2\varrho} \cos \varrho\pi \\ -\sigma_3 \chi^{\varrho+2} \cos\dfrac{(\varrho+2)\pi}{2} - \sigma_4 \chi^{\varrho+1} \cos\dfrac{(\varrho+1)\pi}{2} - \sigma_5 \sigma^{\varrho} \cos\dfrac{\varrho\pi}{2} \\ +\sigma_6 \chi^2 - \sigma_8, \\ \mathscr{A}_4 = -\chi^{2\varrho+2} \sin\dfrac{(2\varrho+2)\pi}{2} - \sigma_1 \chi^{2\varrho+1} \sin\dfrac{(2\varrho+1)\pi}{2} - \sigma_2 \chi^{2\varrho} \sin \varrho\pi \\ -\sigma_3 \chi^{\varrho+2} \sin\dfrac{(\varrho+2)\pi}{2} - \sigma_4 \chi^{\varrho+1} \sin\dfrac{(\varrho+1)\pi}{2} - \sigma_5 \sigma^{\varrho} \sin\dfrac{\varrho\pi}{2} \\ -\sigma_7 \chi. \end{cases} \tag{23}$$

It follows from (22) that

$$\cos 2\,\chi\varsigma = \frac{\mathscr{A}_1\mathscr{A}_3 + \mathscr{A}_2\mathscr{A}_4}{\mathscr{A}_1^2 + \mathscr{A}_2^2}, \tag{24}$$

$$\mathscr{A}_1^2 + \mathscr{A}_2^2 = \mathscr{A}_3^2 + \mathscr{A}_4^2. \tag{25}$$

In equation (23), let

$$\begin{cases} \gamma_1 = -\cos\dfrac{(2\varrho+2)\pi}{2}, \\ \gamma_2 = -\sigma_1 \cos\dfrac{(2\varrho+1)\pi}{2}, \\ \gamma_3 = -\sigma_2 \cos \varrho\pi, \\ \gamma_4 = -\sigma_3 \cos\dfrac{(\varrho+2)\pi}{2}, \\ \gamma_5 = -\sigma_4 \cos\dfrac{(\varrho+1)\pi}{2}, \\ \gamma_6 = -\sigma_5 \cos\dfrac{\varrho\pi}{2}, \\ \gamma_7 = \sigma_6, \\ \gamma_8 = -\sigma_8, \\ \rho_1 = -\sin\dfrac{(2\varrho+2)\pi}{2}, \\ \rho_2 = -\sigma_1 \sin\dfrac{(2\varrho+1)\pi}{2}, \\ \rho_3 = -\sigma_2 \sin \varrho\pi, \\ \rho_4 = -\sigma_3 \sin\dfrac{(\varrho+2)\pi}{2}, \\ \rho_5 = -\sigma_4 \sin\dfrac{(\varrho+1)\pi}{2}, \\ \rho_6 = -\sigma_5 \sin\dfrac{\varrho\pi}{2}, \\ \rho_7 = -\sigma_7, \end{cases} \tag{26}$$

and then (21) becomes

$$\begin{cases} \mathscr{A}_1 = \sigma_{11} - \sigma_9\chi^2, \\ \mathscr{A}_2 = \sigma_{10}\chi, \\ \mathscr{A}_3 = \gamma_1\chi^{2\rho+2} + \gamma_2\chi^{2\rho+1} + \gamma_3\chi^{2\rho} + \gamma_4\chi^{\rho+2} \\ +\gamma_5\chi^{\rho+1} + \gamma_6\chi^{\rho} + \gamma_7\chi^2 + \gamma_8, \\ \mathscr{A}_4 = \rho_1\chi^{2\rho+2} + \rho_2\chi^{2\rho+1} + \rho_3\chi^{2\rho} + \rho_4\chi^{\rho+2} \\ +\rho_5\chi^{\rho+1} + \rho_6\chi^{\rho} + \rho_7\chi. \end{cases} \tag{27}$$

By virtue of (25) and (27), one gets

$$\begin{aligned} &\vartheta_1\chi^{4\varrho+4} + \vartheta_2\chi^{4\varrho+3} + \vartheta_3\chi^{4\varrho+2} + \vartheta_4\chi^{4\varrho+1} + \vartheta_5\chi^{4\varrho} + \vartheta_6\chi^{3\varrho+4} \\ &\quad + \vartheta_7\chi^{3\varrho+3} + \vartheta_8\chi^{3\varrho+2} + \vartheta_9\chi^{3\varrho+1} + \vartheta_{10}\chi^{3\varrho} \\ &\quad + \vartheta_{11}\chi^{2\varrho+4} + \vartheta_{12}\chi^{2\varrho+3} \\ &\quad + \vartheta_{13}\chi^{2\varrho+2} + \vartheta_{14}\chi^{2\varrho+1} + \vartheta_{15}\chi^{2\varrho} + \vartheta_{16}\chi^{\varrho+4} \\ &\quad + \vartheta_{17}\chi^{\varrho+3} + \vartheta_{18}\chi^{\varrho+2} \\ &\quad + \vartheta_{19}\chi^{\varrho+1} + \vartheta_{20}\chi^{\varrho} + \vartheta_{21}\chi^4 + \vartheta_{22}\chi^2 + \vartheta_{23} = 0, \end{aligned} \tag{28}$$

where

$$\begin{cases} \vartheta_1 = \gamma_1^2 + \rho_1^2, \\ \vartheta_2 = 2(\gamma_1\gamma_2 + \rho_1\rho_2), \\ \vartheta_3 = \gamma_2^2 + \rho_2^2 + 2(\gamma_1\gamma_3 + \rho_1\rho_3), \\ \vartheta_4 = 2(\gamma_2\gamma_3 + \rho_2\rho_3), \\ \vartheta_5 = \gamma_3^2 + \rho_3^2, \\ \vartheta_6 = 2(\gamma_1\gamma_4 + \rho_1\rho_4), \\ \vartheta_7 = 2(\gamma_1\gamma_5 + \rho_1\rho_5), \\ \vartheta_8 = 2(\gamma_1\gamma_6 + \gamma_2\gamma_5 + \gamma_3\gamma_4 + \rho_1\rho_6 + \rho_2\rho_5 + \rho_3\rho_4), \\ \vartheta_9 = 2(\gamma_2\gamma_6 + \gamma_3\gamma_5 + \rho_2\rho_6 + \rho_3\rho_5), \\ \vartheta_{10} = 2(\gamma_3\gamma_6 + \rho_3\rho_6), \\ \vartheta_{11} = \gamma_4^2 + \rho_4^2 + 2(\gamma_1\gamma_7 + \rho_1\rho_7), \\ \vartheta_{12} = 2(\gamma_4\gamma_5 + \gamma_2\gamma_7 + \rho_4\rho_5 + \rho_1\rho_7), \\ \vartheta_{13} = \gamma_5^2 + \rho_5^2 + 2(\gamma_4\gamma_6 + \gamma_3\gamma_7 + \gamma_1\gamma_8 + \rho_4\rho_6 + \rho_2\rho_7), \\ \vartheta_{14} = 2(\gamma_5\gamma_6 + \gamma_2\gamma_8 + \rho_5\rho_6 + \rho_3\rho_7), \\ \vartheta_{15} = \gamma_6^2 + \rho_6^2, \\ \vartheta_{16} = 2\gamma_4\gamma_7, \\ \vartheta_{17} = 2(\gamma_5\gamma_7 + \rho_4\rho_7), \\ \vartheta_{18} = 2(\gamma_6\gamma_7 + \gamma_4\gamma_8 + \rho_5\rho_7), \\ \vartheta_{19} = 2(\gamma_5\gamma_8 + \rho_6\rho_7), \\ \vartheta_{20} = 2\gamma_6\gamma_8, \\ \vartheta_{21} = \gamma_7^2 - \sigma_9^2, \\ \vartheta_{22} = 2\gamma_7\gamma_8 - \rho_7^2 - \sigma_{11}^2 + 2\sigma_9\sigma_{11}, \\ \vartheta_{23} = \gamma_8^2 - \sigma_{11}^2. \end{cases} \tag{29}$$

Denote

$$\begin{aligned} \mathscr{Q}(\chi) = {} &\vartheta_1\chi^{4\varrho+4} + \vartheta_2\chi^{4\varrho+3} + \vartheta_3\chi^{4\varrho+2} + \vartheta_4\chi^{4\varrho+1} + \vartheta_5\chi^{4\varrho} + \vartheta_6\chi^{3\varrho+4} \\ &+ \vartheta_7\chi^{3\varrho+3} + \vartheta_8\chi^{3\varrho+2} + \vartheta_9\chi^{3\varrho+1} + \vartheta_{10}\chi^{3\varrho} + \vartheta_{11}\chi^{2\varrho+4} + \vartheta_{12}\chi^{2\varrho+3} \\ &+ \vartheta_{13}\chi^{2\varrho+2} + \vartheta_{14}\chi^{2\varrho+1} + \vartheta_{15}\chi^{2\varrho} + \vartheta_{16}\chi^{\varrho+4} + \vartheta_{17}\chi^{\varrho+3} + \vartheta_{18}\chi^{\varrho+2} \\ &+ \vartheta_{19}\chi^{\varrho+1} + \vartheta_{20}\chi^{\varrho} + \vartheta_{21}\chi^4 + \vartheta_{22}\chi^2 + \vartheta_{23} = 0. \end{aligned} \tag{30}$$

Now the following assumption is given.
($\mathscr{H}2$) $\vartheta_{23} < 0$, where ϑ_{23} is defined by (29). □

Lemma 5. *Assume that ($\mathscr{H}2$) is fulfilled; then, equation (19) possesses at least a pair of purely imaginary roots.*

Proof. It is easy to see that $\mathscr{Q}(0) = \vartheta_{23} < 0$ and $\lim_{\chi\longrightarrow\infty}\mathscr{Q}(\chi) = +\infty$. Then, one can conclude that equation (28) owns at least one positive root, which implies that equation (19) owns at least a pair of purely imaginary roots.

In equation (28), because the parameter ϱ is a fractional number, it is not inconvenient to solve the solution of equation (28). So, we are to change equation (28) to an isovalent equation with the powers involving integer number. Let $x = \chi^{1/\varepsilon}$; then, $\chi = x^{\varepsilon}$. It follows from equation (28) that

$$\begin{aligned} &\vartheta_1 x^{4\epsilon+4\varepsilon} + \vartheta_2 x^{4\epsilon+3\varepsilon} + \vartheta_3 x^{4\epsilon+2\varepsilon} + \vartheta_4 x^{4\epsilon+\varepsilon} + \vartheta_5 x^{4\epsilon} + \vartheta_6 x^{3\epsilon+4\varepsilon} \\ &\quad + \vartheta_7 x^{3\epsilon+3\varepsilon} + \vartheta_8 x^{3\epsilon+2\varepsilon} + \vartheta_9 y^{3\epsilon+\varepsilon} + \vartheta_{10} x^{3\epsilon} \\ &\quad + \vartheta_{11} x^{2\epsilon+4\varepsilon} + \vartheta_{12} x^{2\epsilon+3\varepsilon} \\ &\quad + \vartheta_{13} x^{2\epsilon+2\varepsilon} + \vartheta_{14} x^{2\epsilon+\varepsilon} + \vartheta_{15} y^{2\epsilon} + \vartheta_{16} x^{\epsilon+4\varepsilon} \\ &\quad + \vartheta_{17} x^{\epsilon+3\varepsilon} + \vartheta_{18} x^{\epsilon+2\varepsilon} \\ &\quad + \vartheta_{19} x^{\epsilon+\varepsilon} + \vartheta_{20} x^{\epsilon} + \vartheta_{21} x^{4\varepsilon} + \vartheta_{22} x^{2\varepsilon} + \vartheta_{23} = 0. \end{aligned} \tag{31}$$

By virtue of computer, we can easily find the roots of equation (31). Suppose that equation (31) owns the positive root which is denoted by x_j; then, equation (28) owns the root $\chi_j = x_j^{\varepsilon} > 0$. Suppose that equation (31) owns h positive roots x_j, $j = 1, 2, \ldots, h$. It follows from (24) that

$$\varsigma_j^l = \frac{1}{2\chi_j}\left[\arccos\frac{\mathscr{A}_1\mathscr{A}_3 + \mathscr{A}_2\mathscr{A}_4}{\mathscr{A}_1^2 + \mathscr{A}_2^2} + 2l\pi\right], \quad j = 1, 2, \ldots, h;\ l = 0, 1, 2, \ldots. \tag{32}$$

Set

$$\begin{aligned} \varsigma_0 &= \varsigma_{j0}^{(0)} \min_{j=1,2,\ldots,h}\{\varsigma_j^0\}, \\ \chi_0 &= \chi|_{\varsigma=\varsigma_0}. \end{aligned} \tag{33}$$

Next, the following hypothesis is prepared as follows.
($\mathscr{H}3$)$\mathscr{D}_{11}\mathscr{D}_{21} + \mathscr{D}_{12}\mathscr{D}_{22} > 0$, where

$$\begin{cases}\mathscr{D}_{11} = (2\varrho + 2)\chi_0^{2\varrho+1}\cos\dfrac{(2\varrho + 1)\pi}{2} + \sigma_1(2\varrho + 1)\chi_0^{2\varrho}\cos\varrho\pi \\ +2\rho\sigma_2\chi_0^{2\varrho-1}\cos\dfrac{(2\varrho - 1)\pi}{2} + 2\sigma_3(\varrho + 2)\sigma_0^{\varrho+1}\cos\dfrac{(\varrho + 1)\pi}{2} \\ +\sigma_4(\varrho + 1)\sigma_0^{\varrho}\cos\dfrac{\varrho\pi}{2} + \sigma_5\varrho\sigma_0^{\varrho-1}\cos\dfrac{(\varrho - 1)\pi}{2} + \sigma_7 \\ +\sigma_{10}\cos 2\chi_0\varsigma_0 + 2\sigma_9\sin 2\chi_0\varsigma_0, \\ \mathscr{D}_{12} = (2\varrho + 2)\chi_0^{2\varrho+1}\sin\dfrac{(2\varrho + 1)\pi}{2} + \sigma_1(2\varrho + 1)\chi_0^{2\varrho}\sin\varrho\pi \\ +2\rho\sigma_2\chi_0^{2\varrho-1}\sin\dfrac{(2\varrho - 1)\pi}{2} + 2\sigma_3(\varrho + 2)\sigma_0^{\varrho+1}\sin\dfrac{(\varrho + 1)\pi}{2} \\ +\sigma_4(\varrho + 1)\sigma_0^{\varrho}\sin\dfrac{\varrho\pi}{2} + \sigma_5\varrho\sigma_0^{\varrho-1}\sin\dfrac{(\varrho - 1)\pi}{2} + 2\sigma_6 \\ -\sigma_{10}\sin 2\chi_0\varsigma_0 + 2\sigma_9\cos 2\chi_0\varsigma_0, \\ \mathscr{D}_{21} = 2\chi_0\left(\sigma_{11} - \sigma_9\chi_0^2\right)\cos 2\chi_0\varsigma_0 + 2\chi_0^2\sigma_{10}\sin 2\chi_0\varsigma_0, \\ \mathscr{D}_{22} = 2\chi_0^2\sigma_{10}\cos 2\chi_0\varsigma_0 - 2\chi_0\left(\sigma_{11} - \sigma_9\chi_0^2\right)\sin 2\chi_0\varsigma_0.\end{cases} \tag{34}$$

Lemma 6. *Let* $s(\varsigma) = \psi_1(\varsigma) + i\psi_2(\varsigma)$ *be the root of equation (19) around* $\varsigma = \varsigma_0$ *satisfying* $\psi_1(\varsigma_0) = 0, \psi_2(\varsigma_0) = \chi_0$*; then, one gets* $\mathrm{Re}[ds/d\varsigma]|_{\varsigma=\varsigma_0,\chi=\chi_0} > 0$.

Proof. By virtue of equation (19), we get

$$\begin{aligned}&\Big[(2\varrho + 2)s^{2\varrho+1} + \sigma_1(2\varrho + 1)s^{2\varrho} + 2\varrho\sigma_2 s^{2\varrho-1} + \sigma_3(\varrho + 2)s^{\varrho+1} \\ &\quad + \sigma_4(\varrho + 1)s^{\varrho} + \sigma_5\varrho s^{\varrho-1} + 2\sigma_6 s + \sigma_7\Big]\frac{ds}{d\varsigma} \\ &\quad + (2\sigma_9 s + \sigma_{10})e^{-2s\varsigma}\frac{ds}{d\varsigma} \\ &\quad - 2e^{-2s\varsigma}\left(\frac{ds}{d\varsigma}\varsigma + s\right)\left(\sigma_9 s^2 + \sigma_{10}s + \sigma_{11}\right) = 0.\end{aligned} \tag{35}$$

It follows from (35) that

$$\left[\frac{ds}{d\varsigma}\right]^{-1} = \frac{\mathscr{D}_1(s)}{\mathscr{D}_2(s)} - \frac{\varsigma}{s}, \tag{36}$$

where

$$\begin{cases}\mathscr{D}_1(s) = (2\rho + 2)s^{2\rho+1} + \sigma_1(2\rho + 1)s^{2\rho} + 2\rho\sigma_2 s^{2\rho-1} + 2\sigma_3(\rho + 2)s^{\rho+1} \\ +\sigma_4(\rho + 1)s^{\rho} + \sigma_5\rho s^{\rho-1} + 2\sigma_6 s + \sigma_7 + (2\sigma_9 s + \sigma_{10})e^{-2s\varsigma}, \\ \mathscr{D}_2(s) = 2se^{-2s\varsigma}\left(\sigma_9 s^2 + \sigma_{10}s + \sigma_{11}\right).\end{cases} \tag{37}$$

Then,

$$\mathrm{Re}\left[\frac{ds}{d\varsigma}\right]^{-1}_{\varsigma=\varsigma_0,\chi=\chi_0} = \mathrm{Re}\left[\frac{\mathscr{D}_1(s)}{\mathscr{D}_2(s)}\right]_{\varsigma=\varsigma_0,\chi=\chi_0} = \frac{\mathscr{D}_{11}\mathscr{D}_{21} + \mathscr{D}_{12}\mathscr{D}_{22}}{\mathscr{D}_{21}^2 + \mathscr{D}_{22}^2}. \tag{38}$$

By $(\mathscr{H}3)$, one has

$$\mathrm{Re}\left[\frac{ds}{d\varsigma}\right]^{-1}_{\varsigma=\varsigma_0,\chi=\chi_0} > 0, \tag{39}$$

which completes the proof.

According to the study above, one gets the following conclusion. ☐

Theorem 1. *Suppose that* $(\mathscr{H}1)$–$(\mathscr{H}3)$ *are fulfilled and every root* λ *for equation (18) satisfies* $|\arg(\lambda)| > (\pi/2\varepsilon)$*; then, the positive equilibrium point of model (2) is locally asymptotically stable provided that* $0 \le \varsigma < \varsigma_0$ *and a Hopf bifurcation takes place around the positive equilibrium point provided that* $\varsigma = \varsigma_0$.

Remark 1. In 2020, Rahman et al. [37] dealt with the Hopf bifurcation for integer-order predator-prey model involving discrete and distributed delay. In this article, we have dealt with Hopf bifurcation for fractional-order predator-prey model involving discrete and distributed delay. The method of research in [37] cannot be used to investigate the fractional-order case. From this viewpoint, we say that the research is a good complement of the work of [37].

Remark 3. For many works on Hopf bifurcation of fractional-order predator-prey systems, numerous scholars focus on the fractional-order predator-prey models involving discrete time delay and do not involve the distributed time delay. In this article, we are concerned with the fractional-order predator-prey model involving discrete time delay and distributed time delay. After a suitable variable substitution, we obtain an isovalent fractional-order predator-prey model which includes integer-order operator and fractional-order operator. The discussion on the characteristic equation of the isovalent fractional-order predator-prey model has become more complex. So, we think that our works enrich and develop the stability and bifurcation theory of fractional-order dynamical system.

4. Simulation Figures

Given the following predator-prey model:

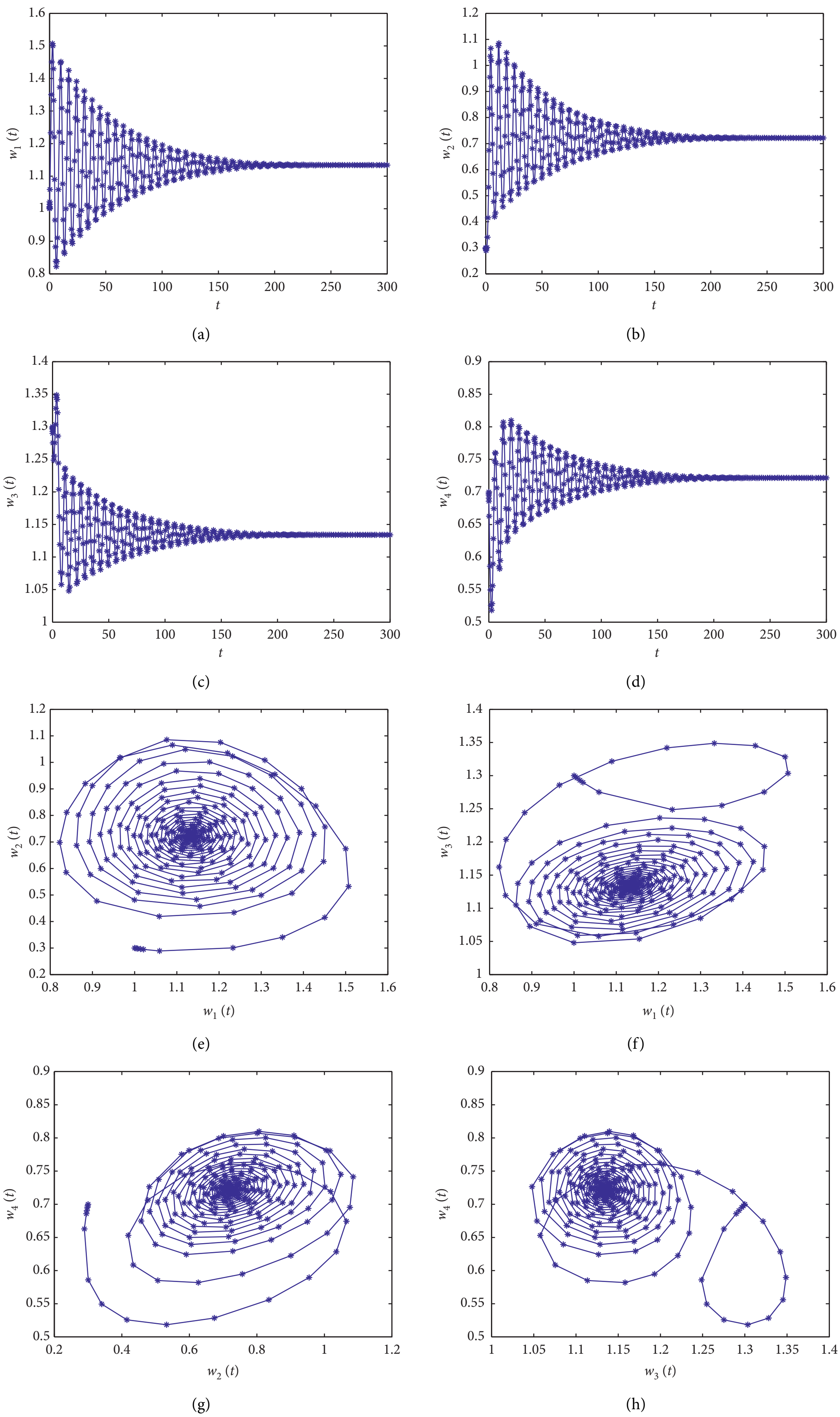

Figure 1: Continued.

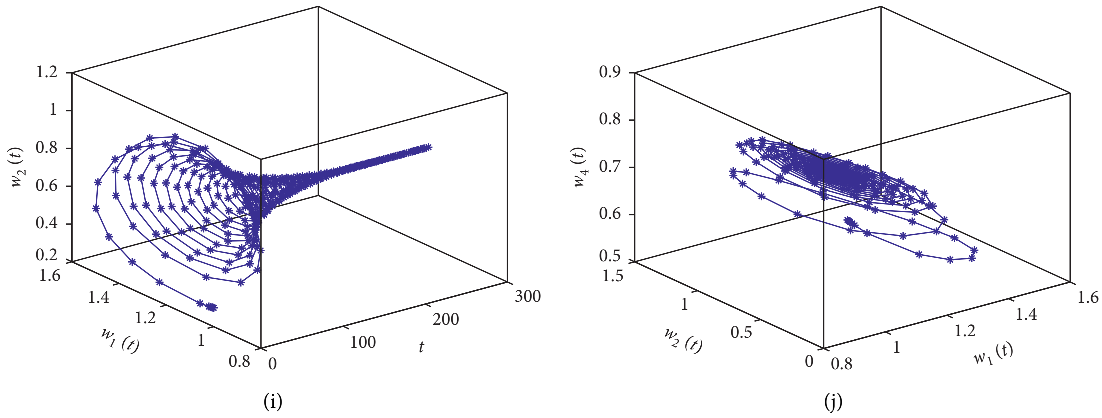

Figure 1: The stable behavior of system (40) when $\varsigma = 0.04 < \varsigma_0 = 0.05$.

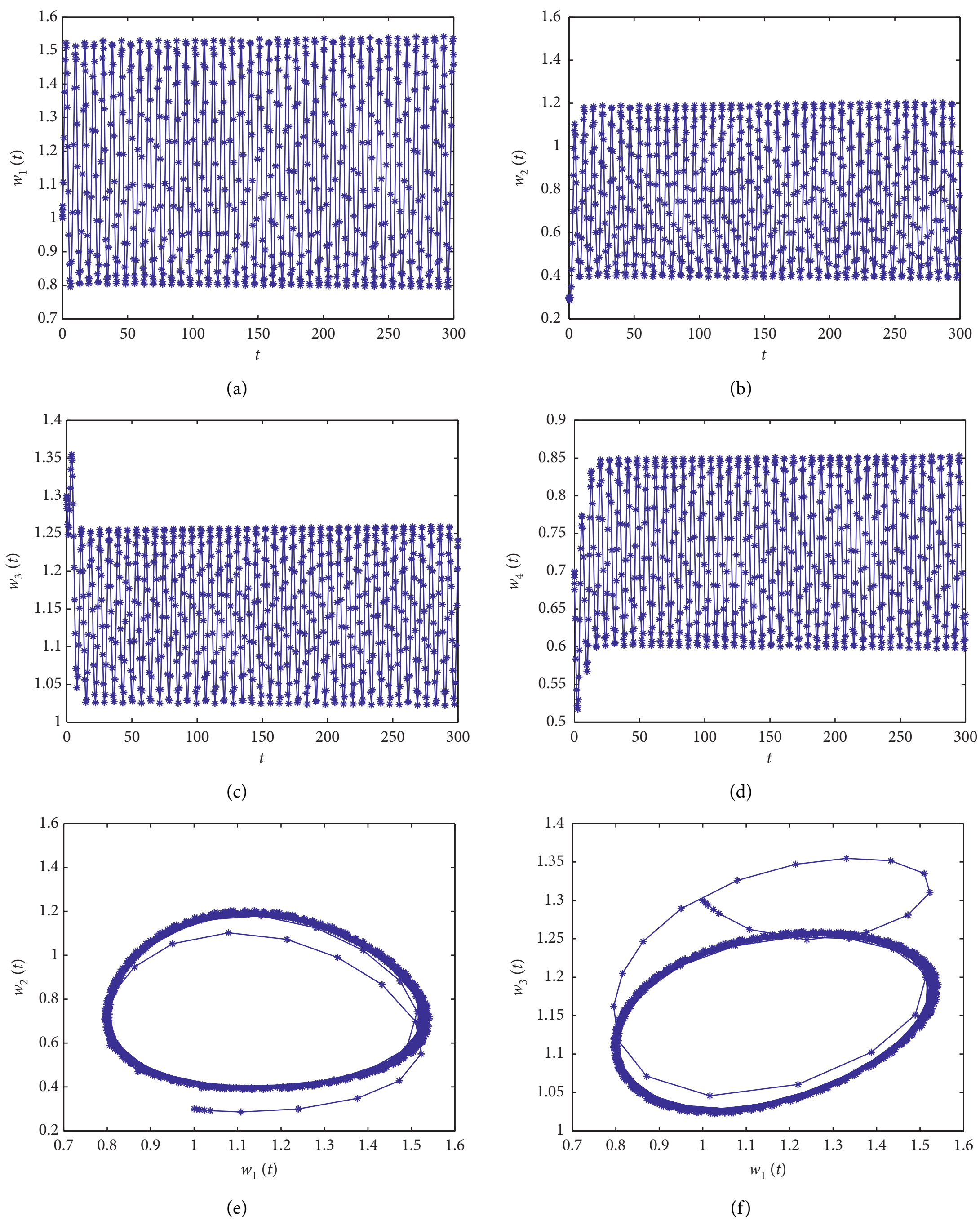

Figure 2: Continued.

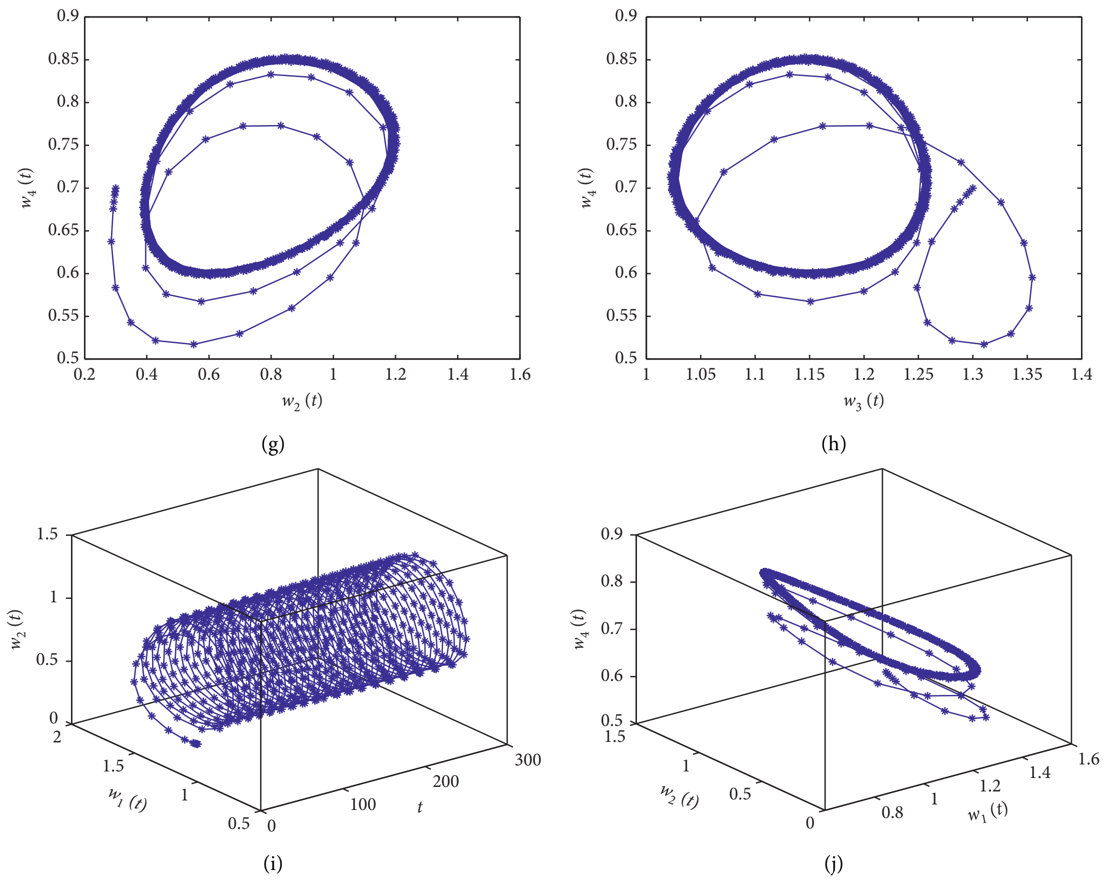

Figure 2: Hopf bifurcation phenomenon of system (40) when $\varsigma = 0.07 > \varsigma_0 = 0.05$.

$$\begin{cases} \dfrac{dw_1^{\varrho}(t)}{dt^{\varrho}} = w_1(t)\left[1 - 0.5w_3(t) - 0.6w_2(t-\varsigma)\right], \\ \dfrac{dw_2^{\varrho}(t)}{dt^{\varrho}} = w_2(t)\left[-1 + 1.2w_1(t-\varsigma) - 0.5w_4(t)\right], \\ \dfrac{dw_3(t)}{dt} = -0.3w_3(t) + 0.3w_1(t), \\ \dfrac{dw_4(t)}{dt} = -0.3w_4(t) + 0.3w_2(t), \end{cases} \quad (40)$$

it is not difficult to derive that model (40) owns a unique positive equilibrium point $\mathscr{E}_3^0(w_1^0, w_2^0, w_3^0, w_4^0) = (1.1340, 0.7216, 1.1340, 0.7216)$. Select $\varrho = 0.93 = (93/100)$. Then, $\epsilon = 93, \varepsilon = 100$. By means of computer software, one derives $\chi_0 = 1.8722, \varsigma_0 = 0.05, \vartheta_{23} = -0.8347, \mathscr{D}_{11} = 0.5091$, $\mathscr{D}_{12} = 0.6783, \mathscr{D}_{21} = -0.4377, \mathscr{D}_{22} = 0.7901$. Then, $\mathscr{D}_{11}\mathscr{D}_{21} + \mathscr{D}_{12}\mathscr{D}_{22} = 0.3131 > 0$. Furthermore, each root λ of equation (18) satisfies $|\arg(\lambda)| > (\pi/200)$. So, all the assumptions of Theorem 1 are satisfied. Therefore, $\mathscr{E}_3^0(1.1340, 0.7216, 1.1340, 0.7216)$ of model (40) is locally asymptotically stable for $0 \le \varsigma < 0.5$. To illustrate this situation, we select $\varsigma = 0.04 < \varsigma_0 = 0.05$. The computer simulation plots are presented in Figure 1. Figure 1 indicates that the variables $w_1(t), w_2(t), w_3(t), w_4(t)$ tend to $1.1340, 0.7216, 1.1340, 0.7216$, respectively. When ς passes through $\varsigma_0 = 0.05$, then a family of Hopf bifurcation caused by time delay ς takes place in the neighborhood of $\mathscr{E}_3^0(1.1340, 0.7216, 1.1340, 0.7216)$. To verify this situation, we select $\varsigma = 0.067 > \eta_0 = 0.05$. The computer simulation plots are shown in Figure 2. Figure 2 suggests that the variables $w_1(t), w_2(t), w_3(t), w_4(t)$ will exhibit periodic oscillatory phenomenon. Moreover, the bifurcation plots are drawn in Figures 3–6. Figures 3–6 show that the bifurcation value of model (40) is roughly equal to 0.05.

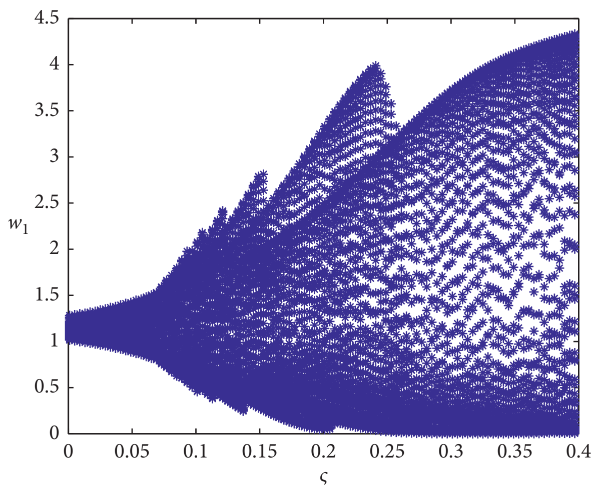

Figure 3: Bifurcation graph of system (40): $\varsigma - w_1$.

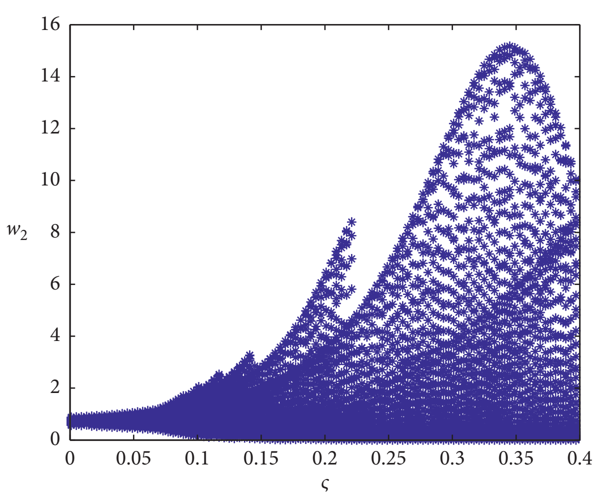

Figure 4: Bifurcation graph of system (40): $\varsigma - w_2$.

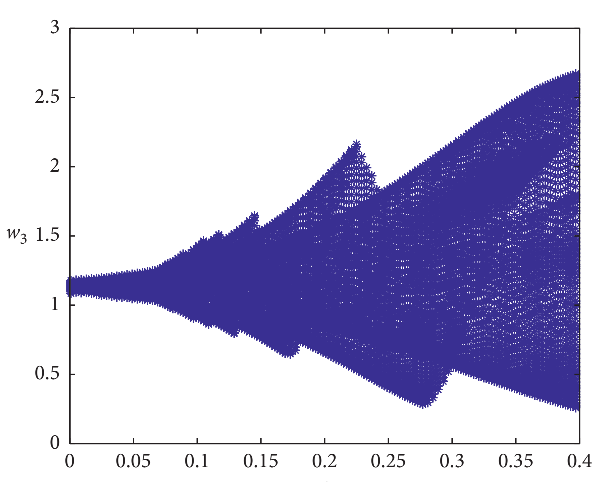

Figure 5: Bifurcation graph of system (40): $\varsigma - w_3$.

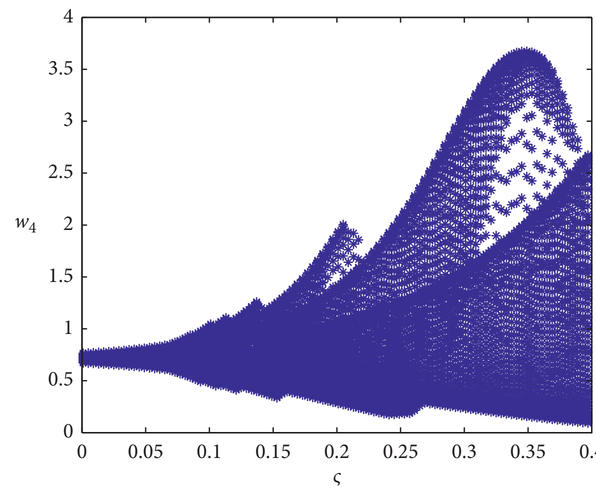

Figure 6: Bifurcation graph of system (40): $\varsigma - w_4$.

5. Conclusions

The investigation on the stability and bifurcation peculiarity of delayed predator-prey models plays a vital role in maintaining ecological balance in real world. In this work, based on the work of [37], we set up a new fractional-order predator-prey model concerning discrete delay and distributed delay. By virtue of apposite change of variable, we derive an equivalent fractional-order predator-prey model concerning one delay. By analyzing the characteristic equation of the equivalent fractional-order predator-prey model and regarding the time delay as bifurcation parameter, the stability and bifurcation condition for the involved predator-prey model is established. The influence of time delay on the stability and bifurcation of the involved predator-prey model has been revealed. The computer simulation plots and bifurcation diagrams are displayed to sustain the validity of the derived main results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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