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Research Article

First General Zagreb Index of Generalized *F*-sum Graphs

H. M. Awais,[1] Muhammad Javaid,[1] and Akbar Ali[2]

[1]*Department of Mathematics, School of Science, University of Management and Technology, Lahore, Pakistan*
[2]*Department of Mathematics, Faculty of Science, University of Ha'il, Ha'il, Saudi Arabia*

Correspondence should be addressed to Akbar Ali; akbarali.maths@gmail.com

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The first general Zagreb (FGZ) index (also known as the general zeroth-order Randić index) of a graph G can be defined as $M^{\gamma}(G) = \sum_{uv\in E(G)} [d_G^{c-1}(u) + d_G^{c-1}(v)]$, where γ is a real number. As $M^{\gamma}(G)$ is equal to the order and size of G when $\gamma = 0$ and $\gamma = 1$, respectively, γ is usually assumed to be different from 0 to 1. In this paper, for every integer $\gamma \geq 2$, the FGZ index M^{γ} is computed for the generalized F-sums graphs which are obtained by applying the different operations of subdivision and Cartesian product. The obtained results can be considered as the generalizations of the results appeared in (IEEE Access; 7 (2019) 47494–47502) and (IEEE Access 7 (2019) 105479–105488).

1. Introduction

Graph theory concepts are being utilized to model and study the several problems in different fields of science, including chemistry and computer science. A topological index (TI) of a (molecular) graph is a numeric quantity that remained unchanged under graph isomorphism [1,2]. Many topological indices have found applications in chemistry, especially in the quantitative structure-activity/property relationships studies; for detail, see [3–13].

Wiener index is the first TI introduced by Harry Wiener in 1947, when he was working on the boiling point of paraffin [14]. In 1972, Trinajstić and Gutman [15] obtained a formula concerning the total energy of π electrons of molecules where the sum of square of valences of the vertices of a molecular structure was appeared. This sum is nowadays known as the first Zagreb index. In this paper, we are concerned with a generalized version of the first Zagreb index, known as the general first Zagreb index as well as the general zeroth-order Randić index.

There are several operations in graph theory such as product, complement, addition, switching, subdivision, and deletion. In many cases, graph operations may be helpful in finding graph quantities of more complicated graphs by considering the less complicated ones. In chemical graph theory, by using different graph operations, one can develop large molecular structures from the simple and basic structures. Recently, many classes of molecular structures are studied with the assistance of graph operations.

In 2007, Yan et al. [6] listed the five subdivision operations with the help of their vertices and edges. They also discussed the different features of Wiener index of graphs under these operations. After that, Eliasi and Taeri [16] introduced the F_1-sum graphs $\Gamma_{1+F_1}\Gamma_2$ with the assistance of Cartesian product on graphs $F_1(\Gamma_1)$ and Γ_2, where $F_1(\Gamma_1)$ is obtained by applying the subdivision operations S_1, R_1, Q_1, and T_1. They also defined the Wiener indices of these resulting graphs $\Gamma_{1+S_1}\Gamma_2$, $\Gamma_{1+R_1}\Gamma_2$, $\Gamma_{1+Q_1}\Gamma_2$, and $\Gamma_{1+T_1}\Gamma_2$. Later on, Deng et al. [17] calculated the 1st and 2nd Zagreb topological indices, and Imran and Akhtar [18] calculated the forgotten topological index of the F_1-sums graph. In 2019, Liu et al. [19] computed the first general Zagreb index of F_1-sums graphs.

Recently, Liu et al. [20] introduced the generalized version of the aforesaid subdivided operations of graphs denoted by S_k, R_k, Q_k, and T_k, where $k \geq 1$ is counting number. They also defined the generalized F-sums graphs using these generalized operations and calculated their 1st and 2nd Zagreb indices. In the present work, we compute the 1st general Zagreb index of the generalized F-sums graphs

$\Gamma_{1+F_k}\Gamma_2$ for $F_k \in \{S_k, R_k, Q_k, T_k\}$. The remaining work is arranged as follows: Section 2 contains some basic definitions, Section 3 contains the key outcomes, and Section 4 contains the some particular applications. Conclusions of the obtained results are presented in Section 5.

2. Preliminaries

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a simple graph having $|V(\Gamma)|$ the order and $|E(\Gamma)|$ the size of a graph, where $V(\Gamma)$ is considered as node set and $E(\Gamma) \subseteq V(\Gamma) \times V(\Gamma)$ is a bond set. Every vertex is considered as an atom in a graph, and bonding within the two atoms is known as edge. The valency or degree of any node is the number of total edges which are incident to the node. Now, few useful TI's are explained given below:

Definition 1. If Γ be a connected graph, then the 1st and 2nd Zagreb topological indices as

$$M_1(\Gamma) = \sum_{uv \in V(\Gamma)} [d_\Gamma(u) + d_\Gamma(v)],$$
$$M_2(\Gamma) = \sum_{uv \in V(\Gamma)} [d_\Gamma(u) d_\Gamma(v)]. \quad (1)$$

These two descriptors of the graph were introduced by Trinajsti and Gutman [15]. Such type of TI's have been utilized to discuss the QSAR/QSPR of the different chemical structures such as chirality, complexity, hetero-system, ZE-isomers, π electron energy, and branching [9, 10].

Definition 2. If R is the real number, $\gamma \in R - \{0, 1\}$, and Γ be a connected graph, so the 1st general Zagreb topological index is given as

$$M^\gamma(\Gamma) = \sum_{uv \in E(\Gamma)} \left[d_\Gamma^{\gamma-1}(u) + d_\Gamma^{\gamma-1}(v)\right]. \quad (2)$$

Definition 3. If R is the real number, $\gamma \in R$, and Γ be a connected graph, so the general Randić is given as

$$R_\gamma(\Gamma) = \sum_{uv \in E(\Gamma)} \left[d_G(u) d_{(v)}\right]^\gamma, \quad (3)$$

where $R_{-(1/2)}$ is considered as the classical Randić connectivity topological index.

The generalized F-sums graph is defined in [20] as follows:

(i) $S_k(G)$ graph is obtained by inserting k vertices in each edge of G.

(ii) $R_k(G)$ is obtained from $S_k(G)$ by joining the old vertices which are adjacent G.

(iii) $Q_k(G)$ is obtained from $S_k(G)$ by joining the new vertices lying on edge to the corresponding new vertices of other edge, if these edges have some common vertex in G.

(iv) $T_k(G)$ is union of $R_k(G)$ and $Q_k(G)$ graphs. For further details, see Figure 1.

Definition 4. If $\Gamma_1 \& \Gamma_2$ be two connected molecular structures, $F_k \in \{S_k, R_k, Q_k, T_k\}$ and $F_k(\Gamma_1)$ be a structure obtained after using F_k on Γ_1 with bonds (edges) $E(F_k(\Gamma_1))$ and nodes (vertices) $V(F_k(\Gamma_1))$. So, the generalized F-sums graph $(\Gamma_{1+F_k}\Gamma_2)$ is a structure with nodes:

$$V\left(\Gamma_{1+F_k}\Gamma_2\right) = V(F_k(\Gamma_1)) \times V(\Gamma_2),$$
$$= (V(\Gamma_1) \cup E(\Gamma_1)) \times V(\Gamma_2), \quad (4)$$

in such a way two nodes $(a_1, b_1) \& (a_2, b_2)$ of $V(\Gamma_{1+F_k}\Gamma_2)$ are adjacent if $[a_1 = a_2 \in V(\Gamma_1) \& (b_1, b_2) \in E(\Gamma_2)]$ or $[b_1 = b_2 \in V(\Gamma_2) \& (a_1, a_2) \in E(F_k(\Gamma_1))]$. For more details, see Figures 2 and 3.

Lemma 1. *For $F_k \in \{S_k, R_k, Q_k, T_k\}$ and $(x, y) \in \Gamma_1 +_{F_k} \Gamma_2$, the degree of (x, y) in $\Gamma_1 +_{F_k} \Gamma_2$ is*

$$\text{(i) } d\left(d_{\Gamma_1 +_{S_k} \Gamma_2}(x, y)\right) = \begin{cases} d_{\Gamma_1}(x) + d_{\Gamma_2}(y), & \text{if } x \in V(\Gamma_1) \wedge y \in V(\Gamma_2), \\ 2, & \text{if } x \in V(S_k(\Gamma_1)) - V(\Gamma_1) \wedge y \in V(\Gamma_2), \end{cases}$$
$$\text{(ii) } d\left(d_{\Gamma_1 +_{R_k} \Gamma_2}(x, y)\right) = \begin{cases} d_{R_k(\Gamma_1)}(x) + d_{\Gamma_2}(y), & \text{if } x \in V(\Gamma_1) \wedge y \in V(\Gamma_2), \\ 2, & \text{if } x \in V(S_k(\Gamma_1)) - V(\Gamma_1) \wedge y \in V(\Gamma_2), \end{cases}$$
$$\text{(iii) } d\left(d_{\Gamma_1 +_{Q_k} \Gamma_2}(x, y)\right) = \begin{cases} d_{\Gamma_1}(x) + d_{\Gamma_2}(y), & \text{if } x \in V(\Gamma_1) \wedge y \in V(\Gamma_2), \\ d_{Q_k(\Gamma_1)}(x), & \text{if } x \in V(Q_k(\Gamma_1)) - V(\Gamma_1) \wedge y \in V(\Gamma_2), \end{cases} \quad (5)$$
$$\text{(iv) } d\left(d_{\Gamma_1 +_{T_k} \Gamma_2}(x, y)\right) = \begin{cases} d_{T_k(\Gamma_1)}(x) + d_{\Gamma_2}(y), & \text{if } x \in V(\Gamma_1) \wedge y \in V(\Gamma_2), \\ d_{T_k(\Gamma_1)}(x), & \text{if } x \in V(T_k(\Gamma_1)) - V(\Gamma_1) \wedge y \in V(\Gamma_2). \end{cases}$$

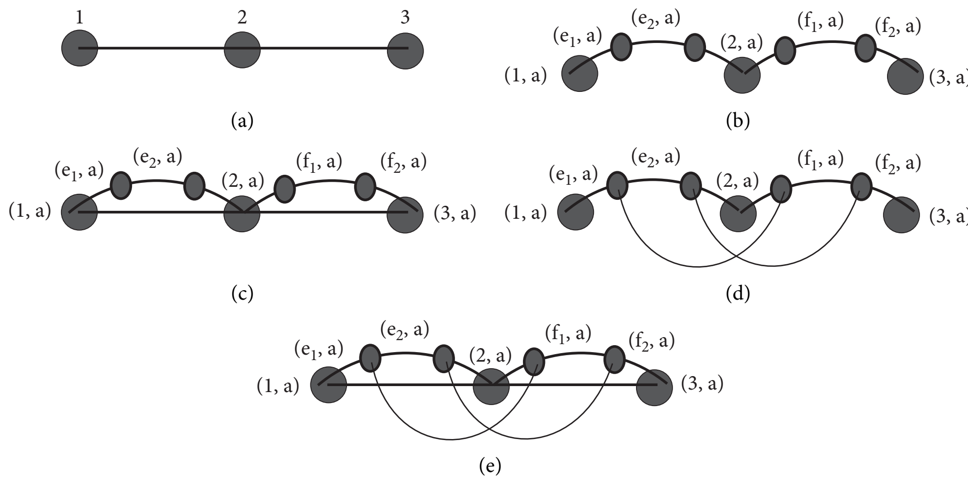

Figure 1: (a) Γ, (b). $S_2(\Gamma)$, (c) $R_2(\Gamma)$, (d) $Q_2(\Gamma)$, and (e) $T_2(\Gamma)$.

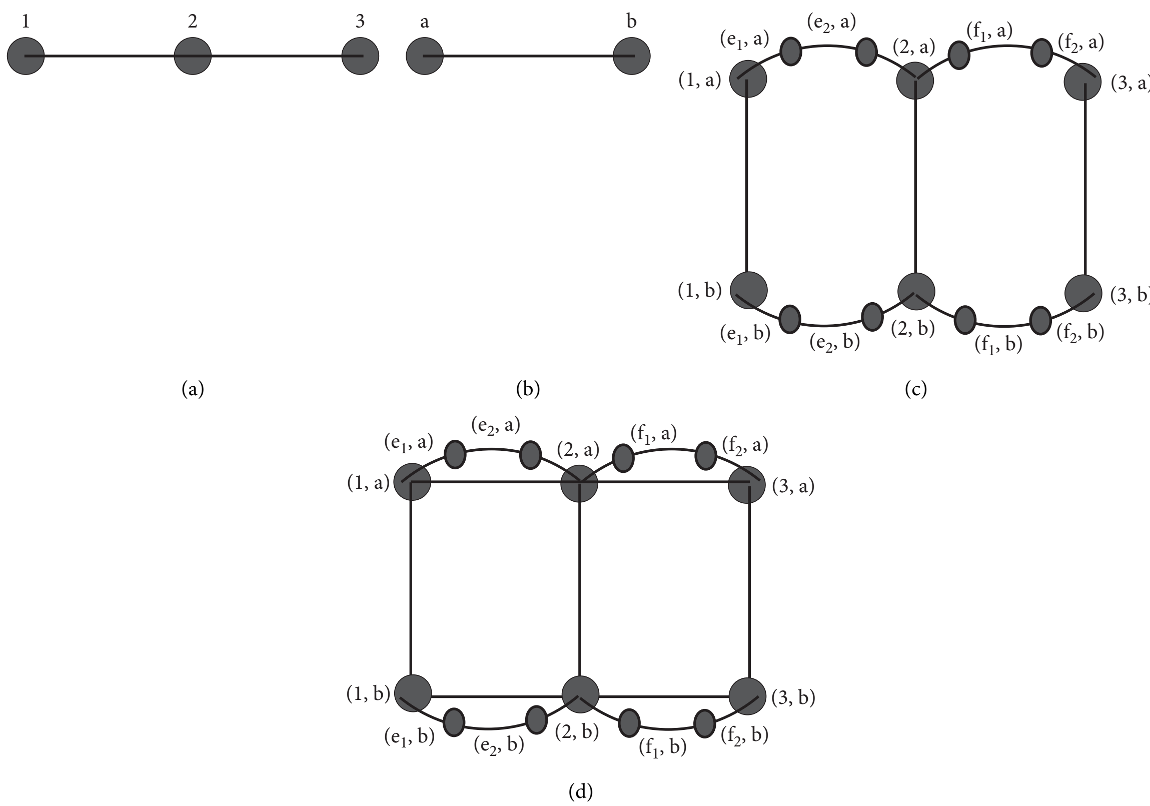

Figure 2: (a) $\Gamma_1 \cong P_3$. (b) $\Gamma_2 \cong P_2$. (c) $\Gamma_{1+S_2}\Gamma_2$. (d) $\Gamma_{1+R_2}\Gamma_2$.

3. Main Results

The main results of FGZ index of the generalized F-sum graphs are presented in this section.

Theorem 1. *Let Γ_1 and Γ_2 be two simple graphs and $\gamma \in N - \{0,1\}$. The FGZ index of the generalized S-sum graph $\Gamma_1 +_{S_k} \Gamma_2$ is*

$$M^{\gamma}\left(\Gamma_1 +_{S_k} \Gamma_2\right) = \sum_{i=0}^{\alpha} \binom{\alpha}{i} \left(M_{\Gamma_1}^{\alpha-i}\right)\left(M_{\Gamma_2}^{i+1}\right) + n_{\Gamma_2} M_{S_k(\Gamma_1)}^{\gamma} + \sum_{i=1}^{\alpha} \binom{\alpha}{i} M_{\Gamma_2}^{i} M_{\Gamma_1}^{\gamma-i} + 2^{\alpha+1}(k-1) n_{\Gamma_2} e_{\Gamma_1}, \quad (6)$$

where N is the set of natural numbers and $\alpha = \gamma - 1$.

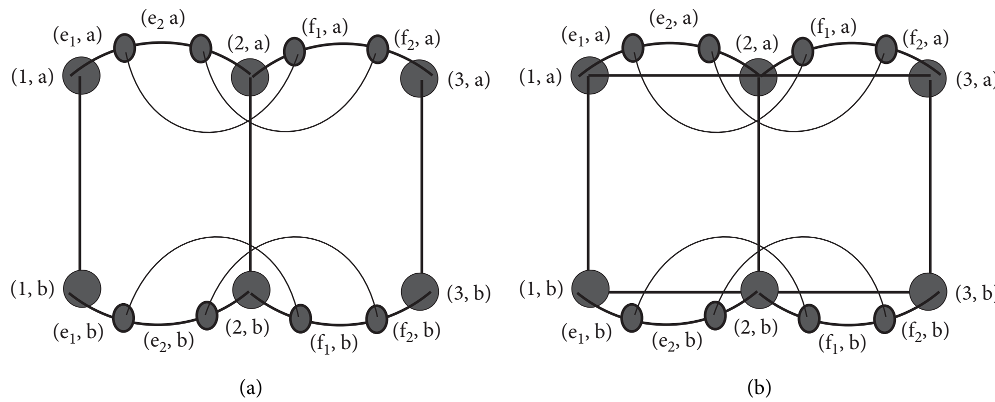

Figure 3: (a) $\Gamma_{1+Q_2}\Gamma_2$. (b) $\Gamma_{1+T_2}\Gamma_2$.

Proof. Let

$$M^{\gamma}\left(\Gamma_1 +_{S_k} \Gamma_2\right) = \sum_{(a,b)\in V\left(\Gamma_1 +_{S_k}\Gamma_2\right)} d^{\gamma}_{\Gamma_1 +_{S_k}\Gamma_2}(a,b). \tag{7}$$

For $\alpha = \gamma - 1$, then the above equation is considered as

$$\begin{aligned}
M^{\gamma}\left(\Gamma_1 +_{S_k}\Gamma_2\right) &= \sum_{(a,b)(c,d)\in E\left(\Gamma_1 +_{S_k}\Gamma_2\right)} \left[d^{\alpha}_{\Gamma_1 +_{S_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1 +_{S_k}\Gamma_2}(c,d)\right] \\
&= \sum_{a\in V(\Gamma_1)} \sum_{bd\in E(\Gamma_2)} \left[d^{\alpha}_{\Gamma_1 +_{S_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1 +_{S_k}\Gamma_2}(a,d)\right] + \sum_{b\in V(\Gamma_2)} \sum_{ac\in E(S_k(\Gamma_1))} \left[d^{\alpha}_{\Gamma_1 +_{S_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1 +_{S_k}\Gamma_2}(b,c)\right] \\
&= \sum_{a\in V(\Gamma_1)} \sum_{bd\in E(\Gamma_2)} \left[d^{\alpha}_{\Gamma_1 +_{S_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1 +_{S_k}\Gamma_2}(a,d)\right] + \sum_{b\in V(\Gamma_2)} \sum_{ac\in E(S_k(\Gamma_1)) a\in V(\Gamma_1), c\in V(S_k(\Gamma_1))-V(\Gamma_1)} \left[d^{\alpha}_{\Gamma_1 +_{S_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1 +_{S_k}\Gamma_2}(b,c)\right] \\
&\quad + \sum_{b\in V(\Gamma_2)} \sum_{ac\in E(S_k(\Gamma_1)) a,c\in V(S_k(\Gamma_1))-V(\Gamma_1)} \left[d^{\alpha}_{\Gamma_1 +_{S_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1 +_{S_k}\Gamma_2}(b,c)\right].
\end{aligned} \tag{8}$$

For every vertex $a \in V(\Gamma_1)$ and edge $bd \in E(\Gamma_2)$, then 1st term of (8) will be

$$\begin{aligned}
&\sum_{a\in V(\Gamma_1)} \sum_{bd\in E(\Gamma_2)} \left[d^{\alpha}_{\Gamma_1 +_{S_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1 +_{S_k}\Gamma_2}(a,d)\right] \\
&\quad = \sum_{a\in V(\Gamma_1)} \sum_{bd\in E(\Gamma_2)} \left[\sum_{i=0}^{\alpha} \binom{\alpha}{i} d^{\alpha-i}_{\Gamma_1}(a) d^{i}_{\Gamma_2}(b) + \sum_{i=0}^{\alpha} \binom{\alpha}{i} d^{\alpha-i}_{\Gamma_1}(a) d^{i}_{\Gamma_2}(d)\right] \\
&\quad = \sum_{a\in V(\Gamma_1)} \sum_{bd\in E(\Gamma_2)} \left[\sum_{i=0}^{\alpha} \binom{\alpha}{i} d^{\alpha-i}_{\Gamma_1}(a)\left[d^{i}_{\Gamma_2}(b) + d^{i}_{\Gamma_2}(d)\right]\right] \\
&\quad = \sum_{a\in V(\Gamma_1)} \left[\sum_{i=0}^{\alpha} \binom{\alpha}{i} d^{\alpha-i}_{\Gamma_1}(a)\right]\left[\sum_{bd\in E(\Gamma_2)} \left[d^{i}_{\Gamma_2}(b) + d^{i}_{\Gamma_2}(d)\right]\right] \\
&\quad = \sum_{a\in V(\Gamma_1)} \left[\sum_{i=0}^{\alpha} \binom{\alpha}{i} d^{\alpha-i}_{\Gamma_1}(a)]\left(M^{i+1}_{\Gamma_2}\right)\right] \\
&\quad = \sum_{i=0}^{\alpha} \binom{\alpha}{i} \left(M^{\alpha-i}_{\Gamma_1}\right)\left(M^{i+1}_{\Gamma_2}\right).
\end{aligned} \tag{9}$$

Since $|E(S_k(\Gamma_1))| = 2|E(\Gamma_1)|$. So, for every $b \in V(\Gamma_2)$ and $ac \in E(S_k(\Gamma_1))$ with $a \in V(\Gamma_1)$, and $c \in V(S_k(\Gamma_1)) - V(\Gamma_1)$; then the 2nd term of (8) is

$$\begin{aligned}
&\sum_{b\in V(\Gamma_2)}\sum_{ac\in E(S_k(\Gamma_1))a\in V(\Gamma_1),c\in V(S_k(\Gamma_1))-V(\Gamma_1)}\left[d^{\alpha}_{\Gamma_1+_{S_k}\Gamma_2}(a,b)+d^{\alpha}_{\Gamma_1+_{S_k}\Gamma_2}(b,c)\right]\\
&=\sum_{b\in V(\Gamma_2)}\sum_{ac\in E(S_k(\Gamma_1))}\left[d^{\alpha}_{\Gamma_1+_{S_k}\Gamma_2}\right](a,b)+d^{\alpha}_{\Gamma_1+_{S_k}\Gamma_2}(b,c)\\
&=\sum_{b\in V(\Gamma_2)}\sum_{ac\in E(S_k(\Gamma_1))}\left[\sum_{i=0}^{\alpha}\binom{\alpha}{i}d^{\alpha-i}_{S_k(\Gamma_1)}(a)d^{i}_{\Gamma_2}(b)+d^{\alpha}_{S_k(\Gamma_1)}(c)\right]\\
&=\sum_{b\in V(\Gamma_2)}\sum_{ac\in E(S_k(\Gamma_1))}\left[d^{\alpha}_{S_k(\Gamma_1)}(a)+\sum_{i=0}^{\alpha}\binom{\alpha}{i}d^{\alpha-i}_{S_k(\Gamma_1)}(a).d^{i}_{\Gamma_2}(b)+d^{\alpha}_{S_k(\Gamma_1)}(c)\right]\\
&=\sum_{b\in V(\Gamma_2)}\sum_{ac\in E(S_k(\Gamma_1))}\left[d^{\alpha}_{S_k(\Gamma_1)}(a)+d^{\alpha}_{S_k(\Gamma_1)}(c)+\sum_{i=1}^{\alpha}\binom{\alpha}{i}d^{\alpha-i}_{S_k(\Gamma_1)}(a)d^{i}_{\Gamma_2}(b)\right]\\
&=\sum_{b\in V(\Gamma_2)}\sum_{ac\in E(S_k(\Gamma_1))}\left[d^{\alpha}_{S_k(\Gamma_1)}(a)+d^{\alpha}_{S_k(\Gamma_1)}(c)\right]+\sum_{b\in V(\Gamma_2)}\sum_{ac\in E(S_k(\Gamma_1))}\left[\sum_{i=1}^{\alpha}\binom{\alpha}{i}d^{\alpha-i}_{S_k(\Gamma_1)}(a)d^{i}_{\Gamma_2}(b)\right]\\
&=\sum_{b\in V(\Gamma_2)}\left[M^{\alpha+1}_{S_k(\Gamma_1)}\right]+\sum_{i=1}^{\alpha}\binom{\alpha}{i}\left[M^{i}_{\Gamma_2}\right]\left[M^{\gamma-i}_{\Gamma_1}\right]\\
&=n_{\Gamma_2}\left[M^{\gamma}_{S_k(\Gamma_1)}\right]+\sum_{i=1}^{\alpha}\binom{\alpha}{i}\left[M^{i}_{\Gamma_2}\right]\left[M^{\gamma-i}_{\Gamma_1}\right],
\end{aligned}\tag{10}$$

and the 3rd term of equation (8) will be

$$\sum_{b\epsilon V(\Gamma_2)}\sum_{ac\epsilon E(S_k(\Gamma_1))a,c\epsilon V(S_k(\Gamma_1))-V(\Gamma_1)}\left[d^{\alpha}_{\Gamma_1+_{S_k}\Gamma_2}(a,b)+d^{\alpha}_{\Gamma_1+_{S_k}\Gamma_2}(c,b)\right]=\sum_{b\epsilon V(\Gamma_2)}\sum_{ac\epsilon E(S_k(\Gamma_1))a,c\epsilon V(S_k(\Gamma_1))-V(\Gamma_1)}\left[2^{\alpha}+2^{\alpha}\right].\tag{11}$$

Since in this case $|E(S_k(\Gamma_1))| = (k-1)|e_{\Gamma_1}$, we have

$$= 2^{\alpha+1}(k-1)n_{\Gamma_2}e_{\Gamma_1}.\tag{12}$$

By using (9), (10), & (12) in (8), we get

$$M^{\gamma}\left(\Gamma_1+_{S_k}\Gamma_2\right)=\sum_{i=0}^{\alpha}\binom{\alpha}{i}\left(M^{\alpha-i}_{\Gamma_1}\right)\left(M^{i+1}_{\Gamma_2}\right)+n_{\Gamma_2}M^{\beta}_{S_k(\Gamma_1)}+\sum_{i=1}^{\alpha}\binom{\alpha}{i}M^{i}_{\Gamma_2}M^{\gamma-i}_{\Gamma_1}+2^{\alpha+1}(k-1)n_{\Gamma_2}e_{\Gamma_1}.\tag{13}$$

Theorem 2. *Let Γ_1 and Γ_2 be two simple graphs and $\gamma \in N - \{0, 1\}$. The FGZ index of the generalized R-sum $\Gamma_1+_{R_k}\Gamma_2$ graph is*

$$\begin{aligned}M^{\gamma}\left(\Gamma_1+_{R_k}\Gamma_2\right)&=\sum_{i=0}^{\alpha}\binom{\alpha}{i}2^{\alpha-i}M^{\alpha-i}_{\Gamma_1}M^{i+1}_{\Gamma_2}+2\sum_{i=0}^{\alpha}\binom{\alpha}{i}2^{\alpha-i}M^{\gamma-i}_{\Gamma_1}M^{i}_{\Gamma_2}\\&\quad+2^{\alpha}e_{\Gamma_1}n_{\Gamma_2}+2^{\alpha+1}(k-1)n_{\Gamma_2}e_{\Gamma_1},\end{aligned}\tag{14}$$

where N is the set of natural numbers and $\alpha = \gamma - 1$.

Proof. Then by definition, we have,

$$M^{\gamma}\left(\Gamma_1+_{R_k}\Gamma_2\right)=\sum_{(a,b)\in\left(V\left(\Gamma_1+_{R_k}\Gamma_2\right)\right)}d^{\gamma}_{\Gamma_1+_{R_k}\Gamma_2}(a,b).\tag{15}$$

For $\alpha = \gamma - 1$, the above equation is consider as

$$\begin{aligned}
M^{\gamma}\left(\Gamma_1 +_{R_k} \Gamma_2\right) &= \sum_{(a,b)(c,d)\in\left(E\left(\Gamma_1+_{R_k}\Gamma_2\right)\right)} \left[d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(c,d)\right] \\
&= \sum_{a\in V(\Gamma_1)} \sum_{bd\in(E(\Gamma_2))} \left[d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,d)\right] + \sum_{b\in V(\Gamma_2)} \sum_{ac\in(E(R_k(\Gamma_1)))} \left[d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(b,c)\right] \\
&= \sum_{a\in V(\Gamma_1)} \sum_{bd\in(E(\Gamma_2))} \left[d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,d)\right] + \sum_{b\in V(\Gamma_2)} \sum_{ac\in(E(R_k(\Gamma_1)))a,c\in V(\Gamma_1)} \left[d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(b,c)\right] \\
&\quad + \sum_{b\in V(\Gamma_2)} \sum_{ac\in(E(R_k(\Gamma_1)))a\in V(\Gamma_1)c\in V(R_k(\Gamma_1))-V(\Gamma_1)} \left[d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(b,c)\right] \\
&= \sum_{a\in V(\Gamma_1)} \sum_{bd\in(E(\Gamma_2))} \left[d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,d)\right] + \sum_{b\in V(\Gamma_2)} \sum_{ac\in(E(R_k(\Gamma_1)))a,c\in V(\Gamma_1)} \left[d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(b,c)\right] \\
&\quad + \sum_{b\in V(\Gamma_2)} \sum_{ac\in(E(R_k(\Gamma_1)))a\in V(\Gamma_1)c\in V(R_k(\Gamma_1))-V(\Gamma_1)} \left[d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(b,c)\right] \sum_{b\varepsilon V(\Gamma_2)} \\
&\quad \cdot \sum_{ac\varepsilon E(R_k(\Gamma_1))a,c\varepsilon V(R_k(\Gamma_1))-V(\Gamma_1)} \left[d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(c,b)\right].
\end{aligned} \tag{16}$$

For every vertex $a \in V(\Gamma_1)$ & edge $b\,d \in E(\Gamma_2)$, then the 1st term of (16) is

$$\begin{aligned}
\sum_{a\in V(\Gamma_1)} \sum_{bd\in(E(\Gamma_2))} \left[d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,d)\right] &= \sum_{a\in V(\Gamma_1)} \sum_{bd\in(E(\Gamma_2))} \left[\left[\sum_{i=0}^{\alpha}\binom{\alpha}{i} d^{\alpha-i}_{R_k(\Gamma_1)}(a).d^{i}_{\Gamma_2}(b)\right] + \left[\sum_{i=0}^{\alpha}\binom{\alpha}{i} d^{\alpha-i}_{R_k(\Gamma_1)}(a).d^{i}_{\Gamma_2}(d)\right]\right] \\
&= \sum_{i=0}^{\alpha}\binom{\alpha}{i} \sum_{a\in V(\Gamma_1)} d^{\alpha-i}_{R_k(\Gamma_1)}(a) \sum_{bd\in(E(\Gamma_2))} d^{i}_{\Gamma_2}(b) + \sum_{i=0}^{\alpha}\binom{\alpha}{i} \sum_{a\in V(\Gamma_1)} d^{\alpha-i}_{R_k(\Gamma_1)}(a) \\
&\quad \cdot \sum_{bd\in(E(\Gamma_2))} d^{i}_{\Gamma_2}(d) \\
&= \sum_{i=0}^{\alpha}\binom{\alpha}{i} \sum_{a\in V(\Gamma_1)} \left(2d_{\Gamma_1}(a)\right)^{\alpha-i} \sum_{bd\in(E(\Gamma_2))} d^{i}_{\Gamma_2}(b) + \sum_{i=0}^{\alpha}\binom{\alpha}{i} \sum_{a\in V(\Gamma_1)} \left(2d_{\Gamma_1}(a)\right)^{\alpha-i} \\
&\quad \cdot \sum_{bd\in(E(\Gamma_2))} d^{i}_{\Gamma_2}(d) \\
&= \sum_{i=0}^{\alpha}\binom{\alpha}{i} (2)^{\alpha-i} M^{\alpha-i}_{\Gamma_1} \sum_{bd\in(E(\Gamma_2))} \left[d^{i}_{\Gamma_2}(b) + d^{i}_{\Gamma_2}(d)\right] \\
&= \sum_{i=0}^{\alpha}\binom{\alpha}{i} (2)^{\alpha-i} M^{\alpha-i}_{\Gamma_1} M^{i+1}_{\Gamma_2}.
\end{aligned} \tag{17}$$

For every vertex $b \in V(\Gamma_2)$ & edge $ac \in E(R_k(\Gamma_1))\, a, c \in V(\Gamma_1)$, then the 2nd term of (16) will be

$$\begin{aligned}
&\sum_{b\in V(\Gamma_2)}\sum_{ac\in(E(R_k(\Gamma_1)))}\left[d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,b)+d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(b,c)\right]\\
&=\sum_{b\in V(\Gamma_2)}\sum_{ac\in(E(R_k(\Gamma_1)))}\left[\left[\sum_{i=0}^{\alpha}\binom{\alpha}{i}d^{\alpha-i}_{R_k(\Gamma_1)}(a).d^{i}_{\Gamma_2}(b)\right]+\left[\sum_{i=0}^{\alpha}\binom{\alpha}{i}d^{\alpha-i}_{R_k(\Gamma_1)}(c).d^{i}_{\Gamma_2}(b)\right]\right]\\
&=\sum_{b\in V(\Gamma_2)}\sum_{ac\in(E(R_k(\Gamma_1)))}\left[\sum_{i=0}^{\alpha}\binom{\alpha}{i}d^{i}_{\Gamma_2}(b)\left[d^{\alpha-i}_{R_k(\Gamma_1)}(a)+d^{\alpha-i}_{R_k(\Gamma_1)}(c)\right]\right]\\
&=\sum_{ac\in(E(R_k(\Gamma_1)))a,c\in V(\Gamma_1)}\left[\left[\sum_{i=0}^{\alpha}\binom{\alpha}{i}\left[\sum_{b\in V(\Gamma_2)}d^{i}_{\Gamma_2}(b)\left[d^{\alpha-i}_{R_k(\Gamma_1)}(a)+d^{\alpha-i}_{R_k(\Gamma_1)}(c)\right]\right]\right]\right]\\
&=\sum_{ac\in(E(R_k(\Gamma_1)))a,c\in V(\Gamma_1)}\left[\left[\sum_{i=0}^{\alpha}\binom{\alpha}{i}M^{i}_{\Gamma_2}\left[\left(2d_{\Gamma_1}(a)\right)^{\alpha-i}+\left(2d_{\Gamma_1}(c)\right)^{\alpha-i}\right]\right]\right]\\
&=\sum_{ac\in(E(R_k(\Gamma_1)))a,c\in V(\Gamma_1)}\sum_{i=0}^{\alpha}\binom{\alpha}{i}M^{i}_{\Gamma_2}\left[(2)^{\alpha-i}\left(d^{\alpha-i}_{\Gamma_1}(a)\right)+(2)^{\alpha-i}\left(d^{\alpha-i}_{\Gamma_1}(c)\right)\right]\\
&=\sum_{i=0}^{\alpha}\binom{\alpha}{i}M^{i}_{\Gamma_2}\sum_{ac\in(E(R_k(\Gamma_1)))a,c\in V(\Gamma_1)}\left[(2)^{\alpha-i}\left(d^{\alpha-i}_{\Gamma_1}(a)\right)+(2)^{\alpha-i}\left(d^{\alpha-i}_{\Gamma_1}(c)\right)\right]\\
&=\sum_{i=0}^{\alpha}\binom{\alpha}{i}M^{i}_{\Gamma_2}\sum_{ac\in(E(R_k(\Gamma_1)))a,c\in V(\Gamma_1)}\left[(2)^{\alpha-i}\left(d^{\alpha-i}_{\Gamma_1}(a)\right)+(2)^{\alpha-i}\left(d^{\alpha-i}_{\Gamma_1}(c)\right)\right]\\
&=\sum_{i=0}^{\alpha}\binom{\alpha}{i}M^{i}_{\Gamma_2}(2)^{\alpha-i}\sum_{ac\in(E(R_k(\Gamma_1)))a,c\in V(\Gamma_1)}\left[d^{\alpha-i}_{\Gamma_1}(a)+d^{\alpha-i}_{\Gamma_1}(c)\right]\\
&=\sum_{i=0}^{\alpha}\binom{\alpha}{i}M^{i}_{\Gamma_2}(2)^{\alpha-1}\left[M^{\gamma-i}_{\Gamma_1}\right]\\
&=\sum_{i=0}^{\alpha}\binom{\alpha}{i}M^{i}_{\Gamma_2}(2)^{\alpha-1}M^{\gamma-i}_{\Gamma_1}.
\end{aligned}\tag{18}$$

For every vertex $b \in V(\Gamma_2)$ & edge $ac \in E(R_k(\Gamma_1))\, a \in V(\Gamma_1), c \in v(R_k(\Gamma_1)) - V(\Gamma_1)$. Since we have $d_{R_k(\Gamma_1)}(a) = 2d_{\Gamma_1}(a)\,\forall\, a \in V(\Gamma_1)$ also $d_{R_k(\Gamma_1)}(c) = 2\,\forall\, c \in V(R_k(\Gamma_1)) - V(\Gamma_1)$. So the 3rd term of (16) will be

$$\begin{aligned}
&=\sum_{b\in V(\Gamma_2)}\sum_{ac\in(E(R_k(\Gamma_1)))a\in V(\Gamma_1)c\in V(R_k(\Gamma_1))-V(\Gamma_1)}\left[d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,b)+d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(b,c)\right]\\
&=\sum_{b\in V(\Gamma_2)}\sum_{ac\in(E(R_k(\Gamma_1)))a\in V(\Gamma_1)c\in V(R_k(\Gamma_1))-V(\Gamma_1)}\left[\sum_{i=0}^{\alpha}\binom{\alpha}{i}d^{\alpha-i}_{R_k(\Gamma_1)}(a).d^{i}_{\Gamma_2}(b)+d^{\alpha}_{R_k(\Gamma_1)}(c)\right],
\end{aligned}\tag{19}$$

Here $d^{\alpha}_{R_k(\Gamma_1)} = 2^{\alpha}$ and $d^{\alpha-i}_{R_k(\Gamma_1)}(a) = (2d_{\Gamma_1}(a))^{\alpha-i}$:

$$\begin{aligned}
&= \sum_{b\in V(\Gamma_2)} \sum_{ac\in(E(R_k(\Gamma_1)))a\in V(\Gamma_1)c\in V(R_k(\Gamma_1))-V(\Gamma_1)} \left[\sum_{i=0}^{\alpha}\binom{\alpha}{i}\left(2d_{\Gamma_1}(a)\right)^{\alpha-i} d^{i}_{\Gamma_2}(b) + 2^{\alpha}\right]\\
&= \sum_{b\in V(\Gamma_2)} \sum_{ac\in(E(R_k(\Gamma_1)))a\in V(\Gamma_1)c\in V(R_k(\Gamma_1))-V(\Gamma_1)} \left[\sum_{i=0}^{\alpha}\binom{\alpha}{i}(2)^{\alpha-i}\left(d_{\Gamma_1}(a)\right)^{\alpha-i} d^{i}_{\Gamma_2}(b) + 2^{\alpha}\right]\\
&= \sum_{i=0}^{\alpha}\binom{\alpha}{i}\left[\sum_{b\in V(\Gamma_2)} \sum_{ac\in(E(R_k(\Gamma_1)))a\in V(\Gamma_1)c\in V(R_k(\Gamma_1))-V(\Gamma_1)} (2)^{\alpha-i}\left(d^{\alpha-i}_{\Gamma_1}(a)\right)d^{i}_{\Gamma_2}(b)\right]\\
&\quad+\left[\sum_{b\in V(\Gamma_2)} \sum_{ac\in(E(R_k(\Gamma_1)))a\in V(\Gamma_1)c\in V(R_k(\Gamma_1))-V(\Gamma_1)} 2^{\alpha}\right]\\
&= \sum_{i=0}^{\alpha}\binom{\alpha}{i}(2)^{\alpha-i}.M^{i}_{\Gamma_2}.M^{\gamma-i}_{\Gamma_1} + 2^{\alpha}e_{\Gamma_1}n_{\Gamma_2}\\
&= \sum_{i=0}^{\alpha}\binom{\alpha}{i}(2)^{\alpha-i}.M^{i}_{\Gamma_2}.M^{\gamma-i}_{\Gamma_1} + 2^{\alpha}e_{\Gamma_1}n_{\Gamma_2},
\end{aligned} \tag{20}$$

and the 4th term of (5) is

$$\sum_{b\epsilon V(\Gamma_2)} \sum_{ac\epsilon E(R_k(\Gamma_1))a,c\epsilon V(R_k(\Gamma_1))-V(\Gamma_1)} \left[d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1+_{R_k}\Gamma_2}(c,b)\right] = \sum_{b\epsilon V(\Gamma_2)} \sum_{ac\epsilon E(R_k(\Gamma_1))a,c\epsilon V(R_k(\Gamma_1))-V(\Gamma_1)} \left[2^{\alpha} + 2^{\alpha}\right]. \tag{21}$$

Since in this case $|E(S_k(\Gamma_1))| = (k-1)|e_{\Gamma_1}$, we have

$$= 2^{\alpha+1}(k-1)n_{\Gamma_2}e_{\Gamma_1}. \tag{22}$$

Using (17), (18), (20), and (22) in (16), then we have

$$M\left(\Gamma_1+_{R_k}\Gamma_2\right) = \sum_{i=0}^{\alpha}\binom{\alpha}{i}2^{\alpha-i}M^{\alpha-i}_{\Gamma_1}M^{i+1}_{\Gamma_2} + 2\left[\sum_{i=0}^{\alpha}\binom{\alpha}{i}2^{\alpha-i}M^{\gamma-i}_{\Gamma_1}M^{i}_{\Gamma_2}\right] + 2^{\alpha}e_{\Gamma_1}n_{\Gamma_2} + 2^{\alpha+1}(k-1)n_{\Gamma_2}e_{\Gamma_1}. \tag{23}$$

□

Theorem 3. *Let Γ_1 and Γ_2 be two simple graphs and $\gamma \in N - \{0,1\}$. The FGZ index of the generalized Q-sum $\Gamma_1+_{Q_k}\Gamma_2$ graph is*

$$\begin{aligned}
M^{\gamma}\left(\Gamma_1+_{Q_k}\Gamma_2\right) &= \sum_{i=0}^{\alpha}\binom{\alpha}{i}\left(M^{\alpha-i}_{\Gamma_1}\right)\left(M^{i+1}_{\Gamma_2}\right) + \sum_{i=0}^{\alpha}\binom{\alpha}{i}\left(M^{\gamma-i}_{\Gamma_1}\right)\left(M^{i}_{\Gamma_2}\right) + 2n_{\Gamma_2}\sum_{i=0}^{\alpha}\binom{\alpha}{i}\left(d^{\alpha-i}_{\Gamma_1}(u).d^{i}_{\Gamma_1}(v)\right)\\
&\quad+ n_{\Gamma_2}\sum_{uv\in E(\Gamma_1)vw\in E(\Gamma_1)}\left[\left[\sum_{i=0}^{\alpha}\binom{\alpha}{i}d^{\alpha-i}_{\Gamma_1}(u).d^{i}_{\Gamma_1}(v)\right] + \left[\sum_{i=0}^{\alpha}\binom{\alpha}{i}d^{\alpha-i}_{\Gamma_1}(v).d^{i}_{\Gamma_1}(w)\right]\right]\\
&\quad+ 2(k-1)n_{\Gamma_2}\sum_{uv\epsilon E(\Gamma_1)}\left[d^{\alpha}_{\Gamma_1}(u) + d^{\alpha}_{\Gamma_1}(v)\right],
\end{aligned} \tag{24}$$

where N is the set of natural numbers and $\alpha = \gamma - 1$.

Proof. Then by definition, we have

$$M^{\gamma}\left(\Gamma_1 +_{Q_k} \Gamma_2\right) = \sum_{(a,b)\in V\left(\Gamma_1 +_{Q_k}\Gamma_2\right)} d^{\gamma}_{\Gamma_1 +_{Q_k}\Gamma_2}(a,b) = \sum_{(a,b)\,(c,d)\in E\left(\Gamma_1 +_{Q_k}\Gamma_2\right)} \left[d^{\alpha}_{\Gamma_1 +_{Q_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1 +_{Q_k}\Gamma_2}(c,d)\right]$$
$$= \sum_{a\in V(\Gamma_1)} \sum_{b\,d\in E(\Gamma_2)} \left[d^{\alpha}_{\Gamma_1 +_{Q_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1 +_{Q_k}\Gamma_2}(a,d)\right] + \sum_{b\in V(\Gamma_2)} \sum_{ac\in E(Q_k(\Gamma_1))} \left[d^{\alpha}_{\Gamma_1 +_{Q_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1 +_{Q_k}\Gamma_2}(b,c)\right]. \tag{25}$$

For every vertex $a \in V(\Gamma_1)$ & edge $b\,d \in E(\Gamma_2)$, then the 1st term of (25) will be

$$\begin{aligned}
&\sum_{a\in V(\Gamma_1)} \sum_{b\,d\in E(\Gamma_2)} \left[d^{\alpha}_{\Gamma_1 +_{Q_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1 +_{Q_k}\Gamma_2}(a,d)\right] \\
&= \sum_{a\in V(\Gamma_1)} \sum_{bd\in E(\Gamma_2)} \left[\sum_{i=0}^{\alpha} \binom{\alpha}{i} d^{\alpha-i}_{Q_k(\Gamma_1)}(a) d^{i}_{\Gamma_2}(b) + \sum_{i=0}^{\alpha} \binom{\alpha}{i} d^{\alpha-i}_{Q_k(\Gamma_1)}(a) d^{i}_{\Gamma_2}(d)\right] \\
&= \sum_{a\in V(\Gamma_1)} \sum_{bd\in E(\Gamma_2)} \left[\sum_{i=0}^{\alpha} \binom{\alpha}{i} d^{\alpha-i}_{\Gamma_1}(a) \left[d^{i}_{\Gamma_2}(b) + d^{i}_{\Gamma_2}(d)\right]\right] \\
&= \sum_{a\in V(\Gamma_1)} \left[\sum_{i=0}^{\alpha} \binom{\alpha}{i} d^{\alpha-i}_{\Gamma_1}(a)\right] \sum_{bd\in E(\Gamma_1)} \left[d^{i}_{\Gamma_2}(b) + d^{i}_{\Gamma_2}(d)\right] \\
&= \sum_{a\in V(\Gamma_1)} \left[\sum_{i=0}^{\alpha} \binom{\alpha}{i} d^{\alpha-i}_{\Gamma_1}(a)\right] \left(M^{i+1}_{\Gamma_2}\right) \\
&= \sum_{i=0}^{\alpha} \binom{\alpha}{i} \left(M^{\alpha-i}_{\Gamma_1}\right)\left(M^{i+1}_{\Gamma_2}\right).
\end{aligned} \tag{26}$$

For every vertex $b \in V(\Gamma_2)$ & edge $ac \in E(Q_k(\Gamma_1))\, a, c \in V(\Gamma_1)$, then the 2nd term of equation (25) will be

$$\begin{aligned}
&\sum_{b\in V(\Gamma_2)} \sum_{ac\in E(Q_k(\Gamma_1))} \left[d^{\alpha}_{\Gamma_1 +_{Q_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1 +_{Q_k}\Gamma_2}(b,c)\right] \\
&= \sum_{b\in V(\Gamma_2)} \sum_{ac\in E(Q_k(\Gamma_1)) a\in V(\Gamma_1) c\in V(Q_k(\Gamma_1))-V(\Gamma_1)} \left[d^{\alpha}_{\Gamma_1 +_{Q_k}\Gamma_2}(a,b) + d^{\alpha}_{\Gamma_1 +_{Q_k}\Gamma_2}(b,c)\right] \\
&\quad + \sum_{b\in V(\Gamma_2)} \sum_{ac\in E(Q_k(\Gamma_1))\backslash\backslash a,c\in V(Q_k(\Gamma_1))-V(\Gamma_1)} \left[d^{\alpha}_{\Gamma_1 +_{Q_k}\Gamma_2}(a,b) + \left[d^{\alpha}_{\Gamma_1 +_{Q_k}\Gamma_2}(b,c)\right]\right].
\end{aligned} \tag{27}$$

Now $\forall b \in V(\Gamma_2)$, $ac \in E(Q_k(\Gamma_1))$ if $a \in V(\Gamma_1)$ and $c \in V(Q_k(\Gamma_1)) - V(\Gamma_1)$; the 1st term of (27) will be

$$\begin{aligned}
&\sum_{b\in V(\Gamma_2)} \sum_{ac\in E(Q_k(\Gamma_1)) a\in V(\Gamma_1) c\in V(Q_k(\Gamma_1))-V(\Gamma_1)} \left[\left[d^{\alpha}_{\Gamma_1+_{Q_k}\Gamma_2}(a,b)\right] + \left[d^{\alpha}_{\Gamma_1+_{Q_k}\Gamma_2}(b,c)\right]\right] \\
&= \sum_{b\in V(\Gamma_2)} \sum_{ac\in E(Q_k(\Gamma_1)) a\in V(\Gamma_1) c\in V(Q_k(\Gamma_1))-V(\Gamma_1)} \left[\left[d_{Q_k(\Gamma_1)}(a) + d_{\Gamma_2}(b)\right]^{\alpha} + \left[d^{\alpha}_{Q_k(\Gamma_1)}(c)\right]\right] \\
&= \sum_{b\in V(\Gamma_2)} \sum_{ac\in E(Q_k(\Gamma_1)) a\in V(\Gamma_1) c\in V(Q_k(\Gamma_1))-V(\Gamma_1)} \left[\sum_{i=0}^{\alpha}\binom{\alpha}{i} d^{\alpha-i}_{Q_k(\Gamma_1)}(a).d^{i}_{\Gamma_2}(b) + d^{\alpha}_{Q_k(\Gamma_1)}(c)\right] \\
&= \sum_{b\in V(\Gamma_2)} \sum_{ac\in E(Q_k(\Gamma_1)) a\in V(\Gamma_1) c\in V(Q_k(\Gamma_1))-V(\Gamma_1)} \left[\sum_{i=0}^{\alpha}\binom{\alpha}{i} d^{\alpha-i}_{Q_k(\Gamma_1)}(a).d^{i}_{\Gamma_2}(b)\right] \\
&\quad + \sum_{b\in V(\Gamma_2)} \sum_{ac\in E(Q_k(\Gamma_1)) a\in V(\Gamma_1) c\in V(Q_k(\Gamma_1))-V(\Gamma_1)} \left[d^{\alpha}_{Q_k(\Gamma_1)}(c)\right] \\
&= \sum_{i=0}^{\alpha}\binom{\alpha}{i} \sum_{b\in V(\Gamma_2)} d^{i}_{\Gamma_2}(b) \sum_{ac\in E(Q_k(\Gamma_1)) a\in V(\Gamma_1) c\in V(Q_k(\Gamma_1))-V(\Gamma_1)} d^{\alpha-i}_{Q_k(\Gamma_1)}(a) \\
&\quad + \sum_{b\in V(\Gamma_2)} \sum_{ac\in E(Q_k(\Gamma_1)) a\in V(\Gamma_1) c\in V(Q_k(\Gamma_1))-V(\Gamma_1)} \left[d^{\alpha}_{Q_k(\Gamma_1)}(c)\right] \\
&= \sum_{i=0}^{\alpha}\binom{\alpha}{i} M^{i}_{\Gamma_2} M^{\gamma-i}_{\Gamma_1} + n_{\Gamma_2}. \sum_{ac\in E(Q_k(\Gamma_1)) a\in V_{\Gamma_1} c\in V(Q_k(\Gamma_1))-V(\Gamma_1)} \left[d^{\alpha}_{Q_k(\Gamma_1)}(c)\right] \\
&= \sum_{i=0}^{\alpha}\binom{\alpha}{i} M^{i}_{\Gamma_2} M^{\gamma-i}_{\Gamma_1} + n_{\Gamma_2}\left[2\sum_{uv\in E(\Gamma_1)} \left(d_{\Gamma_1}(u) + d_{\Gamma_1}(v)\right)^{\alpha}\right] \\
&= \sum_{i=0}^{\alpha}\binom{\alpha}{i} M^{i}_{\Gamma_2} M^{\gamma-i}_{\Gamma_1} + 2n_{\Gamma_2}\left[\sum_{i=0}^{\alpha}\binom{\alpha}{i} \sum_{uv\in E_{\Gamma_1}} \left(d^{\alpha-i}_{\Gamma_1}(u).d^{i}_{\Gamma_1}(v)\right)\right].
\end{aligned} \tag{28}$$

Now $\forall b \in V(\Gamma_2)$ & edge $ac \in E(Q_k(\Gamma_1))$ if the vertex $a, c \in V(Q_k(\Gamma_1)) - V(\Gamma_1)$. Then the 2nd term of equation (27) splits into two parts for the vertices a and c, then the equation will be

$$
\begin{aligned}
&= \sum_{b\in V(\Gamma_2)} \sum_{ac\in E(Q_k(\Gamma_1)) a,c\in V(Q_k(\Gamma_1))-V(\Gamma_1)} \left[d_{\Gamma_1+_{Q_k}\Gamma_2}(a,b)^{\alpha} + d_{\Gamma_1+_{Q_k}\Gamma_2}(b,c)^{\alpha}\right] \\
&= \sum_{b\in V(\Gamma_2)} \sum_{ac\in E(Q_k(\Gamma_1)) a,c\in V(Q_k(\Gamma_1))-V(\Gamma_1)} \left[d_{\Gamma_1+_{Q_k}\Gamma_2}(a)^{\alpha} + d_{\Gamma_1+_{Q_k}\Gamma_2}(c)^{\alpha}\right] \\
&= \sum_{b\in V(\Gamma_2)} \sum_{uv\in E(\Gamma_1), vw\in E(\Gamma_1)} \left[d_{\Gamma_1}(u) + d_{\Gamma_1}(v)\right]^{\alpha} + \left[d_{\Gamma_1}(v) + d_{\Gamma_1}(w)\right]^{\alpha} \\
&= c_{\Gamma_2} \sum_{uv\in E(\Gamma_1), vw\in E(\Gamma_1)} \left[\left[\sum_{i=0}^{\alpha}\binom{\alpha}{i} d_{\Gamma_1}^{\alpha-i}(u)\cdot d_{\Gamma_1}^{i}(v)\right] + \left[\sum_{i=0}^{\alpha}\binom{\alpha}{i} d_{\Gamma_1}^{\alpha-i}(v).d_{\Gamma_1}^{i}(w)\right]\right],
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
\sum_{b\epsilon V(\Gamma_2)} \sum_{ac\epsilon E(Q_k(\Gamma_1)) ac\epsilon V(Q_k(\Gamma_1))-V(\Gamma_1)} \left[d_{Q_k(\Gamma_1)}^{\alpha}(a) + d_{Q_k(\Gamma_1)}^{\alpha}(c)\right] &= 2(k-1) \sum_{b\epsilon V(\Gamma_2)} \sum_{uv\epsilon E(\Gamma_1)} \left[d_{\Gamma_1}^{\alpha}(u) + d_{\Gamma_1}^{\alpha}(v)\right] \\
&= 2(k-1) n_{\Gamma_2} \sum_{uv\epsilon E(\Gamma_1)} \left[d_{\Gamma_1}^{\alpha}(u) + d_{\Gamma_1}^{\alpha}(v)\right].
\end{aligned} \tag{30}
$$

Using (26), (28), (29), and (30) in (25), we get the required result:

$$
\begin{aligned}
&= \sum_{i=0}^{\alpha}\binom{\alpha}{i}\left(M_{\Gamma_1}^{\alpha-i}\left(M_{\Gamma_2}^{i+1} + \sum_{i=0}^{\alpha}\binom{\alpha}{i} M_{\Gamma_2}^{i} M_{\Gamma_1}^{\alpha}\right.\right. \\
&\quad + 2n_{\Gamma_2}\sum_{i=0}^{\alpha}\binom{\alpha}{i} \sum_{uv\in E'(\Gamma_1)} \left(d_{\Gamma_1}^{\alpha-i}(u).d_{\Gamma_1}^{i}(v)\right] \\
&\quad + n_{\Gamma_2} \sum_{uv\in E'(\Gamma_1), vw\in E'(\Gamma_1)} \left[\sum_{i=0}^{\alpha}\left[\binom{\alpha}{i} d_{\Gamma_1}^{\alpha-i}(u).d_{\Gamma_1}^{i}(v)\right]\right] + \left[\sum_{i=0}^{\alpha}\binom{\alpha}{i} d_{\Gamma_1}^{\alpha-i}(v).d_{\Gamma_1}^{i}(w)\right] + 2(k-1n_{\Gamma_2} \sum_{uv\epsilon E(\Gamma_1)} \left[d_{\Gamma_1}^{\alpha}(u) + d_{\Gamma_1}^{\alpha}(v)\right] \\
&= \sum_{i=0}^{\alpha}\binom{\alpha}{i}\left(M_{\Gamma_1}^{2\gamma-2-i}\left(M_{\Gamma_2}^{2i+1}\right) + 2n_{\Gamma_2}\sum_{i=0}^{\alpha}\binom{\alpha}{i} \sum_{uv\in E(\Gamma_1)} \left(d_{\Gamma_1}^{\alpha-i}(u).d_{\Gamma_1}^{i}(v)\right] + n_{\Gamma_2} \sum_{uv\in E(\Gamma_1), vw\in E(\Gamma_1)} \left[\left[\sum_{i=0}^{\alpha}\binom{\alpha}{i} d_{\Gamma_1}^{\alpha-i}(u).d_{\Gamma_1}^{i}(v)\right]\right.\right. \\
&\quad + \left[\sum_{i=0}^{\alpha}\binom{\alpha}{i} d_{\Gamma_1}^{\alpha-i}(v).d_{\Gamma_1}^{i}(w)\right]\bigg] + 2(k-1)n_{\Gamma_2} \sum_{uv\epsilon E(\Gamma_1)} \left[d_{\Gamma_1}^{\alpha}(u) + d_{\Gamma_1}^{\alpha}(v)\right].
\end{aligned} \tag{31}
$$

Theorem 4. *Let Γ_1 and Γ_2 be two simple graphs. The FGZ index of the generalized T-sum graph $\Gamma_1 +_{T_k} \Gamma_2$ is*

$$
\begin{aligned}
M^{\gamma}\left(\Gamma_1 +_{T_k} \Gamma_2\right) &= \sum_{i=0}^{\alpha}\binom{\alpha}{i}(2)^{\alpha-i} M_{\Gamma_1}^{\alpha-i} M_{\Gamma_2}^{i+1} + \sum_{i=0}^{\alpha}\binom{\alpha}{i} M_{\Gamma_2}^{i}(2)^{(\alpha-i)} M_{\Gamma_1}^{\gamma-i} + \sum_{i=0}^{\alpha}\binom{\alpha}{i}(2)^{\alpha-i} M_{\Gamma_2}^{i} M_{\Gamma_1}^{\gamma-i} \\
&\quad + 2n_{\Gamma_2}\sum_{i=0}^{\alpha}\binom{\alpha}{i} \sum_{uv\in E(\Gamma_1)} \left(d_{\Gamma_1}^{\alpha-i}(u).d_{\Gamma_1}^{i}(v)\right] + n_{\Gamma_2} \\
&\quad \cdot \sum_{uv\in E(\Gamma_1), vw\in E(\Gamma_1)} \left[\left[\sum_{i=0}^{\alpha}\binom{\alpha}{i}\left(d_{\Gamma_1}^{\alpha-i}(u).d_{\Gamma_1}^{i}(v)\right] + \left[\sum_{i=0}^{\alpha}\binom{\alpha}{i} d_{\Gamma_1}^{\alpha-i}(v) + d_{\Gamma_1}^{i}(w)\right]\right]\right] \\
&\quad + 2(k-1)n_{\Gamma_2} \sum_{uv\epsilon E(\Gamma_1)} \left[d_{\Gamma_1}^{\alpha}(u) + d_{\Gamma_1}^{\alpha}(v)\right],
\end{aligned} \tag{32}
$$

where $\gamma \in N^{+} - \{0, 1\}$ *and* $\alpha = \gamma - 1$.

Proof. Since we have $d_{\Gamma_1 +_{T_k} \Gamma_2}(a,b) = d_{\Gamma_1 +_{R_k} \Gamma_2}(a,b)$ for every vertex $a \in V(\Gamma_1)$ and $b \in V(\Gamma_2)$, also $d_{\Gamma_1 +_{T_k} \Gamma_2}(a,b) = d_{\Gamma_1 +_{Q_k} \Gamma_2}(a,b)$ for every vertex $a \in V(T_k(\Gamma_1)) - V(\Gamma_1)$ and $b \in V(\Gamma_2)$, the result follows by the proof of Theorems 2 and 3.

Theorem 5. *Assume that* Γ_1 *and* Γ_2 *are two simple graphs and* $\alpha = \gamma - 1$, *where* $\gamma \in \mathfrak{R} - \{0, N^{+}\}$ *and* $\mathfrak{R}$ *is a set of real number. Then, the FGZ index of generalized F-sum graphs* ($\Gamma_1 +_{S_k} \Gamma_2$, $\Gamma_1 +_{R_k} \Gamma_2$, $\Gamma_1 +_{Q_k} \Gamma_2$, *and* $\Gamma_1 +_{T_k} \Gamma_2$) *are*

$$\text{(i) } M^{\gamma}\left(\Gamma_1 +_{S_k} \Gamma_2\right) = \sum_{i=0}^{\infty} \binom{\alpha}{i} \left(M_{\Gamma_1}^{i}\right)\left(M_{\Gamma_2}^{\alpha-i+1}\right) + n_{\Gamma_2} M_{S_k(\Gamma_1)}^{i} + \sum_{i=1}^{\infty} \binom{\alpha}{i} M_{\Gamma_2}^{\alpha-1} M_{\Gamma_1}^{i+1} + 2^{\gamma+1}(k-1) n_{\Gamma_2} e_{\Gamma_1}$$

$$\text{(ii) } M^{\gamma}\left(\Gamma_1 +_{R_k} \Gamma_2\right) = \sum_{i=0}^{\infty} \binom{\alpha}{i} 2^{i} M_{\Gamma_1}^{i} M_{\Gamma_2}^{\alpha-i+1} + 2\left[\sum_{i=0}^{\infty}\left[\binom{\alpha}{i} 2^{i} M_{\Gamma_1}^{i+1} M_{\Gamma_2}^{\alpha-i}\right]\right] + 2^{\gamma} e_{\Gamma_1} n_{\Gamma_2} + 2^{\gamma+1}(k-1) n_{\Gamma_2} e_{\Gamma_1}$$

$$\begin{aligned}\text{(iii) } M_{\Gamma_1 +_{Q_k} \Gamma_2}^{\gamma} = {} & \sum_{i=0}^{\infty} \binom{\alpha}{i} \left(M_{\Gamma_1}^{i}\right)\left(M_{\Gamma_2}^{\alpha-i+1}\right) + \sum_{i=0}^{\infty} \binom{\alpha}{i} \left(M_{\Gamma_1}^{i+1}\right)\left(M_{\Gamma_2}^{\alpha-i}\right) + 2 n_{\Gamma_2}\left[\sum_{i=0}^{\infty} \binom{\alpha}{i} \left(d_{\Gamma_1}^{i}(u) \cdot d_{\Gamma_1}^{\alpha-i}(v)\right]\right. \\ & + n_{\Gamma_2} \sum_{uv \in E(\Gamma_1), vw \in E(\Gamma_1)} \left[\left[\sum_{i=0}^{\infty} \binom{\alpha}{i} d_{\Gamma_1}^{i}(u) \cdot d_{\Gamma_1}^{\alpha-i}(v)\right] + \left[\sum_{i=0}^{\infty}\left[\binom{\alpha}{i} d_{\Gamma_1}^{i}(v) \cdot d_{\Gamma_1}^{\alpha-i}(w)\right]\right]\right] \\ & + 2(k-1) n_{\Gamma_2} \sum_{uv \varepsilon E(\Gamma_1)} \left[d_{\Gamma_1}^{\gamma}(u) + d_{\Gamma_1}^{\gamma}(v)\right]\end{aligned} \tag{33}$$

$$\begin{aligned}\text{(iv) } M^{\gamma}\left(\Gamma_1 +_{T_k} \Gamma_2\right) = {} & \sum_{i=0}^{\infty} \binom{\alpha}{i} 2^{i} M_{\Gamma_1}^{i} M_{\Gamma_2}^{\alpha-i+1} + \left[\sum_{i=0}^{\infty} \binom{\alpha}{i} 2^{i} M_{\Gamma_1}^{i+1} M_{\Gamma_2}^{\alpha-i}\right] \\ & + \sum_{i=0}^{\infty} \binom{\alpha}{i} \left(M_{\Gamma_1}^{i+1}\right)\left(M_{\Gamma_2}^{\alpha-i}\right) + 2 n_{\Gamma_2} \sum_{i=0}^{\infty} \binom{\alpha}{i} \sum_{uv \in E(\Gamma_1)} \left(d_{\Gamma_1}^{i}(u) \cdot d_{\Gamma_1}^{\alpha-i}(v)\right) \\ & + n_{\Gamma_2} \sum_{uv \in E(\Gamma_1), vw \in E(\Gamma_1)} \left[\left[\sum_{i=0}^{\infty} \binom{\alpha}{i} d_{\Gamma_1}^{i}(u) \cdot d_{\Gamma_1}^{\alpha-i}(v)\right] + \left[\sum_{i=0}^{\infty} \binom{\alpha}{i} d_{\Gamma_1}^{i}(v) \cdot d_{\Gamma_1}^{\alpha-i}(w)\right]\right] \\ & + 2(k-1) n_{\Gamma_2} \sum_{uv \varepsilon E(\Gamma_1)} \left[d_{\Gamma_1}^{\gamma}(u) + d_{\Gamma_1}^{\gamma}(v)\right].\end{aligned}$$

Proof. The above proof is similar as of Theorems 1–4.

Let Γ_1 be a negative integer, so from Theorem 5, Corollary 1 is obtained.

Corollary 1. *Assume that Γ_1 & Γ_2 are two simple graphs and $\alpha = \gamma - 1$, where γ is a negative real number. The FGZ index of the generalized F-sums graphs ($\Gamma_1 +_{S_k} \Gamma_2$, $\Gamma_1 +_{R_k} \Gamma_2$, $\Gamma_1 +_{Q_k} \Gamma_2$, and $\Gamma_1 +_{T_k} \Gamma_2$) are*

$$\begin{aligned}
&(i) M^{\gamma}\left(\Gamma_1 +_{S_k} \Gamma_2\right) = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i-1}{i} \left(M_{\Gamma_1}^{i}\right)\left(M_{\Gamma_2}^{\alpha-i+1}\right) + n_{\Gamma_2} M_{S_k(\Gamma_1)}^{i} + \sum_{i=1}^{\infty} (-1)^i \binom{\alpha+i-1}{i} M_{\Gamma_2}^{\alpha-1} M_{\Gamma_1}^{i+1} + 2^{\gamma+1}(k-1) n_{\Gamma_2} e_{\Gamma_1}\\
&(ii) M^{\gamma}\left(\Gamma_1 +_{R_k} \Gamma_2\right) = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i-1}{i} 2^i M_{\Gamma_1}^{i} M_{\Gamma_2}^{\alpha-i+1} + 2\left[\sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i-1}{i} 2^i M_{\Gamma_1}^{i+1} M_{\Gamma_2}^{\alpha-i}\right] + 2^{\gamma} e_{\Gamma_1} n_{\Gamma_2} + 2^{\gamma+1}(k-1) n_{\Gamma_2} e_{\Gamma_1}\\
&(iii) M_{\Gamma_1 +_{Q_k} \Gamma_2}^{\gamma} = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i-1}{i} \left(M_{\Gamma_1}^{i}\right)\left(M_{\Gamma_2}^{\alpha-i+1}\right) + \sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i-1}{i} \left(M_{\Gamma_1}^{i+1}\right)\left(M_{\Gamma_2}^{\alpha-i}\right)\\
&\quad + 2n_{\Gamma_2} \sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i-1}{i} \sum_{uv\in E(\Gamma_1)} \left(d_{\Gamma_1}^{i}(u).d_{\Gamma_1}^{\alpha-i}(v)\right) + n_{\Gamma_2} \sum_{uv\in E(\Gamma_1), vw\in E(\Gamma_1)} \left[\left[\sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i-1}{i} d_{\Gamma_1}^{i}(u).d_{\Gamma_1}^{\alpha-i}(v)\right]\right.\\
&\quad \left. + \left[\sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i-1}{i} d_{\Gamma_1}^{i}(v).d_{\Gamma_1}^{\alpha-i}(w)\right]\right] + 2(k-1) n_{\Gamma_2} \sum_{uv\epsilon E(\Gamma_1)} \left[d_{\Gamma_1}^{\gamma}(u) + d_{\Gamma_1}^{\gamma}(v)\right]\\
&(iv) M^{\gamma}\left(\Gamma_1 +_{T_k} \Gamma_2\right) = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i-1}{i} 2^i M_{\Gamma_1}^{i} M_{\Gamma_2}^{\alpha-i+1} + \left[\sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i-1}{i} 2^i M_{\Gamma_1}^{i+1} M_{\Gamma_2}^{\alpha-i}\right]\\
&\quad + \sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i-1}{i} \left(M_{\Gamma_1}^{i+1}\right)\left(M_{\Gamma_2}^{\alpha-i}\right) + 2n_{\Gamma_2} \sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i-1}{i} \sum_{uv\in E(\Gamma_1)} \left(d_{\Gamma_1}^{i}(u) \cdot d_{\Gamma_1}^{\alpha-i}(v)\right)\\
&\quad + n_{\Gamma_2} \sum_{uv\in E(\Gamma_1), vw\in E(\Gamma_1)} \left[\left[\sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i-1}{i} d_{\Gamma_1}^{i}(u) \cdot d_{\Gamma_1}^{\alpha-i}(v)\right] + \left[\sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i-1}{i} d_{\Gamma_1}^{i}(v) \cdot d_{\Gamma_1}^{\alpha-i}(w)\right]\right]\\
&\quad + 2(k-1) n_{\Gamma_2} \sum_{uv\epsilon E(\Gamma_1)} \left[d_{\Gamma_1}^{\gamma}(u) + d_{\Gamma_1}^{\gamma}(v)\right].
\end{aligned} \tag{34}$$

4. Applications

Now, we present some examples as applications of the obtained results Theorems 1–4. Also the numerical comparisons are represented in Tables 1–4, and the graphical representations are depicted in Figures 4–7.

Example 1. Let P_m and P_n be two simple graphs with $m \geq 2$ and $n \geq 2$. Then, we have

$$\begin{aligned}
1.\ M^{\gamma}\left(P_{m+S_k} P_n\right) &= \sum_{t=0}^{\gamma} C_T^{\gamma-1}\left[2^{\gamma-1-t}(m-2)+2\right]\left[2^{t+1}(n-2)+2\right] + \sum_{t=0}^{\gamma} C_T^{\gamma-1}\left[2^{\gamma-t}(m-2)+2\right]\left[2^{t}(n-2)+2\right]\\
&\quad + n\left(2^{\gamma}(2m-3)+2\right) + 4(k-1)n(m-1).
\end{aligned} \tag{35}$$

From Figure 4, it is clear that the behavior of FGZ index of the generalized S-sum graph $\Gamma_1 +_{S_k} \Gamma_2$ at $t = 2$ is more better than $t = 0$ and $t = 1$:

$$\begin{aligned}
2.\ M^{\gamma}\left(P_{m+R_k} P_n\right) &= \sum_{t=0}^{\gamma} C_T^{\gamma-1} 2^{\gamma-1-t}\left[2^{\gamma-1-t}(m-2)+2\right].\left[2^{t+1}(n-2)+2\right] + 2\sum_{t=0}^{\gamma} C_T^{\gamma-1} 2^{\gamma-1-t}\left[2^{\gamma-t}(m-2)+2\right]\left[2^{t}(n-2)+2\right]\\
&\quad + 2(m-1)n + 4(k-1)n(m-1).
\end{aligned} \tag{36}$$

Table 1: Numerical comparison for $M^{\gamma}(P_{m+S_k}P_n)$.

[m, n, k]	$T=0$	$T=1$	$T=2$
[1, 1, 1]	−4	−4	−13
[2, 2, 2]	28	28	28
[3, 3, 3]	124	124	133
[4, 4, 4]	308	308	326
[5, 5, 5]	604	604	631
[6, 6, 6]	1036	1036	1072
[7, 7, 7]	1628	1628	1673
[8, 8, 8]	2404	2404	2458
[9, 9, 9]	3388	3388	3451
[10, 10, 10]	4604	4604	4676

Table 2: Numerical comparison for $M^{\gamma}(P_{m+R_k}P_n)$.

[m, n, k]	$T=0$	$T=1$	$T=2$
[1, 1, 1]	−8	−2	−6.5000
[2, 2, 2]	36	24	18
[3, 3, 3]	164	110	90.5000
[4, 4, 4]	400	280	235
[5, 5, 5]	768	558	475.000
[6, 6, 6]	1292	968	836
[7, 7, 7]	1996	1534	1340.5
[8, 8, 8]	2904	2280	2013
[9, 9, 9]	4040	3230	2877.5
[10, 10, 10]	5428	4408	3958

Table 3: Numerical comparison for $M^{\gamma}(P_{m+Q_k}P_n)$.

[m, n, k]	$T=0$	$T=1$	$T=2$
[1, 1, 1]	3	3	−1.5000
[2, 2, 2]	38	38	47
[3, 3, 3]	157	157	179.5000
[4, 4, 4]	408	408	444
[5, 5, 5]	839	839	888.5000
[6, 6, 6]	1498	1498	1561
[7, 7, 7]	2433	2433	2509.5
[8, 8, 8]	3692	3692	3782
[9, 9, 9]	5323	5323	5426.5
[10, 10, 10]	7374	7374	7491

Table 4: Numerical comparison for $M^{\gamma}(P_{m+T_k}P_n)$.

[m, n, k]	$T=0$	$T=1$	$T=2$
[1, 1, 1]	−3	3	3
[2, 2, 2]	54	42	45
[3, 3, 3]	227	173	167
[4, 4, 4]	564	444	417
[5, 5, 5]	1113	903	843
[6, 6, 6]	1922	1598	1493
[7, 7, 7]	3039	2577	2415
[8, 8, 8]	4512	3888	3657
[9, 9, 9]	6389	5579	5267
[10, 10, 10]	8718	7698	7293

From Figure 5, it is clear that the behavior of FGZ index of the generalized R-sum graph $\Gamma_1 +_{R_k} \Gamma_2$ at $t=0$ is more better than $t=1$ and $t=2$:

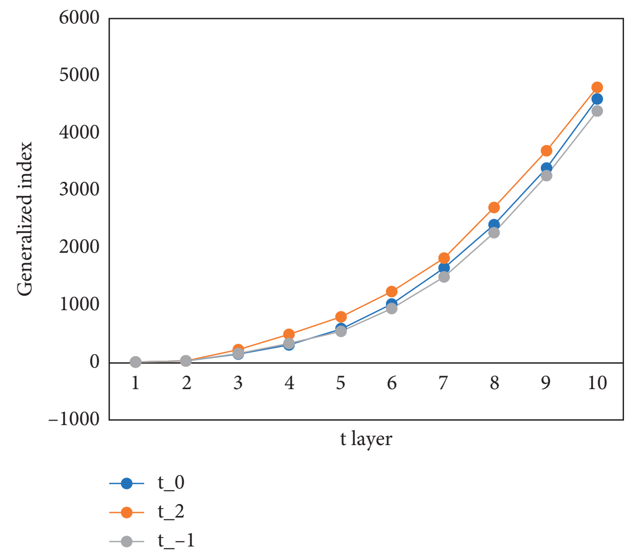

Figure 4: Numerical behavior of $M^{\gamma}(P_{m+S_k}P_n)$ using Table 1.

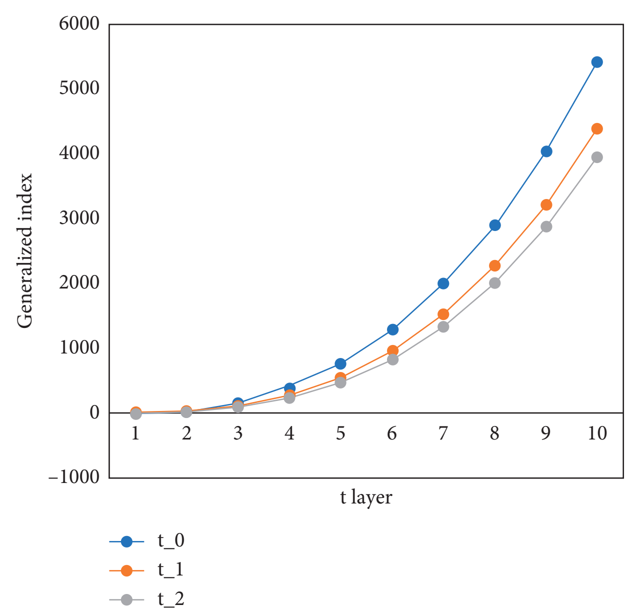

Figure 5: Numerical behavior of $M^{\gamma}(P_{m+R_k}P_n)$ using Table 2.

$$
\begin{aligned}
3.\ M^{\gamma}\left(P_{m+Q_k}P_n\right) = & \sum_{t=0}^{\gamma} C_T^{\gamma-1}\left[2^{\gamma-1-t}(m-2)+2\right]\left[2^{t+1}(n-2)+2\right] \\
& + \sum_{t=0}^{\gamma} C_T^{\gamma-1}\left[2^{\gamma-t}(m-2)+2\right]\left[2^{t}(n-2)+2\right] \\
& + 2n\sum_{t=0}^{\gamma} C_T^{\gamma-1}\left[2^{\gamma-1-t}+(m-2)2^{\gamma-1}+2^{t}\right] \\
& + n\sum_{t=0}^{\gamma} C_T^{\gamma-1}\left[2^{t}+2(m-1)2^{\gamma-1}+2^{\gamma-1-t}\right] \\
& \cdot 2(k-1)n\left[2^{\alpha+1}(m-2)+2\right].
\end{aligned}
\tag{37}
$$

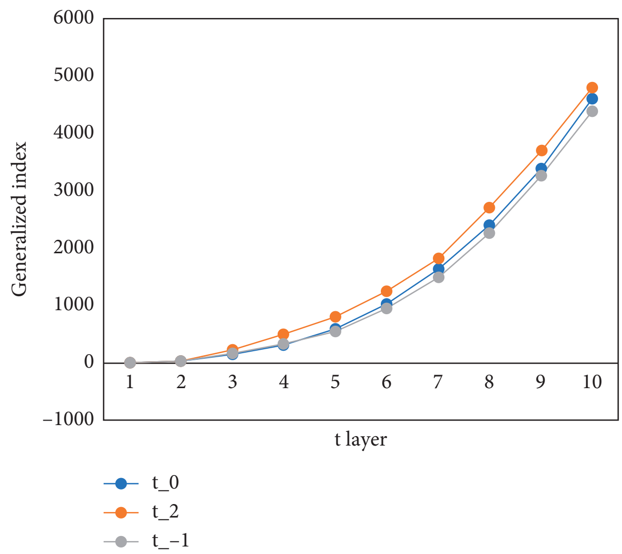

FIGURE 6: Numerical behavior of $M^{\gamma}(P_{m+Q_k}P_n)$ using Table 3.

From Figure 6, it is clear that the behavior of FGZ index of the generalized Q-sum graph $\Gamma_1 +_{Q_k} \Gamma_2$ at $t = 2$ is more better than $t = 0$ and $t = 1$:

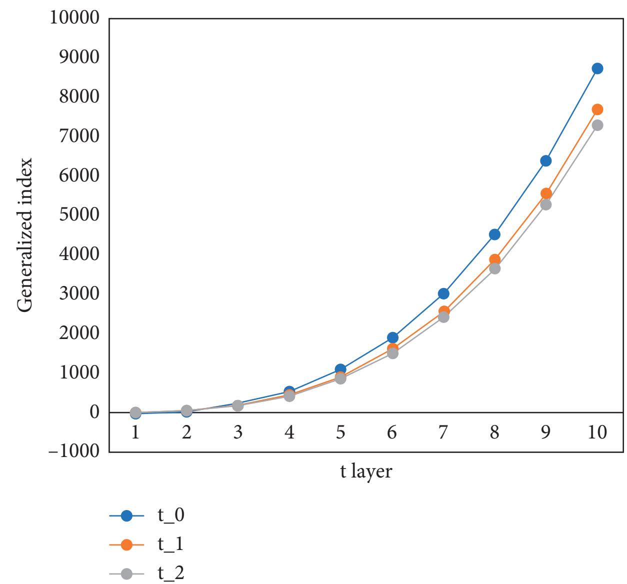

FIGURE 7: Numerical behavior of $M^{\gamma}(P_{m+T_k}P_n)$.

$$4.\ M^{\gamma}\left(P_{m+T_k}P_n\right) = \sum_{t=0}^{\gamma} C_T^{\gamma-1} 2^{\gamma-1-t}\left[2^{\gamma-1-t}(m-2)+2\right]\left[2^{t+1}(n-2)+2\right] + 2\sum_{t=0}^{\gamma} C_T^{\gamma-1} 2^{\gamma-1-t}\left[2^{\gamma-t}(m-2)+2\right]\left[2^{t}(n-2)+2\right]$$
$$+ 2n\sum_{t=0}^{\gamma} C_T^{\gamma-1}\left[2^{\gamma-1-t}+(m-2)2^{\gamma-1}+2^{t}\right] + n\sum_{t=0}^{\gamma} C_T^{\gamma-1}\left[2^{t}+2(m-1)2^{\gamma-1}+2^{\gamma-1-t}\right]2(k-1)n\left[2^{\alpha+1}(m-2)+2\right]. \quad (38)$$

From Figure 7, it is clear that the behavior of FGZ index of the generalized T-sum graph $\Gamma_1 +_{T_k} \Gamma_2$ at $t = 0$ is more better than $t = 1$ and $t = 2$.

5. Conclusions

Now, we close our discussion with the following remarks:

(i) For positive integer k and two graphs Γ_1 & Γ_2, we have computed FGZ index of the generalized F-sums graphs $\Gamma_{1+F_k}\Gamma_2$, where generalized F-sums graphs are obtained by the different operations of subdivision and Cartesian product on Γ_1 & Γ_2.

(ii) The obtained results are also verified and illustrated for the particular classes of graphs.

(iii) The behavior of FGZ index is also analyzed with the help of numerical and graphical presentations.

(iv) However, the problem is still open to compute the different topological indices (degree and distance based) for the generalized F-sum graphs.

Data Availability

All the data are included within this paper. However, the reader may contact the corresponding author for more details of the data.

Conflicts of Interest

The authors have no conflicts of interest.

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