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Research Article

Exploring the Fractal Parameters of Urban Growth and Form with Wave-Spectrum Analysis

Yanguang Chen

Department of Geography, College of Urban and Environmental Sciences, Peking University, Beijing 100871, China

Correspondence should be addressed to Yanguang Chen, chenyg@pku.edu.cn

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The Fourier transform and spectral analysis are employed to estimate the fractal dimension and explore the fractal parameter relations of urban growth and form using mathematical experiments and empirical analyses. Based on the models of urban density, two kinds of fractal dimensions of urban form can be evaluated with the scaling relations between the wave number and the spectral density. One is the *radial dimension* of self-similar distribution indicating the macro-urban patterns, and the other, the profile dimension of self-affine tracks indicating the micro-urban evolution. If a city's growth follows the power law, the summation of the two dimension values may be a constant under certain condition. The estimated results of the radial dimension suggest a new fractal dimension, which can be termed "image dimension". A dual-structure model named *particle-ripple model* (PRM) is proposed to explain the connections and differences between the macro and micro levels of urban form.

1. Introduction

Measurement is the basic link between mathematics and empirical research in any factual science [1]. However, for urban studies, the conventional measures based on Euclidean geometry, such as length, area, and density, are sometimes of no effect due to the scale-free property of urban form and growth. Fortunately, fractal geometry provides us with effective measurements based on fractal dimensions for spatial analysis. Since the concepts of fractals were introduced into urban studies by pioneers, such as Arlinghaus [2], Batty and Longley [3], Benguigui and Daoud [4], Frankhauser and Sadler [5], Goodchild and Mark [6], and Fotheringham et al. [7], many of our theories of urban geography have been reinterpreted using ideas from scaling invariance. Batty and Longley [8] and Frankhauser [9] once summarized the models and theories of fractal cities systematically. From then on, research on fractal cities has progressed in various aspects, including urban forms, structures, transportation, and dynamics of urban evolution (e.g., [10–20]). Because of the development of the cellular automata (CA) theory, fractal geometry and computer-simulated experiment of cities became two principal approaches to researching complex urban systems (e.g., [21–25]).

Despite all the above-mentioned achievements, however, we often run into some difficult problems in urban analysis. The theory on the fractal dimensions of urban space is less developed. We have varied fractal parameters on cities, but we seldom relate them with each other to form a systematic framework. Moreover, the estimation methods of fractal dimensions remain in need of further development. The common approaches to the fractal analysis of cities are limited by self-affine structures. In this instance, three methods, including scaling analysis, spectral analysis, and spatial correlation analysis, are helpful for us to evaluate fractal parameters. The mathematical models of urban density are significant in our research of the fractal form of cities. A density distribution model is usually a spatial correlation function of the distance from city center [26]. In the theory of spectral analysis, the correlation function and energy spectrum can be converted into one another using Fourier transform [27]. Using spectral analysis based on correlation functions, we can find the relations among different fractal parameters, which in turn help us understand urban structure and evolution.

This paper is devoted to exploring the relation between the radial dimension and the self-affine record dimension. The rest of the paper is arranged as follows. In the second section, the wave-spectrum scaling equations for estimating fractal dimensions of urban form are presented. In the third section, two mathematical experiments are implemented to determine the error-correction formula of fractal dimension estimation, and an empirical analysis of Beijing, China, is performed to validate the models and method presented in the text. In the fourth section, a new model of dual structure is proposed to explain urban evolution. Finally, the paper is concluded with a brief summary of this study.

2. Mathematical Models and Fractal Dimension Relations

2.1. Urban Density Functions—Special Spatial Correlation Functions

A fractal is a scale-free phenomenon, but a fractal dimension seems to be a measurement with a characteristic scale. Urban growth and form take on several features of scaling invariance, which can be characterized with fractal dimensions. Three basic concepts about city fractals and fractal dimensions can be outlined here. First, the models of fractal cities are defined in the 2-dimensional Euclidean plane. That is, we investigate the fractal structure of cities through 2D remotely sensed images, digital maps, and so forth. In short, the Euclidean dimension of the embedding space is $d = 2$ [8]. On the other hand, the smallest image-forming units of a city figure can be theoretically treated as points, so the topological dimension of a city form is generally considered to be $d_T = 0$. In terms of the original definition of simple fractals [28], the fractal dimension value of urban form ranges from $d_T = 0$ to $d = 2$. Empirically, the dimension of fractal cities is between 1 and 2. Second, the center of the circles for measuring radial dimension should be the center of a city. The *box dimension* of fractal cities is affirmatively restricted to the interval $1 \sim 2$. However, the *radial dimension* denominated by Frankhauser and Sadler [5] can go beyond the upper limit confined by a Euclidean space. If the measurement center is the centroid of a fractal body, the dimension will not exceed $d = 2$. Otherwise, the radial dimension value may be greater than 2 [23]. Third, for the isotropic growing fractals of cities, the radial dimension is close to the *box dimension* or the *grid dimension* [29]. The radial dimension of a regular self-similar growing fractal equals its box dimension (see [8]). As for cities, if the measurement center is properly located within an urban figure on the digital map, the box dimension will be close to the radial dimension.

Fractal research on urban growth and form is related to the concepts of size, scale, shape, and dimension [30, 31]. Two functions are basic and all-important for these kinds of studies. One is the negative exponential function, and the other is the inverse power function, both of which are associated with fractal cities. They are often employed as density models to describe urban landscapes. The former is mainly used to reflect a city's population density [32–34] while the latter is usually employed to characterize the urban land use density [8, 9]. In fact, the inverse power law can be sometimes applied to describing a city population's spatial distribution [35]. If the fractal structure of a city degenerates to some extent, the land use density also follows exponential distribution. The negative exponential model can be written in the form

$$\rho(r) = \rho_0 e^{-r/r_0}, \tag{2.1}$$

where $\rho(r)$ denotes the population density at the distance r from the center of the city ($r = 0$), ρ_0 refers to a constant coefficient, which theoretically equals the central density $\rho(0)$, and r_0 is the characteristic radius of the population distribution. The reciprocal of r_0 reflects the rate at which the effect of distance decays.

The inverse power law is significant in the spatial analysis of urban form and structure. Formally, given $r > 0$, the power function of urban density can be expressed as

$$\rho(r) = \rho_1 r^{-(d-D_f)}, \tag{2.2}$$

in which $\rho(r)$ and r fulfill the same roles as in (2.1), ρ_1 denotes a proportionality constant, $d = 2$ is the dimension of the embedding space, and D_f is the radial dimension of city form. When $r = 0$, there is a discontinuity and the urban density can be specially defined as ρ_0. Equation (2.1) is the well-known Clark's [34] model and (2.2) Smeed's [36] model.

Urban density functions are in fact special correlation functions that reflect the spatial correlation between a city center and the areas around the center. In theory, almost all fractal dimensions can be regarded as a correlation dimension in a broad sense. For urban growth and form, the D_f can be demonstrated as a one-point correlation dimension (the *zero-order correlation dimension*) while the spectral exponent, β, of the power-law density function can be shown to be a point-point correlation dimension (the *second-order correlation dimension*). These two dimensions can be found within the continuous spectrum of generalized dimensions. By comparing the values of the two correlation dimensions, we can obtain useful information on urban evolution. A fractal dimension is a measurement of space-use extent. Both the box dimension and the D_f can act as two indices for a city. One is the index of uniformity for spatial distribution and the other is the index of space filling, indicative of land use intensity and built-up extent. In addition, the box dimension is associated with information entropy while the D_f is associated with the coefficient of spatial autocorrelation [12].

2.2. The Wave-Spectrum Relation of Urban Density

To simplify the analytical process of spatial scaling, a correlation function can be converted into an energy spectrum using Fourier transform [27]. One of the special properties of the Fourier transform is similarity. By this property, a scaling analysis can be made to derive useful relations of fractal parameters. Any function indicative of self-similarity retains scaling symmetry after being transformed. Consider a density function, $f(r)$, that follows the scaling law

$$f(\lambda r) \propto \lambda^{-\alpha} f(r), \tag{2.3}$$

where λ is the scale factor, α denotes the scaling exponent ($\alpha = d - D_f$), and r represents distance variable. Applying the Fourier transform to (2.3) will satisfy the following scaling relation:

$$F(\lambda k) = \digamma\left[f(\lambda r)\right] = \lambda^{-(1-\alpha)}\digamma\left[f(r)\right] = \lambda^{-(1-\alpha)}F(k), \tag{2.4}$$

in which $\digamma$ refers to the Fourier operator, k to the wave number, and $F(k)$ to the image function of the original function $f(r)$. From (2.4), the wave-spectrum relation can be derived as

$$S(k) \propto k^{-2(1-\alpha)}, \tag{2.5}$$

where $S(k) = |F(k)|^2$ denotes the spectral density of "energy", which bears an analogy to the energy concept in engineering mathematics [37].

The numerical relation between the spectral exponent and fractal dimension can be revealed by comparison. Equation (2.1) fails to follow the scaling law under dilation, while (2.2) is a function of scaling symmetry. Thus, (2.2) can be related to the wave-spectrum scaling. Taking $\alpha = d - D_f$ in (2.5) yields

$$S(k) \propto k^{-2(1-d+D_f)} = k^{-2(D_f-1)} = k^{-\beta}. \tag{2.6}$$

Thus, we have

$$\beta = 2\left(D_f - 1\right). \tag{2.7}$$

The precondition of (2.7) is $1 < D_f < 2$. As stated above, the spectral exponent β can be demonstrated to be the point-point correlation dimension. This implies that (2.7) is a dimension equation that shows the relation between the one-point correlation dimension (D_f) and the point-point correlation dimension (β).

The parameter D_f is the fractal dimension of the self-similar form of cities. We can derive another fractal dimension, the self-affine record dimension, D_s, from the wave-spectrum relation by means of dimensional analysis [38–41]. The well-known result is as follows:

$$\beta = 5 - 2D_s = 2H + 1, \tag{2.8}$$

where D_s and H are the fractal dimensions of the self-affine curve and the Hurst exponent, respectively [42]. The concept of the *Hurst exponent* comes from the method of the rescaled

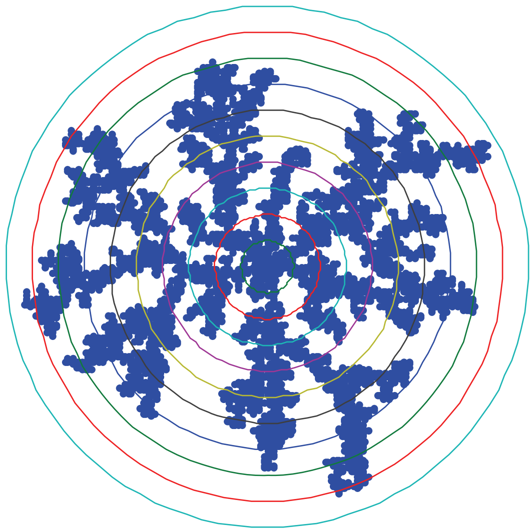

Figure 1: A DLA model showing the particle-ripple duality of city space. (Note that the cluster with a dimension $D \approx 1.7665$ is created in Matlab by using the DLA model. The center of the circles is the origin of growth as the location of the "seed" of DLA.)

range analysis, namely, the R/S analysis [43], which is now widely applied to nonlinear random processes. For the increment series Δx of a space/time series x, H is the scaling exponent of the ratio of the range (R) to the standard deviation (S) versus space/time lag (τ). In other words, H is defined by the power function $R(\tau)/S(\tau) = (\tau/2)^H$ [42].

The parameter D_f is mainly used to analyze the characters of spatial distribution at the macrolevel whereas D_s is used to study the spatial autocorrelation at the microlevel. The latter is termed *profile dimension* because it can be estimated by the profile curve of urban form [37]. The D_s is the local dimension of self-affine fractal records instead of self-similar fractal trails [26, 42]. A useful relation between the D_f and D_s can be derived under certain conditions. Combining (2.7) and (2.8) yields

$$D_f = \frac{7 - 2D_s}{2} = \frac{7}{2} - D_s. \tag{2.9}$$

The question is how to comprehend the relationships and differences between D_f and D_s. Let us look at the diffusion-limited aggregation (DLA) model (Figure 1), which was employed by Batty et al. [44] and Fotheringham et al. [7] to simulate urban growth. In a DLA, each track/trail of a particle has a self-affine record and $D_s = 2$ [42]. However, the final aggregate comprised of countless fine particles takes on the form of statistical self-similarity. In fact, the random walk of the particles in the growing process of DLA is associated with Brownian motion. However, the spatial activity of the "particles" in real urban growth is assumed to be representative of fractional Brownian motion (fBm) rather than standard random walk, thus the D_s of real cities falls between 1 and 2 (see [42, 45] for a discussion on fBm).

Table 1: The numerical relationships between different fractal dimensions, scaling exponents, and autocorrelation coefficients.

Radial dimension (D_f)	Profile dimension (D_s)	Spectral exponent (β)	Hurst exponent (H)	Autocorrelation coefficient (C_Δ)	Correlation function [$C(r)$]
1.00	(2.50)	0.0	(−0.50)	−0.750	$r^{-1.00}$
1.05	(2.45)	0.1	(−0.45)	−0.732	$r^{-0.95}$
1.25	(2.25)	0.5	(−0.25)	−0.646	$r^{-0.75}$
1.50	**2.00**	**1.0**	**0.00**	−0.500	$r^{-0.50}$
1.70	**1.80**	**1.4**	**0.20**	−0.340	$r^{-0.30}$
1.75	**1.75**	**1.5**	**0.25**	−0.293	$r^{-0.25}$
1.95	**1.55**	**1.9**	**0.45**	−0.067	$r^{-0.05}$
2.00	**1.50**	**2.0**	**0.50**	0.000	1.00
(2.25)	1.25	2.5	0.75	0.414	$r^{0.25}$
(2.50)	1.00	3.0	1.00	1.000	$r^{0.50}$

(1) The autocorrelation coefficient (C_Δ) is defined at the micro level and associated with D_s while the correlation function $C(r)$ is defined at the macro level and associated with D_f. (2) The values in the parentheses are meaningless because they go beyond the valid range.

Based on fBm, the relation between H and the autocorrelation coefficient of a increment series can be given as [38, 42]

$$C_\Delta = 2^{2H-1} - 1, \tag{2.10}$$

where C_Δ denotes the autocorrelation coefficient. For urban evolution, C_Δ is a spatial autocorrelation coefficient that is different from Moran's exponent (Moran's I). Moran's I is based on the first-order lag 2-dimensional spatial autocorrelation [46] while C_Δ is based on the multiple-lag 1-dimensional spatial autocorrelation. When $H = 1/2$, $C_\Delta = 0$, indicating Brownian motion (random walk), an independent random process. When $H > 1/2$, $C_\Delta > 0$, indicating positive spatial autocorrelation. Finally, when $H < 1/2$, $C_\Delta < 0$, indicating negative spatial autocorrelation.

In light of (2.8), (2.9), and (2.10), we can reveal the numerical relationships between D_f, D_s, β, H, and C_Δ. The examples are displayed in Table 1. Each parameter has its own valid scale. The D_f, as shown above, ranges from 0 to 2 in theory and 1 to 2 in empirical results. The D_s ranges from 1 to 2, the H ranges from 0 to 1, and the C_Δ ranges from −1 to 1. In sum, only when D_f comes between 1.5 and 2, is the fractal dimension relation, (2.9), theoretically valid. There are two special points in the spectrum of the D_f from 0 to 2. One is $D_f = 1.5$, corresponding to the $1/f$ distribution, and the other is $D_f = 2$, suggesting that a space is occupied and utilized completely. Only within this dimension range, from 1.5 to 2, can the city form be interpreted using the fBm process.

If an urban phenomenon, such as urban land use, follows the inverse power law, it can be characterized by a D_f that varies from 0 to 2. However, what is the dimension of the urban phenomenon that follows the negative exponential law instead of the inverse power law? How can we understand the dimension of urban population if the population density conforms to the negative exponential distribution? These are difficult questions that have puzzled theoretical geographers for a long time. Batty and Kim [35] conducted an interesting discussion about the difference between the exponential function and the power function, and Thomas et al. [20] discussed the fractal question related to the exponential model.

Actually, the spectral density based on the Fourier transform of the negative exponential function approximately follows the inverse power law [37]. The spectral density of the negative exponential distribution meets the scaling relation as follows [38, 47]:

$$S(k) \propto k^{-\beta} = k^{-2}, \tag{2.11}$$

in which $\beta = 2$ is a theoretical value, indicating $D_s = 1.5$. In empirical studies, the calculations may deviate from this standard value and vary from 0 to 3.

The dimension relation, (2.9), can be employed to tackle some difficult problems on cities, including the dimension of urban population departing from self-similar fractal distributions and the scaling exponent of the allometric relation between urban area and population. If urban population density can be described by (2.1), $\beta \to 2$ according to (2.11), and thus we have $D_s \to 3/2$ according to (2.8). Substituting this result into (2.9) yields $D_f \to 7/2 - 3/2 = 2$. This suggests that the dimensions of urban phenomena that satisfy the negative exponential distribution can be treated as $D_f \to d_E = 2$.

To sum up, if we calculate the D_f properly and the value falls between 1.5 and 2, we have a one-point correlation dimension and can estimate the β, D_s, and so forth. Using these fractal parameters, we can conduct spatial correlation analyses of urban evolution. There are often differences between the theoretical results and real calculations because of algorithms among others. However, we can find a formula to correct the errors in computation. For this purpose, a mathematical experiment based on noise-free spatial series is necessary. Moreover, an empirical analysis is essential to support the theoretical relations. The subsequent mathematical experiments consist of two principal parts: one is based on the inverse power law and the other on the negative exponential function. The empirical analysis will involve both the negative exponential distribution and the inverse power-law distribution.

3. Mathematical Experiments and Empirical Analysis

3.1. Mathematical Experiment Based on Inverse Power Law

All the theoretical derivations in Section 2.2 are based on the continuous Fourier transform (CFT), which requires the continuous variable r to vary from negative infinity to infinity ($-\infty < r < \infty$). However, in mathematical experiments or empirical analyses, we can only deal with the discrete sample paths with limited length ($1 \le r < N$). Because of this, the energy spectrum in (2.5), (2.6), and (2.11) should be replaced by the wave spectrum, thus we have

$$W(k) = \frac{S(k)}{N} \propto k^{-\beta}, \tag{3.1}$$

where $W(k)$ refers to the wave-spectral density and N to the length of the sample path. In practice, CFT should be substituted with the discrete Fourier transform (DFT). The calculation error is inevitable owing to the conversion from continuity and infinity to discreteness and finitude.

For the power-law distribution, both D_f and D_s of the urban form can be estimated with the wave-spectrum relation. The procedures in the mathematical experiment are as follows: (1) Create noise-free series of density data for an imaginary land use pattern using (2.2). A real space or time series often consists of trend component, period component, and random component (noise). However, the series produced by theoretical model contain no random component. The D_f value is given in advance ($1 < D_f < 2$). The length of the sample path is taken as $N = 2^z$, where $z = 1, 2, 3 \ldots$ is a positive integer. (2) Implement fast Fourier transform (FFT) on the data. (3) Evaluate β using (3.1). (4) Estimate the fractal dimension value through the spectral exponent and (2.7); the result is notated as D_f^* in contrast to the given value D_f. (5) Compare the difference between the expected value, D_f, and the estimated result, D_f^*. The index of difference can be measured by the squared value of error, $E^2 = (D_f - D_f^*)^2$.

The operation is very simple and all the steps can be carried out in Matlab or MS Excel. Taking $z = 8$, 9, 10, and 11, for example, we have four sample paths of noise-free series of urban land use densities with lengths of $N = 256$, 512, 1024, and 2048, respectively. The length of a sample path is to a space or time series as the size of a sample is to population [48]. It is measured by the number of elements. Given the D_f and ρ_1 values, the data can be produced easily using (2.2). Through spectral analysis, the D_f value can be estimated using (2.7), and the D_s value can be estimated using (2.8). Three conclusions can be drawn from the mathematical experiment. First, the longer the sample path is, the more precise the estimation results will be. The change in accuracy of the fractal dimension estimation over the sample path length is not very remarkable. Second, the closer the fractal dimension value is to $D_f = 1.7$, the better the estimated result will be. For instance, given $N = 512$ and $D_f = 1.05, 1.25, \ldots, 1.95$, the corresponding results of fractal dimension estimation are $D_f^* = 1.4306, 1.5010, \ldots, 1.7693$, respectively. When $D_f = 1.6654$, we have $D_f^* = 1.6654$ and minimal errors are found (Figure 2). This value is very close to $D_f = 1.7$ (Table 2). Third, if we add white noise (a random component) to the data series, the scaling relation between the wave number and the spectral density will not change. The white noise is the simplest series with various frequencies, and the intensity at all frequencies is the same. A formula of error correction can be found by the data in Table 2, that is,

$$D_f \approx \frac{5}{2}\left(D_f^* - 1\right), \tag{3.2}$$

which can be used to reduce the error of the estimated fractal dimension. It is easy to apply the dimension estimation process to the fractal landscape of the DLA model displayed in Figure 1, from which we can abstract a sample of spatial series with random noise.

One of the discoveries is that the estimated result becomes more precise the closer the D_f value approaches 1.7. The relation between the dimension (D_f) and the squared error (E^2) produces a hyperbolic catenary, which can be converted into a concave parabola through the Taylor series expansion. For example, when $N = 2048$, the empirical relation is

$$E^2 = 0.3605D_f^2 - 1.2075D_f + 1.0105. \tag{3.3}$$

The goodness of fit for this relation is $R^2 = 0.9995$. This suggests that when $D_f \approx 1.2075/(2 * 0.3605) \approx 1.675 \rightarrow 1.7$, the square error approaches the minimum ($E^2 \rightarrow 0$).

Another discovery is that the best fit of data to the wave-spectrum relation appears when the fractal dimension approaches $D_f = 1.5$ rather than when $D_f = 1.7$. The relation

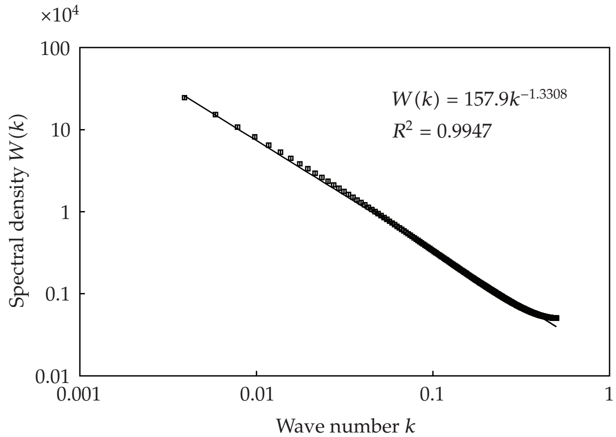

Figure 2: A log-log plot of the wave spectrum relation based on the inverse power function. (Note that a sample, with $N = 512$, can be produced by taking $D_f = 1.6654$ and $\rho_1 = 1000$ in (2.2). The spectral exponent of this data set is computed as $\beta \approx 1.3308$, thus (2.7) yields a dimension estimation $D_f^* \approx 1.6654$.)

between the logarithm of the fractal dimension ($\ln D_f$) and the squared correlation coefficient (R^2) is a convex parabola. For instance, taking $N = 2048$, we have another parabola equation

$$R^2 = -0.0438\left[\ln\left(D_f\right)\right]^2 + 0.0398\ln\left(D_f\right) + 0.986. \tag{3.4}$$

The goodness of fit is $R^2 = 0.9942$. This implies that when $D_f \approx \exp[0.0398/(2 * 0.0438)] \approx 1.575$, the R^2 value approaches the maximum ($R^2 \to 1$). If $D_f = 1.5$, we have $\beta = 1$ (Table 1). In fact, when $\beta \to 3$, the spectrum of short waves becomes divergent; when $\beta \to 0$, the spectrum of long waves becomes divergent. Only when $\beta \to 1$, does the wave spectrum converge in the best way [38].

3.2. Mathematical Experiment Based on Negative Exponential Function

For the negative exponential distribution, the D_f of self-similar urban form does not exist. However, we can estimate the D_s of self-affine curves by means of the wave-spectrum relation. The procedure is comprised of five steps. The first step is to use (2.1) to produce a noise-free series of the urban density by taking certain ρ_0 and r_0 values. The length of the sample path is also taken as 2^z ($z = 1, 2, 3, \ldots$). The next four other steps are similar to those used for estimating the D_f in Section 3.1. The notation of the computed fractal dimension is D_s^*, differing from the given dimension D_s. The expected dimension value is $D_s = 1.5$, and the estimation of the fractal parameter can be illustrated with a log-log plot (Figure 3). The corresponding landscape of exponential distribution can be found in a real urban shape (Figure 4). The longer the sample path is, the closer the spectral exponent value is to $\beta = 2$ and the closer the estimated value of the profile dimension is to $D_s = 1.5$ (Table 3). The length of the spatial series is long enough in theory, so the spectral exponent will be infinitely close to 2 and the D_s^* value will be infinitely close to 1.5.

Random fractal forms can be associated with fBm, with H varying from 0 to 1, thus D_s varying from 1 to 2. If $H = 1/2$, then $C_\Delta = 0$ and $D_s = 1.5$, indicating Brownian motion instead of fBm. This suggests that the city form that satisfies the negative exponential distribution is

Table 2: Comparison between the fractal dimension values of an imaginary city form and its estimated results from the spectral exponent.

Length of sample path (N)	Radial dimension (D_f)	Spectral exponent (β)	Goodness of fit (R^2)	Estimation of D_f (D_f^*)	Estimation of D_s (D_s^*)	Square error (E^2)
	1.0500	0.8684	0.9919	1.4342	2.0658	0.1476
	1.2500	1.0044	0.9943	1.5022	1.9978	0.0636
	1.5000	1.1904	**0.9950**	1.5952	1.9048	0.0091
256	**1.6536**	**1.3072**	**0.9946**	**1.6536**	**1.8464**	**0.0000**
	1.7000	1.3417	0.9944	1.6709	1.8292	0.0008
	1.7500	1.3783	0.9942	1.6892	1.8109	0.0037
	1.9500	1.5126	0.9933	1.7563	1.7437	0.0375
	1.0500	0.8612	0.9903	1.4306	2.0694	0.1449
	1.2500	1.0020	0.9938	1.5010	1.9990	0.0630
	1.5000	1.1974	**0.9950**	1.5987	1.9013	0.0097
512	**1.6654**	**1.3308**	**0.9947**	**1.6654**	**1.8346**	**0.0000**
	1.7000	1.3582	0.9946	1.6791	1.8209	0.0004
	1.7500	1.3970	0.9944	1.6985	1.8015	0.0027
	1.9500	1.5386	0.9934	1.7693	1.7307	0.0327
	1.0500	0.8557	0.9889	1.4279	2.0722	0.1428
	1.2500	0.9998	0.9933	1.4999	2.0001	0.0625
	1.5000	1.2026	**0.9951**	1.6013	1.8987	0.0103
1024	**1.6756**	**1.3512**	**0.9947**	**1.6756**	**1.8244**	**0.0000**
	1.7000	1.3715	0.9946	1.6858	1.8143	0.0002
	1.7500	1.4124	0.9944	1.7062	1.7938	0.0019
	1.9500	1.5605	0.9934	1.7803	1.7198	0.0288
	1.0500	0.8517	0.9877	1.4259	2.0742	0.1413
	1.2500	0.9981	0.9929	1.4991	2.0010	0.0620
	1.5000	1.2066	**0.9951**	1.6033	1.8967	0.0107
2048	**1.6846**	**1.3691**	**0.9947**	**1.6846**	**1.8155**	**0.0000**
	1.7000	1.3825	0.9946	1.6913	1.8088	0.0001
	1.7500	1.4152	0.9944	1.7076	1.7924	0.0018
	1.9500	1.5790	0.9933	1.7895	1.7105	0.0258

based on the Brownian motion process with a self-affine fractal property. The local dimension value of the self-affine fractal record can be estimated as $D_s = 1.5$ by the wave-spectrum relation. In this case, according to (2.9), the dimension of the urban form can be treated as $D_f = 3.5 – D_s = 2$. This is a special dimension value indicative of a self-affine fractal form.

3.3. Empirical Evidence: The Case of Beijing

The spectral analysis can be easily applied to real cities by means of MS Excel, Matlab, or Mathcad. Now, we take the population and land use of Beijing city as an example to show how to make use of the wave spectrum relation in urban studies. The fifth census data of China in 2000 and the land use data of Beijing in 2005 are available. Qianmen, the growth core of Beijing, is taken as the center, and a series of concentric circles are drawn at regular

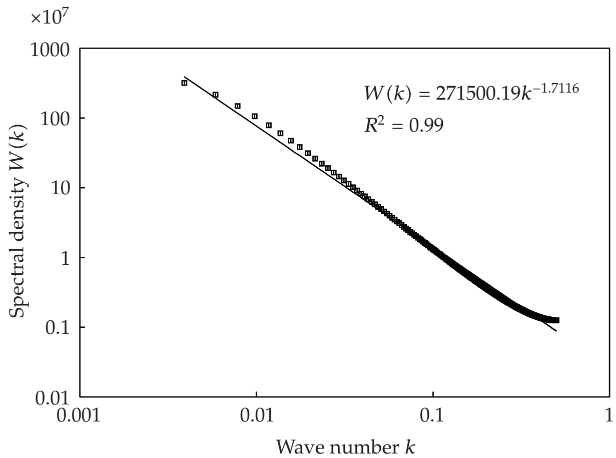

Figure 3: *A log-log plot of a wave-spectrum relation based on negative exponential function.* (Note that taking $\rho_0 = 50000$ and $r_0 = 32$ in (2.1) yields a sample path of $N = 512$. A wave-spectrum analysis of this sample gives $\beta = 1.7116$, which suggests that the fractal dimension of the self-affine record is around $D_s = 1.6442$.)

Table 3: Spectral exponent, fractal dimension, and related parameter values based on the standard exponential distributions (partial results).

Characteristic radius (r_0)	Sample path length (L)	Spectral exponent (β)	Fractal dimension (D_s^*)	Goodness of fit (R^2)
$4 = 2^2$	$64 = 2^6 = 64$	1.3672	1.8164	0.9830
$8 = 2^3$	$128 = 2^7 = 128$	1.5387	1.7307	0.9867
$16 = 2^4$	$256 = 2^8 = 256$	1.6787	1.6607	0.9902
$32 = 2^5$	$512 = 2^9 = 512$	1.7116	1.6442	0.9900
$64 = 2^6$	$1024 = 2^{10} = 1024$	1.7507	1.6247	0.9905
$128 = 2^7$	$2048 = 2^{11} = 2048$	1.7738	1.6131	0.9905
$256 = 2^8$	$4096 = 2^{12} = 4096$	1.7873	1.6064	0.9905

intervals (Figure 4). The width of an interval represents 500 meters on the earth's surface. The land use area between two circles can be measured with the number of pixels on the digital map, and it is not difficult to calculate the area with the aid of ArcGIS software. Thus, the land use density can be determined easily. The population within a ring is hard to estimate because the census is taken in units of *jie-dao* (subdistrict) and each ring runs through different *jie-dao*s. This problem is solved by estimating the weighted average density of the population within a ring [37]. We have 72 circles and thus 72 rings from center to exurb (suburban counties), but only the first 64 data points are adopted because of the algorithmic need of FFT ($N = 2^6$) [27]. The study area is then confined to the field with a radius of 32 kilometers. This is enough for us to study the urban form of Beijing.

The population density distribution of Beijing follows Clark's law and can be fitted to (2.1). An ordinary least squares (OLSs) calculation yields

$$\rho(r) = 30774.8328e^{-r/3.3641}. \tag{3.5}$$

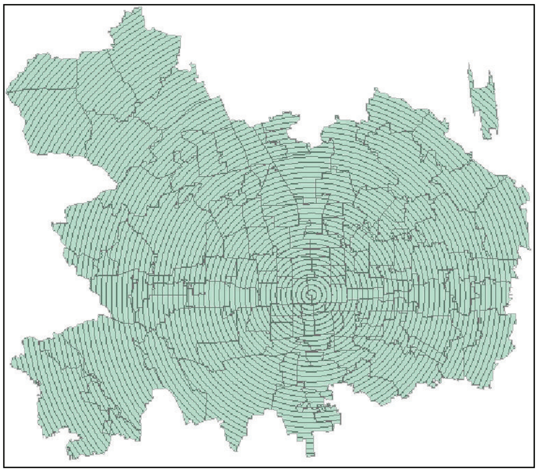

Figure 4: A sketch map of the zonal system of Beijing with a system of concentric circles.

The goodness of fit is about $R^2 = 0.9951$. The population within a certain radius, $P(r)$, does not satisfy the power law. In this instance, Beijing's population distribution cannot be described using the D_f, but it can be depicted by the D_s. That is, the human activities of the city may be based on Brownian motion and contain a set of self-affine fractal records.

The spectral density can be obtained by applying FFT to the population density, involving 64 concentric circles. The relation between the wave number and the spectral density follows the power law. A least squares computation gives the following result:

$$W(k) = 75348.7327k^{-2.0549}. \tag{3.6}$$

The goodness of fit is around $R^2 = 0.9537$ (Figure 5). The estimated value of β (2.0549) is very close to the theoretically expected value ($\beta = 2$). Using (2.8), we can estimate the D_s and have

$$D_s \approx \frac{5 - 2.0549}{2} \approx 1.4726. \tag{3.7}$$

The result approaches the expected value of $D_s(1.5)$. This suggests that the population distribution of Beijing possess some nature of random walk. Then, according to (2.9), the city form's D_f can be estimated to be

$$D_f \approx \frac{2.0549}{2} + 1 \approx 2.0275. \tag{3.8}$$

This value is close to the theoretical value of the Euclidean dimension, $D_f = d = 2$.

Because of underdevelopment of fractal structure, the land use density of Beijing seems to meet the negative exponential distribution rather than the power-law distribution. In a sense, the land use density follows the inverse power law locally. However, as a whole,

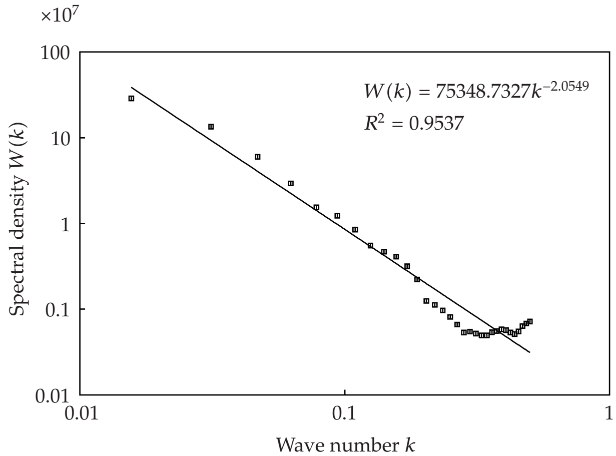

Figure 5: A log-log plot of the wave spectrum relation of Beijing's population density (2000).

the total quantity of land use within a certain radius follows the power law (Figure 6). The integral of (2.2) in the 2-dimensional space is

$$N(r) = N_1 r^{D_f}, \tag{3.9}$$

where $N(r)$ denotes the pixel number indicating the land use area within a radius of r from the city center and N_1 is a constant. Fitting the data of urban land use to (3.9) yields

$$N(r) = 4.2724 r^{1.7827}. \tag{3.10}$$

The goodness of fitness is about $R^2 = 0.985$, and $D_f \approx 1.7827$. Accordingly, $D_s \approx 1.7173$, and $\beta \approx 1.5654$.

For the standard power-law distribution, the D_f of urban form can be estimated by either (2.2) or (3.9). However, as indicated above, the D_f of Beijing cannot be evaluated through (2.2) because the city's land use density fails to follow the inverse power law properly. We can approximately estimate the fractal dimension through spectral analysis based on (2.2). The spectral density is still generated with FFT. The linear relation between the wave number and the spectral density is obvious in the log-log plot (Figure 7). A least squares computation yields

$$W(k) = 0.0009 k^{-1.703}. \tag{3.11}$$

The goodness of fit is about $R^2 = 0.9905$, and $\beta \approx 1.7030$. Correspondingly, the D_f can be estimated as

$$D_f^* \approx \frac{1.7030}{2} + 1 \approx 1.8515, \tag{3.12}$$

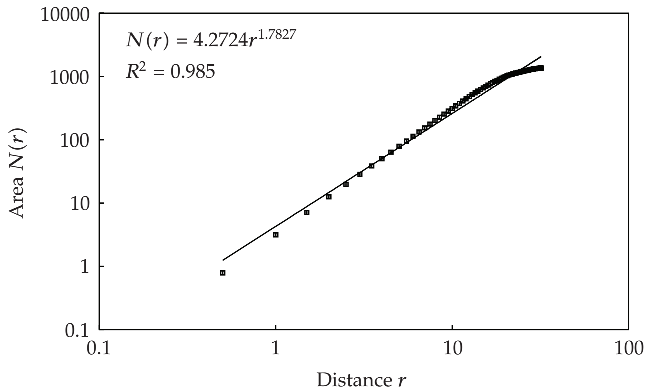

Figure 6: A log-log plot of the relation between radius and corresponding land use quantity of Beijing (2005).

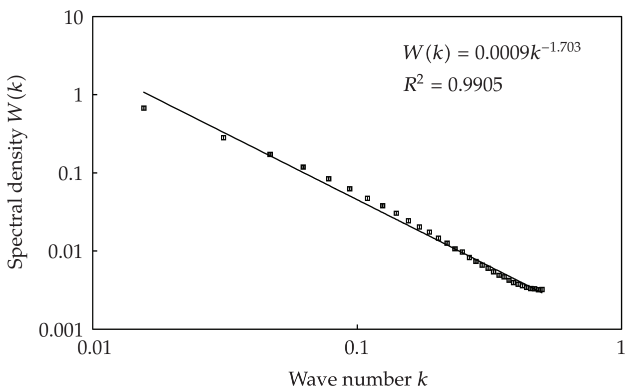

Figure 7: A log-log plot of the wave-spectrum relation of Beijing's land use patterns.

which can be corrected to $D_f \approx 1 + 0.4 * 1.8515 \approx 1.7406$. Accordingly, the D_s is

$$D_s^* \approx \frac{5 - 1.7030}{2} \approx 1.6485. \tag{3.13}$$

This implies that the fractal dimension can be evaluated either by the integral result of (2.2) or by the wave spectrum relation based on (2.2). The former method is more convenient, while the latter approach can be used to reveal the regularity on a large scale due to the filter function of Fourier transform.

To sum up, the D_f of Beijing's city form can be either directly evaluated ($D_f \approx 1.7827$) or indirectly estimated through spectral analysis ($D_f^* \approx 1.8515$). The difference between these two results is due to algorithmic rules and random disturbance among others. The D_s cannot be directly evaluated in this case. The spectral analysis is the most convenient approach to estimating it ($D_s^* \approx 1.6485$). Of course, it can be indirectly estimated with the number-radius scaling ($D_s \approx 1.7173$). The D_f of Beijing's urban population can be treated as $D_f \approx 2$ ($D_f^* \approx 2.0275$), and $D_s \approx 1.5$ ($D_s^* = 1.4727$). The main results are displayed in Table 4, which shows a concise comparison between the parameter values from different approaches.

Table 4: Fractal dimensions, spectral exponents, and related statistics of land use and population distribution in Beijing.

Type	Dimensions evaluated from wave spectrum relation				Dimensions from direct calculation or theoretical derivation		
	β	D_f^*	D_s^*	R^2	D_f	D_s	R^2
Land use (2005)	1.7030	1.8515	1.6485	0.9905	1.7827[a]	1.7173[a]	0.9850
Population (2000)	2.0549	2.0275	1.4726	0.9537	2.0000[b]	1.5000[b]	(1.0000)

Notes. [a]The calculated value from the number-radius scaling; [b]The expected values from the theoretical derivation. For the power-law distribution, the results can be corrected with (3.2); while the results for the xponential distribution need no correction.

From the fractal perspective, the main conclusions about Beijing's population and land use forms can be drawn as follows. First, the population density of Beijing follows Clark's law, so the spatial distribution of the urban population bears no self-similar fractal property. Second, the land uses of this city take on self-similar fractal features, but the fractal structure degenerates to some extent. The quantity of land use within a radius of r from the city center can be approximately modeled with a power function, and the scaling exponent is the radial dimension. Third, the dynamic process of population and land use possesses self-affine fractal properties. Both the population and land use can be associated with self-affine fractal records. The population pattern is possibly based on Brownian motion while the land use patterns are mainly based on fBm. Fourth, the human activity of Beijing is of locality while the land use is associated with action at a distance. The D_s of the population distribution is near $D_s = 1.5$, which suggests that the H is close to 0.5. Therefore, the C_Δ of the spatial increment series is near zero, and this value reminds us of spatial locality [37]. The D_s of land use is around 1.65, and the corresponding H is 0.35. Thus, the C_Δ is estimated to be about $C_\Delta = -0.2$, which suggests a long memory and antipermanence of spatial correlation between the urban core and periphery.

4. Questions and Discussions

The obvious shortcoming of this work is that the wave-spectrum scaling is only applicable to static pictures of urban structures in mathematical experiments and empirical analyses. By means of computer simulation techniques, such as CA and multiagent systems (MASs) [21], perhaps we can base our urban analysis on the continuous process of urban evolution. This is one of the intended directions of spectral analysis for urban growth and form. The focus of this paper is on the theoretical understanding of fractal cities, rather than a case study of real cities. After all, as Hamming [49] pointed out, the purpose of modeling and computing is insight, not numbers.

To reveal the essential properties of fractal cities in a simple way, a new model of monocentric cities, which can be termed the *particle-ripple model* (PRM), is proposed here (Figure 1). A city system can be divided into two levels: the particle layer and the wave layer. At the micro level, the city can be regarded as an irregular aggregate of "particles" taking on random motion. In contrast, at the macro level, the city can be abstracted as some deterministic pattern based on a system of concentric circles and the concept of statistical averages. The former reminds us of the fractal city model, which can be simulated with the DLA model, dielectric breakdown model (DBM), and CA model, among others [7, 21, 44, 50]. The latter remind us of von Thunen's rings and the Burgess's concentric zones, which

Table 5: The similarities and differences between inverse power law and negative exponential distributions.

Distribution	Level	Fractal property	Fractal dimension	Physical base
Power-law distribution	Macro level	Self-similarity	Radial dimension	Dual entropy maximization
	Micro level	Self-affinity	Profile dimension	fBm
Exponential distribution	Macro level	Non-fractality	Euclidean dimension	Entropy maximization
	Micro level	Self-affinity	Profile dimension	Brownian motion

Notes. The physical bases of the inverse power law and the negative exponential law can be found in the work of Chen [12, 37].

can be modeled with (2.1), (2.2), or (2.6). A simple comparison between the power-law and exponential distributions can be made by means of PRM. The main similarities and differences of the two distributions are outlined in Table 5.

The spatial feature of the particle level can be characterized by the fractal models based on the wave layer. In theory, we can use (2.2), (2.6), or (3.9) to estimate the D_f of the cluster in Figure 1. For convenience, we will notate them as $D_f^{(1)}$,$D_f^{(2)}$, and $D_f^{(3)}$, respectively. The results are expected to be the same for each equation (i.e., $D_f^{(1)} = D_f^{(2)} = D_f^{(3)}$). However, the estimated values in empirical analyses are usually different, that is, $D_f^{(1)} \neq D_f^{(2)} \neq D_f^{(3)}$. In most cases, the value of $D_f^{(1)}$ cannot be properly estimated by using the inverse power function. Taking Beijing as an example, the results are as follows: $D_f^{(2)} \approx 1.7828$, $D_f^{(3)} \approx 1.8515$ (Table 4). However, $D_f^{(1)} \approx 0.5036$ is an unacceptable result because the dimensions of Beijing cannot be less than 1.

The three power functions are related to but different from one another. As a special density-density correlation function, (2.2) can capture more details at the micro level (particle layer). Thus the results are usually disturbed to a great extent by random noises. In contrast, as a function of correlation sum, (3.9) omits detailed information and reflects the geographical feature as a whole (wave layer). Equation (2.6) is based on (2.2). The noise and particulars can be filtrated by FFT so that (2.6) catches the main change trend. Both (2.2) and (3.9) characterize the form of the particle layer through the wave layer. Equation (2.6) describes the city form by projecting the particle layer onto the wave layer. The result of projection is defined in the complex number domain rather than in the real number domain.

The D_s can also be used to characterize urban growth and form. A mathematical model is often defined at the macro level, while the parameters of the model, including fractal dimension, always reflect information at the micro level. Both D_f and D_s are the scaling exponents of spatial correlation based on the particle layer, but they are different from each other. The relationships and distinctions between the D_f and D_s can be summarized in several aspects (Table 6). First, the D_f is a measurement of self-similar form while the D_s is one of the measurements of self-affine patterns. Second, the D_f represents the dimension of spatial distribution while the D_s indicates the dimension of a curve or a surface [26]. Third, the D_f represents density-density correlation at the wave layer, while D_s indicates increment-increment correlation at the particle layer. The former is an exponent of spatial correlation of density distribution while the latter is an exponent of spatial autocorrelation of density increments. Finally, if the D_f value falls between 1.5 and 2, the two dimensions can come into contact with each other ($D_f + D_s = 3.5$).

Table 6: Comparison between the radial and profile dimensions.

Fractal dimension	Description object	Related process	Geometrical meaning
Radial dimension (D_f)	Self-similar form	Macro pattern, growth, form, action of core on periphery, and spatial correlation of density series	Extent of spatial uniformity, space filling extent, and spatial correlation at wave layer
Profile dimension (D_s)	Self-affine track	Micro change, aggregation, dynamics, influence of the previous changes on the following changes, and spatial autocorrelation of increment series	Irregularity of spatial pattern, vestige of spatial motion, and autocorrelation at particle layer

By analogy with the fractal growth of DLA, we can understand city forms through their dimensions. Let us examine the DLA model displayed in Figure 1. For the cluster, $D_f \approx 1.7665$ and the goodness of fit is about $R^2 = 0.9924$. In the aggregation process, each particle moves by following a random path until it touches the growing cluster and becomes part of the aggregate. The track of a particle is a self-affine curve, which cannot be recorded directly and does not concern us. What interests us is the final distribution of all the particles with remnant information on the self-affine movements. For a profile from the center to the edge, on the average, $\beta \approx 1.4967$. Thus, $D_s \approx (5 - 1.4967)/2 \approx 1.7517$, and further, we have $D_f^* = 3.5 - D_s \approx 1.7484$. $H = 2 - \beta \approx 0.2484$, so $C_\Delta \approx -0.2945$ as estimated at the micro level. At the macro level, the one-point correlation function is $C(r) = r^{-0.2335}$. The D_f^* may be treated as a new fractal dimension termed the *image dimension* of urban forms because it always differs from D_f in practice. This dimension can act as a complementary measurement of spatial analysis, which remains to be discussed in future work.

5. Conclusions

Spectral analysis based on Fourier transform is one of powerful tools for the studies of fractal cities. First of all, it can help reveal some theoretical equations, such as the relation between D_f and D_s. Next, it can be used to evaluate fractal dimensions, which are hard to calculate directly, such as the D_s indicative of self-affine record of urban evolution. Finally, it can provide us with a supplementary approach to computing the fractal dimension, which can be directly determined by the area-radius scaling. When the urban density fails to follow the inverse power law properly, spectral analysis is an indispensable way of estimating latent fractal dimensions.

Based on the area-radius relation of cities, the main conclusions of this paper are as follows. First, to describe the core-periphery relationships of urban form, we need at least two fractal dimensions, the D_f and the D_s. The D_f can be either directly calculated with the aid of the area-radius scaling or indirectly evaluated by the wave-spectrum relation. The D_s is mainly estimated with the wave spectrum relation. When the D_f ranges from 1.5 to 2, the sum of the two dimension values is a constant. Second, the dimensions of city phenomena satisfying the negative exponential distribution can be treated as $d = 2$. In spatial analysis, it is important to determine the dimensions of a geographical phenomenon. The dimension based on the power-law distribution is easy to evaluate. However, little is known about the dimensions of geographical systems following the exponential distribution. One useful inference of this study is that the dimension of exponential distribution phenomena is 2. If so, a number of theoretical problems, such as the allometric scaling exponent of urban area and population, can be readily solved. Third, city form bears no characteristic scale, but the fractal dimension of city form possesses a characteristic scale. Various fractal parameters, such as D_f, D_s, β, and H, have mathematical relations with one another. However, the rational ranges of these parameter values are not completely consistent with each other. Only when the value of the D_f varies from 1.5 to 2, will all these fractal parameters become valid in value. This seems to suggest that the range of D_f from 1.5 to 2 is a common scale for all these parameters, thus it is a reasonable scale for the D_f. This scale of fractal dimension is revealing for unborn city planning and the spatial optimization of urban structures.

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