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Sociodynamics Applied to the Evolution of Urban and Regional Structures

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The article consists of two parts. In the first section the concepts of sociodynamics are briefly explained. *Sociodynamics* is a general modelling strategy for the quantitative description of dynamic processes in the human society. The central concepts of sociodynamics include transition rates depending on dynamic utilities and the master equation for the probability distribution over macrovariables. From it the set of nonlinear coupled differential equations for the quasi-meanvalues of the macrovariables can be derived.

In the second part this modelling strategy is applied to two "*windows of perception*" of the evolution of settlement structures: one model refers to the relatively fast *urban evolution* on the microscale and the other refers to the relatively slow *regional evolution* on the macroscale.

The *micromodel* considers the urban structure as a system of sites on which different kinds of buildings (dwellings, schools, stores, service-stations, factories...) can be erected. The step by step evolution of the city configuration is treated as a stochastic process guided by utility considerations. The sociodynamic formalization of this concept leads to equations for the evolution of the urban city configuration. Numerical simulations illustrate this urban "microdynamics".

The *macromodel* treats the settlement formation in a region on a more global scale. The evolution of the density of economically active populations who produce and consume goods is considered. The driving force of density changes is the spatial difference of incomes motivating the individuals to migrate to locations of optimal income. This nonlinear process leads to the self-organisation of spatially heterogeneous population distributions forming the settlements. Their micro-structure can thereupon be treated by the micromodel.

Keywords: Urban dynamics, Master equation, Sociodynamics

1 THE MODELLING CONCEPTS OF SOCIODYNAMICS

In the last decades several interdisciplinary efforts have led to a better understanding of the interrelations between different sciences. One of these fruitful interdisciplinary branches of science is the field of *synergetics* which developed out of physics and treats in high generality the spatial, temporal and functional macrostructures arising in multi-component systems [1]. The concepts of synergetics have been applied so far to physical, chemical and biological systems.

Since the *human society* is also a complex multi-component system, it was a challenge to apply the principles and the mathematical algorithms of synergetics to this object of the social sciences, too. However, there arises a fundamental difficulty: in contrast to comparably complex systems of physics and chemistry *there do not exist fundamental dynamic equations of motion for the elements of the human society, namely the individuals.* Therefore some of the mathematical concepts of synergetics could *not* become *directly operational* in the social sciences.

In spite of that dilemma we shall now see that it is possible to develop a conceptual scheme for the mathematical treatment of collective dynamic processes in the society in terms of evolutionary equations. We denote this *modelling strategy* which allows the application of synergetic concepts to the description and explanation of the macrostructure and -dynamics of the *society as sociodynamics* [2,3]. This field therefore provides a further step into interdisciplinarity because it allows for a *transfer of methods and concepts* from statistical physics and synergetics to the social sciences.

So far the modelling procedures of sociodynamics have been applied to the migration of populations [2,4], to political opinion formation [5], to dynamic phenomena in the economy [6,7] and to the evolution of regional settlement structures [8,9]. In this paper we present a sociodynamic modelling approach to the evolution of urban structures (see Section 3).

We now give a brief general description of the steps of the sociodynamic modelling procedure.

Step 1 The Choice of a Configuration Space of Macrovariables

Let us now consider an approximately separable sector of the society. In order to describe its *state* on a macrolevel we have to introduce an appropriate set of *macrovariables* which expand the *configuration space*.

The *socioconfiguration* is a set of macrovariables most directly linked to the members of the society. It describes the distribution of attitudes and actions among the members of the subpopulations of the society with respect to the possible alternatives of a given "choice set". If n_i^α is the number of members of a subpopulation $\mathcal{P}_\alpha$ with attitude or action i, then the socioconfiguration is the multiple of integer variables $\{\dots n_i^\alpha \dots n_j^\alpha \dots n_k^\beta \dots n_l^\beta \dots\} = \mathbf{n}$.

Beyond the socioconfiguration there exists a set of material variables $\mathbf{m}$ depending on the sector under consideration. Amounts of produced and consumed goods, prices, etc. are examples for material variables of the economic sector. In Section 3 we shall introduce variables to characterize the urban evolution state and they will provide another example for material variables. In contrast to the socioconfiguration, the material variables are only *indirectly* linked to the acting and decision making members of the society. Each set of variables $(\mathbf{n}, \mathbf{m})$ is now represented by a point in the multi-dimensional configuration space.

Step 2 Measures for the "Utility" of Configurations

The state of the social system, which can be macroscopically characterized by the variables $(\mathbf{n}, \mathbf{m})$ is now estimated and valuated by the members of this system.

Let $u_i^\alpha(\mathbf{n}, \mathbf{m})$ be a measure of the (subjective) "utility" or "desirability" attributed to the state $(\mathbf{n}, \mathbf{m})$ and valuated by a member of subpopulation $\mathcal{P}_\alpha$ who momentarily adopts the attitude i. (This involves the simplifying assumption that this individual utility only depends on the macrostatevariables $(\mathbf{n}, \mathbf{m})$ and the status (α, i) of the individual.) Increasingly high positive (or low negative) values of $u_i^\alpha(\mathbf{n}, \mathbf{m})$ denote an increasingly positive (negative) estimation of the situation $(\mathbf{n}, \mathbf{m})$ by the individual (α, i).

It should be mentioned, that the "utility" $u_i^\alpha(\mathbf{n}, \mathbf{m})$ not only includes the valuation of "material values" but also (via $\mathbf{n}$) the (positive or negative) estimation of "abstract attitudes and actions" of the other members of the social system).

The choice of $u_i^\alpha(\mathbf{n}, \mathbf{m})$ represents the core of model building because it reflects the social psychology of the members of the society.

Step 3 Elementary Dynamic Processes: Transition Rates

Up to now we have only considered a momentary picture of the social sector: We have introduced its macrostate $(\mathbf{n}, \mathbf{m})$ and its valuation by subjective utilities $u_i^\alpha(\mathbf{n}, \mathbf{m})$ of representative individuals in status (α, i).

Now we introduce *elementary changes of the variables*. If, e.g., a member of $\mathcal{P}_\alpha$ changes his (her) attitude from i to j, this decision induces a change of the socioconfiguration from $\{\ldots n_i^\alpha \ldots n_j^\alpha \ldots n_k^\beta \ldots n_l^\beta \ldots\}$ to $\{\ldots (n_i^\alpha - 1), \ldots (n_j^\alpha + 1), \ldots n_k^\beta \ldots n_l^\beta \ldots\}$. Similar step by step changes (assumed to be small but discrete) may vary the values of any of the material variables. Since the changes are in general effected by direct or indirect decisions of individuals, who act in a *probabilistic manner*, we describe the elementary transitions between neighbouring macrovariables from $\{\mathbf{n}, \mathbf{m}\}$ to $\{\mathbf{n}', \mathbf{m}'\}$ by *probabilistic transition rates* $p\{\mathbf{n}', \mathbf{m}', \mathbf{n}, \mathbf{m}\}$.

The crucial question now arises, which functional form the transition rates should have in terms of the variables $(\mathbf{n}, \mathbf{m})$ of the initial situation and $\{\mathbf{n}', \mathbf{m}'\}$ of the final situation.

We assume, that the *driving force* behind a transition from $\{\mathbf{n}, \mathbf{m}\}$ to $\{\mathbf{n}', \mathbf{m}'\}$ is the difference between the utility of the final state $\{\mathbf{n}', \mathbf{m}'\}$ and initial state $(\mathbf{n}, \mathbf{m})$ as seen by the actors effecting this transition. If that difference is positive, a transition is favoured, and if it is negative, a transition is disfavoured.

In the simplest case, the most plausible form of $p\{\mathbf{n}', \mathbf{m}'; \mathbf{n}, \mathbf{m}\}$ fulfilling all conditions is therefore

$$p\{\mathbf{n}', \mathbf{m}'; \mathbf{n}, \mathbf{m}\} = \nu \exp\{u(\mathbf{n}', \mathbf{m}') - u(\mathbf{n}, \mathbf{m})\},$$

where $u(\mathbf{n}, \mathbf{m})$ is chosen as a weighted mean of the utilities seen by the actors who give rise to the transition $\{\mathbf{n}, \mathbf{m}\} \to \{\mathbf{n}', \mathbf{m}'\}$.

We see that by step (3) the utilities have now obtained a *dynamical meaning*. They are responsible for the magnitude of transition rates and thus for the dynamics of the system. They are therefore also denoted as "*dynamic utilities*" or "*motivation potentials*" to show that they are introduced into a *dynamic framework* which is different from the stationary (equilibrium) framework of neoclassical economics.

Step 4 Evolution Equations for the Macrovariables

The preceding steps are now sufficient to set up a *fully probabilistic description of the macrodynamics* of the system.

The fundamental equation for this is the *master equation* for the temporal evolution of the *probabilistic distribution* $P(\mathbf{n}, \mathbf{m}; t)$ over the variables $(\mathbf{n}, \mathbf{m})$ of the social sector under consideration. The transition rates of step (3) directly re-appear in the master equation.

This equation still comprises all effects due to stochastic fluctuations of the system. Simultaneously it solves in principle the problem of the feedback loop between microdecisions and macrostate of a social system, since on the one side the individual decisions enter the master equation (via the utility dependent transition rates), whereas on the other hand the macrostate of the system evolves according to the master equation. This evolving macrostate in turn influences the utilities and therefore the decisions of the individuals.

It is easy to go over from the master equation to equations for the *quasi-meanvalues of the macrovariables*. These equations are typically a set of *nonlinear autonomous coupled differential equations* to which thereupon the theory of dynamical systems can be applied.

It is evident from the procedure via steps (1)–(4) that we have *not* made use of any "microscopic equations of motion for the individual components of the system". Nevertheless we have arrived via transition rates and the master equation formalism at *stochastic or deterministic equations of motion for the macrovariables* $(\mathbf{n}, \mathbf{m})$ *of the social system.*

2 THE SPACE–TIME WINDOWS OF PERCEPTION OF SETTLEMENT STRUCTURES

We now apply the modelling concepts of sociodynamics to the construction of models for the evolution of human settlements, which belong to the most complex space–time structures in the world. In settlements there exist many different intertwined and interdependent organisational structures; the evolution of these structures takes place on different scales. Therefore the natural question arises whether this manifold of structures and processes can be ordered according to some principles, for instance the space–time scale or level on which they appear.

If this should prove true and if the relation between the levels could be formulated this would provide the justification for considering different windows of perception and for constructing different, nonetheless interrelated, models for each window.

In the following we shall see that indeed a separation of levels is indicated and that it is appropriate to comprehend and to connect the development of settlement structures on different scales with separate models. Only in a final stage the models can and should be fused into one integrated model.

In synergetics there exists the fruitful "slaving principle" set up in high generality by Haken [1]. Verbally it can be formulated as follows: if in a system of nonlinear equations of motion for many variables these variables can be separated into slow ones and fast ones, a few of the slow variables (those with a trend to grow) are predestined to become "order parameters" dominating the dynamics of the whole system on the macroscale.

The reason for this remarkable system behaviour is that the fast variables quickly adapt their values to the momentary state of the slow variables. Since they thereupon depend on the slow variables, the fast variables can be eliminated. As a consequence the slow variables *alone* obey a quasi-autonomous dynamics. Since all other variables depend by adaptation on the few slow variables which rise up to macroscopic size, the latter are denoted as order parameters and determine the macrodynamics of the system.

Let us now somewhat modify and generalize the slaving principle in view of its meaning for urban and regional structures and their dynamics.

In settlements one can easily identify fast and slow processes of change and evolution:

The *fast processes* take place on the *local microlevel* of building sites where e.g. individual buildings are erected or teared down, and where the local traffic infrastructure of streets, subways, etc. is constructed.

The *slow processes* take place on the *regional macrolevel.* They include the slow evolution of whole settlements like villages, towns and cities which can be considered as population agglomerations of different size, density and composition, furthermore the slow development of whole industries.

The relation betwen the fast development of local microstructures and the slow development of global regional macrostructures is rather simple and exhibits a strong similarity to the slaving principle:

On the one side the fast development of local microstructures is driven and guided by the quasi-constant regional macrostructure into which it is embedded. That means the global regional situation serves as the environment and the boundary condition under which each local urban microstructure evolves.

On the other hand, the (slowly developing) regional macrostructure is of course nothing but the global resultant of the many local structures of which an urban settlement is composed. However, similar to the longevity of the body of an animal, whose organs are regenerating on a shorter time scale than the life time of the whole body, the time of persistence of a regional macrostructure as a whole is much higher than the decay – and regeneration times of its local substructures.

Although this relation between urban microstructures and regional macrostructures is rather evident it has an important consequence for model builders: one can *separate* to some extent the microdynamic level from the macrodynamic level and make *separate adequate models for each space–time window of perception*. This means in more detail:

In constructing a model for the *urban microevolution* it is allowed to consider some global regional parameters (e.g. referring to the global regional population or the global regional stage of industrialisation) as *given environmental conditions* and to describe the fast local microdynamics as developing under these global conditions.

On the other hand, in constructing a model for the *regional macroevolution* it is allowed to presume that a corresponding fast microevolution takes place which *adapts the local microstructures* to the respective slow variables of the global development.

In view of this possibility of a separate consideration of the micro- and macroperspective of settlement evolution we shall present in the next sections the design principles of a micromodel for the urban and of a macromodel for the regional evolution.

3 THE DESIGN PRINCIPLES OF A MICROMODEL OF URBAN EVOLUTION ON THE MICROLEVEL

In constructing a model for the urban evolution on the rather detailed level of individual building plots or sites we follow the general modelling strategy described in Section 1. The modelling scheme consists of the following steps:

1. A configuration space of variables characterizing the state of the urban system has to be set up.
2. A measure for the utility of each configuration under given environmental and populational conditions must be found.
3. Transition rates between neighbouring configurations constitute the elements of the system dynamics. The "driving forces" behind these transitions are utility differences between the initial and the final configuration. Therefore the transition rates depend in an appropriate way on these utility differences.
4. Making use of the transition rates, evolution equations for the configurations can be derived on the stochastic and the quasi-deterministic level as well.
5. Selected scenario simulations demonstrate the evolution of characteristic urban structures.

Step 1 The Configuration Space

The city landscape is considered to be tesselated into a square lattice of plots or sites $i(i_1, i_2)$, $j(j_1, j_2)$, where$(i_1, i_2), (j_1, j_2)$ are integer lattice coordinates. One can introduce a distance between sites, for instance by the Manhattan metric

$$d(i, j) = |i_1 - j_1| + |i_2 - j_2|. \qquad (3.1)$$

The sites can either be empty or filled with different kinds of buildings, e.g. x_i lodgings, y_i factories and perhaps other kinds of urban uses (service stations, store houses, parks, etc.). For simplicity we consider only lodgings and factories. The variables $x_i, y_i = 0, 1, 2, \ldots$ are integers denoting the number of (appropriately tailored) building units of the corresponding kind on site i. The city configuration

$$\{\mathbf{x}, \mathbf{y}\} = \{\ldots (x_i, y_i), \ldots (x_j, y_j), \ldots\} \qquad (3.2)$$

characterizes the state of the city with respect to the kind, number and distribution of its buildings over the sites. It is the purpose of the model to give a formal mathematical description of the dynamics of the city configuration.

Step 2 The Utility of Configurations

The utility of a given city configuration has now to be determined. The ansatz for $u(\mathbf{x}, \mathbf{y})$ comprises several terms designed to describe the main effects influencing this utility. The terms contain open coefficients to be calibrated according to the concrete case.

α *The local term* consists of the contributions of local utilities of erecting buildings on each site j:

$$u_L(\mathbf{x}, \mathbf{y}) = \sum_j u_j(x_j, y_j) \tag{3.3}$$

with

$$u_j(x_j, y_j) = p_j^{(x)} \ln(x_j + 1) + p_j^{(y)} \ln(y_j + 1) + p_j^{(z)} \ln(z_j). \tag{3.4}$$

The coefficients $p_j^{(k)} > 0$ are measures of the preferences to build on site j. The first two terms of (3.4) represent the increasing urban utility of site j with growing numbers x_j, y_j, a usefulness which however saturates if x_j, y_j grows to high numbers. The third term describes the capacity constraint of site j. If z_j is the empty disposable space on site j and if one unit of lodging or factory needs one unit of the disposable space, respectively, the capacity C_j of site j is given by

$$C_j = x_j + y_j + z_j. \tag{3.5}$$

If the capacity of site j tends to be exhausted for $z_j = C_j - x_j - y_j \to 0$ the third term of the utility $u_j(x_j, y_j)$ approaches $-\infty$. On the other hand, the first terms of $u_j(x_j, y_j)$ are zero for $x_j = 0$ and $y_j = 0$, respectively, and approach $-\infty$ for $x_j \to -1$ or $y_j \to -1$. As we shall see this has the consequence that states with negative values of x_j or y_j or with values for which $x_j + y_j \geq C_j$ can never be reached. That means, x_j and y_j are confined to values $x_j \geq 0$, $y_j \geq 0$ and $(x_j + y_j) < C_j$.

The value of the capacity C_j on each site j depends on how much this site is opened up for buildings. If the total urban population n_c increases, more sites at the border of the city will be opened. In this way the size of the city area depends on its total population. We choose a Gaussian capacity distribution

$$C_j = C_0 \exp\left[-\frac{d^2(j, j_0)}{2\sigma^2(n_c)}\right], \tag{3.6}$$

where $d(j, j_0)$ is the Manhattan distance of site j from the central site j_0 and $\sigma^2(n_c)$ is the population dependent variance: The factor C_0 has to be calibrated appropriately so that in the equilibrium state the population n_c finds adequate total numbers $\sum_j x_j$ and $\sum_j y_j$ of lodgings and factories, respectively, in the city.

β *The interaction term* describes the – supportive or suppressive – utility influence between buildings on different sites i and j. This term is assumed to have the form

$$u_I(\mathbf{x}, \mathbf{y}) = \sum_{i,j} a_{ij}^{xx} x_i x_j + \sum_{i,j} a_{ij}^{xy} x_i y_j + \sum_{i,j} a_{ij}^{yy} y_i y_j. \tag{3.7}$$

The signs of the coefficients decide about the interaction effect. If one chooses for instance $a_{ii}^{xy} \ll 0$ this means that it is strongly disfavoured and not considered useful to build lodgings *and* factories on the *same* site. If, on the other hand, a_{ij}^{xy} is chosen as a *positive* parameter for $d(i, j) > d_0$ this means that lodgings on site i lead to a high utility of factories on sites j at a distance $d(i, j) \gtrsim d_0$, and vice versa. This is a plausible choice since workers living in the lodgings need working places in a not too distant neighbourhood with $d(i, j) \gtrsim d_0$. On the other hand, the choice of positive coefficients a_{ij}^{xx} and a_{ij}^{yy} for $d(i, j) < d_0$ means that it is considered useful to have further lodgings in the near neighbourhood

($d(i,j) < d_0$) of lodgings, and further factories in the near neighbourhood of factories. In this manner the interaction term represents the dependance of the utility of a city configuration on the location of different kinds of buildings relative to each other.

The total utility of a city configuration $(\mathbf{x}, \mathbf{y})$ is now assumed to be the sum of the two terms (3.3) and (3.7):

$$u(\mathbf{x}, \mathbf{y}) = u_{\mathrm{L}}(\mathbf{x}, \mathbf{y}) + u_{\mathrm{I}}(\mathbf{x}, \mathbf{y}). \tag{3.8}$$

Here, the simplifying tacit assumption has been made in constructing (3.8), that *one objective* utility of a city configuration exists for all those citizens who make decisions about the development of the city.

Step 3 The Transition Rates between Configurations

The transition rates for a transition between the configuration $\mathbf{x}, \mathbf{y}$ and the neighbouring configurations

$$\begin{aligned} \{\mathbf{x}^{j\pm}, \mathbf{y}\} &= \{\ldots, (x_j \pm 1;\ y_j), \ldots\}, \\ \{\mathbf{x}, \mathbf{y}^{j\pm}\} &= \{\ldots, (x_j;\ y_j \pm 1), \ldots\}, \end{aligned} \tag{3.9}$$

must now be set up. Firstly they must be positive definite quantities. Secondly they should depend monotonously on the utility difference between the final and initial configuration, because these utility differences are the "driving forces" behind the activities effecting the transition.

The simplest and mathematically most appealing form for the transition rates fulfilling these conditions is the following:

$$\left.\begin{aligned} \omega_{j\uparrow}^{(x)}(\mathbf{x}, \mathbf{y}) &= \nu_{\uparrow}^{(x)} \cdot \exp\{\Delta_{j+}^{(x)} u(\mathbf{x}, \mathbf{y})\} \\ \omega_{j\uparrow}^{(y)}(\mathbf{x}, \mathbf{y}) &= \nu_{\uparrow}^{(y)} \cdot \exp\{\Delta_{j+}^{(y)} u(\mathbf{x}, \mathbf{y})\} \end{aligned}\right]$$

building up rates for lodgings and factories at site j (3.10)

$$\left.\begin{aligned} \omega_{j\downarrow}^{(x)}(\mathbf{x}, \mathbf{y}) &= \nu_{\downarrow}^{(x)} \cdot \exp\{\Delta_{j-}^{(x)} u(\mathbf{x}, \mathbf{y})\} \\ \omega_{j\downarrow}^{(y)}(\mathbf{x}, \mathbf{y}) &= \nu_{\downarrow}^{(y)} \cdot \exp\{\Delta_{j-}^{(y)} u(\mathbf{x}, \mathbf{y})\} \end{aligned}\right]$$

tearing down rates for lodgings and factories at site j (3.11)

with:

$$\begin{aligned} \Delta_{j\pm}^{(x)} u(\mathbf{x}, \mathbf{y}) &= u(\mathbf{x}^{j\pm}, \mathbf{y}) - u(\mathbf{x}, \mathbf{y}), \\ \Delta_{j\pm}^{(y)} u(\mathbf{x}, \mathbf{y}) &= u(\mathbf{x}, \mathbf{y}^{j\pm}) - u(\mathbf{x}, \mathbf{y}). \end{aligned} \tag{3.12}$$

Here we have taken into account that there will exist different global frequencies $\nu_{\uparrow}^{(x)}, \nu_{\uparrow}^{(y)}$ for building up processes and $\nu_{\downarrow}^{(x)}, \nu_{\downarrow}^{(y)}$ for tearing down processes.

Step 4 Evolution Equations for Configurations

The transition rates which depend on utility differences between neighboring configurations are the starting point for setting up evolution equations for the configurations. Exactly speaking, the rates are probability transition rates per unit of time. The exact equation corresponding to these quantities is the master equation for the probability $P(\mathbf{x}, \mathbf{y}; t)$ to find the configuration $(\mathbf{x}, \mathbf{y})$ at time t. It reads:

$$\begin{aligned} \frac{\mathrm{d}P(\mathbf{x}, \mathbf{y}; t)}{\mathrm{d}t} &= \sum_j [\omega_{j\uparrow}^{(x)}(\mathbf{x}^{j-}, \mathbf{y}) P(\mathbf{x}^{j-}, \mathbf{y}; t) - \omega_{j\uparrow}^{(x)}(\mathbf{x}, \mathbf{y}) P(\mathbf{x}, \mathbf{y}; t)] \\ &+ \sum_j [\omega_{j\downarrow}^{(x)}(\mathbf{x}^{j+}, \mathbf{y}) P(\mathbf{x}^{j+}, \mathbf{y}; t) - \omega_{j\downarrow}^{(x)}(\mathbf{x}, \mathbf{y}) P(\mathbf{x}, \mathbf{y}; t)] \\ &+ \sum_j [\omega_{j\uparrow}^{(y)}(\mathbf{x}, \mathbf{y}^{j-}) P(\mathbf{x}, \mathbf{y}^{j-}; t) - \omega_{j\uparrow}^{(y)}(\mathbf{x}, \mathbf{y}) P(\mathbf{x}, \mathbf{y}; t)] \\ &+ \sum_j [\omega_{j\downarrow}^{(y)}(\mathbf{x}, \mathbf{y}^{j+}) P(\mathbf{x}, \mathbf{y}^{j+}; t) - \omega_{j\downarrow}^{(y)}(\mathbf{x}, \mathbf{y}) P(\mathbf{x}, \mathbf{y}; t)]. \end{aligned} \tag{3.13}$$

From the master equation there can easily be derived exact equations of motion for the meanvalues $\bar{x}_j(t), \bar{y}_j(t)$ of the components x_j, y_j of the city configuration, which are defined by

$$\bar{x}_j(t) = \sum_{\{\mathbf{x},\mathbf{y}\}} x_j P(\mathbf{x},\mathbf{y};t),$$
$$\bar{y}_j(t) = \sum_{\{\mathbf{x},\mathbf{y}\}} y_j P(\mathbf{x},\mathbf{y};t). \quad (3.14)$$

They read:

$$\frac{\mathrm{d}\bar{x}_j(t)}{\mathrm{d}t} = \overline{\omega_{j\uparrow}^{(x)}(\mathbf{x},\mathbf{y})} - \overline{\omega_{j\downarrow}^{(x)}(\mathbf{x},\mathbf{y})},$$
$$\frac{\mathrm{d}\bar{y}_j(t)}{\mathrm{d}t} = \overline{\omega_{j\uparrow}^{(y)}(\mathbf{x},\mathbf{y})} - \overline{\omega_{j\downarrow}^{(y)}(\mathbf{x},\mathbf{y})}, \quad (3.15)$$

where the bars on the right-hand side mean taking meanvalues with the probability distribution $P(\mathbf{x},\mathbf{y};t)$. The quasi-meanvalues $\tilde{x}_j(t), \tilde{y}_j(t)$ obey – in contrast to (3.15) – self-contained autonomous equations of motion which arise from (3.16) by substituting on the right-hand side:

$$\overline{\omega(\mathbf{x},\mathbf{y})} \Rightarrow \omega(\tilde{\mathbf{x}}(t), \tilde{\mathbf{x}}(t)) \quad (3.16)$$

which leads to

$$\frac{\mathrm{d}\tilde{x}_j(t)}{\mathrm{d}t} = \omega_{j\uparrow}^{(x)}(\tilde{\mathbf{x}}(t),\tilde{\mathbf{y}}(t)) - \omega_{j\downarrow}^{(x)}(\tilde{\mathbf{x}}(t),\tilde{\mathbf{y}}(t)),$$
$$\frac{\mathrm{d}\tilde{y}_j(t)}{\mathrm{d}t} = \omega_{j\uparrow}^{(y)}(\tilde{\mathbf{x}}(t),\tilde{\mathbf{y}}(t)) - \omega_{j\downarrow}^{(y)}(\tilde{\mathbf{x}}(t),\tilde{\mathbf{y}}(t)). \quad (3.17)$$

For unimodal probability distributions $P(\mathbf{x},\mathbf{y};t)$ the quasi-meanvalues approximate the true meanvalues; however, in the case of multimodal probability distributions the quasi-meanvalues no longer approximate the meanvalues; instead they approximate the true trajectories of the evolution of the city configuration.

In Figs. 1 and 2 we exhibit a few selected results of simulations based on the model of urban evolution discussed above.

4 THE DESIGN PRINCIPLES OF A MODEL FOR THE REGIONAL EVOLUTION OF SETTLEMENT STRUCTURES ON THE MACROLEVEL

The "space–time window of perception" of a macromodel is open towards the more coarse-grained spatial structures and the slow temporal evolutions. On the other hand, the fast processes on each local site are averaged out here, since we look at the slow evolution on the regional scale only. Therefore global variables are needed which represent the regional processes and structures.

The proposed macromodel is designed according to the following principles (for details see [8, 9]):

1. The economic and the population-dynamic migratory sector are integrated.
2. Populations are described by population densities distributed over the plane.
3. The populations produce goods; the local production costs including fixed costs and production costs, hence the local individual incomes depend on the population densities. The economy is assumed to be in equilibrium with the momentary population distribution.
4. The members of the population migrate between different locations. Driving forces of this nonlinear migration process are income differences between locations.
5. The migration leads to the formation of spatially heterogeneous population distributions; i.e. the settlements.

Let us formulate these principles in mathematical form. We consider A productive populations $\mathcal{P}_\alpha$, $\alpha = 1, 2, \ldots, A$, each producing for simplicity only one kind of commodity composed of units C_α. Furthermore we assume two service populations, the landowners $\mathcal{P}_\lambda$ renting premises to the producers and the transporters $\mathcal{P}_\tau$ dispatching the goods of the producers.

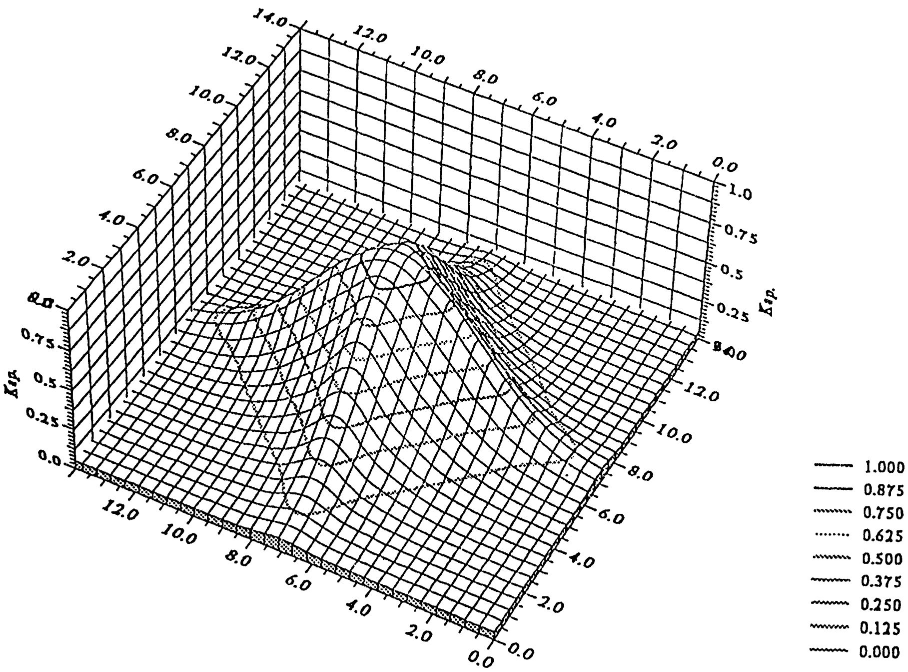

FIGURE 1 Distribution of capacities C_j.

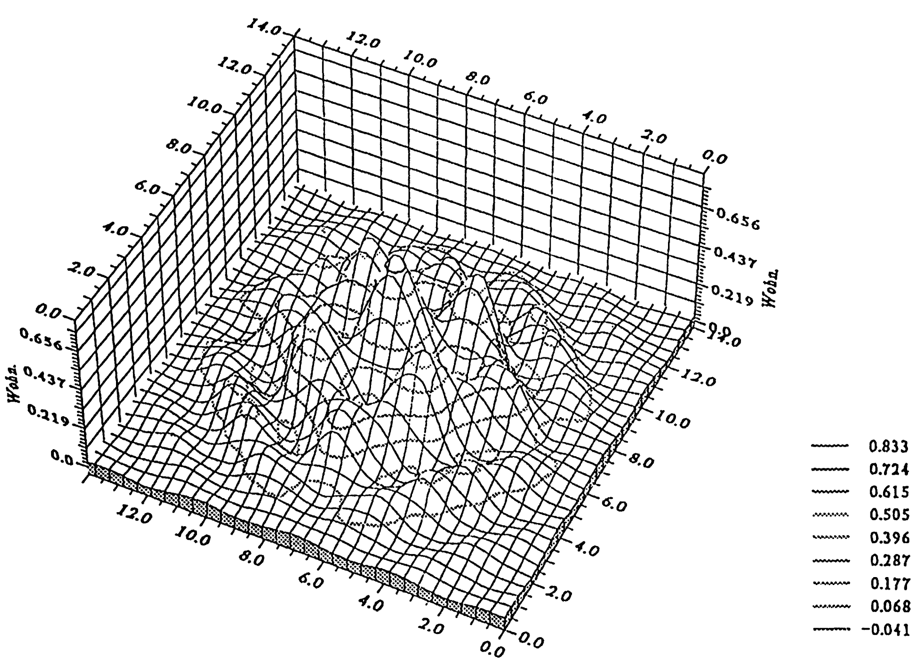

FIGURE 2(a) Distribution of lodgings (x_j).

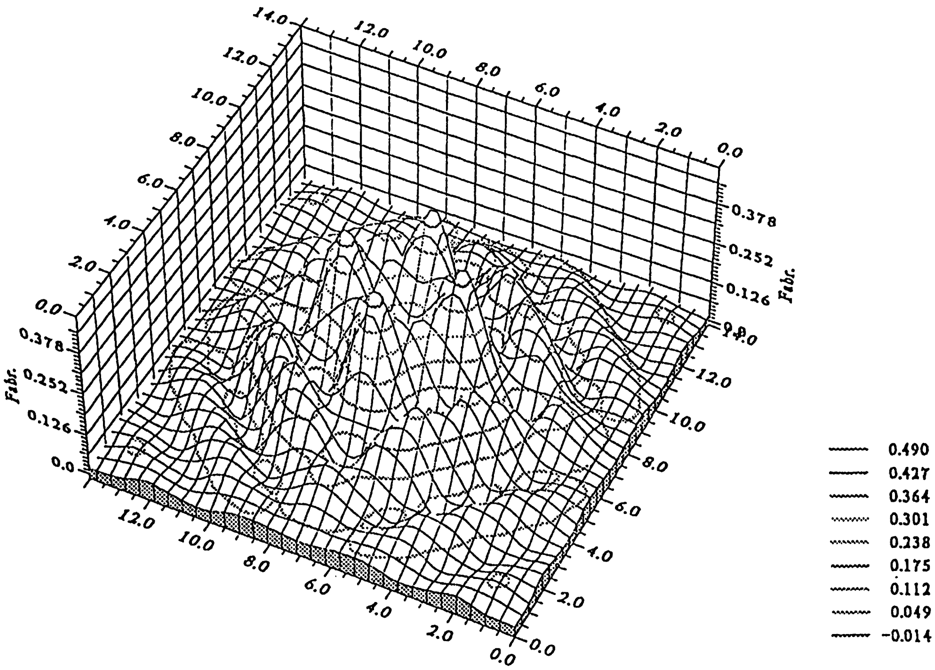

FIGURE 2(b) Distribution of factories (y_j).

Be $n_\alpha(\mathbf{x}, t)$ the *density of population* $\mathcal{P}_\alpha$ at position $\mathbf{x}$ and time t and $c_{\mathrm{pa}}(\mathbf{x}, t)$ the production density of $\mathcal{P}_\alpha$, i.e. the number of units C_α produced per unit of area and time. The *production density* is assumed to have the form

$$c_{\mathrm{pa}}(\mathbf{x}, t) = \gamma_\alpha(\mathbf{x}, t) n_\alpha(\mathbf{x}, t) \tag{4.1}$$

with the *productivity factor*

$$\gamma_\alpha(\mathbf{x}, t) = \bar{\gamma}_\alpha \left[\frac{n_\alpha(\mathbf{x}, t)}{\bar{n}_\alpha}\right]^{a_\alpha}. \tag{4.2}$$

The form (4.2) of γ_α expresses an "*economy of scale*" in the production of commodity C_α. If the productivity exponent is $a_\alpha > 0$, the production density grows more than proportional to $n_\alpha(\mathbf{x}, t)$. This will be true for many industrial goods, whereas for agrarian goods the production density grows less than proportional to $n_\alpha(\mathbf{x}, t)$, which amounts to a productivity exponent $a_\alpha < 0$.

Further densities of economic quantities can now easily be introduced. If $\mathcal{P}_\alpha$ is the price of one unit C_α, then the *gross income density* of population $\mathcal{P}_\alpha$ is given by

$$e_\alpha(\mathbf{x}, t) = P_\alpha c_{\mathrm{pa}}(\mathbf{x}, t); \quad \alpha = 1, 2, \ldots, A. \tag{4.3}$$

The *net income density* of $\mathcal{P}_\alpha$

$$w_\alpha(\mathbf{x}, t) = e_\alpha(\mathbf{x}, t) - k_\alpha(\mathbf{x}, t) - t_\alpha(\mathbf{x}, t) \tag{4.4}$$

follows by *deducting* the *fixed costs density*

$$k_\alpha(\mathbf{x}, t) = \rho_\alpha P_\alpha \bar{\gamma}_\alpha n_\alpha(\mathbf{x}, t) \times \left[\nu_{\alpha 0} + \nu_{\alpha 1}\left(\frac{n_\pi(\mathbf{x}, t)}{\bar{n}_\pi}\right) + \nu_{\alpha 2}\left(\frac{n_\pi(\mathbf{x}, t)}{\bar{n}_\pi}\right)^2\right] \tag{4.5}$$

and the *transport density*

$$t_\alpha(\mathbf{x}, t) = \sigma_\alpha P_\alpha L^{-1} c_{\mathrm{pa}}(\mathbf{x}, t) L^{-1} \bar{d}(\mathbf{x}, t) \tag{4.6}$$

from the *gross income density*.

In (4.5) and (4.6) a reasonable ansatz has been made for the dependance of the fixed costs and the transport costs on the partial and total population densities $n_\alpha(\mathbf{x}, t)$ and $n_\pi(\mathbf{x}, t)$, where

$$n_\pi(\mathbf{x}, t) = \sum_{\alpha=1}^{A} n_\alpha(\mathbf{x}, t). \tag{4.7}$$

The *fixed costs*, with ρ_α as fixed share coefficient, grow according to (4.5) over-proportionally with the total density $n_\pi(\mathbf{x}, t)$ of the local population, and the *transport costs*, with σ_α as transport share coefficient, are proportional to the mean transport distance $\bar{d}_\alpha(\mathbf{x}, t)$ of good C_α from the place $\mathbf{x}$ of production.

The fixed costs (for renting premises) and transport costs (for dispatching goods) for the producing populations $\mathcal{P}_\alpha$ are simultaneously the *net incomes* $w_\lambda(\mathbf{x}, t)$ and $w_\tau(\mathbf{x}, t)$ for *the service populations* $\mathcal{P}_\lambda$ and $\mathcal{P}_\tau$, respectively:

$$w_\lambda(\mathbf{x}, t) = \sum_{\alpha=1}^{A} k_\alpha(\mathbf{x}, t) \tag{4.8}$$

and

$$w_\tau(\mathbf{x}, t) = \sum_{\alpha=1}^{A} t_\alpha(\mathbf{x}, t). \tag{4.9}$$

The relative prices P_α of the commodity units C_α can now be determined by taking into account that the goods are not only *produced* but also *consumed* by the same populations $\mathcal{P}_1, \ldots, \mathcal{P}_A, \mathcal{P}_\lambda, \mathcal{P}_\tau$ in the total area $\mathcal{A}$ under consideration: if $c_{c\alpha}(\mathbf{x}, t)$ is the *consumption density* of commodity C_α, which corresponds to its production density, and which can be expressed by the local net incomes, then the equilibrium between production and consumption in the assumed closed economy of area $\mathcal{A}$ can be expressed by

$$\int_{\mathcal{A}} c_{p\alpha}(\mathbf{x}, t)\mathrm{d}^2x = \int_{\mathcal{A}} c_{c\alpha}(\mathbf{x}, t)\mathrm{d}^2x;$$

$$\alpha = 1, 2, \ldots, A. \tag{4.10}$$

From (4.10) there follow the relative prices (for details see [8,9]).

Finally, the *local net incomes per individual* are easily obtained:

$$\omega_\alpha(\mathbf{x}, t) = \frac{w_\alpha(\mathbf{x}, t)}{n_\alpha(\mathbf{x}, t)} \quad \text{with } \alpha = 1, \ldots, A; \ \lambda, \tau. \tag{4.11}$$

It is important to note that all economic quantities introduced so far, in particular the local individual net income (4.11), are expressed as functions of the population densities.

This means that the state of the simple economy described here (with production – income – consumption – densities, etc.) is well defined if the population densities are known. We now assume that this still holds if the $n_\alpha(\mathbf{x}, t)$ slowly evolve with time. This assumption implies that the adaptation of the economy to the momentary values of the $n_\alpha(\mathbf{x}, t)$ is *fast and flexible enough* to keep it always in *momentary equilibrium* with the population distribution.

We shall now see that a slow migration process governed by nonlinear migratory equations sets in if we let depend the motivations of the individuals to change their location on simple economic considerations. The result of the migration process is that a (perhaps initially existing) homogeneous density distribution of the populations becomes instable and that the different subpopulaions $\mathcal{P}_\alpha$ segregate into agglomerations of different size and density. We conclude that already simple assumptions about economic and migratory structures lead to the self-organisation of settlements.

The equations of motion for the population densities are obvious. They read

$$\frac{\mathrm{d}n_\alpha(\mathbf{x}', t)}{\mathrm{d}t} = \int_{\mathcal{A}} r_\alpha(\mathbf{x}', \mathbf{x}; t) n_\alpha(\mathbf{x}, t)\mathrm{d}^2x - \int_{\mathcal{A}} r_\alpha(\mathbf{x}, \mathbf{x}'; t) n_\alpha(\mathbf{x}', t)\mathrm{d}^2x$$

$$\text{for } \alpha = 1, 2, \ldots, A, \lambda, \tau, \tag{4.12}$$

where the decisive quantity is

$\mathrm{d}^2x' r_\alpha(\mathbf{x}', \mathbf{x}; t) =$
transition rate of a member of $\mathcal{P}_\alpha$
from $\mathbf{x}$ to $\mathbf{x}'$ into the area element d^2x'. (4.13)

The form of this rate, which has been substantiated in [5], is

$$r_\alpha(\mathbf{x}', \mathbf{x}; t) = \mu_\alpha \exp[u_\alpha(\mathbf{x}', t) - u_\alpha(\mathbf{x}, t)], \quad (4.14)$$

where the "dynamic utility" $u_\alpha(\mathbf{x}, t)$, which is sometimes also denoted as motivation potential, is a measure of the attraction of location $\mathbf{x}'$ to a member of population $\mathcal{P}_\alpha$ at time t.

A plausible assumption in the frame of our simple model is that $u_\alpha(\mathbf{x}, t)$ is proportional to the local individual net income $\omega_\alpha(\mathbf{x}, t)$, i.e.

$$u_\alpha(\mathbf{x}, t) = \beta\omega_\alpha(\mathbf{x}, t), \quad (4.15)$$

where β is a sensitivity factor calibrating the strength of migratory reactions to space-dependent income variations.

If (4.15) is inserted into (4.14) and (4.12), where $\omega_\alpha(\mathbf{x}, t)$ is to be expressed in terms of the population densities, Eq. (4.12) takes the form of a nonlinear integro-differential equation which can be solved numerically (see Section 5).

At the end of this short presentation of the macromodel we exhibit the interrelation of its construction elements in schematic form (Fig. 3). A final remark should be made about the connection between the macromodel of Section 4 and the micromodel of Section 3. As the following simulations show the macromodel demonstrates that global population agglomerations on a regional scale can arise by migration from a hinterland into an urban area taking shape due to certain economic production laws and migration decisions.

Thereupon the micromodel considers the decision mechanisms how the – slowly varying – total urban population exerts a population pressure which in turn leads on the local level of sites to the organisation of differentiated urban substructures.

We now present the result of numerical solutions of Eq. (4.12) in the case of only two productive populations which can be interpreted as "peasants" ($\alpha = p$) and "craftsmen" ($\alpha = c$).

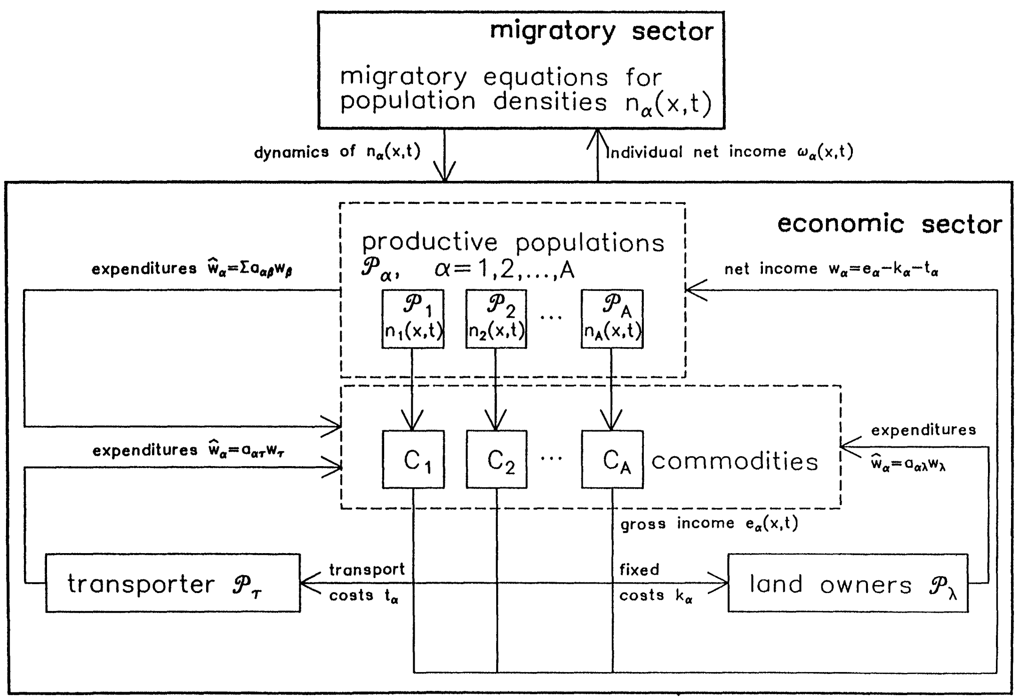

FIGURE 3 The interrelation of the elements of the macromodel for the formation of settlement structures.

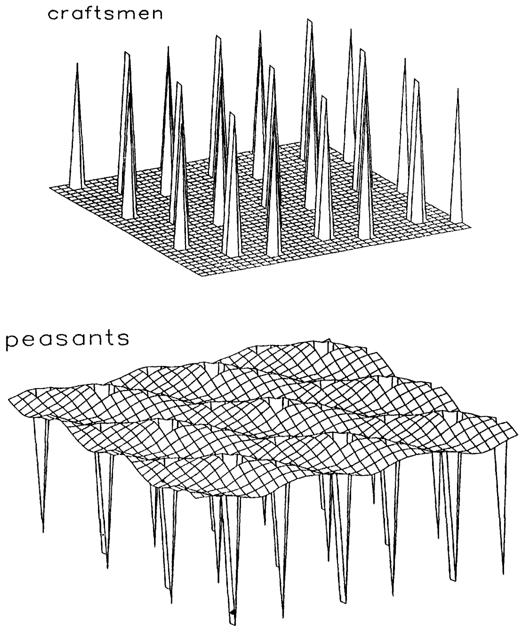

FIGURE 4 Parameters: inclusion of fixed costs ($\rho > 0$) and of transportation costs ($\sigma > 0$). (a) Stationary formation of more than one "town" per unit area in spatial neighbourhood, settled by "craftsmen". (b) Stationary ring-shaped rural settlements of "peasants" around the "towns".

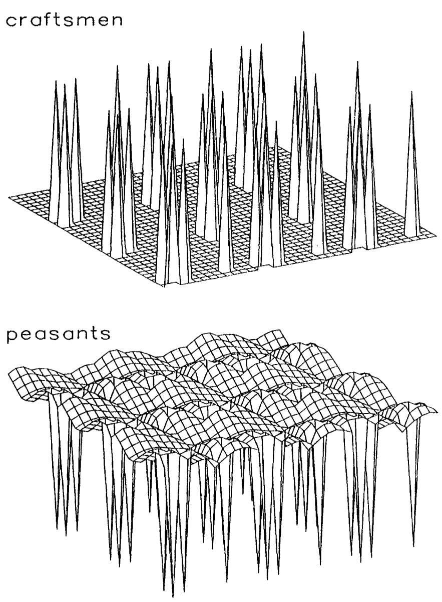

FIGURE 5 Parameters: inclusion of fixed costs ($\rho > 0$) and of transportation costs ($\sigma > 0$). (a) Stationary formation of differentiated "town-structures" settled by "craftsmen" in each unit area. (b) Differentiated ring-shaped rural settlements of "peasants" around the "town-structures".

They are characterized by different productivity exponents (see Eq. (4.2)), namely

$$a_p = -0.1 < 0; \qquad a_c = 0.25 > 0. \tag{4.16}$$

Figures 4 and 5 depict nine equivalent unit areas (in which by construction the population distribution is periodically repeated). The densities in all figures are scaled to their maximum values (for details of the calibration of the model see [9]).

5 CONCLUSION AND OUTLOOK

The two models of Sections 3 and 4 have demonstrated how the concepts of sociodynamics can be applied to the modelling of evolution processes of settlements on different spatial scales. Refinements and extensions are possible and remain to be done. A natural task for further work consists in the concrete matching of the regional and the urban model: the regional macromodel yields the global evolution of population densities, whereas the micromodel is capable of describing the evolution of detailed urban structures under the influence of growing population pressure. This matching problem will be treated in a forthcoming paper. Refined versions of the models should also take into account important empirical regularities of settlement evolution like the Leo Klaassen cyclic stages of urbanisation, suburbanisation, disurbanisation and reurbanisation (see [10,11]).

References

[1] H. Haken (1978). *Synergetics – An Introduction*. Springer, Berlin.
[2] W. Weidlich (1991). Physics and social science – The approach of synergetics. *Physics Reports* **204**, 1–163.
[3] D. Helbing and W. Weidlich (1995). Quantitative Soziodynamik. *Kölner Zeitschrift fü Soziologie und Sozialpsychologie* **47**, 114–140.
[4] W. Weidlich and G. Haag (Eds.) (1988). *Interregional Migration – Dynamic Theory and Comparative Analysis*. Springer, Berlin.
[5] W. Weidlich (1994). Synergetic modelling concepts for sociodynamics with application to collective political

opinion formation. *Journal of Mathematical Sociology* **18**, 267–291.

[6] W. Weidlich and M. Braun (1992). The master equation approach to non-linear economics. *Journal of Evolutionary Economics* **2**, 233–265.

[7] W. Weidlich and T. Brenner (1995). Dynamics of demand including fashion effects for interacting consumer groups. *Studien zur Evolutorischen konomik III*, pp. 79–115; A. Wagner and H.W. Lorenz (Eds.), Duncker & Humblot (Berlin).

[8] W. Weidlich and M. Munz (1990). Settlement formation I, a dynamic theory *Ann. Reg. Sci.* **24**, 83–106.

[9] W. Munz and W. Weidlich (1990). Settlement formation II, numerical simulation *Ann. Reg. Sci.* **24**, 177–196.

[10] L.H. Klassen, J.A. Bourdrez and J. Volmuller (1981). *Transport and Reurbanization*, Gower and Aldershot, England.

[11] L. van den Berg, L.S. Burns and L.H. Klaassen (1986). *Spatial Cycles*, Glower and Aldershot, England.


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