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Research Article

The Cauchy Problem for the Incompressible 2D-MHD with Power Law-Type Nonlinear Viscous Fluid

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We investigate a motion of the incompressible 2D-MHD with power law-type nonlinear viscous fluid. In this paper, we establish the global existence and uniqueness of a weak solution (u, b) depending on a number q in $\mathbb{R}^2$. Moreover, the energy norm of the weak solutions to the fluid flows has decay rate $(1+t)^{-1/2}$.

1. Introduction

In this paper, we study the weak solutions to the incompressible 2D-MHD with power law-type nonlinear viscous fluid:

$$\begin{cases} u_t - \nabla\cdot S + (u\cdot\nabla)u + \nabla\pi = (b\cdot\nabla)b, & \\ b_t - \triangle b + (u\cdot\nabla)b - (b\cdot\nabla)u = 0, & \text{in } Q_T \coloneqq \mathbb{R}^2\times(0,T). \\ \operatorname{div}\ u = 0,\quad \operatorname{div}\ b = 0, & \end{cases} \tag{1}$$

Here, $u : \mathbb{R}^2\times(0,T)\to\mathbb{R}^2$ is the flow velocity vector, $b : \mathbb{R}^2\times(0,T)\to\mathbb{R}^2$ is the magnetic vector, and $\pi : \mathbb{R}^2\times(0, T)\to\mathbb{R}$ is the total pressure. We consider the initial value problem of (1), which requires initial conditions:

$$\begin{aligned} u(x,0) &= u_0(x), \\ b(x,0) &= b_0(x)\ x\in\mathbb{R}^2. \end{aligned} \tag{2}$$

We assume that the initial data $u_0(x), b_0(x)\in L^2(\mathbb{R}^2)$ hold the incompressibility, i.e., $\nabla\cdot u_0(x)=0$ and $\nabla\cdot b_0(x)=0$, respectively. In this paper, we deal with S given as

$$S \coloneqq S(Du) = \left(\mu_0 + \mu_1|Du|^2\right)^{(q-2)/2}Du, \quad Du = \frac{\nabla u + \nabla u^{\perp}}{2}, \tag{3}$$

where $\mu_0\ge 0$ and $\mu_1>0$ are constants (see, e.g., [1, 2]).

In modern industrial application, non-Newtonian fluids play an important role (see [3–5]). In particular, equation (1) is the simplest self-consistent model which describes the dynamics of electrically conducting liquid with involved rheological structure in a magnetic field.

Some examples of non-Newtonian fluids are coal-water, glues, soaps, etc. (see, e.g., [6]). One class of non-Newtonian fluids can be defined by $S=\mu(|D|)D$ (D is the rate of the strain tensor, $\mu(\cdot)>0$ a real function). That is, the relation between the shear stress and the strain rate is nonlinear. In this paper, we study the case $\mu(s)=\mu_0+\mu_1 s^{q-2}$ which is called power law fluids. Commonly, the case of $q>2$ describes dilatant (or shear thickening) fluids whose viscosity increases with the rate of shear (see, e.g., [6]). On the other hand, pseudoplastic (or shear thinning) fluids correspond to the case of $1<q<2$, where viscosity decreases with the increasing rate of shear (see, e.g., [1]).

In what follows, we review some known results related to our concerns. For incompressible Navier-Stokes equation for a non-Newtonian type, namely, $b=0$ in (1), the existence of weak solutions for $(3n+2)/(n+2)\le q$ was first obtained in [7, 8], which is unique for $(n+2)/2\le q$ for any dimension

n (cf. [9]). Later, the existence of weak solutions was investigated for $2n/(n+2) < q$ in [10, 11]. On the other hand, in the case of $q = 2$, that is, $S(Du) = Du$ and $n = 3$, numerous results are known. Among them, we only mention that Ferreira and Villamizar-Roa [12] showed well-posedness, time decay, and stability for 3D magnetohydrodynamic equations.

In [13, 14], Samokhin first studied a nonstationary system of equations describing the motion of the Ostwald-de Waele media type and showed a unique existence of a generalized solution for $q \geq 1 + (2n/(n+2))$ to the problem based on the Faedo-Galerkin method and the monotone operator method. Later on, Gunzburger et al. in [15] proved the global unique solvability of the initial boundary value problem for the modified Navier-Stokes equations coupled with the Maxwell equations. Here, the authors use the strain tension containing the diffusion operator; that is, they do not deal with the degenerate power law fluids. Recently, Razafimandimby [16] proved the existence of weak solutions for $q \in (1, (2n+6)/(n+2)]$ to this model of bipolar type.

In this paper, we will prove the global-in-time existence and uniqueness of the weak solutions for the incompressible 2D-MHD with power law-type nonlinear viscous fluid (1)–(2) under a condition on the range of q.

Our results are based on the standard Galerkin method and some uniform estimates.

Denote $\mathbb{M}^2_{\text{sym}}$ by the vector space of all symmetric 2×2 matrices $\zeta = (\zeta_{ij})_{1\leq i,j\leq 2}$. Let $S := |Du|^{q-2}Du$ and $1 \leq q < \infty$. The deviatoric stress tensor $S = (S_{ij})$, $i, j = 1, 2$, satisfies the following conditions:

(i) $S : Q_T \times \mathbb{M}^2_{\text{sym}} \to \mathbb{M}^3_{\text{sym}}$ is a Carathéodory function

(ii) Symmetry: $S_{ij} = S_{ji}$

(iii) Polynomial growth:

$$\left|S_{ij}(\xi)\right| \leq \left(\mu_0 + \mu_1|\xi|^{q-2}\right)|\xi|. \tag{4}$$

(iv) Coercivity condition: there exists $c_1 > 1$ such that

$$\left(\mu_0 + \mu_1|\xi|^{q-2}\right)|\eta|^2 \leq \frac{\partial S_{ij}}{\partial \xi_{kl}}\eta_{kl}\eta_{ij} \leq c_1\left(\mu_0 + \mu_1|\xi|^{q-2}\right)|\eta|^2. \tag{5}$$

(v) Strict monotonicity: for all $\zeta, \eta \in \mathbb{M}^2_{\text{sym}}(\zeta \neq \eta)$, $S(\zeta) - S(\eta)\colon (\zeta - \eta) > 0$

By the weak solution of the incompressible 2D-MHD with power law-type nonlinear viscous fluid, we mean solutions satisfying the following definitions:

Definition 1 (weak solution). Let $\mu_0 \geq 0$, $\mu_1 > 0$, and $q \in (1,\infty)$. Suppose that $u_0, b_0 \in L^2(\mathbb{R}^2)$. We say that (u, b) is a weak solution of the incompressible 2D-MHD with power law-type nonlinear viscous fluid (1)–(2) if u and b satisfy the following:

$$\begin{aligned} u &\in L^\infty\left([0,T); L^2\left(\mathbb{R}^2\right)\right) \cap L^2\left([0,T); W^{1,q}\left(\mathbb{R}^2\right)\right), \\ b &\in L^\infty\left([0,T); L^2\left(\mathbb{R}^2\right)\right) \cap L^2\left([0,T); H^1\left(\mathbb{R}^2\right)\right). \end{aligned} \tag{6}$$

(i) (u, b) satisfies (1) in the sense of distribution; that is,

$$\begin{aligned} &\int_0^T\int_{\mathbb{R}^2}\left(\frac{\partial\phi}{\partial t} + (u\cdot\nabla)\phi\right)u\,dxdt + \int_0^T\int_{\mathbb{R}^2} S(Du)\colon \nabla\phi\,dxdt \\ &\quad = \int_{\mathbb{R}^2} u_0\phi(x,0)\,dx + \int_0^T\int_{\mathbb{R}^2}(b\cdot\nabla)\phi\, b\,dxdt, \\ &\int_0^T\int_{\mathbb{R}^2}\left(\frac{\partial\phi}{\partial t} + \Delta\phi + (u\cdot\nabla)\phi\right)b\,dxdt \\ &\quad = \int_{\mathbb{R}^2} b_0\phi(x,0)\,dx + \int_0^T\int_{\mathbb{R}^2}(b\cdot\nabla)\phi\, u\,dxdt, \end{aligned} \tag{7}$$

for all $\phi \in C_0^\infty(\mathbb{R}^2 \times [0,T))$ with div $\phi = 0$, and

$$\int_{\mathbb{R}^2} u\cdot\nabla\psi dx = 0, \int_{\mathbb{R}^2} b\cdot\nabla\psi dx = 0, \tag{8}$$

for every $\psi \in C_0^\infty(\mathbb{R}^2)$.

Theorem 2. *Let $2 < q < \infty$ and $\mu_0 \geq 0$ and $\mu_1 > 0$. Assume that $u_0, b_0 \in L^2(\mathbb{R}^2)$. A weak solution (u, b) of the incompressible 2D-MHD with power law-type nonlinear viscous fluid (1)–(2) exists. In particular, in the case $\mu_0 > 0$ and $\mu_1 > 0$, the weak solution (u, b) is unique. Moreover, we obtain the following decay rate of the weak solution:*

$$\|u(t)\|_{L^2} + \|b(t)\|_{L^2} \leq C(1+t)^{-1/2}. \tag{9}$$

2. Preliminaries

In this section, we introduce the notation. Let I be a finite time interval. For $1 \leq q \leq \infty$, we denote by $W^{k,q}(\mathbb{R}^2)$ the usual Sobolev spaces, namely, $W^{k,q}(\mathbb{R}^2) = \{f \in L^q(\mathbb{R}^2)\colon D^\alpha f \in L^q(\mathbb{R}^2), 0 \leq |\alpha| \leq k\}$. The set of the q-th power Lebesgue integrable functions on $\mathbb{R}^2$ is denoted by $L^q(\mathbb{R}^2)$, and $L^q_{\text{loc}}(\mathbb{R}^2)$ indicates the set of the locally q-th power Lebesgue integrable functions defined on $\mathbb{R}^2$. For a function $f(x,t)$, $\mathcal{O} \subset \mathbb{R}^2$, and $J \subset I$, we denote $\|f\|_{L^{p,q}_{x,t}(\mathcal{O}\times J)} = \|\|f\|_{L^p(\mathcal{O})}\|_{L^q(J)}$. For vector fields u, v, we write $(u_i v_j)_{i,j=1,2,3}$ as $u \otimes v$. We denote $A : B = a_{ij}b_{ij}$ for 3×3 matrices $A = (a_{ij})$, $B = (b_{ij})$. The letter C is used to represent a generic constant, which may change from line to line.

Before looking for a solution for the system (1), we give a lemma.

Lemma 3. *Let (u, b) be a solution to the initial value problem of (1)–(2) with the initial data $u_0, b_0 \in L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Then, we have for $2 \le p$*

$$|\widehat{u}(\xi,t)| + \left|\widehat{b}(\xi,t)\right| \le C(\widehat{u}_0(\xi) + \left|\widehat{b}_0(\xi)\right| + |\xi| \int_0^t \cdot \left(\|u(s)\|^2_{L^2(\mathbb{R}^2)} + \|b(s)\|^2_{L^2(\mathbb{R}^2)} \right) ds + C|\xi| \left(\int_0^t \|u(s)\|_{L^2(\mathbb{R}^2)} ds \right)^{1/(p-1)}, \tag{10}$$

where C depends only on the $(L^2 \cap L^1)(\mathbb{R}^2)$-norm of u_0 and b_0 .

Proof. The proof is easily checked. Indeed, it is almost the same as that in [17] replacing (2.5) in [17] by

$$\begin{aligned} \int_0^t \|\nabla u(s)\|^{p-1}_{L^{p-1}(\mathbb{R}^2)} ds &\le \int_0^t \|u(s)\|^{1/(p-1)}_{L^2(\mathbb{R}^2)} \|\nabla u(s)\|^{p(p-2)/(p-1)}_{L^p(\mathbb{R}^2)} ds \\ &\le C \left(\int_0^t \|u(s)\|_{L^2(\mathbb{R}^2)} ds \right)^{1/(p-1)} \cdot \left(\int_0^t \|\nabla u(s)\|^p_{L^p(\mathbb{R}^2)} dt \right)^{(p-2)/(p-1)} \\ &\le C \left(\int_0^t \|u(s)\|_{L^2(\mathbb{R}^2)} ds \right)^{1/(p-1)}, \quad p > 2. \end{aligned} \tag{11}$$

3. Proof of Theorem 2

In this paper, we assume that $\mu_0 = 0$ and $\mu_1 = 1$ for convenience. Let

$$V_q := \left\{ \varphi \in D'\left(\mathbb{R}^2\right)^2 : \nabla \cdot \varphi = 0 \right\}, \tag{12}$$

with $\|\varphi\|_{V_q} := \|D\varphi\|_{L^q(\mathbb{R}^2)}$. Now, we will construct the existence of a weak solution to the system (1) via the standard Galerkin method. For this, first of all, we need to find a countable dense subset of the space $\{\varphi \in \mathcal{D}(\mathbb{R}^2): \nabla\cdot\varphi = 0\}$ in $W^{2,2}(\mathbb{R}^2) \cap V_q$ in Lemma 3.10 of [18].

Now, we consider Galerkin approximate solutions $u^m(t) = \Sigma_{i=1}^m g_j^m(t)\varphi_j$ and $b^m(t) = \Sigma_{i=1}^m h_j^m(t)\psi_j$, where the φ_j, ψ_j are the eigenfunctions which are chosen by using Lemma 3.10 of [18].

$$(u_t^m - \nabla \cdot S(Du) + (u^m \cdot \nabla)u^m - (b^m \cdot \nabla)b^m, \varphi) = 0, \tag{13}$$

$$(b_t^m - \triangle b^m + (u^m \cdot \nabla)b^m - (b^m \cdot \nabla)u^m, \psi) = 0, \tag{14}$$

for $\varphi \in \text{span}\{\varphi_1, \varphi_2, \cdots \varphi_m\}$ and $\psi \in \text{span}\{\psi_1, \psi_2, \cdots \psi_m\}$. The initial conditions were

$$u^m(x,0) = \Sigma_{i=1}^m a_i \varphi^i(x), b^m(x,0) = \Sigma_{i=1}^m c_i \psi^i(x), \tag{15}$$

where $a_i = \int_{\mathbb{R}^2} u^m(x,0) \cdot \varphi^i(x)$ and $c_i = \int_{\mathbb{R}^2} b^m(x,0) \cdot \psi^i(x)$. Indeed, the functions $g_j^m(t)$ and $h_j^m(t)$ satisfy the following ordinary differential equations as follows:

$$\dot{g}_j^m(t) + \lambda_u g_j^m(t) + (g_k^m(t))^{p-1} \int_{\mathbb{R}^2} \left|\nabla \varphi^k\right|^{p-2} \nabla\varphi^k \cdot \nabla\varphi^j + g_k^m(t) g_l^m(t) \int_{\mathbb{R}^2} \left(\varphi^k \cdot \nabla\varphi^l\right)\varphi^j - h_k^m(t) h_l^m(t) \int_{\mathbb{R}^2} \left(\psi^k \cdot \nabla\psi^l\right)\varphi^j = 0, \tag{16}$$

$$\dot{h}_j^m(t) + \lambda_b h_j^m(t) + g_k^m(t) h_l^m(t) \int_{\mathbb{R}^2} \left(\varphi^k \cdot \nabla\psi^l\right)\psi^j - h_k^m(t) g_l^m(t) \int_{\mathbb{R}^2} \left(\psi^k \cdot \nabla\varphi^l\right)\psi^j = 0. \tag{17}$$

By the Carathéodory theorem (see [19], Theorem 3.4 in Appendix), there exist T_m such that equation (16) has unique solutions on $[0, T_m)$. Now set $T_m = T$, $T < \infty$.

Proof of Theorem 2. For a proof of existence for a weak solution, we assume that $\mu_0 = 0$ because it is easier for $\mu_0 > 0$.

Part A: existence

Multiplying equation (13) by u^m and equation (14) by b^m and summing up the equations, we have

$$\|u^m(T)\|^2_{L^2(\mathbb{R}^2)} + \|b^m(T)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla u^m\|^q_{L^q(Q_T)} + \|\nabla b^m\|^2_{L^2(Q_T)} \le \|u^m(0)\|^2_{L^2(\mathbb{R}^2)} + \|b^m(0)\|^2_{L^2(\mathbb{R}^2)}, \tag{18}$$

where we use the divergence free condition, Korn's inequality, and vector identity for the magnetic vector field b. For the distributive time derivative du^m/dt, we have $(du^m/dt) \in L^{q'}(0, T; (W^{1,q})^*) + L^q(0, T; (W^{1,q'})^*) + L^2(0, T; (W^{1,2})^*)$. Here, q' is the conjugate of p, and $(W^{1,q'}(\mathbb{R}^2))^*$ is the dual space for $W^{1,q'}(\mathbb{R}^2)$. Indeed, for $\phi \in L^q(0, T; W^{1,q}) \cap L^{q'}(0, T; (W^{1,q'}(\mathbb{R}^2))) \cap L^2(0, T; (W^{1,2}(\mathbb{R}^2)))$ with $\nabla\cdot\phi = 0$,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^2} \frac{du^m}{dt} \cdot \phi dx dt &= \int_0^T \int_{\mathbb{R}^2} (\nabla\cdot S(Du^m) dx - \nabla \cdot (u^m \otimes u^m) + \nabla \cdot (b^m \otimes b^m)) \cdot \phi dx dt \\ &= -\int_0^T \int_{\mathbb{R}^2} |Du^m|^{q-2} Du^m : \nabla\phi dx dt + \int_0^T \int_{\mathbb{R}^2} (u^m \otimes u^m) : \nabla\phi dx dt \\ &\quad - \int_0^T \int_{\mathbb{R}^2} (b^m \otimes b^m) : \nabla\phi dx d := \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned} \tag{19}$$

(i) Estimate of $\mathcal{I}_1$: using Hölder's inequality and the energy estimate (18), we have

$$|I_1| \le \left\| |Du^m|^{q-1} \right\|_{L^{q'}\left(0,T;L^{q'}\right)} \|\nabla\phi\|_{L^q(0,T;L^q)} \le \|Du^m\|^{q-1}_{L^q(0,T;L^q)} \|\nabla\phi\|_{L^q(0,T;L^q)} \le C\left(\|u_0\|^2_{L^2(\mathbb{R}^2)}, T, q\right) \|\nabla\phi\|_{L^q(0,T;L^q)}. \tag{20}$$

(ii) Estimate of $\mathcal{I}_2$: since u^m belongs to $L^{2q}(0, T; L^{2q})$, we have

$$|I_2| \le \|u^m \otimes u^m\|_{L^q(0,T;L^q)} \|\nabla\phi\|_{L^{q/(q-1)}\left(0,T;L^{q/(q-1)}\right)} \le \|u^m\|_{L^{2q}\left(0,T;L^{2q}\right)} \|u^m\|_{L^{2q}\left(0,T;L^{2q}\right)} \|\nabla\phi\|_{L^{q/(q-1)}\left(0,T;L^{q/(q-1)}\right)} \le C. \tag{21}$$

(iii) Estimate of $\mathcal{I}_3$: using Hölder's inequality, we have

$$|I_3| \le \|b^m \otimes b^m\|_{L^2\left(0,T;L^2\right)} \|\nabla\phi\|_{L^2\left(0,T;L^2\right)} \le \|b^m\|_{L^4\left(0,T;L^4\right)} \|b^m\|_{L^4\left(0,T;L^4\right)} \|\nabla\phi\|_{L^2\left(0,T;L^2\right)} \le C. \tag{22}$$

We combine (20), (21), and (22) to get

$$\frac{du^m}{dt} \in L^{q'}\left(0, T; \left(W^{1,q}\right)^*\right) + L^q\left(0, T; \left(W^{1,q'}\right)^*\right) + L^2\left(0, T; \left(W^{1,2}\right)^*\right). \tag{23}$$

To obtain the distributive time derivative db^m/dt, using the similar argument above, we have

$$\frac{db^m}{dt} \in L^2\left(0, T; \left(W^{1,2}\right)^*\right) + L^2\left(0, T; \left(W^{1,4}\right)^*\right). \tag{24}$$

Indeed, for $\phi \in L^2(0, T; W^{1,2}) \cap L^2(0, T; W^{1,4})$ with $\nabla\cdot\phi = 0$,

$$\int_0^T \int_{\mathbb{R}^2} \frac{du^m}{dt} \cdot \phi dxdt = -\int_0^T \int_{\mathbb{R}^2} \nabla b^m : \nabla\phi dxdt + \int_0^T \int_{\mathbb{R}^2} (u^m \otimes b^m) : \nabla\phi dxdt := \mathcal{J}_1 + \mathcal{J}_2. \tag{25}$$

(iv) Estimate of $\mathcal{J}_1$: using Hölder's inequality and the estimate (18), we have

$$|\mathcal{J}_1| \le \|\nabla b^m\|_{L^2\left(0,T;L^2\right)} \|\nabla\phi\|_{L^2\left(0,T;L^2\right)} \le C. \tag{26}$$

(v) Estimate of $\mathcal{J}_2$: using Hölder's inequality, we have

$$|\mathcal{J}_2| \le C\|u^m\|_{L^{2q}\left(0,T;L^{2q}\right)} \|b^m\|_{L^4\left(0,T;L^4\right)} \|\nabla\phi\|_{L^{\frac{4q}{3q-2}}\left(0,T;L^{\frac{4q}{3q-2}}\right)} \le C. \tag{27}$$

Due to the energy estimate (18) and time derivative class for u^m and b^m, we can choose subsequences u^{m_k} and b^{m_k} such that

$$\begin{aligned}
&u^{m_k} \rightharpoonup u \text{ weakly in } L^\infty\left(I; L^2\left(\mathbb{R}^2\right)\right) \cap L^q\left(I; W^{1,q}\left(\mathbb{R}^2\right)\right),\\
&b^{m_k} \rightharpoonup b \text{ weakly in } L^\infty\left(I; L^2\left(\mathbb{R}^2\right)\right) \cap L^2\left(I; W^{1,2}\left(\mathbb{R}^2\right)\right),\\
&\partial_t u^{m_k} \rightharpoonup \partial_t u \text{ weakly in } L^{q'}\left(0, T; \left(W^{1,q}\right)^*\right) + L^q\left(0, T; \left(W^{1,q'}\right)^*\right) + L^2\left(0, T; \left(W^{1,2}\right)^*\right),\\
&\partial_t b^{m_k} \rightharpoonup \partial_t b \text{ weakly in } L^2\left(0, T; \left(W^{1,2}\right)^*\right) + L^2\left(0, T; \left(W^{1,4}\right)^*\right),
\end{aligned} \tag{28}$$

when k goes to ∞. From the class of u^{m_k} and b^{m_k} in the convergence above and by the Aubin-Lions lemma (e.g., [20], Lemma 3.1), we have

$$\begin{aligned}
&u^{m_k} \rightharpoonup u \text{ strongly in } L^p_{\text{loc}}\left(\mathbb{R}^2 \times I\right), p \in [1, 2q),\\
&b^{m_k} \rightharpoonup b \text{ strongly in } L^{\tilde{p}}_{\text{loc}}\left(\mathbb{R}^2 \times I\right), \tilde{p} \in [1, 4).
\end{aligned} \tag{29}$$

Thus, we have

$$\begin{aligned}
&u^{m_k} \to u \text{ strongly in } L^2_{\text{loc}}\left(\mathbb{R}^2 \times I\right),\\
&b^{m_k} \to b \text{ strongly in } L^2_{\text{loc}}\left(\mathbb{R}^2 \times I\right),
\end{aligned} \tag{30}$$

as $k \to \infty$. So then, due to the weak and strong convergence above, it is possible to pass to the limit in the nonlinear terms (see, e.g., [21]). Moreover, $S(Du^m)$ is uniformly bounded in $L^{q'}(\mathbb{R}^2 \times (0, T))$, and so $S(Du) \rightharpoonup A$ in this class. Hence, we will check $A = S(Du)$ which is shown by monotonicity trick (see [13], pp. 635-636). For this, we note that for $q \ge 2$,

$$\begin{aligned}
&t \to \int_{\mathbb{R}^2} (u \cdot \nabla u) \cdot u dx \in L^1(0, T), t \to \int_{\mathbb{R}^2} (u \cdot \nabla b) \cdot b dx \in L^1(0, T)\\
&t \to \int_{\mathbb{R}^2} (b \cdot \nabla b) \cdot u dx \in L^1(0, T), t \to \int_{\mathbb{R}^2} (b \cdot \nabla u) \cdot b dx \in L^1(0, T).
\end{aligned} \tag{31}$$

From the energy equality, we have for $0 \le s \le T$

$$\frac{1}{2}\left(\|u(s)\|^2_{L^2} + \|b(s)\|^2_{L^2}\right) + \int_0^s \|\nabla b\|_{L^2} dt + \int_0^s A \cdot Du dt = \frac{1}{2}\left(\|u_0\|^2_{L^2} + \|b_0\|^2_{L^2}\right). \tag{32}$$

Define

$$X_s^m = \int_0^s (S(Du^m) - S(D\phi), Du^m - D\phi)dt + \frac{1}{2}\|u(s)\|_{L^2}^2, \phi \in L^q(0, T; W_\sigma^{1,q}). \tag{33}$$

Here, $W_\sigma^{1,q} := \{v \in W^{1,q}(\mathbb{R}^2): \nabla \cdot v = 0\}$. So, due to the property of the monotone operator S and the semicontinuity of the norm, we obtain

$$\liminf_{m\to\infty} X_s^m \geq \frac{1}{2}\|u(s)\|^2, \tag{34}$$

and also

$$\lim_{m\to\infty} X_s^m = \int_0^s \frac{1}{2}\|u_0\|_{L^2}^2 + \int_0^s (b\cdot\nabla b)\cdot u dt - \int_0^s (A, D\phi)dt - \int_0^s (S(D\phi), Du - D\phi)dt \tag{35}$$

Then, due to the equality (32), we have

$$\int_0^s (A - S(D\phi))\cdot(Du - D\phi)dt \geq 0, \text{a.e.} s \in [0, T]. \tag{36}$$

Putting $\phi = u - \lambda w$ for $\lambda > 0$ and $w \in L^q(0, T; W_\sigma^{1,q})$, we obtain

$$\int_0^s (A - S(Du - \lambda w))\cdot w dt \geq 0. \tag{37}$$

As $\lambda \to 0$, we deduce

$$\int_0^s (A - S(Du))\cdot w dt \geq 0, \tag{38}$$

which means that $A = S(Du)$ for a.e. $s \in [0, T]$. Hence, the proof of existence for weak solutions is completed.

Part B: uniqueness

For this part, we consider the equation for $v = u^1 - u^2$, $h = b^1 - b^2$, and $\tilde{\pi} = \pi^1 - \pi^2$:

$$\begin{aligned}&\partial_t v - \nabla\cdot\left(\left(1+\left|Du^1\right|\right)^{q-2}Du^1\right) + \nabla\cdot\left(\left(1+\left|Du^2\right|\right)^{q-2}Du^2\right) + \left(u^1\cdot\nabla\right)v\\ &\quad + (v\cdot\nabla)u^2 - \left(b^1\cdot\nabla\right)h - (h\cdot\nabla)b^2 + \nabla\tilde{\pi} = 0,\\ &\partial_t h - \Delta\tilde{h} + \left(u^1\cdot\nabla\right)h + (v\cdot\nabla)b^2 - \left(b^1\cdot\nabla\right)v - (h\cdot\nabla)u^2 = 0,\end{aligned} \tag{39}$$

with div $v = 0$ and div $h = 0$. Testing v and h to the equations above, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|v\|_{L^2(\mathbb{R}^2)}^2 + \|h\|_{L^2(\mathbb{R}^2)}^2\right) + \left(\|\nabla v\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla h\|_{L^2(\mathbb{R}^2)}^2\right)\\ &\quad \leq \int_{\mathbb{R}^2}(v\cdot\nabla)u^2\cdot v - \int_{\mathbb{R}^2}(h\cdot\nabla)b^2\cdot v + \int_{\mathbb{R}^2}(v\cdot\nabla)b^2\cdot h - \int_{\mathbb{R}^2}(h\cdot\nabla)u^2\cdot h\\ &\quad \leq \|v\|_{L^2}^2\|u^2\|_{L^2}^2\|\nabla u^2\|_{L^2}^2 + \|h\|_{L^2}^2\|b^2\|_{L^2}^2\|\nabla b^2\|_{L^2}^2 + \|v\|_{L^2}^2\|b^2\|_{L^2}^2\|\nabla b^2\|_{L^2}^2\\ &\qquad + \|h\|_{L^2}^2\|u^2\|_{L^2}^2\|\nabla u^2\|_{L^2}^2 + \frac{1}{2}\left(\|\nabla v\|_{L^2}^2 + \|\nabla h\|_{L^2}^2\right),\end{aligned} \tag{40}$$

that is,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|v\|_{L^2(\mathbb{R}^2)}^2 + \|h\|_{L^2(\mathbb{R}^2)}^2\right) + \left(\|\nabla v\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla h\|_{L^2(\mathbb{R}^2)}^2\right)\\ &\quad \leq \left(\|v\|_{L^2}^2 + \|h\|_{L^2}^2\right)\left(\left\|u^2\right\|_{L^2}^2\left\|\nabla u^2\right\|_{L^2}^2 + \left\|b^2\right\|_{L^2}^2\left\|\nabla b^2\right\|_{L^2}^2\right).\end{aligned} \tag{41}$$

Applying Gronwall's inequality, we obtain $v = 0$ and $h = 0$ in $\mathbb{R}^2$ and hence $u_1 = u_2$ and $b_1 = b_2$.

Part C: decay rate

A proof of this part is almost the same as that in [17]. For the convenience of the reader, it gives a proof. From the L^2 -energy inequality and Korn's inequality, it follows that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left(|u|^2 + |b|^2\right)dx + \min\{C, 1\}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + |\nabla b|^2\right) \leq 0, \tag{42}$$

where $C > 0$ is a Korn-type constant. Applying Plancherel's theorem to (42) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left(|u\wedge(\xi,t)|^2 + |b\wedge(\xi,t)|^2\right)d\xi + \min\{C,1\}\int_{\mathbb{R}^3}|\xi|^2\\ &\quad \cdot\left(|u\wedge(\xi,t)|^2 + |b\wedge(\xi,t)|^2\right)d\xi \leq 0.\end{aligned} \tag{43}$$

Put $\Phi(\xi, t) := |u\wedge(\xi,t)|^2 + |b\wedge(\xi,t)|^2$. Let $f(t)$ be a smooth function of t with $f(0) = 1, f(t) > 0$ and $f'(t) > 0$. Set $S(t) = \{\xi \in \mathbb{R}^n : 2\min\{C,1\}f(t)|\xi|^2 \leq f'(t)\}$. Then,

$$\begin{aligned}&2\min\{C,1\}f(t)\int_{\mathbb{R}^3}|\xi|^2|\Phi(\xi,t)|^2 d\xi\\ &\quad \geq f'(t)\int_{\mathbb{R}^3}|\Phi(\xi,t)|^2 d\xi - f'(t)\int_{S(t)}|\Phi(\xi,t)|^2 d\xi.\end{aligned} \tag{44}$$

Since

$$\begin{aligned}&\frac{d}{dt}\left(f(t)\int_{\mathbb{R}^3}|\Phi(\xi,t)|^2 d\xi\right) + 2\min\{C,1\}f(t)\int_{\mathbb{R}^3}|\xi|^2|\Phi(\xi,t)|^2 d\xi\\ &\quad \leq f'(t)\int_{\mathbb{R}^3}|\Phi(\xi,t)|^2 d\xi,\end{aligned} \tag{45}$$

we have

$$\frac{d}{dt}\left(f(t)\int_{\mathbb{R}^n}|u\wedge(\xi,t)|^2 d\xi\right) \le f'(t)\int_{S(t)}|u\wedge(\xi,t)|^2 d\xi. \quad (46)$$

Integrating in time, we get

$$\begin{aligned} &f(t)\int_{\mathbb{R}^n}\left(|u\wedge(\xi,t)|^2+|b\wedge(\xi,t)|^2\right)d\xi \\ &\quad\le \int_{\mathbb{R}^n}\left(|u\wedge_0(\xi)|^2+|b\wedge_0(\xi)|^2\right)d\xi + C\int_0^t f'(s)\int_{S(s)} \\ &\qquad \times\left(|u\wedge(\xi,s)|^2+|b\wedge(\xi,s)|^2\right)d\xi ds. \end{aligned} \quad (47)$$

Set $f(t)=(1+t)^2$. From Lemma 3 with Young's inequality and the energy estimate, we have

$$\begin{aligned} &(1+t)^2\int_{\mathbb{R}^n}\left(|u\wedge(\xi,t)|^2+|b\wedge(\xi,t)|^2\right)d\xi \\ &\quad\le \int_{\mathbb{R}^n}\left(|u\wedge_0(\xi)|^2+|b\wedge_0(\xi)|^2\right)d\xi + C\int_0^t(1+s)\int_{S(s)} \\ &\qquad\times\left(|u\wedge_0(\xi)|^2+|b\wedge_0(\xi)|^2\right)d\xi ds + C\int_0^t(1+s)\int_{S(s)}|\xi|^2 \\ &\qquad\times\left(\int_0^t\left(\|u(s)\|_{L^2}^2+\|b(s)\|_{L^2}^2\right)ds\right)^2 d\xi ds \\ &\qquad + C\int_0^t(1+s)\int_{S(s)}|\xi|^2\left(\int_0^t\|u(s)\|_{L^2(\mathbb{R}^2)}^2 ds\right)^{1/(p-1)} d\xi ds \\ &\quad\le \int_{\mathbb{R}^n}\left(|u\wedge_0(\xi)|^2+|b\wedge_0(\xi)|^2\right)d\xi + C\int_0^t(1+s)\int_{S(s)} \\ &\qquad\times\left(|u\wedge_0(\xi)|^2+|b\wedge_0(\xi)|^2\right)d\xi ds + C\int_0^t(1+s)\int_{S(s)}s|\xi|^2 \\ &\qquad\times\left(\int_0^t\left(\|u(s)\|_{L^2}^4+\|b(s)\|_{L^2}^4\right)ds\right)d\xi ds \\ &\qquad + C\int_0^t(1+s)\int_{S(s)}s|\xi|^2\left(\int_0^t\|u(s)\|_{L^2(\mathbb{R}^2)}^2 ds + C\right)d\xi d \\ &\quad\le \int_{\mathbb{R}^n}\left(|u\wedge_0(\xi)|^2+|b\wedge_0(\xi)|^2\right)d\xi + C\int_0^t(1+s)\int_{S(s)} \\ &\qquad\times\left(|u\wedge_0(\xi)|^2+|b\wedge_0(\xi)|^2\right)d\xi d \\ &\qquad + C\int_0^t(1+s)^2\int_{S(s)}|\xi|^2\int_0^t\left(\|u(s)\|_{L^2}^4+\|b(s)\|_{L^2}^4\right)ds\, d\xi ds \\ &\qquad + C\int_0^t(1+s)^2\int_{S(s)}|\xi|^2\int_0^t\|u(s)\|_{L^2(\mathbb{R}^2)}^2 ds\, d\xi ds \\ &\quad\le C + C(1+t) + \left(\int_0^t\left(\|u(s)\|_{L^2}^2+\|b(s)\|_{L^2}^2\right)ds\right). \end{aligned} \quad (48)$$

Thus, we get

$$\begin{aligned} &(1+t)\int_{\mathbb{R}^3}\left(|u(\xi,t)|^2+|b(\xi,t)|^2\right)d\xi \\ &\quad\le C + C\int_0^t(1+s)\left(\|u(s)\|^2+\|b(s)\|^2\right)(1+s)^{-1}ds. \end{aligned} \quad (49)$$

Applying Gronwall's inequality, we immediately deduce that

$$\|u(t)\|_{L^2}+\|b(t)\|_{L^2}\le C(1+t)^{-1/2}, \quad (50)$$

thus, we finally obtain the desired result.

Appendix

Here, we mention the existence of unique strong solution for (1)–(2). Its proof is easily checked from the argument in [15] or [22]. And thus, we omit the proof.

Definition A.1. Let $2<q<\infty$ and $\mu_0\ge 0$ and $\mu_1>0$. Suppose that $u_0\in(W^{1,2}\cap W^{1,q})(\mathbb{R}^2)$ and $b_0\in W^{1,2}(\mathbb{R}^2)$. We say that a weak solution (u,b) is a strong solution to the incompressible 2D-MHD equations of non-Newtonian fluids (1)–(2) if

$$\begin{aligned} &\nabla u\in L^\infty\left(0,T;L^q\cap L^2\left(\mathbb{R}^2\right)\right), \\ &b\in L^\infty\left(0,T;W^{1,2}\left(\mathbb{R}^2\right)\right)\cap L^2\left(0,T;W^{2,2}\left(\mathbb{R}^2\right)\right), \\ &u_t,b_t\in L^2(Q_T), S(Du)\in L^{q'}\left(0,T;W^{1,q'}_{\mathrm{loc}}\left(\mathbb{R}^2\right)\right). \\ &\int_0^T\int_{\mathbb{R}^2}|Du|^{q-2}\left|D^2u\right|^2 dxdt<\infty. \end{aligned} \quad (A1)$$

Here, q' means the Hölder conjugate of q.

Theorem A.2. *Let $2<q<\infty$ and $\mu_0\ge 0$ and $\mu_1>0$. Suppose that $u_0\in(W^{1,2}\cap W^{1,q})(\mathbb{R}^2)$ and $b_0\in W^{1,2}(\mathbb{R}^2)$. Then, there exists a strong solution (u,b) of the incompressible 2D-MHD equations of the non-Newtonian type (1)–(2) in the sense of Definition A.1.*

Data Availability

This paper uses the method of theoretical analysis.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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