

Research Article

On Existence of Multiplicity of Weak Solutions for a New Class of Nonlinear Fractional Boundary Value Systems via Variational Approach

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This paper deals with the existence of solutions for a new class of nonlinear fractional boundary value systems involving the left and right Riemann-Liouville fractional derivatives. More precisely, we establish the existence of at least three weak solutions for the problem using variational methods combined with the critical point theorem due to Bonano and Marano. In addition, some examples in \mathbb{R}^3 and \mathbb{R}^4 are given to illustrate the theoretical results.

1. Introduction

Fractional differential equations (FDEs) are a generalization of ordinary differential equations (ODEs), as they contain fractional derivatives whose degree is not necessarily an integer. This is what makes it receive great attention from researchers due to its ability to model some difficult and complex phenomena in many fields, including engineering, science, biology, economics, and physics (for more information, see [1–22]). One of the most investigated issues is the existence of solutions for the fractional initial and boundary value problems by using some fixed point theorems, coincidence degree theory, and monotone interactive method. Among the most important of these are the works mentioned in Oldham and Spanier and Podlubny's books (see [13, 23]) and the work of Metzler

and Klafter (see [24]). Furthermore, the first to use the critical point theorem was Jiao and Zhou in [6] to study the following problem:

$$\begin{cases} {}_t D_T^\alpha ({}_0 D_t^\alpha u(t)) = \nabla F(t, u(t)), & \text{a.e } t \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \quad (1)$$

where ${}_0 D_T^\alpha$ and ${}_t D_T^\alpha$ are the left and right Riemann-Liouville fractional derivatives with $0 < \alpha \leq 1$, respectively, and $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^n$ is a suitable function satisfying some hypothesis and $F(t, x)$ is the gradient of F with respect to x .

In [22], the authors have used variational methods to investigate the existence of weak solutions for the following system:

$$\begin{cases} {}_t D_T^\alpha (a(t) {}_0 D_t^\alpha u(t)) = \lambda F_u(t, u(t), v(t)), & \text{a.e } t \in [0, T], \\ {}_t D_T^\beta (b(t) {}_0 D_t^\beta v(t)) = \lambda F_v(t, u(t), v(t)), & \text{a.e } t \in [0, T], \\ u(0) = u(T) = 0, v(0) = v(T) = 0, \end{cases} \quad (2)$$

for ${}_0 D_T^\alpha$ and ${}_t D_T^\alpha$ are the left and right Riemann-Liouville fractional derivatives with $0 < \alpha \leq 1$ and F_s denotes the par-

$$\begin{cases} {}_t D_T^{\alpha_i} (a_i(t) {}_0 D_t^{\alpha_i} u_i(t)) = \lambda F_{u_i}(t, u_1(t), u_2(t), \dots, u_n(t)) + h_i(u_i(t)), & \text{a.e } t \in [0, T], \\ u_i(0) = u_i(T) = 0, \end{cases} \quad (3)$$

for $1 \leq i \leq n$, where $\alpha_i \in (0; 1]$, ${}_0 D_T^{\alpha_i}$ and ${}_t D_T^{\alpha_i}$ are the left and right Riemann-Liouville fractional derivatives of order α_i , respectively, $a_i \in L^\infty([0, T])$ with

$$a_{i0} = \text{ess inf}_{[0, T]} a_i > 0, \quad \text{for } 1 \leq i \leq n, \quad (4)$$

$\lambda > 0$, $F : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a measurable function for all $(x_1, \dots, x_n) \in \mathbb{R}^n$ and is C^1 with respect to $(x_1, \dots, x_n) \in \mathbb{R}^n$ for a.e. $t \in [0, T]$, F_{u_i} denotes the partial derivative of F with respect to u_i , respectively, and $h_i : \mathbb{R} \rightarrow \mathbb{R}$ are Lipschitz continuous functions with the Lipschitz constants $L_i > 0$, for $1 \leq i \leq n$, i.e.,

$$|h_i(x_1) - h_i(x_2)| \leq L_i |x_1 - x_2|, \quad (5)$$

for all $x_1, x_2 \in \mathbb{R}$ and $h_i(0) = 0$, for $1 \leq i \leq n$. In order to state the main results, we introduce the following conditions:

(F0) For all $C > 0$ and any $1 \leq i \leq n$

$$\sup_{|(x_1, \dots, x_n)| \leq C} |F_{u_i}(t, x_1, \dots, x_n)| \in L^1([0, T]). \quad (6)$$

(F1) $F(t; 0, \dots, 0) = 0$, for a.e. $t \in [0; T]$.

In the present study, motivated by the results introduced in [12, 13, 25], using the three critical point theorems due to Ricceri ([26], see Theorem 2.6 in the next section), we ensure the existence of at least three solutions for system (3). For other applications of Ricceri's result for perturbed boundary value problems, the interested readers are referred to the papers [11–13, 23–25, 27].

We divided the paper as follows: in the second section, we put some preliminary facts, while in the third section we presented the main result and its proof. Finally, we proposed two practical examples of our theorem.

2. Preliminaries

In this section, introducing some necessary definitions and preliminary facts.

tial derivative of F with respect to s . In [?], Zhao et al. obtained the existence of infinitely many solutions for system (2) with perturbed functions h_i , $i = 1, 2$.

Yet, there are a few findings for fractional boundary value problems which were established exploiting this approach due to its difficulty in establishing a suitable space and variational functional for fractional problems.

In this work, we shall study the existence of three weak solutions for the following system:

Definition 1 [28]. Let u be a function defined on $[0, T]$ and $\alpha_i > 0$ for $1 \leq i \leq n$. The left and right Riemann-Liouville fractional integrals of order α_i for the function u are defined by

$$\begin{aligned} {}_0 D_t^{-\alpha_i} u(t) &= \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} u(s) ds, & t \in [0, T], \\ {}_0 D_t^{-\alpha_i} u(t) &= \frac{1}{\Gamma(\alpha_i)} \int_t^T (s-t)^{\alpha_i-1} u(s) ds, & t \in [0, T], \end{aligned} \quad (7)$$

for $1 \leq i \leq n$, provided the RHS are pointwise given on $[0, T]$, where $\Gamma(\alpha_i)$ is the standard gamma function defined by

$$\Gamma(z) = \int_0^{+\infty} z^{\alpha_i-1} e^{-z} dz. \quad (8)$$

Definition 2 [25]. Let $0 < \alpha_i \leq 1$ for $1 \leq i \leq n$. The fractional derivative space $H_0^{\alpha_i}$ is given by the closure $C_0^\infty([0, T], \mathbb{R})$, that is

$$H_0^{\alpha_i} = C_0^\infty([0, T], \mathbb{R}), \quad (9)$$

with the norm

$$\|u_i\|_{\alpha_i} = \left(\int_0^T a_i(t) |{}_0 D_t^{\alpha_i} u_i(t)|^2 dt + \int_0^T |u_i(t)|^2 dt \right)^{1/2}, \quad (10)$$

for every $u_i \in H_0^{\alpha_i}$ and for $1 \leq i \leq n$.

We point out that $H_0^{\alpha_i}$ ($0 < \alpha_i \leq 1$) is a reflexive and separable Banach space (see [22], Proposition 3.1) for details.

For every $u_i \in H_0^{\alpha_i}$, set

$$\begin{aligned} \|u_i\|_{L^s} &:= \left(\int_0^T |u_i(t)|^s dt \right)^{1/s}, & s \geq 1 \\ \|u_i\|_\infty &= \max_{t \in [0, T]} |u_i(t)|. \end{aligned} \quad (11)$$

Definition 3 [27]. We mean by a weak solution of system (3), any $u = (u_1, u_2, \dots, u_n) \in X$ such that for all $v = (v_1, v_2, \dots, v_n) \in X$,

$$\begin{aligned} & \int_0^T \sum_{i=1}^n a_i(t) {}_0D_t^{\alpha_i} u_i(t) {}_0D_t^{\alpha_i} v_i(t) dt \\ & - \lambda \int_0^T \sum_{i=1}^n F_{u_i}(t, u_1(t), u_2(t), \dots, u_n(t)) v_i(t) dt \quad (12) \\ & - \int_0^T \sum_{i=1}^n h_i(u_i) v_i(t) dt = 0. \end{aligned}$$

Lemma 4 [27]. Let $0 < \alpha_i \leq 1$, for $1 \leq i \leq n$. $\forall u_i \in H_0^{\alpha_i}$, we have

$$\|u_i\|_{(L^2)} \leq \frac{T^{\alpha_i}}{\Gamma(\alpha_i + 1)} \|{}_0D_t^{\alpha_i} u_i\|_{L^2}. \quad (13)$$

Moreover,

$$\|u_i\|_{\infty} \leq \frac{T^{\alpha_i}}{\Gamma(\alpha_i) \sqrt{(2\alpha_i - 1)}} \|{}_0D_t^{\alpha_i} u_i\|_{L^2}. \quad (14)$$

From Lemma 4, we easily observe that

$$\|u_i\|_{L^2} \leq \frac{T^{\alpha_i}}{\Gamma(\alpha_i + 1) \sqrt{\alpha_{i0}}} \left(\int_0^T a_i(t) |{}_0D_t^{\alpha_i} u_i(t)|^2 dt \right)^{1/2}. \quad (15)$$

for $0 < \alpha_i \leq 1$, and

$$\|u_i\|_{\infty} \leq \frac{T^{\alpha_i - (1/2)}}{\Gamma(\alpha_i) \sqrt{a_{i0}(2\alpha_i - 1)}} \left(\int_0^T a_i(t) |{}_0D_t^{\alpha_i} u_i(t)|^2 dt \right)^{1/2}. \quad (16)$$

By using (15), the norm of (10) is equivalent to

$$\|u_i\|_{\alpha_i} \left(\int_0^T a_i(t) |{}_0D_t^{\alpha_i} u_i(t)|^2 dt \right)^{1/2}, \quad \forall u_i \in H_0^{\alpha_i}. \quad (17)$$

Throughout this paper, let X be the Cartesian product of the n spaces $H_0^{\alpha_i}$ for $1 \leq i \leq n$, i.e., $X = H_0^{\alpha_1} \times H_0^{\alpha_2} \times \dots \times H_0^{\alpha_n}$; we equip X with the norm defined by

$$\|u\| = \sum_{i=1}^n \|u_i\|_{H_0^{\alpha_i}}, \quad u = (u_1, u_2, \dots, u_n), \quad (18)$$

where $\|u_i\|_{H_0^{\alpha_i}}$ is given in (17). We have X compactly embedded in $C([0, T], \mathbb{R})^n$.

Theorem 5 [25]. Let X be a reflexive real Banach space and $\Phi : X \rightarrow \mathbb{R}$ be a coercive, continuously Gâteaux differentiable sequentially weakly lower semicontinuous functional

whose Gâteaux derivative admits a continuous inverse on X^* , bounded on bounded subsets of $X, \Psi : X \rightarrow \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that

$$\Phi(0) = \Psi(0) = 0. \quad (19)$$

Suppose that $\exists r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, satisfying

$$(a_1) \sup_{\Phi(u) \leq r} (\Psi(u)/r) < (\Phi(\bar{x})/\Psi(\bar{x})).$$

$$(a_2) \text{ For each } \lambda \in \Lambda_\lambda := (\Phi(\bar{x})/\Psi(\bar{x}), r / \sup_{\Phi(u) \leq r} \Psi(u)), \text{ the}$$

functional $\Phi - \lambda\Psi$ is coercive.

Hence, $\forall \lambda \in \Lambda_\lambda$, the functional $\Phi - \lambda\Psi$ has at least three critical points in the space X .

3. Main Results

In this section, by applying Theorem 5, we examine the existence of multiple solutions for system (3). For any $\sigma > 0$, let us define

$$\pi(\sigma) = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \frac{1}{2} \sum_{i=1}^n |x_i|^2 \leq \sigma \right\}. \quad (20)$$

This set will be used in some of our hypotheses with appropriate choices of σ . For $u = (u_1, u_2, \dots, u_n) \in X$, we define

$$Y(u) := \sum_{i=1}^n Y_i(u_i) \quad (21)$$

where $Y_i(x) = \int_0^T H_i(x(s)) ds$ and $H_i(x) = \int_0^x h_i(z) dz$ $1 \leq i \leq n$, $\forall t \in [0; T]$ and $x \in \mathbb{R}$.

Furthermore, let

$$\begin{aligned} k &:= \max_{1 \leq i \leq n} \left\{ \frac{T^{2\alpha_i - 1}}{(\Gamma(\alpha_i))^2 a_{i0} (2\alpha_i - 1)} \right\}, \\ M &:= \min_{1 \leq i \leq n} \left\{ 1 - \frac{L_i T^{2\alpha_i}}{(\Gamma(\alpha_i + 1))^2 a_{i0}} \right\}, \\ \tilde{k} &:= \max_{1 \leq i \leq n} \left\{ 1 + \frac{L_i T^{2\alpha_i}}{(\Gamma(\alpha_i + 1))^2 a_{i0}} \right\}. \end{aligned} \quad (22)$$

Theorem 6. Let $1/2 < \alpha_i \leq 1$, for $1 \leq i \leq n$, and suppose that $M > 0$ and the conditions (F0) and (F1) are satisfied. Furthermore, assume that $\exists r > 0$ and a function $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in X$ satisfying

$$(i) \sum_{i=1}^n \frac{\|\omega_i\|_{\alpha_i}^2}{2} > \frac{r}{M},$$

$$(ii) \quad 2r \frac{\int_0^T F(t, \omega_1, \omega_2, \dots, \omega_n) dt}{\sum_{i=1}^n \|\omega_i\|_{\alpha_i}^2 - 2Y(\omega_1, \omega_2, \dots, \omega_n)} - \int_0^T \max_{(x_1, \dots, x_n) \in \pi(\frac{M}{r})} F(t, x_1, \dots, x_n) dt > 0,$$

$$(iii) \quad \lim_{(|x_1|, \dots, |x_n|) \rightarrow (+\infty, \dots, +\infty)} \sup_{t \in [0, T]} \frac{F(t, x_1, \dots, x_n)}{\sum_{i=1}^n |x_i|^2 / 2} \leq 0. \tag{23}$$

Then, setting

$$\Lambda = \left(\frac{\sum_{i=1}^n (\|\omega_i\|_{\alpha_i}^2 / 2) - Y(\omega_1, \omega_2, \dots, \omega_n)}{\int_0^T F(t, \omega_1(t), \omega_2(t), \dots, \omega_n(t)) dt}, \frac{r}{\int_0^T \max_{(x_1, \dots, x_n) \in \pi(M/kr)} F(t, x_1, \dots, x_n) dt} \right). \tag{24}$$

$\forall \lambda \in \Lambda$ system (3) admits at least 3 weak solutions in X .

Proof. For each $u = (u_1, u_2, \dots, u_n) \in X$, we introduce the functionals $\Phi, \Psi : X \rightarrow \mathbb{R}$ as

$$\Phi(u) = \sum_{i=1}^n \frac{\|u_i\|_{\alpha_i}^2}{2} - Y(u), \tag{25}$$

$$\Psi(u) = \int_0^T F(t, u_1(t), u_2(t), \dots, u_n(t)) dt. \tag{26}$$

It is clear that Φ and Ψ are continuously Gâteaux differentiable functionals whose Gâteaux derivatives at the point $u \in X$ are defined by

$$\Phi'(u)(v) = \int_0^T \sum_{i=1}^n a_i(t) D_i^{\alpha_i} u_i(t) D_i^{\alpha_i} v_i(t) dt - \int_0^T \sum_{i=1}^n h_i(u_i(t)) v_i(t) dt$$

$$\Psi'(u)(v) = \int_0^T \sum_{i=1}^n F_{u_i}(t, u_1(t), u_2(t), \dots, u_n(t)) v_i(t) dt, \tag{27}$$

for every $v = (v_1, v_2, \dots, v_n) \in X$.

We have $\Phi'(u), \Psi'(u) \in X^*$, where X^* is the dual space of X . And the functional Φ is sequentially weakly lower semicontinuous and its Gâteaux derivative admits a continuous inverse on X^* ; also $\lim_{\|u\|_X \rightarrow +\infty} \Phi(u) = +\infty$ it is coercive.

Now, we show that the functional Ψ is sequentially weakly upper semicontinuous and its derivative $\Psi' : X \rightarrow X^*$ is a compact operator. Let $u_m \rightharpoonup u$ in X , where $u_m(t) = (u_{m,1}(t), u_{m,2}(t), \dots, u_{m,n}(t))$; then certainly u_m converges uniformly to u on the interval $[0, T]$. Then,

$$\limsup_{m \rightarrow +\infty} \Psi(u_m) \leq \int_0^T \limsup_{m \rightarrow +\infty} F(t, u_{m,1}(t), u_{m,2}(t), \dots, u_{m,n}(t)) dt = \int_0^T F(t, u_1(t), u_2(t), \dots, u_n(t)) dt = \Psi(u), \tag{28}$$

which gets that Ψ is sequentially weakly upper semicontinuous.

Moreover, we have

$$\lim_{m \rightarrow +\infty} F(t, u_{m,1}(t), u_{m,2}(t), \dots, u_{m,n}(t)) = F(t, u_1(t), u_2(t), \dots, u_n(t)), \quad \text{for all } t \in [0, T]. \tag{29}$$

Note that $F(t, \cdot, \dots, \cdot) \in C^1(\mathbb{R}^n)$. The Lebesgue control convergence theorem implies that $\Psi'_m(u) \rightarrow \Psi'(u)$ strongly, hence yielding that Ψ' is strongly continuous on X . Then, $\Psi' : X \rightarrow X^*$ is a compact operator.

We show that required hypothesis $\Phi(\bar{x}) > r$ follows from (i) and the definition of Φ by taking $\bar{x} = \omega$. Indeed, as (5) holds for all $x_1, x_2 \in \mathbb{R}$ and $h_1(0) = \dots = h_n(0) = 0$; one has $|h_i(x)| \leq L_i |x|$, $1 \leq i \leq n$, for any $x \in \mathbb{R}$. It follows from (15) that

$$\Phi(\omega) \geq \frac{\sum_{i=1}^n \|\omega_i\|_{\alpha_i}^2}{2} - \left| \int_0^T \sum_{i=1}^n H_i(\omega_i(t)) dt \right| \geq \frac{\sum_{i=1}^n \|\omega_i\|_{\alpha_i}^2}{2} - \sum_{i=1}^n \frac{L_i}{2} \int_0^T |\omega_i(t)|^2 dt \geq \sum_{i=1}^n \left(\frac{1}{2} - \frac{L_i T^{2\alpha_i}}{(\Gamma(\alpha_i + 1))^2 a_{i0}} \right) \|\omega_i\|_{\alpha_i}^2 \geq \frac{M}{2} \sum_{i=1}^n \|\omega_i\|_{\alpha_i}^2. \tag{30}$$

From (16), for every $u_i \in H_0^{\alpha_i}$, we have

$$\max_{t \in [0, T]} |u_i(t)|^2 \leq k \|u_i\|_{\alpha_i}, \quad (31)$$

for $1 \leq i \leq n$. Hence,

$$\max_{t \in [0, T]} \sum_{i=1}^n |u_i(t)|^2 \leq k \sum_{i=1}^n \|u_i\|_{\alpha_i}. \quad (32)$$

Assume that $u_0(t) = (0, \dots, 0)$ and the supposition (i) deduces that $0 < r < \Phi(\omega)$ and they hold $\Phi(u_0(t)) = \Psi(u_0(t)) = 0$ from definitions (25) and (26), which are required assumptions in Theorem 5. Applying relations (16), (17), and (22) gives the following relation:

$$\begin{aligned} \Phi^{-1}((-\infty; r]) &= \{u = (u_1, u_2, \dots, u_n) \in X : \Phi(u) \leq r\} \\ &= \left\{ u = (u_1, u_2, \dots, u_n) \in X : \sum_{i=1}^n \frac{\|u_i\|_{\alpha_i}^2}{2} \leq \frac{r}{M} \right\} \\ &\subseteq \left\{ u = (u_1, u_2, \dots, u_n) \in X : \sum_{i=1}^n \frac{(\Gamma(\alpha_i))^2 a_{10} (2\alpha_i - 1)}{2T^{2\alpha_i - 1}} \|u_i\|_{\infty}^2 \leq \frac{r}{M} \right\} \\ &\subseteq \left\{ u = (u_1, u_2, \dots, u_n) \in X : \frac{1}{2} \sum_{i=1}^n |u_i(t)|^2 \leq \frac{kr}{M} \right\}, \end{aligned} \quad (33)$$

which implies that

$$\begin{aligned} \sup_{u \in \Phi^{-1}((-\infty; r])} \Psi(u) &= \sup_{u \in \Phi^{-1}((-\infty; r])} \int_0^T F(t, u_1(t), u_2(t), \dots, u_n(t)) dt \\ &\leq \int_0^T \max_{(x_1, \dots, x_n) \in \pi(\frac{kr}{M})} F(t, x_1, \dots, x_n) dt. \end{aligned} \quad (34)$$

Hence, under the condition (ii), we get the following inequality

$$\begin{aligned} \frac{\sup_{u \in \Phi^{-1}((-\infty; r])} \Psi(u)}{r} &\leq \int_0^T \max_{(x_1, \dots, x_n) \in \pi(\frac{kr}{M})} F(t, x_1, \dots, x_n) dt \\ &< 2r \frac{\int_0^T F(t, \omega_1, \omega_2, \dots, \omega_n) dt}{\sum_{i=1}^n \|\omega_i\|_{\alpha_i}^2 - 2Y(\omega_1, \omega_2, \dots, \omega_n)} \\ &= r \frac{\int_0^T F(t, \omega_1, \omega_2, \dots, \omega_n) dt}{\sum_{i=1}^n (\|\omega_i\|_{\alpha_i}^2 / 2) - Y(\omega_1, \omega_2, \dots, \omega_n)} \\ &= \frac{\Psi(\omega)}{\Phi(\omega)}. \end{aligned} \quad (35)$$

Thus, the hypothesis (a_1) of Theorem 5 holds.

On the other hand, fix $0 < \varepsilon < (1/2Tk\lambda)$. From (iii) into account, there exist constants $\tau_\varepsilon \in \mathbb{R}$ such that

$$F(t, x_1, \dots, x_n) \leq \varepsilon \sum_{i=1}^n |x_i|^2 + \tau_\varepsilon, \quad (36)$$

for any $t \in [0, T]$ and $(x_1, \dots, x_n) \in \mathbb{R}^n$, by using (36) and (15) yields, it follows that, for each $u \in X$,

$$\begin{aligned} \Phi(u) - \lambda\Psi(u) &= \frac{1}{2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - \lambda \int_0^T F(t, u_1(t), u_2(t), \dots, u_n(t)) dt \\ &\geq \frac{1}{2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - T\lambda k\varepsilon \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - \lambda\tau_\varepsilon \\ &\geq \left(\frac{1}{2} - T\lambda k\varepsilon\right) \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - \lambda\tau_\varepsilon. \end{aligned} \quad (37)$$

And from him,

$$\lim_{\|u\|_X \rightarrow +\infty} \Phi(u) - \lambda\Psi(u) = +\infty. \quad (38)$$

Moreover, analogous to the case of $\tau_\varepsilon > 0$, we imply that $\Phi(u) - \lambda\Psi(u) \rightarrow +\infty$ as $\|u\|_X \rightarrow +\infty$ with $\tau_\varepsilon \leq 0$. Then, the hypotheses of Theorem 5 hold, which means that system (3) admits at least 3 weak solutions in X , which completes the proof.

Now, we present some notations, before the corollary of Theorem 6.

Put

$$\begin{aligned} A_i(\alpha_i) &= \frac{16}{T^2} \left\{ \int_0^T a_i(t) t^{2(1-\alpha_i)} dt + \int_{T/4}^T a_i(t) \left(t - \frac{T}{4}\right)^{2(1-\alpha_i)} dt \right. \\ &\quad + \int_{3T/4}^T a_i(t) \left(t - \frac{3T}{4}\right)^{2(1-\alpha_i)} dt - 2 \int_{T/4}^T a_i(t) \\ &\quad \cdot \left(t^2 - \frac{T}{4}t\right)^{1-\alpha_i} dt - 2 \int_{3T/4}^T a_i(t) \left(t^2 - \frac{3T}{4}t\right)^{1-\alpha_i} dt \\ &\quad \left. + 2 \int_{3T/4}^T a_i(t) \left(t^2 - Tt + \frac{3T^2}{16}t\right)^{1-\alpha_i} dt \right\}, \end{aligned}$$

$$\Delta_1 = \min_{1 \leq i \leq n} \{A_i(\alpha_i) : \text{for } 1 \leq i \leq n\},$$

$$\Delta_2 = \max_{1 \leq i \leq n} \{A_i(\alpha_i) : \text{for } 1 \leq i \leq n\}. \quad (39)$$

Corollary 7. Let $1/2 < \alpha_i \leq 1$, $1 \leq i \leq n$ and supposition (iii) in Theorem 6 holds. Suppose that $\exists \tau > 0$ and d such that $(\tau/\Delta_1 kMn) < d^2$, and also

$$\begin{aligned} (i') \quad &F(t, x_1, \dots, x_n) \geq 0, \quad \text{for } (t, x_1, \dots, x_n) \\ &\in \left(\left[0, \frac{T}{4}\right] \cup \left[\frac{3T}{4}, T\right] \right) \\ &\quad \times [0, +\infty) \times \dots \times [0, +\infty), \end{aligned}$$

$$(ii') \frac{\int_0^T \max_{(x_1, \dots, x_n) \in \pi(\tau)} F(t, x_1, \dots, x_n) dt}{\tau M} < \frac{\int_{T/4}^{3T/4} F(t, \Gamma(2-\alpha_1)d, \Gamma(2-\alpha_2)d, \dots, \Gamma(2-\alpha_n)d) dt}{nk\tilde{\Delta}_2 d^2},$$

$$(iii') \lim_{(|x_1|, \dots, |x_n|) \rightarrow (+\infty, \dots, +\infty)} \sup_{t \in [0, T]} \frac{F(t, x_1, \dots, x_n)}{\sum_{i=1}^n |x_i|^2 / 2} \leq 0. \tag{40}$$

Then, setting

$$\lambda \in \Lambda' = \left(\frac{n\tilde{k}\Delta_2 d^2}{\int_{T/4}^{3T/4} F(t, \Gamma(2-\alpha_1)d, \Gamma(2-\alpha_2)d, \dots, \Gamma(2-\alpha_n)d) dt}, \frac{\tau}{k \int_0^T \max_{(x_1, \dots, x_n) \in \pi(\tau)} F(t, x_1, \dots, x_n) dt} \right) \tag{41}$$

Thus, system (3) admits at least three weak solutions in X

By (25), for $1 \leq i \leq n$, imply that

Proof. Choose

$$\omega_i(t) = \begin{cases} \frac{4\Gamma(2-\alpha_i)d}{T} t, & t \in \left[0, \frac{T}{4}\right], \\ \Gamma(2-\alpha_i)d, & t \in \left[\frac{T}{4}, \frac{3T}{4}\right], \\ \frac{4\Gamma(2-\alpha_i)d}{hT} (T-t), & t \in \left[\frac{3T}{4}, T\right]. \end{cases} \tag{42}$$

$$\begin{aligned} \Phi(\omega) &= \Phi(\omega_1, \omega_2, \dots, \omega_n) \\ &= \sum_{i=1}^n \frac{\|\omega_i\|_{\alpha_i}^2}{2} - Y(\omega) \\ &\geq \frac{M}{2} \sum_{i=1}^n \|\omega_i\|_{\alpha_i}^2 \\ &= Md^2 \sum_{i=1}^n A(\alpha_i) \\ &\geq nM\Delta_1 d^2. \end{aligned} \tag{46}$$

We derive

$${}_0D_t^{\alpha_i} \omega_i(t) = \begin{cases} \frac{4d}{T} t^{1-\alpha_i}, & t \in \left[0, \frac{T}{4}\right], \\ \frac{4d}{T} \left(t^{1-\alpha_i} - \left(t - \frac{T}{4} \right)^{1-\alpha_i} \right), & t \in \left[\frac{T}{4}, \frac{3T}{4}\right], \\ \frac{4d}{T} \left(t^{1-\alpha_i} - \left(t - \frac{T}{4} \right)^{1-\alpha_i} - \left(t - \frac{3T}{4} \right)^{1-\alpha_i} \right), & t \in \left[\frac{3T}{4}, T\right]. \end{cases} \tag{43}$$

Similar to (30) and (46), we have $\Phi(\omega) \leq n\tilde{k}\Delta_2 d^2$.
Let $r = \tau M/k$. From $(\tau/\Delta_1 k M n) < d^2$, we have

$$\sum_{i=1}^n \frac{\|\omega_i\|_{\alpha_i}^2}{2} \geq \Phi(\omega) \geq nM\Delta_1 d^2 > nM\Delta_1 \times \frac{\tau}{\Delta_1 k M n} = \frac{r}{M}. \tag{47}$$

Moreover,

$$\begin{aligned} &\int_0^T a_i(t) |{}_0D_t^{\alpha_i} \omega_i(t)|^2 dt \\ &= \int_0^{T/4} + \int_{T/4}^{3T/4} + \int_{3T/4}^T a_i(t) |{}_0D_t^{\alpha_i} \omega_i(t)|^p dt \\ &= 2A_i(\alpha_i) d^2. \end{aligned} \tag{44}$$

Then, $\omega_i(0) = \omega_i(T) = 0$, $\omega_i(t) {}_0D_t^{\alpha_i} \omega_i(t) \in L^2[0, T]$, $i = 1, 2, \dots, n$; hence, $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in X$, and we have

$$\|\omega_i\|_{\alpha_i}^2 = \int_0^T a_i(t) |{}_0D_t^{\alpha_i} \omega_i(t)|^2 dt = 2A_i(\alpha_i) d^2. \tag{45}$$

Thus, the assumption (ii) of Theorem 6 holds.
(i') implies that

$$\begin{aligned} \Psi(\omega) &= \int_0^T F(t, \omega_1, \omega_2, \dots, \omega_n) dt \\ &= \int_0^{T/4} F(t, \omega_1, \omega_2, \dots, \omega_n) dt \\ &\quad + \int_{T/4}^{3T/4} F(t, \omega_1, \omega_2, \dots, \omega_n) dt \\ &\quad + \int_{3T/4}^T F(t, \omega_1, \omega_2, \dots, \omega_n) dt \\ &\geq \int_{T/4}^{3T/4} F(t, \omega_1, \omega_2, \dots, \omega_n) dt. \end{aligned} \tag{48}$$

Moreover, by condition (ii'), we have

$$\begin{aligned} & \frac{\int_0^T \max_{(x_1, \dots, x_n) \in \pi(kr/M)} F(t, x_1, \dots, x_n) dt}{r} \\ &= \frac{k \int_0^T \max_{(x_1, \dots, x_n) \in \pi(kr/M)} F(t, x_1, \dots, x_n) dt}{\tau M} \\ &< \frac{\int_{T/4}^{3T/4} F(t, \Gamma(2 - \alpha_1)d, \Gamma(2 - \alpha_2)d, \dots, \Gamma(2 - \alpha_n)d) dt}{nk\Delta_2 d^2} \\ &\leq \frac{\int_{T/4}^{3T/4} F(t, \Gamma(2 - \alpha_1)d, \Gamma(2 - \alpha_2)d, \dots, \Gamma(2 - \alpha_n)d) dt}{\Phi(\omega)} \\ &\leq \frac{2 \int_0^T F(t, \omega_1, \omega_2, \dots, \omega_n) dt}{\sum_{i=1}^n \|\omega_i\|_{\alpha_i}^2 - 2Y(\omega_1, \omega, \dots, \omega_n)}. \end{aligned} \tag{49}$$

Hence, the supposition (ii) of Theorem 6 is verified.

Moreover, the supposition (iii) of Theorem 6 holds under (iii') from $\Lambda' \subseteq \Lambda$. Theorem 6 is successfully employed to ensure the existence of at least 3 weak solutions for system (3). This completes of the proof.

4. Examples

In this section, we propose two practical examples of Theorem 6.

Example 1. Let $\alpha_1 = 0.7, \alpha_2 = 0.65, \alpha_3 = 0.6, a_1(t) = 1 + t^2, a_2(t) = 0.5 + t, a_3(t) = 1 + t, T = 1$. Then, sys-

tem (3) gets the following form:

$$\begin{cases} {}_t D_1^{0.7}((1+t^2){}_0 D_t^{0.7} u_1(t)) = \lambda F_{u_1}(t, u_1(t), u_2(t), u_3(t)) + h_1(u_1), & t \in [0, 1], \\ {}_t D_1^{0.65}((0.5+t){}_0 D_t^{0.65} u_2(t)) = \lambda F_{u_2}(t, u_1(t), u_2(t), u_3(t)) + h_2(u_2), & t \in [0, 1], \\ {}_t D_1^{0.6}((1+t){}_0 D_t^{0.6} u_3(t)) = \lambda F_{u_3}(t, u_1(t), u_2(t), u_3(t)) + h_3(u_3), & t \in [0, 1], \\ u_1(0) = u_1(1) = 0, u_2(0) = u_2(1) = 0, u_3(0) = u_3(1) = 0, \end{cases} \tag{50}$$

where $h_1(u_1) = 1/4 \sin u_1, h_2(u_2) = u_2/2$, and $h_3(u_3) = 1/20 \arctan u_3$.

Furthermore, $\forall (t; x_1, x_2, x_3) \in [0; 1] \times \mathbb{R}^3$; put

$$F(t, x_1(t), x_2(t), x_3(t)) = (1 + t^2)G(x_1, x_2, x_3), \tag{51}$$

where

$$\begin{aligned} & G(x_1, x_2, x_3) \\ &= \begin{cases} (x_1^2 + x_2^2 + x_3^2)^2, x_1^2 + x_2^2 + x_3^2 \leq 1, \\ 10(x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}} - 9(x_1^2 + x_2^2 + x_3^2)^{\frac{1}{3}}, x_1^2 + x_2^2 + x_3^2 > 1. \end{cases} \end{aligned} \tag{52}$$

Obviously $h_1, h_2, h_3 \rightarrow \mathbb{R}$ are three Lipschitz continuous functions with Lipschitz constants $L_1 = 1/4, L_2 = 1/2, L_3 = 1/20$ and $h_1(0) = h_2(0) = h_3(0) = 0$. Clearly, $F(t, 0, 0, 0) = 0, \forall t \in [0, 1]$, by the direct calculation, we have $a_{10} = 1, a_{20} = 1$, and $a_{30} = 0.5$

$$\begin{aligned} k &= \max \left\{ \frac{1}{(\Gamma(0.7))^2(2 \times 0.7 - 1)}, \frac{1}{(\Gamma(0.65))^2(2 \times 0.65 - 1)}, \frac{1}{(\Gamma(0.6))^2 \times 0.5(2 \times 0.6 - 1)} \right\} \approx 4.509191, \\ M &= \min \left\{ 1 - \frac{L_1}{(\Gamma(0.7) + 1)^2}, 1 - \frac{L_2}{(\Gamma(0.65) + 1)^2}, 1 - \frac{L_3}{(\Gamma(0.6) + 1)^2 \times 0.5} \right\} \approx 0.912084 \end{aligned} \tag{53}$$

Taking

$$\begin{aligned} \omega_1(t) &= \Gamma(1.3)t(1-t), \omega_2(t) \\ &= \Gamma(1.35)t(1-t), \omega_3(t) \\ &= \Gamma(1.4)t(1-t) \\ {}_0 D_t^{0.7} \omega_1(t) &= t^{0.3} - \frac{2\Gamma(1.3)}{\Gamma(2.3)} t^{1.3}, \\ {}_0 D_t^{0.65} \omega_2(t) &= t^{0.35} - \frac{2\Gamma(1.35)}{\Gamma(2.35)} t^{1.35}, \\ {}_0 D_t^{0.6} \omega_3(t) &= t^{0.4} - \frac{2\Gamma(1.4)}{\Gamma(2.4)} t^{1.4}. \end{aligned} \tag{54}$$

By a simple calculation, we obtain

$$\begin{aligned} \|\omega_1(t)\|_{0.7}^2 &\approx 0.130566, \|\omega_2(t)\|_{0.65}^2 \\ &\approx 0.078559, \|\omega_3(t)\|_{0.6}^2 \\ &\approx 0.102638. \end{aligned} \tag{55}$$

Select $r = 1 \times 10^{-3}$, we find

$$\begin{aligned} \|\omega_1(t)\|_{0.7}^2 + \|\omega_2(t)\|_{0.65}^2 + \|\omega_3(t)\|_{0.6}^2 \\ \approx 0.311763 > \frac{2r}{M} \approx 0.002192. \end{aligned} \tag{56}$$

We deduce that the supposition (i) holds, and

$$\begin{aligned} & \frac{\int_0^1 \max_{(x_1, x_2, x_3) \in \pi(kr/M)} F(t, x_1, x_2, x_3) dt}{r} \\ &= \frac{16k^2 r}{3M^2} \approx 0.130355 \\ &< \frac{2 \int_0^1 F(t, \omega_1, \omega_2, \omega_3) dt}{(\|\omega_1(t)\|_{0.7}^2 + \|\omega_2(t)\|_{0.65}^2 + \|\omega_3(t)\|_{0.6}^2) - Y(\omega_1, \omega_2, \omega_3)} \\ &\approx 0.365517, \end{aligned}$$

$$\lim_{(|x_1|, |x_2|, |x_3|) \rightarrow (+\infty, +\infty, +\infty)} \sup_{t \in [0, T]} \frac{F(t, x_1, x_2, x_3)}{(|x_1|^2/2) + (|x_2|^2/2) + (|x_3|^2/2)} = 0. \tag{57}$$

Then, suppositions (ii) and (iii) are verified. Hence, in view of Theorem 6 for every $\lambda \in]2.7359, 7.6714[$, system (50) has at least 3 weak solutions in the space $X = H_0^{0.7} \times H_0^{0.65} \times H_0^{0.6}$.

Example 2. Let $\alpha_1 = 0.65, \alpha_2 = 0.75, \alpha_3 = 0.85, \alpha_4 = 0.95, a_1(t) = 1 + t^3, a_2(t) = 1 + t^2, a_3(t) = 0.5 + t, a_4(t) = 1 + t, T = 1$. Hence, system (3) gives

$$\begin{cases} {}_t D_1^{0.65} ((1 + t^3)_0 D_t^{0.65} u_1(t)) = \lambda F_{u_1}(t, u_1(t), u_2(t), u_3(t), u_4(t)) + h_1(u_1), & t \in [0, 1], \\ {}_t D_1^{0.75} ((1 + t^2)_0 D_t^{0.75} u_2(t)) = \lambda F_{u_2}(t, u_1(t), u_2(t), u_3(t), u_4(t)) + h_2(u_2), & t \in [0, 1], \\ {}_t D_1^{0.85} ((0.5 + t)_0 D_t^{0.85} u_3(t)) = \lambda F_{u_3}(t, u_1(t), u_2(t), u_3(t), u_4(t)) + h_3(u_3), & t \in [0, 1], \\ {}_t D_1^{0.95} ((1 + t)_0 D_t^{0.95} u_4(t)) = \lambda F_{u_4}(t, u_1(t), u_2(t), u_3(t), u_4(t)) + h_4(u_4), & t \in [0, 1], \\ u_1(0) = u_1(1) = 0, u_2(0) = u_2(1) = 0, u_3(0) = u_3(1) = 0, u_4(0) = u_4(1) = 0. \end{cases} \tag{58}$$

Taking

$$\begin{aligned} \omega_1(t) &= \Gamma(1.35)t(1 - t), \omega_2(t) \\ &= \Gamma(1.25)t(1 - t), \omega_3(t) \\ &= \Gamma(1.15)t(1 - t), \omega_4(t) \\ &= \Gamma(1.05)t(1 - t). \end{aligned} \tag{59}$$

Moreover, for all $(t; x_1, x_2, x_3, x_4) \in [0; 1] \times R^4$, put

$$F(t, x_1(t), x_2(t), x_3(t), x_4(t)) = (1 + t^2)G(x_1, x_2, x_3, x_4), \tag{60}$$

where

$$G(x_1, x_2, x_3, x_4) = \begin{cases} (x_1^2 + x_2^2 + x_3^2 + x_4^2)^2, & x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1, \\ 10(x_1^2 + x_2^2 + x_3^2 + x_4^2)^{\frac{1}{2}} - 9(x_1^2 + x_2^2 + x_3^2 + x_4^2)^{\frac{1}{3}}, & x_1^2 + x_2^2 + x_3^2 + x_4^2 > 1. \end{cases} \tag{61}$$

Obviously $h_1, h_2, h_3, h_4 \rightarrow \mathbb{R}$ are three Lipschitz continuous functions, $h_1(u_1) = 1/4 \sin u_1, h_2(u_2) = u_2/20$ and $h_3(u_3) = 1/100 \arctan u_3, h_4(u_4) = 1/10 \ln(u_4 + 1)$ for all $u_1, u_2, u_3, u_4 \in \mathbb{R}$ with Lipschitz constants $L_1 = 1/4, L_2 = 1/20, L_3 = 1/100, L_4 = 1/10$ and $h_1(0) = h_2(0) = h_3(0) = h_4(0) = 0$. Clearly, $F(t, 0, 0, 0, 0) = 0$ for any $t \in [0, 1], a_{10} = 1, a_{20} = 0.5, a_{30} = 1$, and $a_{40} = 1$

The direct calculation, gives

$$k = \max \left\{ \frac{1}{(\Gamma(0.65))^2(2 \times 0.65 - 1)}, \frac{1}{(\Gamma(0.75))^2 \times 0.5(2 \times 0.75 - 1)}, \frac{1}{(\Gamma(0.85))^2(2 \times 0.85 - 1)}, \frac{1}{(\Gamma(0.95))^2(2 \times 0.95 - 1)} \right\} \approx 2.663742,$$

$$M = \min \left\{ 1 - \frac{L_1}{(\Gamma(0.65) + 1)^2}, 1 - \frac{L_2}{(\Gamma(0.75) + 1)^2 \times 0.5}, 1 - \frac{L_3}{(\Gamma(0.85) + 1)^2}, 1 - \frac{L_4}{(\Gamma(0.95) + 1)^2} \right\} \approx 0.956042,$$

$${}_0 D_t^{0.65} \omega_1(t) = t^{0.35} - \frac{2\Gamma(1.35)}{\Gamma(2.35)} t^{1.35},$$

$${}_0 D_t^{0.75} \omega_2(t) = t^{0.25} - \frac{2\Gamma(1.25)}{\Gamma(2.25)} t^{1.25},$$

$${}_0 D_t^{0.85} \omega_3(t) = t^{0.15} - \frac{2\Gamma(1.15)}{\Gamma(2.15)} t^{1.15},$$

$${}_0D_t^{0.95} \omega_3(t) = t^{0.05} - \frac{2\Gamma(1.05)}{\Gamma(2.05)} t^{1.05}. \tag{62}$$

So that

$$\begin{aligned} \|\omega_1(t)\|_{0.65}^2 &\approx 0.104555, \|\omega_2(t)\|_{0.75}^2 \\ &\approx 0.158153, \|\omega_3(t)\|_{0.85}^2 \\ &\approx 0.170894, \|\omega_4(t)\|_{0.95}^2 \\ &\approx 0.397611. \end{aligned} \tag{63}$$

Select $r = 1 \times 10^{-3}$; we find

$$\begin{aligned} &\|\omega_1(t)\|_{0.65}^2 + \|\omega_2(t)\|_{0.75}^2 \\ &\quad + \|\omega_3(t)\|_{0.85}^2 + \|\omega_4(t)\|_{0.95}^2 \\ &\approx 0.831213 > \frac{2r}{M} \approx 0.002092. \end{aligned} \tag{64}$$

We deduce that the supposition (i) holds, and

$$\begin{aligned} &\frac{\int_0^1 \max_{(x_1, x_2, x_3, x_4) \in \pi(kr/M)} F(t, x_1, x_2, x_3, x_4) dt}{r} = \frac{16k^2r}{3M^2} \approx 0.041403 \\ &< \frac{2 \int_0^1 F(t, \omega_1, \omega_2, \omega_3, \omega_4) dt}{(\|\omega_1(t)\|_{0.65}^2 + \|\omega_2(t)\|_{0.75}^2 + \|\omega_3(t)\|_{0.85}^2 + \|\omega_4(t)\|_{0.95}^2) - Y(\omega_1, \omega_2, \omega_3, \omega_4)} \approx 0.135277, \\ &\lim_{(|x_1|, |x_2|, |x_3|, |x_4|) \rightarrow (+\infty, +\infty, +\infty, +\infty)} \sup_{t \in [0, T]} \frac{F(t, x_1, x_2, x_3, x_4)}{(|x_1|^2/2) + (|x_2|^2/2) + (|x_3|^2/2) + (|x_4|^2/2)} = 0. \end{aligned} \tag{65}$$

Then, suppositions (ii) and (iii) are verified. Hence, in view of Theorem 6 for every $\lambda \in]7.3922, 24.1528[$, system (58) has at least 3 weak solutions in the space $X = H_0^{0.65} \times H_0^{0.75} \times H_0^{0.85} \times H_0^{0.95}$.

5. Conclusion

In this work, at least 3 weak solutions were obtained for a new class of nonlinear fractional BVPs using a critical three-point theorem due to Bonano and Marano. Some appropriate function spaces and variational frameworks were successfully created for system (3). Finally, we suggested two practical examples of Theorem 6 with a special case discussion \mathbb{R}^3 . As for case \mathbb{R}^4 , it was discussed. This makes our results prominent and distinct than previous ones. In the next work, we extend our recent work to the coupled system for this important problem. Also some numerical examples will be given in order to ensure the theory study by using some famous algorithms which are presented in ([28, 29]).

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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