Research Article

Representations and Deformations of Hom-Lie-Yamaguti Superalgebras

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Let (L, α) be a Hom-Lie-Yamaguti superalgebra. We first introduce the representation and cohomology theory of Hom-Lie-Yamaguti superalgebras. Also, we introduce the notions of generalized derivations and representations of (L, α) and present some properties. Finally, we investigate the deformations of (L, α) by choosing some suitable cohomology.

1. Introduction

Lie triple systems arose initially in Cartan's study of Riemannian geometry. Jacobson [1] first introduced Lie triple systems and Jordan triple systems in connection with problems from Jordan theory and quantum mechanics, viewing Lie triple systems as subspaces of Lie algebras that are closed relative to the ternary product. Lie-Yamaguti algebras were introduced by Yamaguti in [2] to give an algebraic interpretation of the characteristic properties of the torsion and curvature of homogeneous spaces with canonical connection in [3]. He called them generalized Lie triple systems at first, which were later called "Lie triple algebras". Recently, they were renamed as "Lie-Yamaguti algebras" in [4].

The theory of Hom-algebra started from Hom-Lie algebras introduced and discussed in [5], motivated by quasi-deformations of Lie algebras of vector fields, in particular q-deformations of Witt and Virasoro algebras. More precisely, Hom-Lie algebras are different from Lie algebras as the Jacobi identity is replaced by a twisted form using morphism. This twisted Jacobi identity is called Hom-Jacobi identity given by

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0. \qquad (1)$$

So far, many authors have studied Hom-type algebras motivated in part for their applications in physics ([6–10]).

In [11], Gaparayi and Issa introduced the concept of Hom-Lie-Yamaguti algebras, which can be viewed as a Hom-type generalization of Lie-Yamaguti algebras. In [12], Ma et al. studied the formal deformations of Hom-Lie-Yamaguti algebras. Recently, in [13], Lin et al. introduced the quasi-derivations of Lie-Yamaguti algebras. In [14], Zhang and Li introduced the representation and cohomology theory of Hom-Lie-Yamaguti algebras and studied deformations and extensions of Hom-Lie-Yamaguti algebras as an application, generalizing the results of [15]. In [16], Zhang et al. introduced the notion of crossed modules for Hom-Lie-Yamaguti algebras and studied their construction of Hom-Lie-Yamaguti algebras.

In [17], Gaparayi et al. introduced the concept of Hom-Lie-Yamaguti superalgebras and gave some examples of Hom-Lie-Yamaguti superalgebras. Later, in [18], Gaparayi et al. studied the relation between Hom-Leibniz superalgebras and Hom-Lie-Yamaguti superalgebras.

The purpose of this paper is to study the representations and deformations of Hom-Lie-Yamaguti superalgebras. This paper is organized as follows. In Section 2, we recall the definitions of Hom-Lie-superalgebras and Hom-Lie supertriple systems. In Section 3, we introduce the representation and cohomology theory of Hom-Lie-Yamaguti superalgebras. In Section 4, we introduce the notions of generalized derivations and representations of a Hom-Lie-Yamaguti superalgebra

and present some properties. In Section 5, we consider the theory of deformations of a Hom-Lie-Yamaguti superalgebra by choosing a suitable cohomology.

2. Preliminaries

Throughout this paper, we work on an algebraically closed field $\mathbb{K}$ of characteristic different from 2 and 3, all elements like x, y, z, u, v, w should be homogeneous unless otherwise state. We recall the definitions of Hom-Lie-superalgebras and Hom-Lie supertriple systems from [6, 9].

Definition 1. A Hom-Lie superalgebra L is a $\mathbb{Z}_2$-graded algebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$, endowed with an even bilinear mapping $[\cdot, \cdot] : L \times L \longrightarrow L$ and a homomorphism $\alpha : L \longrightarrow L$ satisfying the following conditions, for all $x \in L_{\bar{i}}, y \in L_{\bar{j}}$, $z \in L_{\bar{k}}$ and $\bar{i}, \bar{j}, \bar{k} \in \mathbb{Z}_2$:

$$[x, y] \subset L_{\bar{i}+\bar{j}},$$

$$[x, y] = -(-1)^{\bar{i}\bar{j}}[y, x],$$

$$(-1)^{\bar{k}\bar{i}}[\alpha(x), [y, z]] + (-1)^{\bar{i}\bar{j}}[\alpha(y), [z, x]] \tag{2}$$

$$+ (-1)^{\bar{j}\bar{k}}[\alpha(z), [x, y]] = 0.$$

Definition 2. A Hom-Lie supertriple system is a Z_2-graded vector space L, endowed with an even trilinear mapping $[\cdot, \cdot, \cdot] : L \times L \times L \longrightarrow L$ and a homomorphism $\alpha : L \longrightarrow L$ satisfying the following conditions,

(1) $|[x, y, z]| = (|x| + |y| + |z|)(\mathrm{mod}\ 2)$

(2) $[y, x, z] = -(-1)^{|x||y|}[x, y, z]$

(3) $(-1)^{|x||z|}[x, y, z] + (-1)^{|y||x|}[y, z, x] + (-1)^{|z||y|}[z, x, y]$
    $= 0$

(4) $[\alpha(x), \alpha(y), [z, u, v]] = [[x, y, z], \alpha(u), \alpha(v)] +$
    $(-1)^{|z|(|x|+|y|)}[\alpha(z), [x, y, u], \alpha(v)] + (-1)^{(|x|+|y|)(|z|+|u|)}$
    $[\alpha(z), \alpha(u), [x, y, v]]$

for all $x, y, z, u, v \in L$, and $|x|$ denotes the degree of the element $x \in L$.

3. Representations of Hom-Lie-Yamaguti Superalgebras

We recall the basic definition of Hom-Lie-Yamaguti superalgebras from [17].

Definition 3. A Hom-Lie-Yamaguti superalgebra (Hom-LY superalgebra for short) is a quadruple $(L, [\cdot, \cdot], \{\cdot, \cdot, \cdot\}, \alpha)$ in which L is $\mathbb{K}$-vector superspace, $[\cdot, \cdot]$ a binary superoperation and $\{\cdot, \cdot, \cdot\}$ a ternary superoperation on L, and $\alpha : L \longrightarrow L$ an even linear map such that
(SHLY1) $\alpha([x, y]) = [\alpha(x), \alpha(y)]$
(SHLY2) $\alpha(\{x, y, z\}) = \{\alpha(x), \alpha(y), \alpha(z)\}$
(SHLY3) $[x, y] = -(-1)^{|x||y|}[y, x]$

(SHLY4) $\{x, y, z\} = -(-1)^{|x||y|}\{y, x, z\}$
(SHLY5) $\circlearrowleft_{(x,y,z)} (-1)^{|x||z|}([[x, y], \alpha(z)] + \{x, y, z\}) = 0$
(SHLY6) $\circlearrowleft_{(x,y,z)} (-1)^{|x||z|}(\{[x, y], \alpha(z), \alpha(u)\}) = 0$
(SHLY7) $\{\alpha(x), \alpha(y), [u, v]\} = [\{x, y, u\}, \alpha^2(v)] +$
$(-1)^{|u|(|x|+|y|)}[\alpha^2(u), \{x, y, v\}]$
(SHLY8) $\{\alpha^2(x), \alpha^2(y), \{u, v, w\}\} = \{\{x, y, u\}, \alpha^2(v), \alpha^2(w)\} + (-1)^{|u|(|x|+|y|)}\{\alpha^2(u), \{x, y, v\}, \alpha^2(w)\} + (-1)^{(|u|+|v|)(|x|+|y|)}\{\alpha^2(u), \alpha^2(v), \{x, y, w\}\}$ for all $x, y, z, u, v, w \in L$ and where $\circlearrowleft_{(x,y,z)}$ denotes the sum over cyclic permutation of x, y, z, and $|x|$ denotes the degree of the element $x \in L$. We denote a Hom-LY superalgebra by (L, α).

Remark 4.

(1) If $\alpha = Id$, then the Hom-LY superalgebra $(L, [\cdot, \cdot], \{\cdot, \cdot, \cdot\}, \alpha)$ reduces to a LY superalgebra $(L, [\cdot, \cdot], \{\cdot, \cdot, \cdot\})$ *(see (SLY 1)-(SLY 6))*.

(2) If $[x, y] = 0$, for all $x, y \in L$, then $(L, [\cdot, \cdot], \{\cdot, \cdot, \cdot\}, \alpha)$ becomes a Hom-Lie supertriple system $(L, \{\cdot, \cdot, \cdot\}, \alpha^2)$.

(3) If $\{x, y, z\} = 0$ for all $x, y, z \in L$, then the Hom-LY superalgebra $(L, [\cdot, \cdot], \{\cdot, \cdot, \cdot\}, \alpha)$ becomes a Hom-Lie superalgebra $(L, [\cdot, \cdot], \alpha)$.

A homomorphism between two Hom-LY superalgebras (L, α) and (L', α') is a linear map $\varphi : L \longrightarrow L'$ satisfying $\varphi \circ \alpha = \alpha' \circ \varphi$ and

$$\varphi([x, y]) = [\varphi(x), \varphi(y)]', \varphi(\{x, y, z\}) = \{\varphi(x), \varphi(y), \varphi(z)\}'. \tag{3}$$

Example 5. Consider the 5-dimensional $\mathbb{Z}_2$-graded vector space $L = L_{\bar{0}} \oplus L_{\bar{1}}$, over an arbitrary base filed $\mathbb{K}$ of characteristic different from 2, with basis $\{u_1, u_2, u_3\}$ of $L_{\bar{0}}$ and $\{e_1, e_2\}$ of $L_{\bar{1}}$, and the nonzero products on these elements are induced by the following relations:

$$u_2 * u_1 = -u_3, u_1 * u_2 = u_3, u_1 * u_3 = -2u_1,$$

$$u_3 * u_1 = 2u_1, u_3 * u_2 = -2u_2, u_2 * u_3 = 2u_2, \tag{4}$$

$$e_1 * u_2 = e_2, e_1 * u_3 = -e_1, e_2 * u_1 = e_1, e_2 * u_3 = e_2.$$

Define the superspace homomorphisms $\alpha : L \longrightarrow L$ by

$$\alpha(u_i) = u_i, i = 1, 2, 3, \alpha(e_1) = -e_1, \alpha(e_2) = -e_2. \tag{5}$$

It is not hard to check that $(L, *, \alpha)$ is a Hom-Leibniz superalgebra. By [18], we can define $[\cdot, \cdot]$ and $\{\cdot, \cdot, \cdot\}$, and the nonzero products on these elements are induced by the following relations:

$$[u_1, u_2] = 2u_3, [u_1, u_3] = -4u_1, [u_2, u_3] = 4u_2,$$

$$[e_1, u_2] = e_2, [e_1, u_3] = -e_1, [e_2, u_1] = e_1, [e_2, u_3] = e_2,$$

$$\{u_1, u_3, u_2\} = 2u_3, \{u_2, u_3, u_1\}$$

$$= 2u_3, \{e_1, u_2, u_3\} = -\frac{1}{2}e_2,$$

$$\{e_1, u_2, u_1\} = -\frac{1}{2}e_1, \{e_1, u_3, u_2\}$$

$$= \frac{1}{2}e_2, \{e_1, u_3, u_3\} = -\frac{1}{2}e_1,$$

$$\{e_2, u_1, u_2\} = -\frac{1}{2}e_2, \{e_2, u_1, u_3\}$$

$$= \frac{1}{2}e_1, \{e_2, u_3, u_1\} = -\frac{1}{2}e_1,$$

$$\{e_2, u_3, u_3\} = -\frac{1}{2}e_2. \tag{6}$$

Then, $(L, [\cdot, \cdot], \{\cdot, \cdot, \cdot\}, \alpha)$ becomes a Hom-LY superalgebra.

Definition 6. Let (L, α) be a Hom-LY superalgebra and (V, β) be a Hom-vector super-space. A representation of L on V consists of an even linear map $\rho : L \longrightarrow \mathrm{End}(V)$ and even bilinear maps $D, \theta : L \times L \longrightarrow \mathrm{End}(V)$ such that the following conditions are satisfied:

(SHR1) $\rho(\alpha(x)) \circ \beta = \beta \circ \rho(x)$
(SHR2) $D(\alpha(x), \alpha(y)) \circ \beta = \beta \circ D(x, y)$
(SHR3) $\theta(\alpha(x), \alpha(y)) \circ \beta = \beta \circ \theta(x, y)$
(SHR4) $D(x, y) - (-1)^{|x||y|}\theta(y, x) + \theta(x, y) + \rho([x, y]) \circ \beta - \rho(\alpha(x))\rho(y) + (-1)^{|x||y|}\rho(\alpha(y))\rho(x) = 0$
(SHR5) $D([x, y], \alpha(z)) + (-1)^{|x|(|y|+|z|)}D([y, z], \alpha(x)) + (-1)^{|z|(|x|+|y|)}D([z, x], \alpha(y)) = 0$
(SHR6) $\theta([x, y], \alpha(z)) \circ \beta = (-1)^{|y||z|}\theta(\alpha(x), \alpha(z))\rho(y) - (-1)^{|x|(|y|+|z|)}\theta(\alpha(y), \alpha(z))\rho(x)$
(SHR7) $D(\alpha(x), \alpha(y))\rho(z) = (-1)^{|z|(|x|+|y|)}\rho(\alpha^2(z))D(x, y) + \rho(\{x, y, z\}) \circ \beta^2$
(SHR8) $\theta(\alpha(x), [y, z]) \circ \beta = (-1)^{|x||y|}\rho(\alpha^2(y))\theta(x, z) - (-1)^{|z|(|x|+|y|)}\rho(\alpha^2(z))\theta(x, y)$
(SHR9) $D(\alpha^2(x), \alpha^2(y))\theta(u, v) = (-1)^{(|u|+|v|)(|x|+|y|)}\theta(\alpha^2(u), \alpha^2(v))D(x, y) + \theta(\{x, y, u\}, \alpha^2(v)) \circ \beta^2 + (-1)^{|u|(|x|+|y|)}\theta(\alpha^2(u), \{x, y, v\}) \circ \beta^2$
(SHR10) $\theta(\alpha^2(x), \{y, z, u\}) \circ \beta^2 = (-1)^{(|z|+|u|)(|x|+|y|)}\theta(\alpha^2(z), \alpha^2(u))\theta(x, y) - (-1)^{|y||z|}\theta(\alpha^2(y), \alpha^2(u))\theta(x, z) + (-1)^{|x|(|y|+|z|)}D(\alpha^2(y), \alpha^2(z))\theta(x, u)$ for any $x, y, z, u, v \in L$. In this case, V is also called an L-module.

Proposition 7. *Let (L, α) be a Hom-LY superalgebra and V be a Hom-vector superspace. Assume we have a map ρ from L to $\mathrm{End}(V)$ and maps $D, \theta : L \times L \longrightarrow \mathrm{End}(V)$ satisfying (SHR1)–(SHR10). Then, (ρ, D, θ) is a representation of (L, α) on (V, β) if and only if $L \oplus V$ is a Hom-LY superalgebra under the following maps:*

$$(\alpha + \beta)(x + u) := \alpha(x) + \beta(u),$$

$$[x + u, y + v] := [x, y] + \rho(x)(v) - (-1)^{|x||y|}\rho(y)(u),$$

$$\{x + u, y + v, z + w\} := \{x, y, z\} + D(x, y)(w)$$
$$- (-1)^{|y||z|}\theta(x, z)(v)$$
$$+ (-1)^{|x|(|y|+|z|)}\theta(y, z)(u), \tag{7}$$

for any $x, y, z \in L$ and $u, v, w \in V$.

Proof. It is easy to check that the conditions (SHLY1)–(SHLY4) hold, we only verify that conditions (SHLY5)–(SHLY8) hold for maps defined on $L \oplus V$.

For (SHLY5), we have

$$\{x + u, y + v, z + w\} + c.p.$$
$$= \Big[\{x, y, z\} + D(x, y)(w) - (-1)^{|y||z|}\theta(x, z)(v) \tag{8}$$
$$+ (-1)^{|x|(|y|+|z|)}\theta(y, z)(u) \Big] + c.p.$$

and

$$[[x + u, y + v], \alpha(z) + \beta(w)] + c.p.$$
$$= \Big[[x, y] + \rho(x)(v) - (-1)^{|x||y|}\rho(y)(u), \alpha(z) + \beta(w) \Big] + c.p.$$
$$= \Big([[x, y], \alpha(z)] + \rho([x, y])\beta(w) - (-1)^{|x||z|}\rho(\alpha(z))\rho(x)(v)$$
$$+ (-1)^{(|x|+|z|)|y|}\rho(\alpha(z))\rho(y)(u)0 + c.p. \tag{9}$$

Thus by (SHR4), the condition (SHLY5) holds.

For (SHLY6), we have

$$(-1)^{|x||z|}([x + u, y + v], \alpha(z) + \beta(w), \alpha(p) + \beta(t)) + c.p.$$
$$= (-1)^{|x||z|}\Big(\{[x, y], \alpha(z), \alpha(p)\} + D([x, y], \alpha(z))(\beta(t))$$
$$- \theta([x, y], \alpha(p))(\beta(w)) + (-1)^{(|z|+|p|+|y|)|x|}\theta(\alpha(z), \alpha(p))$$
$$\cdot (\rho(x)(v)) - (-1)^{(|z|+|p|+|y|)|x|}\theta(\alpha(z), \alpha(p))(\rho(y)(u)) \Big)$$
$$+ c.p. = 0. \tag{10}$$

Thus by (SHR5), the condition (SHLY6) holds.

For (SHLY7), we have

$$\{\alpha(x) + \beta(u), \alpha(y) + \beta(v), [z + w, p + t]\}$$
$$= \{\alpha(x), \alpha(y), [z, p]\} + D(\alpha(x), \alpha(y))(\rho(z)(t))$$
$$- (-1)^{|z||p|}D(\alpha(x), \alpha(y))(\rho(p)(w))$$
$$- (-1)^{(|z|+|p|)|y|}\theta(x, [z, p])(v)$$
$$+ (-1)^{(|z|+|p|+|y|)|x|}\theta(y, [z, p])(u),$$

$$
\begin{aligned}
&\left[\{x+u, y+v, z+w\}, \alpha^2(p)+\beta^2(t)\right] \\
&\quad + (-1)^{|z|(|x|+|y|)}\left[\alpha^2(z)+\beta^2(w), \{x+u, y+v, p+t\}\right] \\
&= \left[\{x, y, z\}, \alpha^2(p)\right] + \rho(\{x, y, z\})(t) \\
&\quad - (-1)^{(|x|+|y|+|z|)|p|}\rho(p)(D(x, y)(w)) \\
&\quad - (-1)^{(|x|+|y|+|z|)|p|+|y||z|}\rho(p)(\theta(x, z)(v)) \\
&\quad + (-1)^{(|z|+|y|)(|p|+|x|)+|p||x|}\rho(p)(\theta(y, z)(u)) \\
&\quad + (-1)^{|z|(|x|+|y|)}\left[\alpha^2(z), \{x, y, p\}\right] \\
&\quad + (-1)^{|z|(|x|+|y|)}\rho(z)(D(x, y)(t)) \\
&\quad - (-1)^{|x||z|+(|x|+|p|)|y|}\rho(z)(\theta(x, p)(v)) \\
&\quad + (-1)^{|z||p|+|x|(|z|+|y|+|p|)}\rho(z)(\theta(y, p)(u)) \\
&\quad - (-1)^{|p||z|}\rho(\{x, y, p\})(w).
\end{aligned}
$$

$$(11)$$

Thus by (SHR6), the condition (SHLY7) holds.

Now it suffices to verify (SHLY8). By the definition of the Hom-LY superalgebra, we have

$$
\begin{aligned}
&\left\{\alpha^2(x_1)+\beta^2(u_1), \alpha^2(x_2)+\beta^2(u_2), \{y_1+v_1, y_2+v_2, y_3+v_3\}\right\} \\
&= \left\{\alpha^2(x_1), \alpha^2(x_2), \{y_1, y_2, y_3\}\right\} \\
&\quad - (-1)^{|x_2|(|y_1|+|y_2|+|y_3|)}\theta(\alpha^2(x_1), \{y_1, y_2, y_3\})(\beta^2(u_2)) \\
&\quad + (-1)^{|x_1|(|x_2|+|y_1|+|y_2|+|y_3|)}\theta(\alpha^2(x_2), \{y_1, y_2, y_3\})(\beta^2(u_1)) \\
&\quad + D(\alpha^2(x_1), \alpha^2(x_2))(D(y_1, y_2)(v_3)) \\
&\quad - (-1)^{|y_2||y_3|}D(\alpha^2(x_1), \alpha^2(x_2))(\theta(y_1, y_3)(v_2)) \\
&\quad + (-1)^{|y_1|(|y_2|+|y_3|)}D(\alpha^2(x_1), \alpha^2(x_2)) \\
&\quad \cdot (\theta(y_2, y_3)(v_1)), \{\{x_1+u_1, x_2+u_2, y_1+v_1\}, \\
&\quad \alpha^2(y_2)+\beta^2(v_2), \alpha^2(y_3)+\beta^2(v_3)\} \\
&= \left\{\{x_1, x_2, y_1\}, \alpha^2(y_2)+\alpha^2(y_3)\right\} \\
&\quad + D(\{x_1, x_2, y_1\}, \alpha^2(y_2))(\beta^2(v_3)) \\
&\quad + (-1)^{|y_2||y_3|}\theta(\{x_1, x_2, y_1\}, \alpha^2(y_3))(\beta^2(u_1)) \\
&\quad + (-1)^{(|y_2|+|y_3|)(|x_1|+|x_2|+|y_1|)}\theta(\alpha^2(y_2), \alpha^2(y_3)) \\
&\quad (D(x_1, x_2)(v_1)) - (-1)^{(|y_1|+|y_2|+|y_3|)|x_2|+(|x_1|+|y_1|)(|y_2|+|y_3|)} \\
&\quad \cdot \theta(\alpha^2(y_2), \alpha^2(y_3))(\theta(x_1, y_1)(u_2)) \\
&\quad + (-1)^{(|y_2|+|y_3|)(|x_2|+|y_3|)+|x_1|(|y_2|+|y_3|+|x_2|+|y_3|)}\theta(\alpha^2(y_2), \\
&\quad \alpha^2(y_3))(\theta(x_2, y_1)(u_1)), (-1)^{|y_1|(|x_1|+|x_2|)}\{\alpha^2(y_1) \\
&\quad + \beta^2(v_1), \{x_1+u_1, x_2+u_2, y_2+v_2\}, \alpha^2(y_3)+\beta^2(v_3)\} \\
&= (-1)^{|y_1|(|x_1|+|x_2|)}\left\{\alpha^2(y_1), \{x_1, x_2, y_2\}, \alpha^2(y_3)\right\} \\
&\quad + (-1)^{|y_1|(|x_1|+|x_2|)}D(\alpha^2(y_1), \{x_1, x_2, y_2\})(\beta^2(v_3)) \\
&\quad + (-1)^{|y_1|(|y_2|+|y_3|)}\theta(\{x_1, x_2, y_2\}, \alpha^2(y_3))(\beta^2(v_1)) \\
&\quad - (-1)^{|y_2||y_3|}\theta(\alpha^2(y_1), \alpha^2(y_3))(D(x_1, x_2)(v_3)) \\
&\quad + (-1)^{(|y_1|+|y_3|)(|x_1|+|y_2|)+|x_2|(|y_1|+|y_2|+|y_3|)+|y_1||y_2|}
\end{aligned}
$$

$$
\begin{aligned}
&\quad \cdot \theta(\alpha^2(y_1), \alpha^2(y_3))(\theta(x_1, y_2)(u_2)) \\
&\quad - \theta(\alpha^2(y_1), \alpha^2(y_3))(\theta(x_2, y_2)(u_1)) \\
&\quad \cdot (-1)^{(|y_1|+|y_2|)(|x_1|+|x_2|)}\{\alpha^2(y_1)+\beta^2(v_1), \alpha^2(y_2) \\
&\quad + \beta^2(v_2), \{x_1+u_1, x_2+u_2, y_3+v_3\}\} \\
&= (-1)^{(|y_1|+|y_2|)(|x_1|+|x_2|)}\left[\alpha^2(y_1), \alpha^2(y_2), \{x_1, x_2, y_3\}\right] \\
&\quad - (-1)^{|y_1|(|x_1|+|x_2|)+|y_2||y_3|} - \theta(\alpha^2(y_1), \{x_1, x_2, y_3\})(\beta^2(v_2)) \\
&\quad + (-1)^{|y_2|(|x_1|+|x_2|)+|y_1|(|y_2|+|y_3|)}\theta(\alpha^2(y_2), \{x_1, x_2, y_3\})(\beta^2(v_1)) \\
&\quad + (-1)^{(|y_1|+|y_2|)(|x_1|+|x_2|)}D(\alpha^2(y_1), \alpha^2(y_2))(D(x_1, x_2)(v_3)) \\
&\quad - (-1)^{(|y_1|+|y_2|)(|x_1|+|x_3|)+|x_2||y_3|}D(\alpha^2(y_1), \alpha^2(y_2)) \\
&\quad \cdot (\theta(x_1, y_3)(u_2)) + (-1)^{(|y_1|+|y_2|)(|x_2|+|y_3|)+|x_1|(|x_2|+|y_1|+|y_2|+|y_3|)} \\
&\quad \cdot D(\alpha^2(y_1), \alpha^2(y_2))(\theta(x_2, y_3)(u_1)).
\end{aligned}
$$

$$(12)$$

Thus by (SHR7), the condition (SHLY8) holds. Therefore, we obtain that $L \oplus V$ is a Hom-LY superalgebra.

Let V be a representation of Hom-LY superalgebra L. Let us define the cohomology group of L with coefficients in V. Let $f: L \times L \times \cdots \times L \longrightarrow V$ be an n-linear map such that the following conditions are satisfied:

$$
\begin{aligned}
&f(\alpha(x_1), \cdots, \alpha(x_n)) = \beta(f(x_1, \cdots, x_n)), \\
&f(x_1, \cdots, x_{2i-1}, x_{2i}, \cdots, x_n) = 0, \text{ if } x_{2i-1} = x_{2i}.
\end{aligned}
$$

$$(13)$$

The vector space spanned by such linear maps is called an n-cochain of L, which is denoted by $C^n(L, V)$ for $n \geq 1$.

Definition 8. For any $(f, g) \in C^{2n}(L, V) \times C^{2n+1}(L, V)$, the coboundary operator $\delta: (f, g) \longrightarrow (\delta_I f, \delta_{II} g)$ is a mapping from $C^{2n}(L, V) \times C^{2n+1}(L, V)$ into $C^{2n+2}(L, V) \times C^{2n+3}(L, V)$ defined as follows:

$$
\begin{aligned}
&(\delta_I f)(x_1, x_2, \cdots, x_{2n+2}) \\
&= (-1)^{|x_{2n+1}|(|x_1|+|x_2|+\cdots+|x_{2n}|)}\rho(\alpha^{2n}(x_{2n+1})) \\
&\quad \cdot g(x_1, x_2, \cdots, x_{2n}, x_{2n+2}) - (-1)^{|x_{2n+2}|(|x_1|+|x_2|+\cdots+|x_{2n+1}|)} \\
&\quad \cdot \rho(\alpha^{2n}(x_{2n+2}))g(x_1, x_2, \cdots, x_{2n}, x_{2n+1}) \\
&\quad - g(\alpha(x_1), \alpha(x_2), \cdots, \alpha(x_{2n}), [x_{2n+1}, x_{2n+2}]) \\
&\quad + \sum_{k=1}^{n}(-1)^{n+k+1+(|x_{2k-1}|+|x_{2k}|)(|x_1|+|x_2|+\cdots+|x_{2k-2}|)} \\
&\quad D(\alpha^{2n-1}(x_{2k-1}), \alpha^{2n-1}(x_{2k}))f(x_1, \cdots, \widehat{x}_{2k-1}, \widehat{x}_{2k}, \cdots, x_{2n+2}) \\
&\quad + \sum_{k=1}^{n}\sum_{j=2k+1}^{2n+2}(-1)^{n+k+(|x_{2k-1}|+|x_{2k}|)(|x_{2k+1}|+|x_{2k+2}|+\cdots+|x_{j-1}|)}
\end{aligned}
$$

$$f\left(\alpha^2(x_1), \cdots, \widehat{x}_{2k-1}, \widehat{x}_{2k}, \cdots, \{x_{2k-1}, x_{2k}, x_j\}, \cdots, \alpha^2(x_{2n+2})\right),$$

$$= (-1)^{(|x_{2n+2}|+|x_{2n+3}|)(|g|+|x_1|+|x_2|+\cdots+|x_{2n+1}|)}$$

$$\cdot \theta\left(\alpha^{2n}(x_{2n+2}), \alpha^{2n}(x_{2n+3})\right) g(x_1, \cdots, x_{2n+1})$$

$$- (-1)^{(|x_{2n+1}|+|x_{2n+3}|)(|g|+|x_1|+|x_2|+\cdots+|x_{2n}|)+|x_{2n+2}||x_{2n+3}|}$$

$$\cdot \theta\left(\alpha^{2n}(x_{2n+1}), \alpha^{2n}(x_{2n+3})\right) g(x_1, \cdots, x_{2n}, x_{2n+1})$$

$$+ \sum_{k=1}^{n+1} (-1)^{n+k+1+(|x_{2k-1}|+|x_{2k}|)(|g|+|x_1|+|x_2|+\cdots+|x_{2k-2}|)}$$

$$D\left(\alpha^{2n}(x_{2k-1}), \alpha^{2n}(x_{2k})\right) g(x_1, \cdots, \widehat{x}_{2k-1}, \widehat{x}_{2k}, \cdots, x_{2n+3})$$

$$+ \sum_{k=1}^{n+1} \sum_{j=2k+1}^{2n+3} (-1)^{n+k+(|x_{2k-1}|+|x_{2k}|)(|x_{2k+1}|+|x_{2k+2}|+\cdots+|x_{j-1}|)}$$

$$g\left(\alpha^2(x_1), \cdots, \widehat{x}_{2k-1}, \widehat{x}_{2k}, \cdots, \{x_{2k-1}, x_{2k}, x_j\}, \cdots, \alpha^2(x_{2n+3})\right). \tag{14}$$

Proposition 9. *The coboundary operator defined above satisfies* $\delta \circ \delta = 0$, *that is* $\delta_I \circ \delta_I = 0$ *and* $\delta_{II} \circ \delta_{II} = 0$.

Proof. Similar to [14].

The subspace $Z^{2n}(L, V) \times Z^{2n+1}(L, V)$ of $C^{2n}(L, V) \times C^{2n+1}(L, V)$ spanned by all the (f, g)'s such that $\delta(f, g) = 0$ is called the space of cocycles while the space $B^{2n}(L, V) \times B^{2n+1}(L, V) =: \delta(C^{2n-2}(L, V) \times C^{2n-1}(L, V))$ is called the space of coboundaries.

Definition 10. For the case $n \geq 2$, the $(2n, 2n+1)$-cohomology group of a Hom-LY superalgebra L with coefficients in V is defined to be the quotient space:

$$H^{2n}(L, V) \times H^{2n+1}(L, V)$$

$$:= \left(Z^{2n}(L, V) \times Z^{2n+1}(L, V)\right) / \left(B^{2n}(L, V) \times B^{2n+1}(L, V)\right). \tag{15}$$

In conclusion, we obtain a cochain complex whose cohomology group is called the cohomology group of a Hom-LY superalgebra L with coefficients in V.

4. α^k-Derivations of Hom-Lie-Yamaguti Superalgebras

In this section, we give the definition of α^k-derivations of Hom-LY superalgebras, then, we study its generalized derivations.

Definition 11. A linear map $D : L \longrightarrow L$ is called an α^k-derivation of L if it satisfies

$$D([x, y]) = (-1)^{|D||x|} \left[\alpha^k(x), D(y)\right] + \left[D(x), \alpha^k(y)\right],$$

$$D(\{x, y, z\}) = \left\{D(x), \alpha^k(y), \alpha^k(z)\right\}$$

$$+ (-1)^{|D||x|} \left\{\alpha^k(x), D(y), \alpha^k(z)\right\}$$

$$+ (-1)^{|D|(|x|+|y|)} \left\{\alpha^k(x), \alpha^k(y), D(z)\right\}, \tag{16}$$

for all $x, y, z \in L$, where $|D|$ denotes the degree of D.

We denote by $\text{Der}(L) = \oplus_{k \geq 0} \text{Der}_{\alpha^k}(L)$, where $\text{Der}_{\alpha^k}(L)$ is the set of all homogeneous α^k-derivations of L. Obviously, $\text{Der}(L)$ is a subalgebra of $\text{End}(L)$.

Theorem 12. $\text{Der}(L)$ *is a Lie superalgebra, where the bracket product is defined as follows:*

$$\left[D, D'\right] = DD' - (-1)^{|D||D'|} D'D. \tag{17}$$

Proof. It is sufficient to prove $[\text{Der}_{\alpha^k}(L), \text{Der}_{\alpha^s}(L)] \subseteq \text{Der}_{\alpha^{k+s}}(L)$. Note that

$$\left[D, D'\right]([x, y])$$

$$= D\left(\left[\alpha^s(x), D'(y)\right] + (-1)^{|D'||x|} \left[D'(x), \alpha^s(y)\right]\right)$$

$$- (-1)^{|D||D'|} D'\left(\left[\alpha^k(x), D(y)\right] + (-1)^{|D||x|} \left[D(x), \alpha^k(y)\right]\right)$$

$$= (-1)^{|D'||x|} \left[D\alpha^s(x), \alpha^k D'(y)\right]$$

$$+ (-1)^{|D||x|+|D'||x|} \left[\alpha^{k+s}(x), DD'(y)\right]$$

$$+ \left[DD'(x), \alpha^{k+s}(y)\right] + (-1)^{|D|(|D'||x|)} \left[\alpha^k D'(x), D\alpha^s(y)\right]$$

$$- (-1)^{|D||x|+|D'||x|} \left[D'\alpha^k(x), \alpha^s D(y)\right]$$

$$- (-1)^{|D'||x|+|D||x|+|D||D'|} \left[\alpha^{k+s}(x), D'D(y)\right]$$

$$- (-1)^{|D||D'|} \left[D'D(x), \alpha^{k+s}(y)\right]$$

$$- (-1)^{|D'|(|x|+|D|)+|D||D'|} \left[\alpha^s D(x), D'\alpha^k(y)\right]$$

$$= \left[\left[D, D'\right](x), \alpha^{k+s}(y)\right]$$

$$+ (-1)^{(|D'|+|D|)|x|} \left[\alpha^{k+s}(x), \left[D, D'\right](y)\right]. \tag{18}$$

Similarly, we can check that

$$
\begin{aligned}
&\left[D, D'\right](\{x, y, z\}) \\
&= \left\{\left[D, D'\right](x), \alpha^{k+s}(y), \alpha^{k+s}(z)\right\} \\
&\quad + (-1)^{\left(\left|D'\right|+|D|\right)|x|}\left\{\alpha^{k+s}(x), \left[D, D'\right](y), \alpha^{k+s}(z)\right\} \\
&\quad + (-1)^{\left(\left|D'\right|+|D|\right)(|x|+|y|)}\left\{\alpha^{k+s}(x), \alpha^{k+s}(y), \left[D, D'\right](z)\right\}.
\end{aligned}
\tag{19}
$$

It follows that $[D, D'] \in Der_{\alpha^{k+s}}(L)$.

Definition 13. Let (L, α) be a Hom-LY superalgebra. $D \in \mathrm{End}_s(L)$ is said to be a homogeneous generalized α^k-derivation of L, if there exist three endomorphisms $D', D'', D''' \in \mathrm{End}_s(L)$ such that

$$
\begin{aligned}
&\left[D(x), \alpha^k(y)\right] + (-1)^{s|x|}\left[\alpha^k(x), D'(y)\right] = D''([x, y]), \\
&\left\{D(x), \alpha^k(y), \alpha^k(z)\right\} + (-1)^{s|x|}\left\{\alpha^k(x), D'(y), \alpha^k(z)\right\} \\
&\quad + (-1)^{s(|x|+|y|)}\left\{\alpha^k(x), \alpha^k(y), D''(z)\right\} = D'''(\{x, y, z\}),
\end{aligned}
\tag{20}
$$

for all $x, y, z \in L$.

Definition 14. Let (L, α) be a Hom-LY superalgebra. $D \in \mathrm{End}_s(L)$ is said to be a homogeneous α^k-quasiderivation of L, if there exist endomorphisms $D', D'' \in \mathrm{End}_s(L)$ such that

$$
\begin{aligned}
&\left[D(x), \alpha^k(y)\right] + (-1)^{s|x|}\left[\alpha^k(x), D(y)\right] = D'([x, y]), \\
&\left\{D(x), \alpha^k(y), \alpha^k(z)\right\} + (-1)^{s|x|}\left\{\alpha^k(x), D(y), \alpha^k(z)\right\} \\
&\quad + (-1)^{s(|x|+|y|)}\left\{\alpha^k(x), \alpha^k(y), D(z)\right\} = D''(\{x, y, z\}),
\end{aligned}
\tag{21}
$$

for all $x, y, z \in L$.

Let $GDer(L)$ and $QDer(L)$ be the sets of homogeneous generalized α^k-derivations and of homogeneous α^k-quasiderivations, respectively. That is,

$$
GDer(L) := \bigoplus_{k \geq 0} GDer_{\alpha^k}(L), \quad QDer(L) := \bigoplus_{k \geq 0} QDer_{\alpha^k}(L). \tag{22}
$$

Definition 15. Let (L, α) be a Hom-LY superalgebra. The α^k-centroid of L is the space of linear transformations on L given by

$$
\begin{aligned}
C_{\alpha^k}(L) &= \left\{D \in \mathrm{End}(L) \,\middle|\, \left[D(x), \alpha^k(y)\right]\right. \\
&= (-1)^{|D||x|}\left[\alpha^k(x), D(y)\right] \\
&= D([x, y]) \text{ and } \left\{D(x), \alpha^k(y), \alpha^k(z)\right\} \\
&= (-1)^{|D||x|}\left\{\alpha^k(x), D(y), \alpha^k(z)\right\} \\
&= (-1)^{|D|(|x|+|y|)}\left\{\alpha^k(x), \alpha^k(y), D(c)\right\} \\
&= \left.D(\{x, y, z\})\right\}.
\end{aligned}
\tag{23}
$$

We denote $C(L) = \bigoplus_{k \geq 0} C_{\alpha^k}(L)$ and call it the centroid of L.

Definition 16. Let (L, α) be a Hom-LY superalgebra. The quasicentroid of L is the space of linear transformations on L given by

$$
\begin{aligned}
QC_{\alpha^k}(L) &= \left\{D \in \mathrm{End}(L) \,\middle|\, \left[D(x), \alpha^k(y)\right]\right. \\
&= (-1)^{|D||x|}\left[\alpha^k(x), D(y)\right] \text{ and } \left\{D(x), \alpha^k(y), \alpha^k(z)\right\} \\
&= (-1)^{|D||x|}\left\{\alpha^k(x), D(y), \alpha^k(z)\right\} \\
&= (-1)^{|D|(|x|+|y|)}\left.\left\{\alpha^k(x), \alpha^k(y), D(z)\right\}\right\},
\end{aligned}
\tag{24}
$$

for all $x, y, z \in L$. We denote $QC(L) = \oplus_{k \geq 0} QC_{\alpha^k}(L)$ and call it the quasicentroid of L.

Remark 17. Let (L, α) be a Hom-LY superalgebra. Then $C(L) \subseteq QC(L)$.

Definition 18. Let (L, α) be a Hom-LY superalgebra. $D \in \mathrm{End}(L)$ is said to be a central α^k-derivation of L if

$$
\begin{aligned}
&\left[D(x), \alpha^k(y)\right] = D([x, y]) = 0, \\
&D([x, y, z]) = \left[D(x), \alpha^k(y), \alpha^k(z)\right] = 0,
\end{aligned}
\tag{25}
$$

for all $x, y, z \in L$. Denote the set of all central α^k-derivations by $ZDer(L)$.

Remark 19. Let (L, α) be a Hom-LY superalgebra. Then

$$
ZDer(L) \subseteq Der(L) \subseteq QDer(L) \subseteq GDer(L) \subseteq \mathrm{End}(L). \tag{26}
$$

Definition 20. Let (L, α) be a Hom-LY superalgebra. If $Z(L) = \{x \in L \mid [x, y, z] = 0, \forall x, y, z \in L\}$, then $Z(L)$ is called the center of L.

Proposition 21. *Let (L, α) be a Hom-LY superalgebra, then the following statements hold:*

(1) *$GDer(L), QDer(L),$ and $C(L)$ are subalgebras of End(L)*

(2) ZDer(L) is an ideal of Der(L).

Proof.

(1) We only prove that $GDer(L)$ is a subalgebra of End (L), and similarly for cases of $QDer(L)$ and $C(L)$. For any $D_1 \in GDer_{\alpha^k}(L), D_2 \in GDer_{\alpha^s}(L)$ and $x, y, z \in L$, we have

$$
\begin{aligned}
&\left\{D_1 D_2(x), \alpha^{k+s}(y), \alpha^{k+s}(z)\right\} \\
&= D''_1 \{D_2(x), \alpha^s(y), \alpha^s(z)\} \\
&\quad - (-1)^{|D_1|(|D_2|+|x|)} \left\{\alpha^k(D_2(x)), D'_1(\alpha^s(y)), \alpha^{k+s}(z)\right\} \\
&\quad - (-1)^{|D_1|(|D_2|+|x|+|y|)} \left\{\alpha^k(D_2(x)), \alpha^{k+s}(y), D''_1(\alpha^s(z))\right\} \\
&= D'''_1 \Big\{D'''_2(\{x,y,z\}) - (-1)^{|D_2||x|}\left\{\alpha^s(x), D'_2(y), \alpha^s(z)\right\} \\
&\quad - (-1)^{|D_2|(|x|+|y|)}\left\{\alpha^s(x), \alpha^s(y), D''_2(z)\right\}\Big\} \\
&\quad - (-1)^{|D_1|(|D_2|+|x|)} \left\{\alpha^k(D_2(x)), D'_1(\alpha^s(y)), \alpha^{k+s}(z)\right\} \\
&\quad - (-1)^{|D_1|(|D_2|+|x|+|y|)} \left\{\alpha^k(D_2(x)), \alpha^{k+s}(y), D''_1(\alpha^s(z))\right\} \\
&= D'''_1 D'''_2(\{x,y,z\}) - (-1)^{|D_2||x|} D'''_1\left\{x, D'_2(y), z\right\} \\
&\quad - (-1)^{|D_2|(|x|+|y|)} D'''_1\left\{x, y, D''_2(z)\right\} \\
&\quad - (-1)^{|D_1|(|D_2|+|x|)} \left\{D_2(x), D'_1(y), z\right\} \\
&\quad - (-1)^{|D_1|(|D_2|+|x|+|y|)} \left\{D_2(x), y, D''_1(z)\right\} \\
&= D'''_1 D'''_2(\{x,y,z\}) - (-1)^{|D_2||x|}\Big\{D_1(\alpha^s(x)), \\
&\quad \alpha^k\left(D'_2(y)\right), \alpha^{k+s}(z)\Big\} - (-1)^{(|D_1|+|D_2|)|x|} \\
&\quad \cdot \left\{\alpha^{k+s}(x), D'_1 D'_2(y), \alpha^{k+s}(z)\right\} - (-1)^{|D_2||x|+|D_1|(|x|+|y|+|D_2|)} \\
&\quad \cdot \left\{\alpha^{k+s}(x), \alpha^k\left(D'_2(y)\right), D''_1(\alpha^s(z))\right\} \\
&\quad - (-1)^{|D_2|(|x|+|y|)}\left\{D_1(\alpha^s(x)), \alpha^{k+s}(y), \alpha^k\left(D''_2(z)\right)\right\} \\
&\quad - (-1)^{|D_2|(|x|+|y|)+|D_1||x|}\left\{\alpha^{k+s}(x), D'_1(\alpha^s(y)), \alpha^k\left(D''_2(z)\right)\right\} \\
&\quad - (-1)^{(|D_1|+|D_2|)(|x|+|y|)}\left\{\alpha^{k+s}(x), \alpha^{k+s}(y), D''_1 D''_2(z)\right\} \\
&\quad - (-1)^{|D_1|(|D_2|+|x|)}\left\{\alpha^k(D_2(x)), D'_1(\alpha^s(y)), \alpha^{k+s}(z)\right\} \\
&\quad - (-1)^{|D_1|(|D_2|+|x|+|y|)}\left\{\alpha^k(D_2(x)), \alpha^{k+s}(y), D''_1(\alpha^s(z))\right\}.
\end{aligned}
\tag{27}
$$

Similarly, we have

$$
\begin{aligned}
&\left\{D_2 D_1(x), \alpha^{k+s}(y), \alpha^{k+s}(z)\right\} \\
&= D'''_2 D'''_1(\{x,y,z\}) - (-1)^{|D_1||x|}
\end{aligned}
$$

$$
\begin{aligned}
&\quad \cdot \left\{D_2\left(\alpha^k(x)\right), \alpha^s D'_1(y), \alpha^{k+s}(z)\right\} \\
&\quad - (-1)^{(|D_1|+|D_2|)|x|}\left\{\alpha^{k+s}(x), D'_2 D'_1(y), \alpha^{k+s}(z)\right\} \\
&\quad - (-1)^{|D_1||x|+|D_2|(|x|+|y|+|D_1|)} \\
&\quad \cdot \left\{\alpha^{k+s}(x), \alpha^s D'_1(y), D''_2\left(\alpha^k(z)\right)\right\} \\
&\quad - (-1)^{|D_1|(|x|+|y|)}\left\{D_2\left(\alpha^k(x)\right), \alpha^{k+s}(y), \alpha^s D''_1(z)\right\} \\
&\quad - (-1)^{|D_1|(|x|+|y|)+|D_2||x|}\left\{\alpha^{k+s}(x), D'_2\left(\alpha^k(y)\right), \alpha^s D''_1(z)\right\} \\
&\quad - (-1)^{(|D_1|+|D_2|)(|x|+|y|)}\left\{\alpha^{k+s}(x), \alpha^{k+s}(y), D''_2 D''_1(z)\right\} \\
&\quad - (-1)^{|D_2|(|D_1|+|x|)}\left\{\alpha^s D_1(x), D'_2\left(\alpha^k(y)\right), \alpha^{k+s}(z)\right\} \\
&\quad - (-1)^{|D_2|(|D_1|+|x|+|y|)}\left\{\alpha^s D_1(x), \alpha^{k+s}(y), D''_2\left(\alpha^k(z)\right)\right\}.
\end{aligned}
\tag{28}
$$

It follows that

$$
\begin{aligned}
&\left\{[D_1, D_2](x), \alpha^{k+s}(y), \alpha^{k+s}(z)\right\} \\
&= \{D_1 D_2(x), y, z\} - (-1)^{|D_1||D_2|}\{D_2 D_1(x), y, z\} \\
&= \left(D'''_1 D'''_2 - (-1)^{|D_1|+|D_2|} D'''_2 D'''_1\right) \\
&\quad \cdot \{x, y, z\} - (-1)^{|x|(|D_1|+|D_2|)}\Big\{\alpha^{k+s}(x), \\
&\quad \left(D'_1 D'_2 - (-1)^{|D_1|+|D_2|} D'_2 D'_1\right)(y), \alpha^{k+s}(z)\Big\} \\
&\quad - (-1)^{(|x|+|y|)(|D_1|+|D_2|)}\Big\{\alpha^{k+s}(x), \alpha^{k+s}(y), \\
&\quad \left(D''_1 D''_2 - (-1)^{|D_1|+|D_2|} D''_2 D''_1\right)(z)\Big\} \\
&= \left[D'''_1, D'''_2\right]\{x, y, z\} - (-1)^{|x|(|D_1|+|D_2|)} \\
&\quad \cdot \left\{\alpha^{k+s}(x), \left[D'_1, D'_2\right](y), \alpha^{k+s}(z)\right\} \\
&\quad - (-1)^{(|x|+|y|)(|D_1|+|D_2|)}\Big\{\alpha^{k+s}(x), \alpha^{k+s}(y), \\
&\quad \left[D''_1, D''_2\right](z)\Big\},
\end{aligned}
\tag{29}
$$

and it is easy to check that

$$
\begin{aligned}
&\left[[D_1, D_2](x), \alpha^{k+s}(y)\right] \\
&= \left[D''_1, D''_2\right][x, y] - (-1)^{|x|(|D_1|+|D_2|)} \\
&\quad \cdot \left[\alpha^{k+s}(x), \left[D'_1, D'_2\right](y)\right].
\end{aligned}
\tag{30}
$$

Obviously, $[D'_1, D'_2], [D''_1, D''_2]$ and $[D'''_1, D'''_2]$ are contained in $End(L)$, thus $[D_1, D_2] \in GDer_{\alpha^{k+l}}(L) \subseteq GDer(L)$, that is, $GDer(L)$ is a subalgebra of $End(L)$.

(2) For any $D_1 \in ZDer_{\alpha^k}(L), D_2 \in Der_{\alpha^s}(L)$ and $x, y, z \in L$, we have

$$[D_1, D_2]([x, y, z]) = D_1 D_2(\{x, y, z\})$$
$$- (-1)^{|D_1||D_2|} D_2 D_1(\{x, y, z\}) = 0. \qquad (31)$$

Also, we have

$$\left\{ [D_1, D_2](x), \alpha^{k+s}(y), \alpha^{k+s}(z) \right\}$$
$$= \left\{ D_1 D_2(x), \alpha^{k+s}(y), \alpha^{k+s}(z) \right\}$$
$$- (-1)^{|D_1||D_2|} \left\{ D_2 D_1(x), \alpha^{k+s}(y), \alpha^{k+s}(z) \right\}$$
$$= 0 - (-1)^{|D_1||D_2|} \left\{ D_2 D_1(x), \alpha^{k+s}(y), \alpha^{k+s}(z) \right\}$$
$$= -(-1)^{|D_1||D_2|} \left(D_2 \left(\left\{ D_1(x), \alpha^k(y), \alpha^k(z) \right\} \right) \right.$$
$$- (-1)^{|D_2|(|D_1|+|x|)} \left\{ \alpha^s D_1(x), D_2\left(\alpha^k(y)\right), \alpha^{k+s}(z) \right\} \right)$$
$$+ (-1)^{|D_1||D_2|} (-1)^{|D_2|(|D_1|+|x|+|y|)}$$
$$\cdot \left\{ \alpha^s D_1(x), \alpha^{k+s}(y), D_2\left(\alpha^k(z)\right) \right\} = 0, \qquad (32)$$

and it is easy to check that

$$\left[[D_1, D_2](x), \alpha^{k+s}(y) \right] = 0. \qquad (33)$$

It follows that $[D_1, D_2] \in ZDer_{\alpha^{k+s}}(L)$. That is, $ZDer(L)$ is an ideal of $Der(L)$.

Lemma 22. *Let (L, α) be a Hom-LY superalgebra, then the following statements hold:*

$$[Der(L), C(L)] \subseteq C(L).$$
$$[QDer(L), QC(L)] \subseteq QC(L).$$
$$[QC(L), QC(L)] \subseteq QDer(L). \qquad (34)$$
$$C(L) \subseteq QDer(L).$$
$$QDer(L) + QC(L) \subseteq GDer(L).$$

Proof. (1)–(4) are easy to prove and we omit them, we only check (5). In fact, let $D_1 \in QDer_{\alpha^k}(L), D_2 \in QC_{\alpha^k}(L)$. Then, there exist $D'_1, D''_1 \in End_s(L)$, for any $x, y, z \in L$, we have

$$\left[D_1(x), \alpha^k(y) \right] + (-1)^{s|x|} \left[\alpha^k(x), D_1(y) \right] = D_1'([x, y]),$$

$$\left\{ D_1(x), \alpha^k(y), \alpha^k(z) \right\} + (-1)^{s|x|} \left\{ \alpha^k(x), D_1(y), \alpha^k(z) \right\}$$
$$+ (-1)^{s(|x|+|y|)} \left\{ \alpha^k(x), \alpha^k(y), D_1(z) \right\} = D_1''(\{x, y, z\}).$$
$$(35)$$

Thus, for any $x, y, z \in L$, we have

$$\left[(D_1 + D_2)(x), \alpha^k(y) \right]$$
$$= \left[D_1(x), \alpha^k(y) \right] + \left[D_2(x), \alpha^k(y) \right]$$
$$= D_1'([x, y]) - (-1)^{s|x|} \left[\alpha^k(x), D_1(y) \right]$$
$$+ (-1)^{s|x|} \left[\alpha^k(x), D_2(y) \right]$$
$$= D_1'([x, y]) - (-1)^{s|x|} \left[\alpha^k(x), (D_1 - D_2)(y) \right],$$

$$\left\{ (D_1 + D_2)(x), \alpha^k(y), \alpha^k(z) \right\}$$
$$= \left\{ D_1(x), \alpha^k(y), \alpha^k(z) \right\} + \left\{ D_2(x), \alpha^k(y), \alpha^k(z) \right\}$$
$$= D_1''(\{x, y, z\}) - (-1)^{s|x|} \left\{ \alpha^k(x), D_1(y), \alpha^k(z) \right\}$$
$$- (-1)^{s(|x|+|y|)} \left\{ \alpha^k(x), \alpha^k(y), D_1(z) \right\}$$
$$+ (-1)^{s|x|} \left\{ \alpha^k(x), D_2(y), \alpha^k(z) \right\}$$
$$= D_1''(\{x, y, z\}) - (-1)^{s|x|} \left\{ \alpha^k(x), (D_1 - D_2)(y), \alpha^k(z) \right\}$$
$$- (-1)^{s(|x|+|y|)} \left\{ \alpha^k(x), \alpha^k(y), D_1(z) \right\}. \qquad (36)$$

Therefore, $D_1 + D_2 \in GDer_{\alpha^k}(L)$.

Proposition 23. *Let (L, α) be a Hom-LY superalgebra, then $QC(L) + [QC(L), QC(L)]$ is a subalgebra of $GDer(L)$.*

Proof. By Lemma 22, (3) and (5), we have

$$QC(L) + [QC(L), QC(L)] \subseteq GDer(L), \qquad (37)$$

and it follows that

$$[QC(L) + [QC(L), QC(L)], QC(L) + [QC(L), QC(L)]]$$
$$\subseteq [QC(L) + GDer(L), QC(L) + [QC(L), QC(L)]]$$
$$\subseteq [QC(L), QC(L)] + [QC(L), [QC(L), QC(L)]]$$
$$+ [QDer(L), QC(L)][QDer(L), [QC(L), QC(L)]]. \qquad (38)$$

It is easy to verify that $[QDer(L), [QC(L), QC(L)]] \subseteq [QC(L), QC(L)]$ by the Jacobi identity of Hom-Lie algebras. Thus,

$$[QC(L) + [QC(L), QC(L)], QC(L) + [QC(L), QC(L)]]$$
$$\subseteq QC(L) + [QC(L), QC(L)] \subseteq GDer(L). \qquad (39)$$

The proof is finished.

Theorem 24. *Let (L, α) be a Hom-Lie-Yamaguti superalgebra, where α is surjective, then $[C(L), QC(L)] \subseteq End(L, Z(L))$. Moreover, if $Z(L) = \{0\}$, then $[C(L), QC(L)] = \{0\}$.*

Proof. For any $D_1 \in C_{\alpha^k}(L), D_2 \in QC_{\alpha^s}(L)$ and $x, y, z \in L$, since α is surjective, there exist $y', z' \in L$ such that $y = \alpha^{k+s}(y'), z = \alpha^{k+s}(z')$, then we have

$$
\begin{aligned}
& [[D_1, D_2](x), y] \\
&= \left[[D_1, D_2](x), \alpha^{k+s}\left(y'\right)\right] = \left[D_1 D_2(x), \alpha^{k+s}\left(y'\right)\right] \\
&\quad - (-1)^{|D_1||D_2|}\left[D_2 D_1(x), \alpha^{k+s}\left(y'\right)\right] \\
&= D_1\left(D_2(x), \alpha^s\left(y'\right)\right) - (-1)^{|D_1||D_2|}\left[\alpha^s D_1(x), D_2 \alpha^k\left(y'\right)\right] \\
&= D_1\left(\left[D_2(x), \alpha^s\left(y'\right)\right]\right) - D_1\left(\left[D_2(x), \alpha^s\left(y'\right)\right]\right) = 0,
\end{aligned}
$$

$$
\begin{aligned}
& \{[D_1, D_2](x), y, z\} \\
&= \left\{D_1 D_2(x), \alpha^{k+s}\left(y'\right), \alpha^{k+s}\left(z'\right)\right\} \\
&\quad - (-1)^{|D_1||D_2|}\left\{D_2 D_1(x), \alpha^{k+s}\left(y'\right), \alpha^{k+s}\left(z'\right)\right\} \\
&= D_1\left(\left\{D_2(x), \alpha^s\left(y'\right), \alpha^s\left(z'\right)\right\}\right) \\
&\quad - (-1)^{|D_1||D_2|}(-1)^{(|D_1|+|x|)|D_2|} \\
&\quad \cdot \left\{\alpha^s D_1(x), D_2\left(\alpha^k\left(y'\right)\right), \alpha^{k+s}\left(z'\right)\right\} \\
&= (-1)^{|x||D_2|} D_1\left(\left\{\alpha^s(x), D_2\left(y'\right), \alpha^s\left(z'\right)\right\}\right) \\
&\quad - (-1)^{|a||D_2|} D_1\left(\left\{\alpha^s(x), D_2\left(y'\right), \alpha^s\left(z'\right)\right\}\right) = 0.
\end{aligned}
$$
(40)

So $[D_1, D_2](x) \subseteq Z(L)$ and therefore, $[C(L), QC(L)] \subseteq \text{End}(L, Z(L))$. Moreover, if $Z(L) = \{0\}$, it is easy to see that $[C(L), QC(L)] = \{0\}$.

5. 1-Parameter Formal Deformations of Hom-Lie-Yamaguti Superalgebras

Let (L, α) be a Hom-LY superalgebra over $\mathbb{K}$ and $\mathbb{K}[[t]]$ the power series ring in one variable t with coefficients in $\mathbb{K}$. Assume that $L[[t]]$ is the set of formal series whose coefficients are elements of the vector space L.

Definition 25. Let (L, α) be a Hom-LY superalgebra. A 1-parameter formal deformations of L is a pair of formal power series (f_t, g_t) of the form

$$
f_t = [\cdot, \cdot] + \sum_{i \geq 1} f_i t^i, \quad g_t = \{\cdot, \cdot, \cdot\} + \sum_{i \geq 1} g_i t^i,
$$
(41)

where each f_i is a $\mathbb{K}$-bilinear map $f_i : L \times L \longrightarrow L$ (extended to be $\mathbb{K}[[t]]$-bilinear) and each g_i is a $\mathbb{K}$-trilinear map $g_i : L \times L \times L \longrightarrow L$ (extended to be $\mathbb{K}[[t]]$-trilinear) such that $(L[[t]], f_t, g_t, \alpha)$ is a Hom-Lie-Yamaguti superalgebra over $\mathbb{K}[[t]]$. Set $f_0 = [\cdot, \cdot]$ and $g_0 = \{\cdot, \cdot, \cdot\}$, then f_t and g_t can be written as $f_t = \sum_{i \geq 0} f_i t^i, g_t = \sum_{i \geq 0} g_i t^i$, respectively.

Since $(L[[t]], f_t, g_t, \alpha)$ is a Hom-LY superalgebra. Then, it satisfies the following axioms:

$$
\alpha \circ f_t(x, y) = f_t(\alpha(x), \alpha(y)),
$$
(42)

$$
\alpha \circ g_t(x, y, z) = g_t(\alpha(x), \alpha(y), \alpha(z)),
$$
(43)

$$
f_t(x, y) = -(-1)^{|x||y|} f_t(y, x),
$$
(44)

$$
g_t(x, y, z) = -(-1)^{|x||y|} g_t(y, x, z),
$$
(45)

$$
\mathcal{O}_{(x,y,z)}(-1)^{|x||z|}(f_t(f_t(x, y), \alpha(z)) + g_t(x, y, z)) = 0,
$$
(46)

$$
\mathcal{O}_{(x,y,z)}(-1)^{|x||z|} g_t(f_t(x, y), \alpha(z), \alpha(u)) = 0,
$$
(47)

$$
\begin{aligned}
g_t(\alpha(x), \alpha(y), f_t(u, v)) \\
= f_t\left(g_t(x, y, u), \alpha^2(v)\right) \\
+ (-1)^{|u|(|x|+|y|)} f_t\left(\alpha^2(u), g_t(x, y, v)\right),
\end{aligned}
$$
(48)

$$
\begin{aligned}
g_t\left(\alpha^2(x), \alpha^2(y), g_t(u, v, w)\right) \\
= g_t\left(g_t(x, y, u), \alpha^2(v), \alpha^2(w)\right) \\
+ (-1)^{|u|(|x|+|y|)} g_t\left(\alpha^2(u), g_t(x, y, v), \alpha^2(w)\right) \\
+ (-1)^{(|u|+|v|)(|x|+|y|)} g_t\left(\alpha^2(u), \alpha^2(v), g_t(x, y, w)\right),
\end{aligned}
$$
(49)

for all $x, y, z, u, v, w \in L$.

Remark 26. Equations (42)–(49) are equivalent to $(n = 0, 1, 2, \cdots)$

$$
\alpha \circ f_n(x, y) = f_n(\alpha(x), \alpha(y)),
$$
(50)

$$
\alpha \circ g_n(x, y, z) = g_n(\alpha(x), \alpha(y), \alpha(z)),
$$
(51)

$$
f_n(x, y) = -(-1)^{|x||y|} f_n(y, x),
$$
(52)

$$
g_n(x, y, z) = -(-1)^{|x||y|} g_n(y, x, z),
$$
(53)

$$
\mathcal{O}_{(x,y,z)}(-1)^{|x||z|}\left(\sum_{i+j=n} f_i\left(f_j(x, y), \alpha(z)\right) + g_n(x, y, z)\right) = 0,
$$
(54)

$$
\mathcal{O}_{(x,y,z)}(-1)^{|x||z|} \sum_{i+j=n} g_i\left(f_j(x, y), \alpha(z), \alpha(u)\right) = 0,
$$
(55)

$$
\begin{aligned}
\sum_{i+j=n} g_i\left(\alpha(x), \alpha(y), f_j(u, v)\right) \\
= \sum_{i+j=n} f_i\left(g_j(x, y, u), \alpha^2(v)\right) \\
+ (-1)^{|u|(|x|+|y|)} f_i\left(\alpha^2(u), g_j(x, y, v)\right),
\end{aligned}
$$
(56)

$$\sum_{i+j=n} g_i\Big(\alpha^2(x), \alpha^2(y), g_j(u,v,w)\Big)$$

$$= \sum_{i+j=n} g_i\Big(g_j(x,y,u), \alpha^2(v), \alpha^2(w)\Big)$$

$$+ (-1)^{|u|(|x|+|y|)} g_i\Big(\alpha^2(u), g_j(x,y,v), \alpha^2(w)\Big) \tag{57}$$

$$+ (-1)^{(|u|+|v|)(|x|+|y|)} g_i\Big(\alpha^2(u), \alpha^2(v), g_j(x,y,w)\Big),$$

for all $x, y, z, u, v, w \in L$. These equations are called the deformation equations of a Hom-LY superalgebra. Equations (50)–(53) imply $(f,g) \in C^2(L,L) \times C^3(L,L)$.

Let $n = 1$ in Equations (50)–(57). Then,

$$O_{(x,y,z)}(-1)^{|x||z|}([f_1(x,y), \alpha(z)]$$
$$+ f_1([x,y], \alpha(z)) + g_1(x,y,z)) = 0,$$

$$O_{(x,y,z)}(-1)^{|x||z|}(\{f_1(x,y), \alpha(z), \alpha(u)\}$$
$$+ g_1([x,y], \alpha(z), \alpha(u))) = 0,$$

$$\{\alpha(x), \alpha(y), f_1(u,v)\} + g_1(\alpha(x), \alpha(y), [u,v])$$
$$- [g_1(x,y,u), \alpha^2(v)] - f_1(\{x,y,u\}, \alpha^2(v))$$
$$- (-1)^{|u|(|x|+|y|)}[\alpha^2(z), g_1(x,y,u)]$$
$$- (-1)^{|u|(|x|+|y|)}f_1(\alpha^2(z), \{x,y,u\}) = 0,$$

$$\{\alpha^2(x), \alpha^2(y), g_1(u,v,w)\} + g_1(\alpha^2(x), \alpha^2(y), \{u,v,w\})$$
$$- \{g_1(x,y,u), \alpha^2(v), \alpha^2(w)\} - g_1(\{x,y,u\}, \alpha^2(v), \alpha^2(w))$$
$$- (-1)^{|u|(|x|+|y|)}\{\alpha^2(u), g_1(x,y,v), \alpha^2(w)\}$$
$$- (-1)^{|u|(|x|+|y|)}g_1(\alpha^2(u), \{x,y,v\}, \alpha^2(w))$$
$$- (-1)^{|u|(|x|+|y|)}\{\alpha^2(u), \alpha^2(v), g_1(x,y,w)\}$$
$$- (-1)^{(|u|+|v|)(|x|+|y|)}g_1(\alpha^2(u), \alpha^2(v), \{x,y,w\})$$
$$- (-1)^{(|u|+|v|)(|x|+|y|)}\{\alpha^2(u), \alpha^2(v), g_1(x,y,w)\} = 0, \tag{58}$$

which imply $(\delta_I^2, \delta_{II}^2)(f_1, g_1) = (0,0)$, i.e.,

$$(f_1, g_1) \in Z^2(L,L) \times Z^3(L,L). \tag{59}$$

The pair (f_1, g_1) is called the infinitesimal deformation of (f_t, g_t).

Definition 27. Let (L, α) be a Hom-LY superalgebra. Two 1-parameter formal deformations (f_t, g_t) and (f'_t, g'_t) of L are said to be equivalent, denoted by $(f_t, g_t) \sim (f'_t, g'_t)$, if there exists a formal isomorphism of $\mathbb{K}[[t]]$-modules

$$\phi_t(x) = \sum_{i \geq 0} \phi_i(x)t^i : (L[[t]], f_t, g_t, \alpha) \longrightarrow \Big(L[[t]], f'_t, g'_t, \alpha\Big), \tag{60}$$

where $\phi_i : L \longrightarrow L$ is a $\mathbb{K}$-linear map (extended to be $\mathbb{K}[[t]]$-linear) such that

$$\phi_0 = id_L, \phi_t \circ \alpha = \alpha \circ \phi_t,$$
$$\phi_t \circ f_t(x,y) = f'_t(\phi_t(x), \phi_t(y)), \phi_t \circ g_t(x,y,z) \tag{61}$$
$$= g'_t(\phi_t(x), \phi_t(y), \phi_t(z)).$$

In particular, if $(f_1, g_1) = (f_2, g_2) = \cdots = (0,0)$, then $(f_t, g_t) = (f_0, g_0)$ is called the null deformation. If $(f_t, g_t) \sim (f_0, g_0)$, then (f_t, g_t) is called the trivial deformation. If every 1-parameter formal deformation (f_t, g_t) is trivial, then L is called an analytically rigid Hom-LY superalgebra.

Theorem 28. *Let (f_t, g_t) and (f'_t, g'_t) be the two equivalent 1-parameter formal deformations of L. Then, the infinitesimal deformations (f_1, g_1) and (f'_1, g'_1) belong to the same cohomology class in $H^2(L,L) \times H^3(L,L)$.*

Proof. By the assumption that (f_1, g_1) and (f'_1, g_1') are equivalent, there exists a formal isomorphism $\phi_t(x) = \sum_{i \geq 0} \phi_i(x)t^i$ of $\mathbb{K}[[t]]$-modules satisfying

$$\sum_{i \geq 0} \phi_i\left(\sum_{j \geq 0} f_j(x_1, x_2)t^j\right)t^i$$
$$= \sum_{i \geq 0} f'_i\left(\sum_{k \geq 0}\phi_k(x_1)t^k, \sum_{l \geq 0}\phi_l(x_2)t^l\right)t^i,$$
$$\sum_{i \geq 0} \phi_i\left(\sum_{j \geq 0} g_j(x_1, x_2, x_3)t^j\right)t^i$$
$$= \sum_{i \geq 0} g'_i\left(\sum_{k \geq 0}\phi_k(x_1)t^k, \sum_{l \geq 0}\phi_l(x_2)t^l, \sum_{m \geq 0}\phi_m(x_3)t^m\right)t^i, \tag{62}$$

for any $x_1, x_2, x_3 \in L$. Comparing the coefficients of t^1 in both sides in each of the equations above, we have

$$f_1(x_1, x_2) + \phi_1([x_1, x_2])$$
$$= f'_1(x_1, x_2) + [\phi_1(x_1), x_2] + [x_1, \phi_1(x_2)],$$
$$g_1(x_1, x_2, x_3) + \phi_1(\{x_1, x_2, x_3\}) \tag{63}$$
$$= g'_1(x_1, x_2, x_3) + \{\phi_1(x_1), x_2, x_3\}$$
$$+ \{x_1, \phi_1(x_2), x_3\} + \{x_1, x_2, \phi_1(x_3)\}.$$

It follows that $(f_1 - f'_1, g_1 - g'_1) = (\delta_I^1, \delta_{II}^1)(\phi_1, \phi_1) \in B^2(L,L) \times B^3(L,L)$, as desired. The proof is completed.

Theorem 29. *Let (L, α) be a Hom-LY superalgebra with $H^2(L,L) \times H^3(L,L) = 0$, then L is analytically rigid.*

Proof. Let (f_t, g_t) be a 1-parameter formal deformation of L. Suppose $f_t = f_0 + \sum_{i \geq r} f_i t^i$, $g_t = g_0 + \sum_{i \geq r} g_i t^i$. Set $n = r$ in Equations (50)–(53), it follows that

$$(f_r, g_r) \in Z^2(L, L) \times Z^3(L, L) = B^2(L, L) \times B^3(L, L). \quad (64)$$

Then, there exists $h_r \in C^1(L, L)$ such that $(f_r, g_r) = (\delta_I^1 h_r, \delta_I^1 h_r)$.

Consider $\phi_t = id_L - h_r t^r$, then $\phi_t : L \longrightarrow L$ is a linear isomorphism and $\phi_t \circ \alpha = \alpha \circ \phi_t$. Thus, we can define another 1-parameter formal deformation by ϕ_t^{-1} in the form of

$$\begin{aligned} f'_t(x, y) &= \phi_t^{-1} f_t(\phi_t(x), \phi_t(y)), g'_t(x, y, z) \\ &= \phi_t^{-1} g_t(\phi_t(x), \phi_t(y), \phi_t(z)). \end{aligned} \quad (65)$$

Set $f'_t = \sum_{i \geq 0} f'_i t^i$ and use the fact $\phi_t f'_t(x, y) = f_t(\phi_t(x), \phi_t(y))$, then we have

$$\begin{aligned} &(id_L - h_r t^r) \sum_{i \geq 0} f'_i(x, y) t^i \\ &= \left(f_0 + \sum_{i \geq 0} f_i t^i \right)(x - h_r(x) t^r, y - h_r(y) t^r), \end{aligned} \quad (66)$$

that is

$$\begin{aligned} &\sum_{i \geq 0} f'_i(x, y) t^i - \sum_{i \geq 0} h_r \circ f'_i(x, y) t^{i+r} \\ &= f_0(x, y) - f_0(h_r(x), y) t^r - f_0(x, h_r(y)) t^r \\ &\quad + f_0(h_r(x), h_r(y)) t^{2r} + \sum_{i \geq r} f_i(x, y) t^i \\ &\quad - \sum_{i \geq r} \{ f_i(h_r(x), y) - f_i(x, h_r(y)) \} t^{i+r} \\ &\quad + \sum_{i \geq r} f_i(h_r(x), h_r(y)) t^{i+2r}. \end{aligned} \quad (67)$$

By the above equation, it follows that

$$\begin{aligned} f'_0(x, y) &= f_0(x, y) = [x, y], \\ f'_1(x, y) &= f'_1(x, y) = \cdots = f'_{r-1}(x, y) = 0, \\ f'_r(x, y) - h_r([x, y]) &= f_r(x, y) - [h_r(x), y] - [x, h_r(y)]. \end{aligned} \quad (68)$$

Therefore, we deduce

$$f'_r(x, y) = f_r(x, y) - \delta_I^1 h_r(x, y) = 0. \quad (69)$$

So $f'_t = [\cdot, \cdot] + \sum_{i \geq r+1} f'_i t^i$. Similarly, we have $g'_t = \{\cdot, \cdot, \cdot\} + \sum_{i \geq r+1} g'_i t^i$. By induction, we have $(f_t, g_t) \sim (f_0, g_0)$, that is, L is analytically rigid. The proof is finished.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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