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Research Article

Characterizations of Trivial Ricci Solitons

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Finding characterizations of trivial solitons is an important problem in geometry of Ricci solitons. In this paper, we find several characterizations of a trivial Ricci soliton. First, on a complete shrinking Ricci soliton, we show that the scalar curvature satisfying a certain inequality gives a characterization of a trivial Ricci soliton. Then, it is shown that the potential field having geodesic flow and length of potential field satisfying certain inequality gives another characterization of a trivial Ricci soliton. Finally, we show that the potential field of constant length satisfying an inequality gives a characterization of a trivial Ricci soliton.

1. Introduction

Recall that Ricci solitons, being self-similar solutions of the Ricci flow (cf. [1]), are a topic of current interest. Moreover, they are models for some singularities which make their geometry very interesting. An n-dimensional Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is a Riemannian manifold (M, g) on which there is a smooth vector field $\mathbf{u}$ (called potential field) satisfying (cf. [1]),

$$\mathrm{Ric} + \frac{1}{2}\pounds_{\mathbf{u}} g = \lambda g, \tag{1}$$

where Ric is the Ricci tensor, $\pounds_{\mathbf{u}} g$ is the Lie derivative of the metric g with respect to $\mathbf{u}$, and λ is a constant. A Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is said to be expanding, stable, or shrinking depending on $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$, respectively. If the potential field $\mathbf{u} = \nabla f$ is a gradient of a smooth function f, then $(M, g, \nabla f, \lambda)$ is called a gradient Ricci soliton, and in this case, equation (1) takes the form

$$\mathrm{Ric} + H_f = \lambda g, \tag{2}$$

where H_f is the Hessian of the function f. Ricci solitons are stable solutions of the Ricci flow (cf. [1]) and have been used in settling Poincare conjecture, and since then, the study of Ricci solitons has picked up immense importance. One of the important findings on Ricci solitons is that if it is compact, the potential field $\mathbf{u}$ is a gradient of a smooth function f, that is, a compact Ricci soliton is a gradient Ricci soliton (cf. [1]). A Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is said to be trivial if $\pounds_{\mathbf{u}} g = 0$, and in this case, the metric g becomes an Einstein metric with λ becoming the Einstein constant. Several authors have studied the geometry of Ricci solitons (cf. [2–4]); in [5–7], Myers-type theorems have been proved for Ricci soliton; similarly in [8], it has been observed that a complete shrinking Ricci soliton $(M, g, \mathbf{u}, \lambda)$ has a finite fundamental group. In [9, 10], Bishop-type volume comparison theorems have been proved for noncompact shrinking Ricci solitons.

As Ricci solitons generalize Einstein metrics, a natural open problem is the existence of triviality results (i.e., conditions under which a Ricci soliton becomes an Einstein manifold). Thus, an important question in the geometry of a Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is to find conditions under which it becomes trivial. Recently in [11, 12], authors have found necessary and sufficient conditions for a compact Ricci soliton to be a trivial Ricci soliton. In this paper, we find necessary and sufficient conditions for compact Ricci solitons as well as noncompact Ricci solitons to be trivial. In our first result, we show that the scalar curvature S of a compact Ricci soliton $(M, g, \mathbf{u}, \lambda)$ satisfying a differential inequality involving the first nonzero eigenvalue λ_1 of the Laplace operator gives a characterization of a trivial Ricci soliton (cf. Theorem 1).

We also show that for a connected Ricci soliton $(M, g, \mathbf{u}, \lambda)$ the flow of potential field $\mathbf{u}$ being geodesic flow with its length $\|\mathbf{u}\|$ satisfying certain inequality gives a characterization of a trivial Ricci soliton (cf. Theorem 2). Finally, it is observed that potential field $\mathbf{u}$ being of constant length satisfying certain inequality on a connected Ricci soliton $(M, g, \mathbf{u}, \lambda)$ also gives a characterization of a trivial Ricci soliton (cf. Theorem 4).

2. Preliminaries

Let $(M, g, \mathbf{u}, \lambda)$ be an n-dimensional Ricci soliton and α be smooth 1-form dual to the potential field $\mathbf{u}$. We define a skew symmetric tensor field ψ on the Ricci soliton $(M, g, \mathbf{u}, \lambda)$ by

$$\frac{1}{2} d\alpha(X, Y) = g(\psi X, Y), X, Y \in \mathfrak{X}(M), \tag{3}$$

where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. We call this tensor field ψ the associated tensor field of the Ricci soliton $(M, g, \mathbf{u}, \lambda)$. The Ricci operator Q on the Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is a symmetric operator defined by $\operatorname{Ric}(X, Y) = g(QX, Y)$, $X, Y \in \mathfrak{X}(M)$. The gradient ∇S of the scalar curvature $S = TrQ$ satisfies

$$\sum (\nabla Q)(e_i, e_i) = \frac{1}{2} \nabla S, \tag{4}$$

where $\{e_1, \cdots, e_n\}$ is a local orthonormal frame and the covariant derivative $(\nabla Q)(X, Y) = \nabla_X QY - Q(\nabla_X Y)$.

Using equations (1) and (3) and Koszul's formula, the covariant derivative of the potential field $\mathbf{u}$ is given by

$$\nabla_X \mathbf{u} = \lambda X - QX + \psi X, \quad X \in \mathfrak{X}(M). \tag{5}$$

Now, using equation (5), we get the following expression for Riemannian curvature tensor of the Ricci soliton $(M, g, \mathbf{u}, \lambda)$:

$$\begin{aligned} R(X, Y)\mathbf{u} = {} & (\nabla Q)(Y, X) - (\nabla Q)(X, Y) \\ & + (\nabla \psi)(X, Y) - (\nabla \psi)(Y, X). \end{aligned} \tag{6}$$

As the operator Q is symmetric and ψ is skew-symmetric, using equations (4) and (6), we obtain

$$\operatorname{Ric}(Y, \mathbf{u}) = Y(S) - \frac{1}{2} Y(S) - g\left(Y, \sum (\nabla \psi)(e_i, e_i)\right), \tag{7}$$

which leads to

$$Q(\mathbf{u}) = \frac{1}{2} \nabla S - \sum (\nabla \psi)(e_i, e_i). \tag{8}$$

We denote by λ_1 the first nonzero eigenvalue of the Laplace operator Δ acting on smooth functions on compact $(M, g, \mathbf{u}, \lambda)$. If $h : M \longrightarrow R$ is a smooth function satisfying

$$\int_M h = 0, \tag{9}$$

then by minimum principle, we have

$$\int_M \|\nabla h\|^2 \geq \lambda_1 \int_M h^2. \tag{10}$$

3. A Characterization of Compact Trivial Ricci Solitons

Now, we prove the first result of this paper.

Theorem 1. *An n-dimensional complete shrinking Ricci soliton $(M, g, \mathbf{u}, \lambda)$ with Ricci curvature bounded below by a constant $c > 0$ and first nonzero eigenvalue λ_1 of the Laplacian operator is trivial if and only if the scalar curvature S satisfies the inequality*

$$(\Delta S)^2 \leq \frac{2n\lambda(\lambda_1 - \lambda)}{(n-1)} (S - n\lambda)^2. \tag{11}$$

Proof. Suppose $(M, g, \mathbf{u}, \lambda)$ is a complete shrinking Ricci soliton with Ricci curvature satisfying $\operatorname{Ric} \geq c > 0$ and the scalar curvature S satisfies the inequality

$$(\Delta S)^2 \leq \frac{2n\lambda(\lambda_1 - \lambda)}{(n-1)} (S - n\lambda)^2. \tag{12}$$

Note that the assumption on the Ricci curvature in view of Myers' theorem implies that M is compact. Thus, $(M, g, \mathbf{u}, \lambda)$ is a compact Ricci soliton, and therefore, it is a gradient Ricci soliton (cf. [1]). Consequently, $\mathbf{u}$ is a closed vector field, that is, $\psi = 0$. Equation (8) takes the form

$$Q(\mathbf{u}) = \frac{1}{2} \nabla S, \tag{13}$$

which gives

$$\operatorname{Ric}(\mathbf{u}, \nabla S) = \frac{1}{2} \|\nabla S\|^2. \tag{14}$$

Moreover equation (5) becomes

$$\nabla_X \mathbf{u} = \lambda X - QX, \tag{15}$$

which we use to compute the divergence of $Q\mathbf{u}$ and obtain

$$\operatorname{div} Q\mathbf{u} = \lambda S - \|Q\|^2 + \frac{1}{2} \mathbf{u}(S). \tag{16}$$

Now, using equation (13) in the above equation leads to

$$\operatorname{div} Q\mathbf{u} = \lambda S - \|Q\|^2 + \operatorname{Ric}(\mathbf{u}, \mathbf{u}), \tag{17}$$

which on integrating gives

$$\int_M \left\{ \left(\|Q\|^2 - \frac{S^2}{n} \right) + \frac{S^2}{n} - \lambda S - \operatorname{Ric}(\mathbf{u}, \mathbf{u}) \right\} = 0. \tag{18}$$

Using equation (15), we have div $\mathbf{u} = n\lambda - S$, which gives

$$\int_M n\lambda = \int_M S, \quad (19)$$

and consequently, we conclude

$$\int_M (S-n\lambda)^2 = \int_M \left(S^2 - n\lambda S\right). \quad (20)$$

Thus, equation (18) takes the form

$$\int_M \left\{ \left(\|Q\|^2 - \frac{S^2}{n} \right) + \frac{1}{n}(S-n\lambda)^2 - \operatorname{Ric}(\mathbf{u},\mathbf{u}) \right\} = 0. \quad (21)$$

Now, equations (13) and (16) imply

$$\Delta S = 2 \operatorname{div} Q\mathbf{u} = 2\left[\lambda S - \|Q\|^2 + \frac{1}{2}\mathbf{u}(S)\right], \quad (22)$$

which together with div $S\mathbf{u} = \mathbf{u}(S) + S(n\lambda - S)$ gives

$$\Delta S - \operatorname{div} S\mathbf{u} = 2\lambda S - 2\|Q\|^2 - S(n\lambda - S). \quad (23)$$

Integrating the above equation, we conclude

$$\int_M \left\{ 2\|Q\|^2 - 2\lambda S + S(n\lambda - S) \right\} = 0, \quad (24)$$

that is,

$$\int_M \left\{ 2\left(\|Q\|^2 - \frac{S^2}{n} \right) + 2\frac{S^2}{n} - S^2 + (n-2)\lambda S \right\} = 0, \quad (25)$$

which gives

$$\int_M \left\{ 2\left(\|Q\|^2 - \frac{S^2}{n} \right) - \left(\frac{n-2}{n} \right)\left(S^2 - n\lambda S\right) \right\} = 0. \quad (26)$$

Now, using equation (20) in the above equation yields

$$\int_M \left(\|Q\|^2 - \frac{S^2}{n} \right) = \left(\frac{n-2}{2n} \right) \int_M (S-n\lambda)^2. \quad (27)$$

Thus, equations (21) and (27) imply

$$\int_M \operatorname{Ric}(\mathbf{u},\mathbf{u}) = \frac{1}{2}\int_M (S-n\lambda)^2. \quad (28)$$

Also, we have Bochner's formula

$$\int_M \left\{ \operatorname{Ric}(\nabla S, \nabla S) + \|A_S\|^2 - (\Delta S)^2 \right\} = 0, \quad (29)$$

where $A_S(X) = \nabla_X \nabla S$ is the Hessian operator of the scalar curvature S. Note that equation (19) implies $\int_M (S - n\lambda) = 0$, which in view of equation (10) gives

$$\int_M \|\nabla S\|^2 \geq \lambda_1 \int_M (S-n\lambda)^2. \quad (30)$$

Now, we use equation (14) to compute

$$\begin{aligned} \operatorname{Ric}(\nabla S - 2\lambda\mathbf{u}, \nabla S - 2\lambda\mathbf{u}) = {} & \operatorname{Ric}(\nabla S, \nabla S) - 2\lambda\|\nabla S\|^2 \\ & + 4\lambda^2 \operatorname{Ric}(\mathbf{u},\mathbf{u}). \end{aligned} \quad (31)$$

Integrating the above equation and using equations (28) and (29), we get

$$\begin{aligned} \int_M \operatorname{Ric}(\nabla S - 2\lambda\mathbf{u}, \nabla S - 2\lambda\mathbf{u}) = \int_M & \left\{ (\Delta S)^2 - \|A_S\|^2 - 2\lambda\|\nabla S\|^2 \right. \\ & \left. + 4\lambda^2 \operatorname{Ric}(\mathbf{u},\mathbf{u}) \right\}, \end{aligned} \quad (32)$$

which on using $\lambda > 0$ (for a shrinking Ricci soliton) and the inequality (30) gives

$$\begin{aligned} & \int_M \operatorname{Ric}(\nabla S - 2\lambda\mathbf{u}, \nabla S - 2\lambda\mathbf{u}) \\ & \leq \int_M \left\{ -\left(\|A_S\|^2 - \frac{1}{n}(\Delta S)^2 \right) + \left(\frac{n-1}{n} \right)(\Delta S)^2 \right. \\ & \quad \left. - 2\lambda\lambda_1 (S-n\lambda)^2 + 2\lambda^2 (S-n\lambda)^2 \right\} \end{aligned} \quad (33)$$

or

$$\begin{aligned} & \int_M \operatorname{Ric}(\nabla S - 2\lambda\mathbf{u}, \nabla S - 2\lambda\mathbf{u}) \\ & \leq \int_M \left\{ -\left(\|A_S\|^2 - \frac{1}{n}(\Delta S)^2 \right) \right. \\ & \quad \left. + \left(\frac{n-1}{n} \right)(\Delta S)^2 - 2\lambda(\lambda_1 - \lambda)(S-n\lambda)^2 \right\}. \end{aligned} \quad (34)$$

Thus,

$$\begin{aligned} & \int_M \operatorname{Ric}(\nabla S - 2\lambda\mathbf{u}, \nabla S - 2\lambda\mathbf{u}) \\ & \leq \int_M \left\{ -\left(\|A_S\|^2 - \frac{1}{n}(\Delta S)^2 \right) \right. \\ & \quad \left. + \left(\frac{n-1}{n} \right)\left[(\Delta S)^2 - \frac{2\lambda n}{(n-1)}(\lambda_1 - \lambda)(S-n\lambda)^2 \right] \right\}. \end{aligned} \quad (35)$$

Since the Ricci curvature satisfies $\operatorname{Ric} \geq c$ for a constant $c > 0$, the above inequality takes the form

$$c\int_M \|\nabla S - 2\lambda \mathbf{u}\|^2 \le \int_M \left\{ -\left(\|A_S\|^2 - \frac{1}{n}(\Delta S)^2 \right) + \left(\frac{n-1}{n} \right) \cdot \left[(\Delta S)^2 - \frac{2\lambda n}{(n-1)}(\lambda_1 - \lambda)(S - n\lambda)^2 \right] \right\}. \tag{36}$$

Using the Schwarz inequality $\|A_S\|^2 \ge (1/n)(\Delta S)^2$, and the inequality (12) in the above inequality, we conclude

$$\nabla S = 2\lambda \mathbf{u}, \quad \|A_S\|^2 = \frac{1}{n}(\Delta S)^2. \tag{37}$$

Also, the equality in the Schwarz inequality holds if and only if $A_S = (\Delta S/n)I$. Moreover, the equation $\nabla S = 2\lambda \mathbf{u}$ in view of equation (15) implies

$$A_S(X) = 2\lambda(\lambda X - QX), \quad \Delta S = 2\lambda(n\lambda - S). \tag{38}$$

Consequently, using $A_S = (\Delta S/n)I$, we get

$$2\lambda(\lambda X - QX) = \frac{2\lambda(n\lambda - S)}{n} X, \tag{39}$$

that is, $Q(X) = (S/n)X$. Now, using $Q(X) = (S/n)X$ with equation (13) and first equation in equation (37), we get $S = n\lambda$, that is, $Q = \lambda I$. Hence, $\mathrm{Ric} = \lambda g$ and the Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is trivial.

Conversely, if $(M, g, \mathbf{u}, \lambda)$ is a trivial soliton, then $\mathrm{Ric} = \lambda g$, $\lambda > 0$ gives $S = n\lambda$ which implies $\Delta S = 0$, and consequently, the equality (12) holds.

It is well known that the odd-dimensional unit sphere S^{2n+1} with induced metric g as a hypersurface of the Euclidean space $(C^{n+1}, \langle , \rangle)$ admits a unit Killing vector field $\mathbf{u}$, and consequently, we have the trivial Ricci soliton $(S^{2n+1}, g, \mathbf{u}, \lambda)$, $\lambda = 2n$, satisfying the hypothesis of Theorem 1.

4. Characterizations of Connected Trivial Ricci Solitons

In this section, we consider a connected Ricci soliton $(M, g, \mathbf{u}, \lambda)$ and find necessary and sufficient conditions under which it is a trivial Ricci soliton. Recall that the local flow $\{\phi_t\}$ of a smooth vector field $\mathbf{u}$ on a Riemannian manifold (M, g) is said to be geodesic flow if the orbits of $\{\phi_t\}$ are geodesics on (M, g). Geodesic flows have been used in studying geometry of foliations on a Riemannian manifold (cf. [7, 13]). Note that a flow consisting of isometries is a geodesic flow and the converse is not true. For example, consider the 3-dimensional unit sphere S^3 which has a Sasakian structure $(\phi, \mathbf{u}, \alpha, g)$ (cf. [14]). Then for a positive function f on S^3, deform the metric g by

$$\bar{g} = fg + (1 - f)\alpha \otimes \alpha. \tag{40}$$

Then, $\mathbf{u}$ is still a unit vector field on the Riemannian manifold $(S^3, \bar{g})$. However, $\mathbf{u}$ is no more a Killing vector field on $(S^3, \bar{g})$ but instead $(\psi, \mathbf{u}, \alpha, \bar{g})$ is a trans-Sasakian structure [15], and the flow of $\mathbf{u}$ on the Riemannian manifold $(S^3 . \bar{g})$ is a geodesic flow.

In the next result, we use this notion of geodesic flow for the potential field $\mathbf{u}$ of the Ricci soliton $(M, g, \mathbf{u}, \lambda)$ to characterize trivial Ricci solitons.

Theorem 2. *Let $(M, g, \mathbf{u}, \lambda)$ be an n-dimensional connected shrinking Ricci soliton with the local flow of potential field $\mathbf{u}$ be the geodesic flow. Then, $(M, g, \mathbf{u}, \lambda)$ is trivial Ricci soliton if and only if the scalar curvature S is a constant along the integral curves of $\mathbf{u}$ and the associated tensor ψ satisfies the inequality*

$$\|\psi\|^2 \le \lambda \|\mathbf{u}\|^2. \tag{41}$$

Proof. Suppose $(M, g, \mathbf{u}, \lambda)$ is connected with local flow of $\mathbf{u}$ a geodesic flow and the scalar curvature S is a constant along the integral curves of $\mathbf{u}$ and the associated tensor ψ satisfies

$$\|\psi\|^2 \le \lambda \|\mathbf{u}\|^2. \tag{42}$$

As the local flow of $\mathbf{u}$ is a geodesic flow, equation (5) gives

$$Q\mathbf{u} = \lambda \mathbf{u} + \psi \mathbf{u}. \tag{43}$$

As the scalar curvature S is a constant along the integral curves of $\mathbf{u}$, using equations (4) and (8), we conclude

$$g\left(\mathbf{u}, \sum (\nabla Q)(e_i, e_i)\right) = 0, \tag{44}$$

$$\mathrm{Ric}(\mathbf{u}, \mathbf{u}) = -g\left(\mathbf{u}, \sum (\nabla \psi)(e_i, e_i)\right). \tag{45}$$

Now, using equations (5) and (44), we find the divergence of the vector field $Q\mathbf{u}$. After some straight forward computations, we get

$$\operatorname{div} Q\mathbf{u} = \lambda S - \|Q\|^2. \tag{46}$$

Similarly, using equations (5) and (45), we get

$$\operatorname{div} \psi \mathbf{u} = -\|\psi\|^2 + \mathrm{Ric}(\mathbf{u}, \mathbf{u}). \tag{47}$$

Equation (43) gives $\mathrm{Ric}(\mathbf{u}, \mathbf{u}) = \lambda \|\mathbf{u}\|^2$, which on inserting in the above equation yields

$$\operatorname{div} \psi \mathbf{u} = -\|\psi\|^2 + \lambda \|\mathbf{u}\|^2. \tag{48}$$

Note that equation (5) gives $\operatorname{div} \mathbf{u} = (n\lambda - S)$. Consequently, on taking divergence in equation (43) and using equations (46) and (48), we conclude

$$\lambda S - \|Q\|^2 = n\lambda^2 - \lambda S - \|\psi\|^2 + \lambda \|\mathbf{u}\|^2, \tag{49}$$

which gives

$$\left(\|Q\|^2 - \frac{1}{n}S^2\right) + \frac{1}{n}(S - n\lambda)^2 + \left(\lambda\|\mathbf{u}\|^2 - \|\psi\|^2\right) = 0. \quad (50)$$

Using the Schwarz inequality $\|Q\|^2 \geq (1/n)S^2$ and inequality (42), in the above equation, we conclude

$$\begin{aligned} \|Q\|^2 &= \frac{1}{n}S^2, \\ S &= n\lambda, \\ \|\psi\|^2 &= \lambda\|\mathbf{u}\|^2. \end{aligned} \quad (51)$$

Since the equality in the Schwarz inequality holds if and only if $Q = (S/n)I$, we get $\mathrm{Ric} = \lambda g$, that is, $(M, g, \mathbf{u}, \lambda)$ is trivial.

Conversely, if $(M, g, \mathbf{u}, \lambda)$ is a trivial Ricci soliton with local flow of $\mathbf{u}$ a geodesic flow, then it follows that S is a constant and equation (5) takes the form

$$\nabla_X \mathbf{u} = \psi X \text{ and } \psi\mathbf{u} = 0. \quad (52)$$

Then finding the divergence of $\psi\mathbf{u}$ using above equation, gives the equality

$$\|\psi\|^2 = \lambda\|\mathbf{u}\|^2. \quad (53)$$

Remark 3.

(1) It is clear that an odd-dimensional unit sphere $(S^{2n+1}, g, \mathbf{u}, \lambda)$ is a trivial Ricci soliton, where $\lambda = 2n$, the potential field $\mathbf{u} = -JN$, J being the complex structure on C^{n+1} and N is the unit normal to the hypersurface S^{2n+1}. The associated tensor ψ is given by $\psi X = (JX)^T$, the tangential component of JX. It follows that $\|\psi\|^2 = 2n = \lambda\|\mathbf{u}\|^2$ holds. Naturally, $\mathbf{u}$ being the Killing vector field, its flow consists of isometries of S^{2n+1}, and therefore, it is a geodesic flow.

(2) Next, we give an example of a nontrivial Ricci soliton with the flow of potential field $\mathbf{u}$ not a geodesic field. Consider the open subset

$$M = \left\{x \in R^n : \|x\| > \sqrt{2}\right\}, \quad n > 3, \quad (54)$$

of the Euclidean space (R^n, g), where g is the Euclidean metric. Consider the vector field $\mathbf{u} \in \mathfrak{X}(M)$ defined by

$$\mathbf{u} = \Psi - x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2}, \quad (55)$$

where

$$\Psi = x^1\frac{\partial}{\partial x^1} + \cdots + x^n\frac{\partial}{\partial x^n} \quad (56)$$

is the position vector field and $x^1, \cdots, x^n$ are the Euclidean coordinates on M. It follows that

$$£_{\mathbf{u}} g = 2g. \quad (57)$$

Hence, we have

$$\mathrm{Ric} + \frac{1}{2}£_{\mathbf{u}} g = g, \quad (58)$$

that is, $(M, g, \mathbf{u}, \lambda)$, $\lambda = 1$ is a nontrivial Ricci soliton with associated tensor field ψ, given by

$$\psi X = -X\left(x^2\right)\frac{\partial}{\partial x^1} + X\left(x^1\right)\frac{\partial}{\partial x^2}. \quad (59)$$

The flow $\{\varphi_t\}$ of $\mathbf{u}$ is given by

$$\begin{aligned} \varphi_t\left(x^1, \cdots, x^n\right) = e^t\left(x^1 \cos t + x^2 \sin t, x^2 \cos t\right. \\ \left. - x^1 \sin t, x^3, \cdots, x^n\right), \end{aligned} \quad (60)$$

which is not a geodesic flow. Moreover, we have $\|\psi\|^2 = 2$ and $\|\mathbf{u}\|^2 = \|\Psi\|^2 + (x^1)^2 + (x^2)^2$, that is, $\|\psi\|^2 < \lambda\|\mathbf{u}\|^2$ holds.

Next, we consider Ricci solitons $(M, g, \mathbf{u}, \lambda)$, with potential field $\mathbf{u}$ of constant length. Note that if M is compact and $\|\mathbf{u}\|$ is a constant, then $(M, g, \mathbf{u}, \lambda)$ is trivial, the argument goes as follows: in this case, $\mathbf{u} = \nabla h$ for a smooth function h, and as M is compact, there is point $p \in M$ (the critical point of h), where $\mathbf{u}_p = 0$. As $\|\mathbf{u}\| = c$, a constant, that will give $\mathbf{u} = 0$, that is, $(M, g, \mathbf{u}, \lambda)$ is trivial.

We get the following characterization of noncompact trivial Ricci solitons with potential field $\mathbf{u}$ having constant length.

Theorem 4. *Let $(M, g, \mathbf{u}, \lambda)$ be an n-dimensional connected noncompact Ricci soliton with a constant length of potential field. Then, $(M, g, \mathbf{u}, \lambda)$ is trivial if and only if the associated tensor ψ satisfies the inequality*

$$\|\psi\|^2 \geq \lambda\|\mathbf{u}\|^2. \quad (61)$$

Proof. Suppose $(M, g, \mathbf{u}, \lambda)$ is an n-dimensional Ricci soliton with $\|\mathbf{u}\|$ a constant and

$$\|\psi\|^2 \geq \lambda\|\mathbf{u}\|^2. \quad (62)$$

As $\|\mathbf{u}\|^2$ is a constant, using equation (5), we conclude

$$\psi\mathbf{u} = \lambda\mathbf{u} - Q\mathbf{u}. \quad (63)$$

Now, $\operatorname{div}\mathbf{u} = n\lambda - S$ and using equations (5) and (8), we get

$$\begin{aligned} \operatorname{div} Q\mathbf{u} &= \lambda S - \|Q\|^2 + \frac{1}{2}\mathbf{u}(S), \\ \operatorname{div} \psi\mathbf{u} &= -\|\psi\|^2 + \mathrm{Ric}(\mathbf{u}, \mathbf{u}) - \frac{1}{2}\mathbf{u}(S). \end{aligned} \quad (64)$$

Taking divergence in equation (63), and using the above equations, we conclude

$$-\|\psi\|^2 + \mathrm{Ric}(\mathbf{u}, \mathbf{u}) = n\lambda^2 - 2\lambda S + \|Q\|^2. \tag{65}$$

Also, the inner product with $\mathbf{u}$ in equation (63) gives $\mathrm{Ric}(\mathbf{u}, \mathbf{u}) = \lambda\|\mathbf{u}\|^2$, and consequently, the above equation becomes

$$\left(\|Q\|^2 - \frac{1}{n}S^2\right) + \frac{1}{n}(S - n\lambda)^2 + \left(\|\psi\|^2 - \lambda\|\mathbf{u}\|^2\right) = 0. \tag{66}$$

Using the Schwarz inequality and the inequality (62), in the above equation, we conclude that

$$\begin{aligned} \|Q\|^2 &= \frac{1}{n}S^2, \\ S &= n\lambda, \\ \|\psi\|^2 &= \lambda\|\mathbf{u}\|^2, \end{aligned} \tag{67}$$

which, as in the proof of Theorem 2, implies that $(M, g, \mathbf{u}, \lambda)$ is trivial.

Converse follows on the similar lines as in Theorem 2.

We construct an example of a nontrivial Ricci soliton with a nonconstant length of potential. Let M be the unit open ball

$$M = \{x \in C^n : \|x\| < 1\} \tag{68}$$

in the Euclidean space (C^n, J, g), where J is the complex structure and g is the Euclidean metric. Consider the smooth vector field $\mathbf{u} \in \mathfrak{X}(M)$ defined by

$$\mathbf{u} = \Psi + J\Psi, \tag{69}$$

where

$$\Psi = x^1 \frac{\partial}{\partial x^1} + \cdots + x^{2n} \frac{\partial}{\partial x^{2n}} \tag{70}$$

is the position vector field. Then, it follows that

$$\pounds_{\mathbf{u}} g = 2g, \tag{71}$$

that is,

$$\mathrm{Ric} + \frac{1}{2}\pounds_{\mathbf{u}} g = g. \tag{72}$$

Hence, $(M, g, \mathbf{u}, \lambda)$ is a nontrivial Ricci soliton with $\lambda = 1$ and associated tensor $\psi = J$. We get $\|\psi\|^2 = 2n$ and $\|\mathbf{u}\|^2 = 2\|\Psi\|^2 < 2$, that is, $\lambda\|\mathbf{u}\|^2 < \|\psi\|^2$.

Data Availability

No data have been used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

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