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Research Article

Theory of Generalized Canonical Transformations for Birkhoff Systems

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Transformation is an important means to study problems in analytical mechanics. It is often difficult to solve dynamic equations, and the use of variable transformation can make the equations easier to solve. The theory of canonical transformations plays an important role in solving Hamilton's canonical equations. Birkhoffian mechanics is a natural generalization of Hamiltonian mechanics. This paper attempts to extend the canonical transformation theory of Hamilton systems to Birkhoff systems and establish the generalized canonical transformation of Birkhoff systems. First, the definition and criterion of the generalized canonical transformation for the Birkhoff system are established. Secondly, based on the criterion equation and considering the generating functions of different forms, six generalized canonical transformation formulas are derived. As special cases, the canonical transformation formulas of classical Hamilton's equations are given. At the end of the paper, two examples are given to illustrate the application of the results.

1. Introduction

Birkhoffian mechanics can be traced back to Birkhoff's monograph Dynamical Systems, which gave a new class of dynamic equations more common than Hamilton's canonical equations and a new class of integral variational principles more common than Hamilton's principle [1]. Santilli [2] studied Birkhoff's equations, the transformation theory of Birkhoff's equations, and the extension of Galileo's relativity and applied Birkhoff's equations to hadron physics. Galiullin et al. [3] studied the inverse problem of Birkhoffian dynamics, the symmetry, and the conformal invariance of Birkhoff systems. Mei et al. have conducted in-depth studies on the dynamics of Birkhoff systems, including Birkhoffian representation of holonomic and nonholonomic systems, integration theory, symmetry theory, inverse problem of dynamics, motion stability, geometric method, and global analysis of Birkhoff systems [4], and extended the results to generalized Birkhoff systems [5]. In recent years, some new advances have been made in the study of dynamics of Birkhoff systems, such as [6–19] and the references therein.

The integration problem of dynamic equations is an important aspect of analytical mechanics. Since it is often difficult to solve the general dynamic equations, the transformation of variables can make the equations easy to solve. The classical Hamilton canonical transformation theory plays an important role in solving dynamic equations. How do we extend Hamilton canonical transformation theory to Birkhoff systems? Santilli first proposed and preliminarily studied the transformation theory of Birkhoff's equations and only gave one kind of generating function and its transformation [2]. In this paper, based on the basic identity of Birkhoff system's generalized canonical transformation, we derive six generalized canonical transformation formulas by selecting different forms of generating functions and give the transformation relations between the old and new variables in each case. The way of selecting the generating function in this paper is different from that in Reference [2]. The canonical transformation formulas of classical Hamilton's equations are the special cases of the generalized canonical transformation formulas of Birkhoff systems. The application of the results is illustrated with two examples.

This paper is organized as follows. In Section 2, we give the definition of the generalized canonical transformation for Birkhoff systems and establish the basic identity for constructing the generalized canonical transformation. In Section 3, we present six generating functions of Birkhoff systems, derive the corresponding generalized regular transformation formulas, and discuss their special cases, namely, the canonical transformation of Hamilton systems. Two examples are given in Section 4. We conclude in Section 5.

2. Definition and Criterion of Generalized Canonical Transformations

We consider a mechanical system described by $2n$ Birkhoff's variables $a^\mu (\mu = 1, 2, \cdots, 2n)$. The differential equations of motion of the system can be expressed as the following Birkhoff's equations:

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = 0 \quad (\mu, \nu = 1, 2, \cdots, 2n), \tag{1}$$

where $B = B(t, a^\nu)$ is the Birkhoffian and $R_\mu = R_\mu(t, a^\nu)$ are Birkhoff's functions.

If there is a contemporaneous transformation

$$t \longrightarrow \bar{t} \equiv t, a^\mu \longrightarrow \bar{a}^\mu(t, a^\nu), \tag{2}$$

where the equations of motion expressed by the new variables $\bar{a}^\mu$ remain in the form of Birkhoff's equations, i.e.,

$$\left(\frac{\partial \bar{R}_\nu}{\partial \bar{a}^\mu} - \frac{\partial \bar{R}_\mu}{\partial \bar{a}^\nu}\right)\dot{\bar{a}}^\nu - \frac{\partial \bar{B}}{\partial \bar{a}^\mu} - \frac{\partial \bar{R}_\mu}{\partial t} = 0 \quad (\mu, \nu = 1, 2, \cdots, 2n), \tag{3}$$

where $\bar{B} = \bar{B}(t, \bar{a}^\nu)$ and $\bar{R}_\mu = \bar{R}_\mu(t, \bar{a}^\nu)$ are the new Birkhoffian and new Birkhoff's functions, then the transformation (2) is the generalized canonical transformation of the Birkhoff system.

Considering that Birkhoff's equations are directly derived from the Pfaff-Birkhoff principle, we give a general definition of the generalized canonical transformation of the Birkhoff system, as follows:

Definition 1. For the Birkhoff system (1), if the contemporaneous transformation (2) preserves the Pfaff-Birkhoff principle in the transition from the old variables

$$\delta\int_{t_0}^{t_1} \left[R_\mu(t, a^\nu)\dot{a}^\mu(t) - B(t, a^\nu)\right]dt = 0 \tag{4}$$

to the new variables

$$\delta\int_{t_0}^{t_1} \left[\bar{R}_\mu(t, \bar{a}^\nu)\dot{\bar{a}}^\mu(t) - \bar{B}(t, \bar{a}^\nu)\right]dt = 0, \tag{5}$$

where the notation $\delta(*)$ represents the isochronous variation of $(*)$, then the transformation is called a generalized canonical transformation of the system.

According to Definition 1, equations (4) and (5) need to be satisfied simultaneously for the generalized canonical transformation, but this does not mean that their integrand functions are exactly the same. In general, they can differ from each other by the total derivative of any function $F(t, a^\nu, \bar{a}^\nu)$ with respect to time t. Due to

$$\int_{t_0}^{t_1} \frac{dF}{dt}dt = F(t, a^\nu, \bar{a}^\nu)|_{t_1} - F(t, a^\nu, \bar{a}^\nu)|_{t_0}. \tag{6}$$

Considering $\delta a^\mu(t_1) = \delta a^\mu(t_0) = \delta\bar{a}^\mu(t_1) = \delta\bar{a}^\mu(t_0) = 0$ $(\mu = 1, 2, \cdots, 2n)$, so the variation of equation (6) is zero, we have

$$\delta\int_{t_0}^{t_1} \frac{dF}{dt}dt = 0. \tag{7}$$

Hence, we obtain the following.

Criterion 2. Transformation (2) is the generalized canonical transformation of Birkhoff system (1), if and only if there is a function $F(t, a^\nu, \bar{a}^\nu)$ such that the following basic identity [2]

$$\begin{aligned} R_\mu(t, a^\nu)da^\mu - B(t, a^\nu)dt - \bar{R}_\mu(t, \bar{a}^\nu)d\bar{a}^\mu + \bar{B}(t, \bar{a}^\nu)dt \\ = dF(t, a^\nu, \bar{a}^\nu) \end{aligned} \tag{8}$$

holds.

Formula (8) is called the criterion equation to judge whether the given transformation of the Birkhoff system is a generalized canonical transformation. The function F is called the generating function.

Since the old variables a^μ, the new variables $\bar{a}^\mu$, and time t are connected by $2n$ transformation equation (2), only $2n$ variables are independent except for the variable . Selecting independent variables usually can have different schemes. Thus, the generating function can also have different forms.

3. Generating Functions and Generalized Canonical Transformations

For the convenience of interpretation, we express Birkhoff's variables as $a = \{a^s, a_s\}$ and Birkhoff's functions as $R = \{R_s, R^s\}$, where $s = 1, 2, \cdots, n$. Then, the criterion equation (8) can be expressed as

$$\begin{aligned} R_s da^s + R^s da_s - Bdt - \bar{R}_s d\bar{a}^s - \bar{R}^s d\bar{a}_s + \bar{B}dt \\ = dF \quad (s = 1, 2, \cdots, n), \end{aligned} \tag{9}$$

where R_s, R^s are functions of the old variables a^k, a_k and $\bar{R}_s$, $\bar{R}^s$ are functions of the new variables $\bar{a}^k, \bar{a}_k$. The generating function F only needs to be a function of any $2n$ dimension subset of variables $a^k, a_k, \bar{a}^k, \bar{a}_k$ and time t. The generalized canonical transformation of Birkhoff system (1) depends on different choices of generating functions. In the following six theorems, we give the transformation relations between new functions and new variables and old functions and old variables and corresponding generating functions of six basic forms.

Theorem 3. *If the old variables a^s and the new variables $\bar{a}^s\ (s = 1, 2, \cdots, n)$ are regarded as 2n independent variables, namely, the generating function is taken as $F_1(t, a^s, \bar{a}^s)$, then the transformation determined by the following equations*

$$
\begin{aligned}
&R_s - \frac{\partial F_1}{\partial a^s} - a_k \frac{\partial R^k}{\partial a^s} - a_k \frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial a^s} + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial a^s} = 0,\\
&\bar{R}_s - \frac{\partial F_1}{\partial \bar{a}^s} - a_k \frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial \bar{a}^s} + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}^s} + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial \bar{a}^s} = 0,\\
&\qquad \bar{B} - B - \frac{\partial F_1}{\partial t} - a_k \frac{\partial R^k}{\partial t} - a_k \frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial t}\\
&\qquad + \bar{a}_k \frac{\partial \bar{R}^k}{\partial t} + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial t} = 0,\\
&\qquad\qquad s, k, j = 1, 2, \cdots, n,
\end{aligned}
\tag{10}
$$

is the generalized canonical transformation of Birkhoff system (1), where $F_1(t, a^s, \bar{a}^s)$ is called the first kind of generating function.

Proof. Let

$$
F = F_1(t, a^s, \bar{a}^s) + R^s a_s - \bar{R}^s \bar{a}_s. \tag{11}
$$

We have

$$
\begin{aligned}
dF = {} & \frac{\partial F_1}{\partial t} dt + \frac{\partial F_1}{\partial a^s} da^s + \frac{\partial F_1}{\partial \bar{a}^s} d\bar{a}^s + R^s da_s\\
& + a_s \left(\frac{\partial R^s}{\partial t} dt + \frac{\partial R^s}{\partial a^k} da^k + \frac{\partial R^s}{\partial a_k} da_k \right) - \bar{R}^s d\bar{a}_s\\
& - \bar{a}_s \left(\frac{\partial \bar{R}^s}{\partial t} dt + \frac{\partial \bar{R}^s}{\partial \bar{a}^k} d\bar{a}^k + \frac{\partial \bar{R}^s}{\partial \bar{a}_k} d\bar{a}_k \right).
\end{aligned}
\tag{12}
$$

Since a^s, $\bar{a}^s\ (s = 1, 2, \cdots, n)$ are independent variables, we have

$$
\begin{aligned}
da_k &= \frac{\partial a_k}{\partial t} dt + \frac{\partial a_k}{\partial a^j} da^j + \frac{\partial a_k}{\partial \bar{a}^j} d\bar{a}^j,\\
da_k &= \frac{\partial a_k}{\partial t} dt + \frac{\partial a_k}{\partial a^j} da^j + \frac{\partial a_k}{\partial \bar{a}^j} d\bar{a}^j.
\end{aligned}
\tag{13}
$$

Substituting formula (12) into the criterion equation (9), and considering the relation (13), we get

$$
\begin{aligned}
&\left(R_s - \frac{\partial F_1}{\partial a^s} - a_k \frac{\partial R^k}{\partial a^s} - a_k \frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial a^s} + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial a^s} \right) da^s\\
&\quad + \left(-\bar{R}_s - \frac{\partial F_1}{\partial \bar{a}^s} - a_k \frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial \bar{a}^s} + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}^s} + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial \bar{a}^s} \right) d\bar{a}^s\\
&\quad + \left(\bar{B} - B - \frac{\partial F_1}{\partial t} - a_k \frac{\partial R^k}{\partial t} - a_k \frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial t} \right.\\
&\quad \left. + \bar{a}_k \frac{\partial \bar{R}^k}{\partial t} + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial t} \right) dt = 0.
\end{aligned}
\tag{14}
$$

By the independence of da^s and $d\bar{a}^s$, we get the results easily. The theorem is proved.

Hamilton's principle is a special case of the Pfaff-Birkhoff principle, and Hamilton's canonical equation is a special case of Birkhoff's equation. Therefore, the generalized canonical transformations of the Birkhoff system are naturally suitable for the Hamilton system. In fact, if we take $a^s = q_s$, $a_s = p_s$, $R_s = p_s$, $R^s = 0$, $B = H$, then equation (1) is reduced to the following Hamilton canonical equations

$$
\begin{aligned}
\dot{q}_s &= \frac{\partial H}{\partial p_s},\\
\dot{p}_s &= -\frac{\partial H}{\partial q_s},\\
s &= 1, 2, \cdots, n.
\end{aligned}
\tag{15}
$$

Equation (8) becomes the basic identity for constructing the canonical transformation of the Hamilton system, i.e.,

$$
p_s dq_s - H dt - \bar{p}_s d\bar{q}_s + \bar{H} dt = dF. \tag{16}
$$

The transformation (10) gives

$$
\begin{aligned}
p_s &= \frac{\partial F_1}{\partial q_s},\\
\bar{p}_s &= -\frac{\partial F_1}{\partial \bar{q}_s},\\
\bar{H} &= H + \frac{\partial F_1}{\partial t}.
\end{aligned}
\tag{17}
$$

So, from Theorem 3, we have the following corollary.

Corollary 4. *If we take the old variables q_s and the new variables $\bar{q}_s\ (s = 1, 2, \cdots, n)$ as 2n independent variables, the transformation determined by equation (17) is the canonical transformation of Hamilton system (15), where $F_1(t, q_s, \bar{q}_s)$ is called the first kind of generating function.*

Theorem 5. *If the old variables a^s and the new variables $\bar{a}_s\ (s = 1, 2, \cdots, n)$ are regarded as 2n independent variables, namely, the generating function is taken as $F_2(t, a^s, \bar{a}_s)$, then the transformation determined by the following equations*

$$
\begin{aligned}
&R_s-\frac{\partial F_2}{\partial a^s}-a_k\frac{\partial R^k}{\partial a^s}-a_k\frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial a^s}+\bar{a}^k\frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial a^s}=0,\\
&-\bar{R}^s-\frac{\partial F_2}{\partial \bar{a}_s}-a_k\frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial \bar{a}_s}+\bar{a}^k\frac{\partial \bar{R}_k}{\partial \bar{a}_s}+\bar{a}^k\frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial \bar{a}_s}=0,\\
&\bar{B}-B-\frac{\partial F_2}{\partial t}-a_k\frac{\partial R^k}{\partial t}-a_k\frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial t}\\
&\quad+\bar{a}^k\frac{\partial \bar{R}_k}{\partial t}+\bar{a}^k\frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial t}=0,\\
&\qquad s,k,j=1,2,\cdots,n,
\end{aligned}
\tag{18}
$$

is the generalized canonical transformation of Birkhoff system (1), where $F_2(t,a^s,\bar{a}_s)$ is called the second kind of generating function.

Proof. Let

$$F=F_2(t,a^s,\bar{a}_s)+R^s a_s-\bar{R}_s\bar{a}^s. \tag{19}$$

We have

$$
\begin{aligned}
dF=&\frac{\partial F_2}{\partial t}dt+\frac{\partial F_2}{\partial a^s}da^s+\frac{\partial F_2}{\partial \bar{a}_s}d\bar{a}_s+R^s da_s\\
&+a_s\left(\frac{\partial R^s}{\partial t}dt+\frac{\partial R^s}{\partial a^k}da^k+\frac{\partial R^s}{\partial a_k}da_k\right)-\bar{R}_s d\bar{a}^s\\
&-\bar{a}^s\left(\frac{\partial \bar{R}_s}{\partial t}dt+\frac{\partial \bar{R}_s}{\partial \bar{a}^k}d\bar{a}^k+\frac{\partial \bar{R}_s}{\partial \bar{a}_k}d\bar{a}_k\right).
\end{aligned}
\tag{20}
$$

Since $a^s,\bar{a}_s\ (s=1,2,\cdots,n)$ are independent variables, we have

$$
\begin{aligned}
da_k&=\frac{\partial a_k}{\partial t}dt+\frac{\partial a_k}{\partial a^j}da^j+\frac{\partial a_k}{\partial \bar{a}_j}d\bar{a}_j,\\
d\bar{a}^k&=\frac{\partial \bar{a}^k}{\partial t}dt+\frac{\partial \bar{a}^k}{\partial a^j}da^j+\frac{\partial \bar{a}^k}{\partial \bar{a}_j}d\bar{a}_j.
\end{aligned}
\tag{21}
$$

Substituting formula (20) into equation (9), and considering the relations (21), we get

$$
\begin{aligned}
&\left(R_s-\frac{\partial F_2}{\partial a^s}-a_k\frac{\partial R^k}{\partial a^s}-a_k\frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial a^s}+\bar{a}^k\frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial a^s}\right)da^s\\
&+\left(-\bar{R}^s-\frac{\partial F_2}{\partial \bar{a}_s}-a_k\frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial \bar{a}_s}+\bar{a}^k\frac{\partial \bar{R}_k}{\partial \bar{a}_s}+\bar{a}^k\frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial \bar{a}_s}\right)d\bar{a}_s\\
&+\left(\bar{B}-B-\frac{\partial F_2}{\partial t}-a_k\frac{\partial R^k}{\partial t}-a_k\frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial t}\right.\\
&\left.+\bar{a}^k\frac{\partial \bar{R}_k}{\partial t}+\bar{a}^k\frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial t}\right)dt=0.
\end{aligned}
\tag{22}
$$

By the independence of da^s and $d\bar{a}_s$, we get the results easily. The theorem is proved.

For the Hamilton system (15), equation (18) gives

$$
\begin{aligned}
p_s&=\frac{\partial F_2}{\partial q_s},\\
\bar{q}_s&=\frac{\partial F_2}{\partial \bar{p}_s},\\
\bar{H}&=H+\frac{\partial F_2}{\partial t}.
\end{aligned}
\tag{23}
$$

So, from Theorem 5, we have the following corollary.

Corollary 6. *If we take the old variables q_s and the new variables $\bar{p}_s\ (s=1,2,\cdots,n)$ as 2n independent variables, the transformation determined by equation (23) is the canonical transformation of Hamilton system (15), where $F_2(t,q_s,\bar{p}_s)$ is called the second kind of generating function.*

Theorem 7. *If the old variables a_s and the new variables $\bar{a}^s\ (s=1,2,\cdots,n)$ are regarded as 2n independent variables, namely, the generating function is taken as $F_3(t,a_s,\bar{a}^s)$, then the transformation determined by the following equations*

$$
\begin{aligned}
&R^s-\frac{\partial F_3}{\partial a_s}-a^k\frac{\partial R_k}{\partial a_s}-a^k\frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial a_s}+\bar{a}_k\frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial a_s}=0,\\
&-\bar{R}_s-\frac{\partial F_3}{\partial \bar{a}^s}-a^k\frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial \bar{a}^s}+\bar{a}_k\frac{\partial \bar{R}^k}{\partial \bar{a}^s}+\bar{a}_k\frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial \bar{a}^s}=0,\\
&\bar{B}-B-\frac{\partial F_3}{\partial t}-a^k\frac{\partial R_k}{\partial t}-a^k\frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial t}\\
&\quad+\bar{a}_k\frac{\partial \bar{R}^k}{\partial t}+\bar{a}_k\frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial t}=0,
\end{aligned}
\tag{24}
$$

is the generalized canonical transformation of Birkhoff system (1), where $F_3(t,a_s,\bar{a}^s)$ is called the third kind of generating function.

Proof. Let

$$F=F_3(t,a_s,\bar{a}^s)+R_s a^s-\bar{R}^s\bar{a}_s. \tag{25}$$

We have

$$
\begin{aligned}
dF=&\frac{\partial F_3}{\partial t}dt+\frac{\partial F_3}{\partial a_s}da_s+\frac{\partial F_3}{\partial \bar{a}^s}d\bar{a}^s+R_s da^s\\
&+a^s\left(\frac{\partial R_s}{\partial t}dt+\frac{\partial R_s}{\partial a^k}da^k+\frac{\partial R_s}{\partial a_k}da_k\right)-\bar{R}^s d\bar{a}_s\\
&-\bar{a}_s\left(\frac{\partial \bar{R}^s}{\partial t}dt+\frac{\partial \bar{R}^s}{\partial \bar{a}^k}d\bar{a}^k+\frac{\partial \bar{R}^s}{\partial \bar{a}_k}d\bar{a}_k\right).
\end{aligned}
\tag{26}
$$

Since a_s, $\bar{a}^s$ $(s = 1, 2, \cdots, n)$ are independent variables, we have

$$da^k = \frac{\partial a^k}{\partial t}dt + \frac{\partial a^k}{\partial a_j}da_j + \frac{\partial a^k}{\partial \bar{a}^j}d\bar{a}^j,$$
$$d\bar{a}_k = \frac{\partial \bar{a}_k}{\partial t}dt + \frac{\partial \bar{a}_k}{\partial a_j}da_j + \frac{\partial \bar{a}_k}{\partial \bar{a}^j}d\bar{a}^j. \quad (27)$$

Substituting formula (26) into equation (9), and considering the relation (27), we get

$$\left(R^s - \frac{\partial F_3}{\partial a_s} - a^k\frac{\partial R_k}{\partial a_s} - a^k\frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial a_s} + \bar{a}_k\frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial a_s}\right)da_s$$
$$+ \left(-\bar{R}_s - \frac{\partial F_3}{\partial \bar{a}^s} - a^k\frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial \bar{a}^s} + \bar{a}_k\frac{\partial \bar{R}^k}{\partial \bar{a}^s} + \bar{a}_k\frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial \bar{a}^s}\right)d\bar{a}^s$$
$$+ \left(\bar{B} - B - \frac{\partial F_3}{\partial t} - a^k\frac{\partial R_k}{\partial t} - a^k\frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial t}\right.$$
$$\left. + \bar{a}_k\frac{\partial \bar{R}^k}{\partial t} + \bar{a}_k\frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial t}\right)dt = 0. \quad (28)$$

By the independence of da_s and $d\bar{a}^s$, we get the results easily. The theorem is proved.

For the Hamilton system (15), equation (24) give

$$q_s = -\frac{\partial F_3}{\partial p_s},$$
$$\bar{p}_s = -\frac{\partial F_3}{\partial \bar{q}_s}, \quad (29)$$
$$\bar{H} = H + \frac{\partial F_3}{\partial t}.$$

So, from Theorem 7, we have the following corollary.

Corollary 8. *If we take the old variables p_s and the new variables $\bar{q}_s$ $(s = 1, 2, \cdots, n)$ as 2n independent variables, the transformation determined by equation (29) is the canonical transformation of Hamilton system (15), where $F_3(t, p_s, \bar{q}_s)$ is called the third kind of generating function.*

Theorem 9. *If the old variables a_s and the new variables $\bar{a}_s$ $(s = 1, 2, \cdots, n)$ are regarded as 2n independent variables, namely, the generating function is taken as $F_4(t, a_s, \bar{a}_s)$, then the transformation determined by the following equations*

$$R^s - \frac{\partial F_4}{\partial a_s} - a^k\frac{\partial R_k}{\partial a_s} - a^k\frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial a_s} + \bar{a}^k\frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial a_s} = 0,$$
$$-\bar{R}^s - \frac{\partial F_4}{\partial \bar{a}_s} - a^k\frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial \bar{a}_s} + \bar{a}^k\frac{\partial \bar{R}_k}{\partial \bar{a}_s} + \bar{a}^k\frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial \bar{a}_s} = 0,$$
$$\bar{B} - B - \frac{\partial F_4}{\partial t} - a^k\frac{\partial R_k}{\partial t} - a^k\frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial t}$$
$$+ \bar{a}^k\frac{\partial \bar{R}_k}{\partial t} + \bar{a}^k\frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial t} = 0 \quad (30)$$

is the generalized canonical transformation of Birkhoff system (1), where $F_4(t, a_s, \bar{a}_s)$ is called the fourth kind of generating function.

Proof. Let

$$F = F_4(t, a_s, \bar{a}_s) + R_s a^s - \bar{R}_s \bar{a}^s. \quad (31)$$

We have

$$dF = \frac{\partial F_4}{\partial t}dt + \frac{\partial F_4}{\partial a_s}da_s + \frac{\partial F_4}{\partial \bar{a}_s}d\bar{a}_s + R_s da^s$$
$$+ a^s\left(\frac{\partial R_s}{\partial t}dt + \frac{\partial R_s}{\partial a^k}da^k + \frac{\partial R_s}{\partial a_k}da_k\right) - \bar{R}_s d\bar{a}^s \quad (32)$$
$$- \bar{a}^s\left(\frac{\partial \bar{R}_s}{\partial t}dt + \frac{\partial \bar{R}_s}{\partial \bar{a}^k}d\bar{a}^k + \frac{\partial \bar{R}_s}{\partial \bar{a}_k}d\bar{a}_k\right).$$

Since a_s, $\bar{a}_s$ $(s = 1, 2, \cdots, n)$ are independent variables, we have

$$da^k = \frac{\partial a^k}{\partial t}dt + \frac{\partial a^k}{\partial a_j}da_j + \frac{\partial a^k}{\partial \bar{a}_j}d\bar{a}_j,$$
$$d\bar{a}^k = \frac{\partial \bar{a}^k}{\partial t}dt + \frac{\partial \bar{a}^k}{\partial a_j}da_j + \frac{\partial \bar{a}^k}{\partial \bar{a}^j}d\bar{a}^j. \quad (33)$$

Substituting formula (32) into equation (9), and considering the relation (33), we get

$$\left(R^s - \frac{\partial F_4}{\partial a_s} - a^k\frac{\partial R_k}{\partial a_s} - a^k\frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial a_s} + \bar{a}^k\frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial a_s}\right)da_s$$
$$+ \left(-\bar{R}^s - \frac{\partial F_4}{\partial \bar{a}_s} - a^k\frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial \bar{a}_s} + \bar{a}^k\frac{\partial \bar{R}_k}{\partial \bar{a}_s} + \bar{a}^k\frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial \bar{a}_s}\right)d\bar{a}_s$$
$$+ \left(\bar{B} - B - \frac{\partial F_4}{\partial t} - a^k\frac{\partial R_k}{\partial t} - a^k\frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial t}\right.$$
$$\left. + \bar{a}^k\frac{\partial \bar{R}_k}{\partial t} + \bar{a}^k\frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial t}\right)dt = 0. \quad (34)$$

By the independence of da_s and $d\bar{a}_s$, we get the results easily. The theorem is proved.

For the Hamilton system (15), equation (30) gives

$$\begin{aligned} q_s &= -\frac{\partial F_4}{\partial p_s}, \\ \bar{q}_s &= \frac{\partial F_4}{\partial \bar{p}_s}, \\ \bar{H} &= H + \frac{\partial F_4}{\partial t}. \end{aligned} \tag{35}$$

So, from Theorem 9, we have the following corollary.

Corollary 10. *If we take the old variables p_s and the new variables $\bar{p}_s$ $(s = 1, 2, \cdots, n)$ as 2n independent variables, the transformation determined by equation (35) is the canonical transformation of Hamilton system (15), where $F_4(t, p_s, \bar{p}_s)$ is called the fourth kind of generating function.*

Theorem 11. *If the old variables a^s and a_s $(s = 1, 2, \cdots, n)$ are regarded as 2n independent variables, namely, the generating function is taken as $F_5(t, a^s, a_s)$, then the transformation determined by the following equations*

$$\begin{aligned} &R_s - \frac{\partial F_5}{\partial a^s} + \bar{a}^k \frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial a^s} + \bar{a}^k \frac{\partial \bar{R}_k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial a^s} \\ &\quad + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial a^s} + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial a^s} = 0, \\ &R^s - \frac{\partial F_5}{\partial a_s} + \bar{a}^k \frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial a_s} + \bar{a}^k \frac{\partial \bar{R}_k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial a_s} \\ &\quad + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial a_s} + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial a_s} = 0, \\ &\bar{B} - B - \frac{\partial F_5}{\partial t} + \bar{a}^k \frac{\partial \bar{R}_k}{\partial t} + \bar{a}^k \frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial t} + \bar{a}^k \frac{\partial \bar{R}_k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial t} \\ &\quad + \bar{a}_k \frac{\partial \bar{R}^k}{\partial t} + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial t} + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial t} = 0 \end{aligned} \tag{36}$$

is the generalized canonical transformation of Birkhoff system (1), where $F_5(t, a^s, a_s)$ is called the fifth kind of generating function.

Proof. Let

$$F = F_5(t, a^s, a_s) - \bar{R}_s \bar{a}^s - \bar{R}^s \bar{a}_s. \tag{37}$$

We have

$$\begin{aligned} dF &= \frac{\partial F_5}{\partial t} dt + \frac{\partial F_5}{\partial a^s} da^s + \frac{\partial F_5}{\partial a_s} da_s - \bar{R}_s d\bar{a}^s \\ &\quad - \bar{a}^s \left(\frac{\partial \bar{R}_s}{\partial t} dt + \frac{\partial \bar{R}_s}{\partial \bar{a}^k} d\bar{a}^k + \frac{\partial \bar{R}_s}{\partial \bar{a}_k} d\bar{a}_k \right) - \bar{R}^s d\bar{a}_s \\ &\quad - \bar{a}_s \left(\frac{\partial \bar{R}^s}{\partial t} dt + \frac{\partial \bar{R}^s}{\partial \bar{a}^k} d\bar{a}^k + \frac{\partial \bar{R}^s}{\partial \bar{a}_k} d\bar{a}_k \right). \end{aligned} \tag{38}$$

Since a^s, a_s $(s = 1, 2, \cdots, n)$ are independent variables, we have

$$\begin{aligned} d\bar{a}^k &= \frac{\partial \bar{a}^k}{\partial t} dt + \frac{\partial \bar{a}^k}{\partial a^j} da^j + \frac{\partial \bar{a}^k}{\partial a_j} da_j, \\ d\bar{a}_k &= \frac{\partial \bar{a}_k}{\partial t} dt + \frac{\partial \bar{a}_k}{\partial a^j} da^j + \frac{\partial \bar{a}_k}{\partial a_j} da_j. \end{aligned} \tag{39}$$

Substituting formula (38) into equation (9), and considering the relation (39), we get

$$\begin{aligned} &\left(R_s - \frac{\partial F_5}{\partial a^s} + \bar{a}^k \frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial a^s} + \bar{a}^k \frac{\partial \bar{R}_k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial a^s} + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial a^s} \right. \\ &\quad \left. + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial a^s} \right) da^s + \left(R^s - \frac{\partial F_5}{\partial a_s} + \bar{a}^k \frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial a_s} \right. \\ &\quad \left. + \bar{a}^k \frac{\partial \bar{R}_k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial a_s} + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial a_s} + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial a_s} \right) da_s \\ &\quad + \left(\bar{B} - B - \frac{\partial F_5}{\partial t} + \bar{a}^k \frac{\partial \bar{R}_k}{\partial t} + \bar{a}^k \frac{\partial \bar{R}_k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial t} + \bar{a}^k \frac{\partial \bar{R}_k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial t} \right. \\ &\quad \left. + \bar{a}_k \frac{\partial \bar{R}^k}{\partial t} + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}^j}\frac{\partial \bar{a}^j}{\partial t} + \bar{a}_k \frac{\partial \bar{R}^k}{\partial \bar{a}_j}\frac{\partial \bar{a}_j}{\partial t} \right) dt = 0. \end{aligned} \tag{40}$$

By the independence of da^s and da_s, we get the results easily. The theorem is proved.

For the Hamilton system (15), equation (36) gives

$$\begin{aligned} p_s &= \frac{\partial F_5}{\partial q_s} - \bar{q}_j \frac{\partial \bar{p}_j}{\partial q_s}, \\ \bar{q}_j \frac{\partial \bar{p}_j}{\partial p_s} &= \frac{\partial F_5}{\partial p_s}, \\ \bar{H} &= H + \frac{\partial F_5}{\partial t} - \bar{q}_j \frac{\partial \bar{p}_j}{\partial t}. \end{aligned} \tag{41}$$

So, from Theorem 11, we have the following corollary.

Corollary 12. *If we take the old variables q_s and p_s $(s = 1, 2, \cdots, n)$ as 2n independent variables, the transformation determined by equation (41) is the canonical transformation of Hamilton system (15), where $F_5(t, q_s, p_s)$ is called the fifth kind of generating function.*

Theorem 13. *If the new variables $\bar{a}^s$ and $\bar{a}_s$ $(s = 1, 2, \cdots, n)$ are regarded as 2n independent variables, namely, the generating function is taken as $F_6(t, \bar{a}^s, \bar{a}_s)$, then the transformation determined by the following equations*

$$\begin{aligned}
&-\bar{R}_s - \frac{\partial F_6}{\partial \bar{a}^s} - a^k \frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial \bar{a}^s} - a^k \frac{\partial R_k}{\partial a_j}\frac{\partial a_j}{\partial \bar{a}^s} - a_k \frac{\partial R^k}{\partial a^j}\frac{\partial a^j}{\partial \bar{a}^s} - a_k \frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial \bar{a}^s} = 0,\\
&-\bar{R}^s - \frac{\partial F_6}{\partial \bar{a}_s} - a^k \frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial \bar{a}_s} - a^k \frac{\partial R_k}{\partial a_j}\frac{\partial a_j}{\partial \bar{a}_s} - a_k \frac{\partial R^k}{\partial a^j}\frac{\partial a^j}{\partial \bar{a}_s} - a_k \frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial \bar{a}_s} = 0,\\
&\bar{B} - B - \frac{\partial F_6}{\partial t} - a^k \frac{\partial R_k}{\partial t} - a^k \frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial t} - a^k \frac{\partial R_k}{\partial a_j}\frac{\partial a_j}{\partial t} - a_k \frac{\partial R^k}{\partial t} - a_k \frac{\partial R^k}{\partial a^j}\frac{\partial a^j}{\partial t} - a_k \frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial t} = 0
\end{aligned} \tag{42}$$

is the generalized canonical transformation of Birkhoff system (1), where $F_6(t, \bar{a}^s, \bar{a}_s)$ is called the sixth kind of generating function.

Proof. Let

$$F = F_6(t, \bar{a}^s, \bar{a}_s) + R_s a^s + R^s a_s. \tag{43}$$

We have

$$\begin{aligned}
dF = {}& \frac{\partial F_6}{\partial t} dt + \frac{\partial F_6}{\partial \bar{a}^s} d\bar{a}^s + \frac{\partial F_6}{\partial \bar{a}_s} d\bar{a}_s + R_s da^s\\
&+ a^s \left(\frac{\partial R_s}{\partial t} dt + \frac{\partial R_s}{\partial a^k} da^k + \frac{\partial R_s}{\partial a_k} da_k\right) + R^s da_s\\
&+ a_s \left(\frac{\partial R^s}{\partial t} dt + \frac{\partial R^s}{\partial a^k} da^k + \frac{\partial R^s}{\partial a_k} da_k\right).
\end{aligned} \tag{44}$$

Since $\bar{a}^s, \bar{a}_s$ $(s = 1, 2, \cdots, n)$ are independent variables, we have

$$\begin{aligned}
da^k &= \frac{\partial a^k}{\partial t} dt + \frac{\partial a^k}{\partial \bar{a}^j} d\bar{a}^j + \frac{\partial a^k}{\partial \bar{a}_j} d\bar{a}_j,\\
da_k &= \frac{\partial a_k}{\partial t} dt + \frac{\partial a_k}{\partial \bar{a}^j} d\bar{a}^j + \frac{\partial a_k}{\partial \bar{a}_j} d\bar{a}_j.
\end{aligned} \tag{45}$$

Substituting formula (44) into equation (9), and considering the relations (45), we get

$$\begin{aligned}
&\left(-\bar{R}_s - \frac{\partial F_6}{\partial \bar{a}^s} - a^k \frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial \bar{a}^s} - a^k \frac{\partial R_k}{\partial a_j}\frac{\partial a_j}{\partial \bar{a}^s} - a_k \frac{\partial R^k}{\partial a^j}\frac{\partial a^j}{\partial \bar{a}^s} - a_k \frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial \bar{a}^s}\right) d\bar{a}^s\\
&+ \left(-\bar{R}^s - \frac{\partial F_6}{\partial \bar{a}_s} - a^k \frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial \bar{a}_s} - a^k \frac{\partial R_k}{\partial a_j}\frac{\partial a_j}{\partial \bar{a}_s} - a_k \frac{\partial R^k}{\partial a^j}\frac{\partial a^j}{\partial \bar{a}_s} - a_k \frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial \bar{a}_s}\right) d\bar{a}_s\\
&+ \left(\bar{B} - B - \frac{\partial F_6}{\partial t} - a^k \frac{\partial R_k}{\partial t} - a^k \frac{\partial R_k}{\partial a^j}\frac{\partial a^j}{\partial t} - a^k \frac{\partial R_k}{\partial a_j}\frac{\partial a_j}{\partial t} - a_k \frac{\partial R^k}{\partial t} - a_k \frac{\partial R^k}{\partial a^j}\frac{\partial a^j}{\partial t} - a_k \frac{\partial R^k}{\partial a_j}\frac{\partial a_j}{\partial t}\right) dt = 0.
\end{aligned} \tag{46}$$

By the independence of $d\bar{a}^s$ and $d\bar{a}_s$, we get the results easily. The theorem is proved.

For the Hamilton system (15), equation (42) gives

$$\begin{aligned}
\bar{p}_s &= -\frac{\partial F_6}{\partial \bar{q}_s} - q_j \frac{\partial p_j}{\partial \bar{q}_s},\\
q_j \frac{\partial p_j}{\partial \bar{p}_s} &= -\frac{\partial F_6}{\partial \bar{p}_s},\\
\bar{H} &= H + \frac{\partial F_6}{\partial t} + q_j \frac{\partial p_j}{\partial t}.
\end{aligned} \tag{47}$$

So, from Theorem 13, we have the following corollary.

Corollary 14. *If we take the new variables $\bar{q}_s$ and $\bar{p}_s$ $(s = 1, 2, \cdots, n)$ as 2n independent variables, the transformation determined by equation (47) is the canonical transformation of Hamilton system (15), where $F_6(t, \bar{q}_s, \bar{p}_s)$ is called the sixth kind of generating function.*

The generating functions $F_1(t, q_s, \bar{q}_s)$, $F_2(t, q_s, \bar{p}_s)$, $F_3(t, p_s, \bar{q}_s)$, and $F_4(t, p_s, \bar{p}_s)$ for Hamilton system (15) are consistent with the classical results [20, 21], while the fifth kind of generating function $F_5(t, q_s, p_s)$ and the sixth kind of generating function $F_6(t, \bar{q}_s, \bar{p}_s)$ and their corresponding canonical transformations (41) and (47) are generally not reported in the classic textbooks, for example, [20, 21].

The application of the theorems given above has two aspects. One is that you can specify the explicit form of any kind of the generating functions. The corresponding generalized canonical transformation can be calculated according to the generating function by using the theorem on implicit functions. Second, if a generalized canonical transformation is specified, the corresponding generating function can be obtained by applying the above transformation formulas.

4. Examples

Example 1. The famous Lane-Emden equation [4, 22] arising in the field of mathematical physics and astrophysics can be expressed as the following Birkhoff system:

$$B = \frac{1}{6}t^2\left(a^1\right)^6 + \frac{1}{2t^2}\left(a^1a^2\right)^2, \quad R_1 = 0, \quad R_2 = -\frac{1}{2}\left(a^1\right)^2. \tag{48}$$

Let us study the generalized canonical transformation of the system.

According to (48), Birkhoff's equations of the system are

$$\begin{aligned} -a^1\dot{a}^2 - t^2\left(a^1\right)^5 - \frac{1}{t^2}a^1\left(a^2\right)^2 &= 0, \\ a^1\dot{a}^1 - \frac{1}{t^2}\left(a^1\right)^2a^2 &= 0. \end{aligned} \tag{49}$$

If we take $F_5 = F_5(t, a^1, a^2)$ as the generating function, by using the transformation formula (36), we have

$$\begin{aligned} &R_1 - \frac{\partial F_5}{\partial a^1} + \bar{a}^1\frac{\partial \bar{R}_1}{\partial \bar{a}^1}\frac{\partial \bar{a}^1}{\partial a^1} + \bar{a}^1\frac{\partial \bar{R}_1}{\partial \bar{a}^2}\frac{\partial \bar{a}^2}{\partial a^1} + \bar{a}^2\frac{\partial \bar{R}_2}{\partial \bar{a}^1}\frac{\partial \bar{a}^1}{\partial a^1} + \bar{a}^2\frac{\partial \bar{R}_2}{\partial \bar{a}^2}\frac{\partial \bar{a}^2}{\partial a^1} = 0, \\ &R_2 - \frac{\partial F_5}{\partial a^2} + \bar{a}^1\frac{\partial \bar{R}_1}{\partial \bar{a}^1}\frac{\partial \bar{a}^1}{\partial a^2} + \bar{a}^1\frac{\partial \bar{R}_1}{\partial \bar{a}^2}\frac{\partial \bar{a}^2}{\partial a^2} + \bar{a}^2\frac{\partial \bar{R}_2}{\partial \bar{a}^1}\frac{\partial \bar{a}^1}{\partial a^2} + \bar{a}^2\frac{\partial \bar{R}_2}{\partial \bar{a}^2}\frac{\partial \bar{a}^2}{\partial a^2} = 0, \\ &\bar{B} - B - \frac{\partial F_5}{\partial t} + \bar{a}^1\frac{\partial \bar{R}_1}{\partial t} + \bar{a}^1\frac{\partial \bar{R}_1}{\partial \bar{a}^1}\frac{\partial \bar{a}^1}{\partial t} + \bar{a}^1\frac{\partial \bar{R}_1}{\partial \bar{a}^2}\frac{\partial \bar{a}^2}{\partial t} + \bar{a}^2\frac{\partial \bar{R}_2}{\partial t} + \bar{a}^2\frac{\partial \bar{R}_2}{\partial \bar{a}^1}\frac{\partial \bar{a}^1}{\partial t} + \bar{a}^2\frac{\partial \bar{R}_2}{\partial \bar{a}^2}\frac{\partial \bar{a}^2}{\partial t} = 0. \end{aligned} \tag{50}$$

Suppose the transformed Birkhoff's functions are

$$\bar{R}_1 = \bar{a}^2, \quad \bar{R}_2 = 0. \tag{51}$$

Substituting (51) into equation (50), we have

$$\begin{aligned} &-\frac{\partial F_5}{\partial a^1} + \bar{a}^1\frac{\partial \bar{a}^2}{\partial a^1} = 0, \\ &-\frac{1}{2}\left(a^1\right)^2 - \frac{\partial F_5}{\partial a^2} + \bar{a}^1\frac{\partial \bar{a}^2}{\partial a^2} = 0, \\ &\bar{B} - \frac{1}{6}t^2\left(a^1\right)^6 - \frac{1}{2t^2}\left(a^1a^2\right)^2 - \frac{\partial F_5}{\partial t} + \bar{a}^1\frac{\partial \bar{a}^2}{\partial t} + \bar{a}^1\frac{\partial \bar{a}^2}{\partial t} = 0. \end{aligned} \tag{52}$$

If we take $F_5 = -(1/2)\left(a^1\right)^2a^2$, from the second equation of (52), we get

$$\bar{a}^2 = f\left(a^1\right), \tag{53}$$

where $f(a^1)$ represents any differentiable function of a^1. Substituting equation (53) and $F_5 = -(1/2)\left(a^1\right)^2a^2$ into the first equation of (52), we get

$$a^1a^2 + \bar{a}^1f'\left(a^1\right) = 0. \tag{54}$$

If we take $f(a^1) = a^1$, then we the following generalized canonical transformation

$$\bar{a}^1 = -a^1a^2, \quad \bar{a}^2 = a^1. \tag{55}$$

According to the third equation of (52), the new Birkhoffian is obtained as

$$\bar{B} = \frac{1}{2t^2}\left(\bar{a}^1\right)^2 + \frac{1}{6}t^2\left(\bar{a}^2\right)^6. \tag{56}$$

From formulas (51) and (56), we get the new Birkhoff's equation as follows:

$$\begin{aligned} -\dot{\bar{a}}^2 - \frac{1}{t^2}\bar{a}^1 &= 0, \\ \dot{\bar{a}}^1 - t^2\left(\bar{a}^2\right)^5 &= 0, \end{aligned} \tag{57}$$

where $\bar{a}^1$ and $\bar{a}^2$ are the new variables. Here, the new equation (57) is simpler than the original equation (49).

Example 2. We now study a nonconservative system [4], whose Birkhoffian and Birkhoff's functions are

$$B = \frac{1}{8}e^{-t}\left(a^1\right)^4 + \frac{\left(a^2\right)^2}{2\left(a^1\right)^2}e^t + \frac{1}{2}a^1a^2, \quad R_1 = a^2, \quad R_2 = 0. \tag{58}$$

Birkhoff's equations of the system can be written as

$$\begin{aligned} -\dot{a}^2 - \frac{1}{2}e^{-t}\left(a^1\right)^3 + \frac{\left(a^2\right)^2}{\left(a^1\right)^3}e^t - \frac{1}{2}a^2 &= 0, \\ \dot{a}^1 - \frac{a^2}{\left(a^1\right)^2}e^t - \frac{1}{2}a^1 &= 0. \end{aligned} \tag{59}$$

According to Birkhoff's equation (59), the second-order differential equation of motion of the system is

$$-e^{-t}\left(a^1\right)^2\ddot{a}^1 - e^{-t}a^1\left(\dot{a}^1\right)^2 + e^{-t}\left(a^1\right)^2\dot{a}^1 - \frac{1}{2}e^{-t}\left(a^1\right)^3 = 0. \tag{60}$$

If we take $F_2(t, a^1, \bar{a}^2)$ as the generating function, by using the transformation formula (18), we have

$$\begin{aligned} a^2 - \frac{\partial F_2}{\partial a^1} + \bar{a}^1\frac{\partial \bar{R}_1}{\partial \bar{a}^1}\frac{\partial \bar{a}^1}{\partial a^1} &= 0, \\ -\bar{R}_2 - \frac{\partial F_2}{\partial \bar{a}^2} + \bar{a}^1\frac{\partial \bar{R}_1}{\partial \bar{a}^2} + \bar{a}^1\frac{\partial \bar{R}_1}{\partial \bar{a}^1}\frac{\partial \bar{a}^1}{\partial \bar{a}^2} &= 0, \\ \bar{B} - \frac{1}{8}e^{-t}\left(a^1\right)^4 - \frac{\left(a^2\right)^2}{2\left(a^1\right)^2}e^t - \frac{1}{2}a^1a^2 & \\ - \frac{\partial F_2}{\partial t} + \bar{a}^1\frac{\partial \bar{R}_1}{\partial t} + \bar{a}^1\frac{\partial \bar{R}_1}{\partial \bar{a}^1}\frac{\partial \bar{a}^1}{\partial t} &= 0. \end{aligned} \tag{61}$$

Suppose new Birkhoff's functions are

$$\begin{aligned} \bar{R}_1 &= \bar{a}^2, \\ \bar{R}_2 &= 0. \end{aligned} \tag{62}$$

Then equation (61) is reduced to

$$\begin{aligned} a^2 - \frac{\partial F_2}{\partial a^1} &= 0, \\ -\frac{\partial F_2}{\partial \bar{a}^2} + \bar{a}^1 &= 0, \\ \bar{B} - \frac{1}{8}e^{-t}\left(a^1\right)^4 - \frac{\left(a^2\right)^2}{2\left(a^1\right)^2}e^t - \frac{1}{2}a^1a^2 - \frac{\partial F_2}{\partial t} &= 0. \end{aligned} \tag{63}$$

If we take the transformation as follows,

$$\begin{aligned} \bar{a}^1 &= \frac{1}{2}e^{-t}\left(a^1\right)^2, \\ a^2 &= e^{-t}a^1\bar{a}^2. \end{aligned} \tag{64}$$

Substitute formula (64) into the first two equations of formula (63), we get

$$F_2 = \frac{1}{2}e^{-t}\left(a^1\right)^2\bar{a}^2. \tag{65}$$

Substituting formula (65) into the third equation of (63), we get

$$\bar{B} = \frac{1}{2}e^t\left(\bar{a}^1\right)^2 + \frac{1}{2}e^{-t}\left(\bar{a}^2\right)^2. \tag{66}$$

From formulas (62) and (66), we obtain the new Birkhoff's equations as follows:

$$\begin{aligned} -\dot{\bar{a}}^2 - \bar{a}^1e^t &= 0, \\ \dot{\bar{a}}^1 - \bar{a}^2e^{-t} &= 0, \end{aligned} \tag{67}$$

where $\bar{a}^1$ and $\bar{a}^2$ are the new variables. Similarly, other forms of generalized canonical transformations given in this paper can be obtained.

5. Conclusions

Birkhoffian mechanics is a natural generalization of Hamiltonian mechanics. It is because Birkhoff systems have many good properties, such as autonomous and semiautonomous Birkhoff systems have a Lie algebraic structure and proper symplectic form and Birkhoff's equations have self-adjoint form, that Birkhoff systems are widely used in physics, mechanics, engineering, and other fields. In addition, the generalized canonical transformation has the property of preserving algebraic and geometric structures, which lays a foundation for the Hamilton-Jacobi method, so it is an important aspect of the integral theory of analytical mechanics. In this paper, we studied the generalized canonical transformations of Birkhoff systems. Our main work consists of three aspects. The first is that we derived the criterion equation (8) or (9) of the generalized canonical transformations of Birkhoff systems. The second is that, according to the selection of $2n$ independent variables, we presented six different forms of generating functions, and the method we constructed generating functions is different from the existing methods, as shown in formula (11), (19), (25), (35), (37), and (43). The third is that, based on the basic identity (9), six kinds of generalized canonical transformation formulas corresponding to the generating functions are derived from the independence of variables, namely, (10), (18), (24), (30), (36), and (42), which are the new results of this paper. The main results are summarized as six theorems. At the end of the paper, we gave two examples to illustrate the validity of the results.

Data Availability

This article has no additional data.

Conflicts of Interest

There are no conflicts of interest regarding this research work.

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