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Research Article

Fixed Point Problems for Nonexpansive Mappings in Bounded Sets of Banach Spaces

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It is well known that nonexpansive mappings do not always have fixed points for bounded sets in Banach space. The purpose of this paper is to establish fixed point theorems of nonexpansive mappings for bounded sets in Banach spaces. We study the existence of fixed points for nonexpansive mappings in bounded sets, and we present the iterative process to approximate fixed points. Some examples are given to support our results.

1. Introduction and Preliminaries

Throughout the paper, we assume that X is a real Banach space and $C \subset X$ is a subset. $T : C \longrightarrow X$ is a mapping, if a point $x* \in C$ such that $Tx* = x*$ we call that $x*$ is a fixed point of T. The set of fixed points of T is denoted by $F(T)$. Let N and N^+ denote the set of natural numbers and the set of positive integers; let Q and R denote the set of rational numbers and the set of real numbers.

We recall that nonexpansive mapping if for every x, $y \in C$ such that $\|Tx - Ty\| \leq \|x - y\|$. A Banach space X is said to have the *fixed point property* if for every closed convex bounded subset $C \subset X$ and for every nonexpansive $T : C \longrightarrow C$, there is a fixed point. Since Browder [1] obtained fixed point theorems for nonexpansive mapping, the fixed point theory of nonexpansive mappings has made great progress. A large number of results are obtained by authors (e.g., see [2–7]). Goebel et al. presented generalized nonexpansive mappings, if for every $x, y \in C$ such that $\|Tx - Ty\| \leq a\|x - y\| + b(\|x - Tx\| + \|Ty - y\|) + c(\|Tx - y\| + \|y - Tx\|)$, where $a + 2b + 2c \leq 1$, $a, b, c \geq 0$. In recent years, generalizations of nonexpansive mappings have received attention, and their fixed point theory has been studied by many authors (see [8–14]).

Amini-Harandi et al. [15] presented (α, β)-nonexpansive mappings and obtained fixed point theorems for (α, β)-nonexpansive mappings in Banach space.

Definition 1. Letting X be a Banach space, we say $T : X \longrightarrow X$ is an (α, β)-nonexpansive mapping, if for every $x, y \in X$, α, $\beta \in R$ we have

$$\begin{aligned}\|Tx - Ty\|^2 \leq{}& \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + \beta\|Tx - x\|^2 \\ &+ \beta\|Ty - y\|^2 + (1 = 2\alpha - 2\beta)\|x - y\|^2.\end{aligned} \tag{1}$$

Theorem 2. *Let X be a Banach space, $C \subset X$ is a bounded subset, $T : C \longrightarrow C$ is an (α, β)-nonexpansive mapping with $\alpha = 0, \beta > 0$, then T has a fixed point.*

Remark 3. Obviously, (α, β)-nonexpansive mappings are more generalized nonexpansive mappings, and many fixed point theorems are extended by Theorem 2, but the conditions of Theorem 2 are inadequate. See Example 1.

Example 1. Letting the set $C = (0, 1]$, take $Tx = (1/2)x$.

We have $T : (0, 1] \longrightarrow (0, 1]$ which is a mapping; moreover,

$$|Tx - Ty|^2 = \frac{1}{4}|x - y|^2 \leq \left|\frac{1}{2}x - y\right|^2 + \left|\frac{1}{2}y - x\right|^2 - |x - y|^2. \tag{2}$$

So, T is a (1,0)-nonexpansive mapping in bounded set (0,1]. Hence, the conditions of Theorem 2 are satisfied, but T has no fixed point. In fact, it is easy that bounded subset

C of conditions of Theorem 2 should be a closed bounded subset C.

Remark 4. If C is a closed bounded set, $T : C \longrightarrow C$ is an (α, β)-nonexpansive mapping, where $\alpha = 0, \beta > 0$. Then, T has no fixed point. See Example 2.

Example 2. Let the set $C = [-1, 1]$,

$$Tx = \begin{cases} 1, & x = 0, \\ -x, & x \neq 0. \end{cases} \tag{3}$$

If $x = 0$ and for every $y \neq 0$, we have

$$\begin{aligned} |T0 - Ty|^2 &= |y + 1|^2 \leq 2 + 2|2y|^2 - 3|0 - y|^2 \\ &= 2|T0 - 0|^2 + 2|Ty - y|^2 - 3|0 - y|^2, \end{aligned} \tag{4}$$

if for all $x, y \in C/\{0\}$, we get

$$\begin{aligned} |Tx - Ty|^2 &= |x - y|^2 \leq 2|2x|^2 + 2|2y|^2 - 3|x - y|^2 \\ &= 2|Tx - x|^2 + 2|Ty - y|^2 - 3|x - y|^2. \end{aligned} \tag{5}$$

So, T is a (0,2)-nonexpansive mapping in [−1,1], which has no fixed point.

Remark 5. For an (α, β)-nonexpansive mapping in mapping, as $\alpha = 0$ and $\beta = 0$, the (α, β)-nonexpansive mapping is the class of nonexpansive mapping. It is important about fixed point theory of the class of nonexpansive mapping, but which is not contained by Theorem 2.

The following examples show that for some nonclosed bounded sets, nonexpansive mappings have fixed points in Banach space, but for some closed bounded sets, nonexpansive mappings have no fixed point.

Example 3. Let the set $C = \{-1, 1\}$, and take $Tx = -x$.

It is easy that $T : C \longrightarrow C$ is a nonexpansive mapping and C is a closed bounded set. But T has not fixed point in C.

Example 4. Let the set $C = [0, 1]$

$$Tx = \begin{cases} \dfrac{1}{4}, & x = 1, \\ \dfrac{7}{8}, & x \neq 1. \end{cases} \tag{6}$$

It is obvious that T is not a nonexpansive mapping in closed bounded set C; however, it is a nonexpansive mapping in nonclosed bounded set $[0, 1)$, and there exists a fixed point $(7/8) \in [0, 1)$.

Example 5. Dirichlet function:

$$D(x) = \begin{cases} 1, & x \in [0, 1] \cap Q, \\ 0, & x \in \dfrac{[0, 1]}{Q}. \end{cases} \tag{7}$$

Obviously, $D : [0, 1] \longrightarrow [0, 1]$ is not a nonexpansive mapping, but D is a nonexpansive mapping in bounded set $[0, 1] \cap Q$, and $1 \in [0, 1] \cap Q$ is a fixed point of D.

Usually, authors study fixed point problems of nonexpansive mappings for closed bounded sets. The above examples show that for some bounded sets, nonexpansive mappings have fixed points in Banach space, and for others of bounded sets, nonexpansive mappings have no fixed point. So it is significant to study fixed point problems of nonexpansive mappings for bounded sets. The goal of this paper is to obtain fixed point theorems of nonexpansive mappings for bounded sets.

2. Main Results

Let l_∞ denote the Banach space of bounded real sequence with the supremum norm. Let ψ be a bounded linear functional on l_∞; ψ is called a Banach limit, if it satisfies $\|\psi\| = \psi(1) = 1$ and $\psi(t_{n+1}) = \psi(t_n)$, $t_n \in l_\infty$. Moreover, suppose ψ is a Banach limit, then the following conclusions are held:

(i) If for all $n \in N$, $s_n \leq t_n$ means $\psi(s_n) \leq \psi(t_n)$, $s_n, t_n \in l_\infty$

(ii) For each $p \in N^+$, $t_n \in l_\infty$, have $\psi(t_n) = \psi(t_{n+p})$

(iii) $\liminf_{n\to\infty} t_n \leq \psi(t_n) \leq \limsup_{n\to\infty} t_n$, $t_n \in l_\infty$

Lemma 6. *[16] Suppose that $\{a_n\}, \{b_n\}$ are two sequences of nonnegative numbers, and $\sum b_n < \infty$, if there exists some number $N_0 \in N$, for all $n \geq N_0$ such that*

$$a_{n+1} \leq a_n + b_n, \tag{8}$$

then $\lim_{n\to\infty} a_n$ exists.

Theorem 7. *Letting X be a Banach space, $C \subset X$ is a bounded subset and $T : C \longrightarrow C$ is a nonexpansive mapping. If the following two conditions are satisfied:*

(i) There exists a sequence $\{s_n\} \subset [0, 1)$ with $\lim_{n\to\infty} s_n = 1$, $\sum |s_{n+p} - s_n|/(1 - s_n) < \infty$, for all $n, p \in N, x \in C$, such that $s_n x \in C$.

(ii) If the sequence $\{x_n\} \subset C$ and $\lim_{k\to\infty} x_{nk} = x*$, we have $x* \in C$, where $k \longrightarrow \infty$ implies $n_k \longrightarrow \infty$.

Then, (1) there is at least a fixed point of T in C; (2) take $x_n = s_{n-1} T x_{n-1}$, then the iterative sequence $\{x_n\}$ approach to a fixed point of T.

Proof. Let $x_0 \in C$, take $x_n = s_{n-1} T x_{n-1}$, where the sequence $\{s_n\} \subset [0,1)$ satisfies the condition (i) of Theorem 7, that is, $\lim_{n\to\infty} s_n = 1$ and $\sum |s_{n+p} - s_n|/(1-s_n) < \infty$, for each n, $p \in N$.

Firstly, we show that there exists a subsequence $\{\chi_{nk}\}$ of $\{\chi_n\}$ which is convergent.

Since $T : C \longrightarrow C$ is a nonexpansive mapping, for each $n, p \in N^+$, we have

$$\begin{aligned}\|x_{n+p} - x_n\| &= \|s_{n+p-1} T x_{n+p-1} - s_{n-1} T x_{n-1}\| \\ &= \|s_{n+p-1} T x_{n+p-1} - s_{n-1} T x_{n+p-1} \\ &\quad + s_{n-1} T x_{n+p-1} - s_{n-1} T x_{n-1}\| \\ &\le |s_{n+p-1} - s_{n-1}| \|T x_{n+p-1}\| \\ &\quad + s_{n-1} \|T x_{n+p-1} - T x_{n-1}\| \\ &\le |s_{n+p-1} - s_{n-1}| \|T x_{n+p-1}\| \\ &\quad + s_{n-1} \|x_{n+p-1} - x_{n-1}\|. \end{aligned} \tag{9}$$

According to the Banach limit, we have

$$\begin{aligned}\psi(\|x_{n+p} - x_n\|) &\le \psi(|s_{n+p-1} - s_{n-1}| \|T x_{n+p-1}\| \\ &\quad + s_{n-1} \|x_{n+p-1} - x_{n-1}\|) \\ &= |s_{n+p-1} - s_{n-1}| \psi(\|T x_{n+p-1}\|) \\ &\quad + s_{n-1} \psi(\|x_{n+p-1} - x_{n-1}\|) \\ &= |s_{n+p-1} - s_{n-1}| \psi(\|T x_{n+p-1}\|) \\ &\quad + s_{n-1} \psi(\|x_{n+p} - x_n\|). \end{aligned} \tag{10}$$

Also, for C is a bounded set, which means that there exists a constant $M > 0$ such that for every $x \in C$, we have $\|x\| \le M$. Thus, by the inequality (10), we have

$$\begin{aligned}\psi(\|x_{n+p} - x_n\|) &\le \frac{|s_{n+p-1} - s_{n-1}|}{1 - s_{n-1}} \psi(\|T x_{n+p-1}\|) \\ &\le \frac{|s_{n+p-1} - s_{n-1}|}{1 - s_{n-1}} \psi(M). \end{aligned} \tag{11}$$

From the condition (ii) of Theorem 7, we have

$$\lim_{n\to\infty} \frac{|s_{n+p-1} - s_{n-1}|}{1 - s_{n-1}} = 0. \tag{12}$$

Let $n \to \infty$ in the inequality (11), by (12), we get

$$\lim_{n\to\infty} \psi(\|x_{n+p} - x_n\|) = 0, \tag{13}$$

that is

$$\lim_{n\to\infty} \psi(\|x_m - x_n\|) = 0, \quad m, n \in N. \tag{14}$$

So

$$0 \le \liminf_{n\to\infty} \|x_m - x_n\| \le \psi(\|x_m - x_n\|) = 0, \tag{15}$$

which implies that there exists a monotonically increasing sequence $\{n_k\}$ (that is, $n_1 \le n_2 \le n_3 \cdots$) such that

$$\lim_{k\to\infty} \|x_{m_k} - x_{n_k}\| = 0, \quad m, n \in N. \tag{16}$$

It means that $\{x_{n_k}\}$ is a Cauchy sequence; thus, there exists some x^* such that

$$\lim_{k\to\infty} x_{n_k} = x^*. \tag{17}$$

From the condition (ii) of Theorem 7, we have $x^* \in C$. Next, we show that x_n is convergent to x^* and $x^* \in F(T)$. From (9), we have

$$\|x_{n+p} - x_n\| \le \|x_{n+p-1} - x_{n-1}\| + |s_{n+p-1} - s_{n-1}| \|T x_{n+p-1}\|. \tag{18}$$

Also, by the condition (i) of Theorem 7, we may obtain $\sum \|s_{n+p} - s_n| < \infty$. Thus, applying Lemma 6 in (18), $\lim_{n\to\infty} \| x_{n+p} - x_n\|$ exists; moreover, from (16), $\lim_{k\to\infty} \|x_{m_k} - x_{n_k}\| = 0$, $m, n \in N$. So it implies

$$\lim_{n\to\infty} \|x_m - x_n\| = 0, \quad m, n \in N. \tag{19}$$

Hence, $\{x_n\}$ is a Cauchy sequence, also from (17), that is, $\lim_{k\to\infty} x_{n_k} = x^*$. Therefore,

$$\lim_{k\to\infty} x_n = x^*. \tag{20}$$

Since T is nonexpansive, then

$$\begin{aligned}\|x^* - T x^*\| &= \|x^* - x_n + x_n - T x^*\| \\ &\le \|x^* - x_n\| + \|x_n - T x^*\| \\ &= \|x^* - x_n\| + \|s_{n-1} T x_{n-1} \\ &\quad - s_{n-1} T x^* - (1 - s_{n-1}) T x^*\| \\ &\le \|x^* - x_n\| + s_{n-1} \|T x_{n-1} - T x^*\| \\ &\quad + (1 - s_{n-1}) \|T x^*\| \le \|x^* - x_n\| \\ &\quad + s_{n-1} \|x_{n-1} - x^*\| + (1 - s_{n-1}) \|T x^*\|. \end{aligned} \tag{21}$$

According to (20) and the condition (i) of Theorem 7, letting $n \to \infty$ in the inequality (21), we have $\|x^* - T x^*\| = 0$; that is, $T x^* = x^*$; therefore, x^* is a fixed point of T.

Corollary 8. *Let X be a Banach space, $C \subset X$ is a bounded closed subset, and $T : C \to C$ is a nonexpansive mapping. If there exists the sequence $\{s_n\} \subset [0,1)$ with $\lim_{n\to\infty} s_n = 1$, $\sum |s_{n+p} - s_n|/(1 - s_n) < \infty$, for all $n, p \in N, x \in C$, such that $s_n x \in C$.*

Then, (1) there is at least a fixed point of T in C; (2) taking $x_n = s_{n-1} T x_{n-1}$, then the iterative sequence $\{x_n\}$ approaches to a fixed point of T.

Let us use the following example to support the above results.

Example 6. Let the set $C = \{\pm(1/n) | n \in N\} \cup \{0\}$

$$Tx = \begin{cases} 0, & x = 0, \\ -x, & x \neq 0. \end{cases} \tag{22}$$

If every $x \in C, y = 0$, we have

$$|Tx - T0| = |x| = |x - 0|. \tag{23}$$

Moreover, if every $x, y \in \{\pm(1/n) | n \in N\}$, we get

$$|Tx - Ty| = |x - y|. \tag{24}$$

Obviously, C is a bounded closed set and $T : C \longrightarrow C$ is a nonexpansive mapping. And for all $x \in C$, that is, for all n, then $x = \pm(1/n)$, or $x = 0$, so there exists the sequence $s_n = n/(n+1)$, which satisfies the following three conditions:

(a) $\lim_{n\to\infty} s_n = 1$

(b) $\sum |s_{n+p} - s_n|/(1 - s_n) < \infty, n, p \in N$

(c) $s_n x \in C$

Thus, all conditions of Corollary 8 are satisfied; therefore, T has at least a fixed point. Next, we use the iterative approximation methods to obtain a fixed point of T. Now, we take $x_0 = 1 \in C$ and $x_{n+1} = s_n T x_n$, then we easily obtain

$$x_1 = -\frac{1}{2}, x_2 = \frac{1}{3}, \cdots, x_n = (-1)^{n-1} \frac{1}{n}. \tag{25}$$

Letting $n \longrightarrow \infty$, we have $\lim_{n\to\infty} x_n = 0$, in which 0 is a fixed point of T.

The following results are extensions of fixed point theorems of nonexpansive mappings for convex bounded sets.

Theorem 9. *Let X be a Banach space, $C \subset X$ is a bounded subset, and $T : C \longrightarrow C$ is a nonexpansive mapping. If the following two conditions are satisfied:*

(i) There exists some $x_0 \in C$ and the sequence $\{s_n\} \subset [0, 1)$ with $\lim_{n\to\infty} s_n = 1$ and $\sum |s_{n+p} - s_n|/(1 - s_n) < \infty$, for all $n, p \in N, x \in C$, such that $(1 - s_n)x_0 + s_n x \in C$

(ii) If the subsequence $\{x_{n_k}\} \subset \{x_{n_k}\} \subset C$ and $\lim_{k\to\infty} x_{n_k} = x^*$, we have$x^* \in C$, where $k \longrightarrow \infty$ implies $n_k \longrightarrow \infty$

Then, (1) there is at least a fixed point of T in C; (2) taking iterative sequence $x_n = (1 - s_{n-1})x_0 + s_{n-1} T x_{n-1}$, the sequence $\{x_n\}$ approaches to a fixed point of T.

Proof. From the condition (i) of Theorem 9, there exists some $x_0 \in C$ and the sequence $s_n \in [0, 1]$, for each $x \in C$, such that $(1 - s_n)x_0 + s_n x \in C$, where the sequence $\{s_n\} \subset [0, 1)$ satisfies $\lim_{n\to\infty} s_n = 1$ and $\sum |s_{n+p} - s_n|/(1 - s_n) < \infty$, for all $n, p \in N$. Now, we take $x_n = (1 - s_{n-1})x_0 + s_{n-1} T x_{n-1}$, which implies that $x_n \in C$; moreover, since $T : C \longrightarrow C$ is a nonexpansive mapping, thus we have

$$\begin{aligned} \|x_{n+p} - x_n\| &= \big\| (1 - s_{n+p-1})x_0 + s_{n+p-1} T x_{n+p-1} \\ &\quad - (1 - s_{n-1})x_0 - s_{n-1} T x_{n-1} \big\| \\ &= \big\| (s_{n+p-1} - s_{n-1})x_0 + s_{n+p-1} T x_{n+p-1} \\ &\quad - s_{n-1} T x_{n+p-1} + s_{n-1} T x_{n+p-1} - s_{n-1} T x_{n-1} \big\| \\ &\leq |s_{n+p-1} - s_{n-1}| \|x_0\| + |s_{n+p-1} - s_{n-1}| \|T x_{n+p-1}\| \\ &\quad + s_{n-1} \|T x_{n+p-1} - T x_{n-1}\| \\ &\leq |s_{n+p-1} - s_{n-1}| (\|x_0\| + \|T x_{n+p-1}\|) \\ &\quad + s_{n-1} \|x_{n+p-1} - x_{n-1}\|. \end{aligned} \tag{26}$$

Based on the Banach limit, we have

$$\begin{aligned} \psi(\|x_{n+p} - x_n\|) &\leq \psi\big(|s_{n+p-1} - s_{n-1}| (\|x_0\| + \|T x_{n+p-1}\|) \\ &\quad + s_{n-1} \|x_{n+p-1} - x_{n-1}\|\big) \\ &= |s_{n+p-1} - s_{n-1}| \psi(\|x_0\| + \|T x_{n+p-1}\|) \\ &\quad + s_{n-1} \psi(\|x_{n+p-1} - x_{n-1}\|) \\ &= |s_{n+p-1} - s_{n-1}| \psi(\|x_0\| + \|T x_{n+p-1}\|) \\ &\quad + s_{n-1} \psi(\|x_{n+p} - x_n\|). \end{aligned} \tag{27}$$

Since C is a bounded set, then there exists a constant $M > 0$ for each $x \in C$ such that $\|x\| \leq M$. So by (27), we may get

$$\begin{aligned} \psi(\|x_{n+p} - x_n\|) &\leq \frac{|s_{n+p-1} - s_{n-1}|}{1 - s_{n-1}} \psi((\|x_0\| + \|T x_{n+p-1}\|)) \\ &\leq \frac{|s_{n+p-1} - s_{n-1}|}{1 - s_{n-1}} \psi(2M). \end{aligned} \tag{28}$$

From the condition (ii) of Theorem 9, we have

$$\lim_{n\to\infty} \frac{|s_{n+p-1} - s_{n-1}|}{1 - s_{n-1}} = 0, \tag{29}$$

Hence, as $n \longrightarrow \infty$ in (28), we get

$$\lim_{n\to\infty} \psi(\|x_{n+p} - x_n\|) = 0, \tag{30}$$

that is

$$\lim_{n\to\infty} \psi(\|x_m - x_n\|) = 0, \quad m, n \in N. \tag{31}$$

This implies that

$$0 \leq \liminf_{n\to\infty} \|x_m - x_n\| \leq \psi(\|x_m - x_n\|) = 0. \tag{32}$$

Hence, there exists a monotonically increasing sequence $\{n_k\}$ (that is, $n_1 \le n_2 \le n_3 \cdots$) such that

$$\lim_{n\to\infty} \|x_{m_k} - x_{n_k}\| = 0, \quad m, n \in N. \tag{33}$$

It means that $\{x_{n_k}\}$ is a Cauchy sequence, so there exists some x^* such that

$$\lim_{k\to\infty} x_{n_k} = x^*. \tag{34}$$

From the condition (ii) of Theorem 9, we have $x^* \in C$.

Next, we show that x_n is convergent to x^* and x^* is a fixed point of T.

From (26), we have

$$\begin{aligned}\|x_{n+p} - x_n\| \le \|x_{n+p-1} - x_{n-1}\| + |s_{n+p-1} - s_{n-1}| \\ \cdot \left(\|Tx_{n+p-1}\| + \|x_0\|\right).\end{aligned} \tag{35}$$

Also, by the condition (i) of Theorem 9, we obtain $\sum\| s_{n+p} - s_n| < \infty$. Thus, applying Lemma 6 in (35), we have $\lim_{n\to\infty}\|x_{n+p} - x_n\|$; moreover, from (33), $\lim_{k\to\infty}\|x_{m_k} - x_{nk}\| = 0$, $m, n \in N$. It implies

$$\lim_{n\to\infty} \|x_m - x_n\| = 0, \quad m, n \in N. \tag{36}$$

So $\{x_n\}$ is a Cauchy sequence, also from (34), that is, $\lim_{k\to\infty} x_{n_k} = x^*$. Therefore,

$$\lim_{k\to\infty} x_n = x^*. \tag{37}$$

By the triangle inequality, we have

$$\begin{aligned}\|x^* - Tx^*\| &= \|x^* - x_n + x_n - Tx^*\| \\ &\le \|x^* - x_n\| + \|x_n - Tx^*\| \\ &\le \|x^* - x_n\| + \|s_{n-1}Tx_{n-1} + (1 - s_{n-1})x_0 \\ &\quad - s_{n-1}Tx^* - (1 - s_{n-1})Tx^*\| \\ &\le \|x^* - x_n\| + s_{n-1}\|Tx_{n-1} - Tx^*\| \\ &\quad + (1 - s_{n-1})\|Tx^* - x_0\| \\ &\le \|x^* - x_n\| + s_{n-1}\|x_{n-1} - x^*\| \\ &\quad + (1 - s_{n-1})\|Tx^* - x_0\|.\end{aligned} \tag{38}$$

Applying (37) and the condition (i) of Theorem 9 and letting $n \longrightarrow \infty$ in (38), we have $\|x^* - Tx^*\| = 0$. That is, $Tx^* = x^*$; therefore, x^* is a fixed point of T.

Corollary 10. *Let X be a Banach space, $C \subset X$ is a bounded closed subset, and $T : C \longrightarrow C$ is a nonexpansive mapping. If there exists some $x_0 \in C$ and sequence $\{s_n\} \subset [0, 1)$ with $\lim_{n\to\infty} s_n = 1$, and $\sum|s_{n+p} - s_n|/(1 - s_n) < \infty$, for all $n, p \in N$, $x \in C$, such that*

$$(1 - s_n)x_0 + s_n x \in C. \tag{39}$$

Then, (1) there is at least a fixed point of T in C; (2) taking iterative sequence $x_n = (1 - s_{n-1})x_0 + s_{n-1}Tx_{n-1}$, the sequence $\{x_n\}$ approaches to a fixed point of T.

Definition 11. Letting X be a Banach space and $C \subset X$ be a subset, if there exists a $x_0 \in C$ such that for every $y \in C$, $\lambda \in [0, 1]$, we have

$$(1 - \lambda)x_0 + \lambda y \in C, \tag{40}$$

then, C is a called a star-shaped set.

Obviously, a star-shaped set satisfies the conditions of which set C of Corollary 10, so we have

Corollary 12. *Let X be a Banach space, $C \subset X$ is a bounded closed star-shaped set, and $T : C \longrightarrow C$ is a nonexpansive mapping. Then, T has at least a fixed point in C.*

Corollary 13. *Let X be a Banach space, $C \subset X$ is a bounded closed convex set, and $T : C \longrightarrow C$ is a nonexpansive mapping. Then, T has at least a fixed pont in C.*

Let us give the example to support our results.

Example 7. Let the set $C = [0, 1] \cap Q$, take $Tx = -x + 1$, $x \in C$.

It means that C be a bounded set and $T : C \longrightarrow C$ is a nonexpansive mapping. Now, we take $x_0 = 1$ and a sequence $s_n = n/(n + 1)$, we have $s_n \in [0, 1)$, and for all n, $p \in N^+$ such that

$$\begin{gathered}\lim_{n\to\infty} s_n = 1, \\ \sum \frac{|s_{n+p} - s_n|}{1 - s_n} < \infty.\end{gathered} \tag{41}$$

Moreover, for all $x \in C$, we have $(1 - s_n)x_0 + s_n x \in C$.

As an overview, all conditions of Theorem 9 are satisfied. So T has a fixed point.

Next, we take

$$x_{n+1} = (1 - s_n)x_0 + s_n Tx_n, \tag{42}$$

then we have

$$x_1 = \frac{1}{2}, x_2 = \frac{2}{3}, x_3 = \frac{1}{2}, x_4 = \frac{3}{5}, x_5 = \frac{1}{2}, x_6 = \frac{4}{7}, \cdots \tag{43}$$

So we have

$$x_n = \begin{cases} \dfrac{1}{2}, & n = 2k + 1, k \in N, \\ \dfrac{n+2}{2n+2}, & n = 2k, k \in N. \end{cases} \tag{44}$$

It implies $x_n \in C$, and $\lim_{n\to\infty} x_n = 1/2 \in C$. It is clear that $1/2$ is a fixed point of T.

Therefore, a fixed point of T be iteratively approximated by the sequence $\{x_n\}$.

From Remark 2.1 of the reference [15], a subset C of Banach space X, $T : C \longrightarrow X$ is an (α, β)-nonexpansive mapping; if $\alpha + \beta < 0$, then T has a fixed point. So by Theorem 7 and Theorem 9, we easily obtain the following two results.

Theorem 14. *Let X be a Banach space, $C \subset X$ is a bounded subset, and $T : C \longrightarrow C$ is an (α, β)-nonexpansive mapping with $\alpha \leq 0, \beta \leq 0$. If the following two conditions are satisfied:*

(i) There exists a sequence $\{s_n\} \subset [0, 1)$ with $\lim_{n\to\infty} s_n = 1$ and $\sum |s_{n+p} - s_n|/(1 - s_n) < \infty$, for all $n, p \in N$, $x \in C$ such that $s_n x \in C$

(ii) If the sequence $\{x_n\} \subset C$ and $\lim_{k\to\infty} x_{n_k} = x^*$, we have $x^* \in C$, where $k \longrightarrow \infty$ implies $n_k \longrightarrow \infty$

Then, (1) there is at least a fixed point of T in C; (2) taking $x_n = s_{n-1} T x_{n-1}$, the iterative sequence $\{x_n\}$ approaches to a fixed point of T.

Theorem 15. *Let X be a Banach space, $C \subset X$ is a bounded subset, and $T : C \longrightarrow C$ is an (α, β)-nonexpansive mapping with $\alpha \leq 0, \beta \leq 0$. If the following two conditions are satisfied:*

(i) There exists some $x_0 \in C$ and the sequence $\{s_n\} \subset [0, 1)$ with $\lim_{n\to\infty} s_n = 1$ and $\sum |s_{n+p} - s_n|/(1 - s_n) < \infty$, for all $n, p \in N, x \in C$, such that $(1 - s_n)x_0 + s_n x \in C$

(ii) If the subsequence $\{x_{n_k}\} \subset \{x_{n_k}\} \subset C$ and $\lim_{k\to\infty} x_{n_k} = x^*$, we have $x^* \in C$, where $k \longrightarrow \infty$ implies $n_k \longrightarrow \infty$

Then, (1) there is at least a fixed point of T in C; (2) taking iterative sequence $x_n = (1 - s_{n-1})x_0 + s_{n-1} T x_{n-1}$, the sequence $\{x_n\}$ approaches to a fixed point of T.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that he has no competing interests.

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