Research Article

Existence Results for a Class of Coupled Hilfer Fractional Pantograph Differential Equations with Nonlocal Integral Boundary Value Conditions

Karim Guida [ID], Khalid Hilal, and Lahcen Ibnelazyz [ID]

Sultan Moulay Slimane University, BP 523 Beni Mellal, Morocco

Correspondence should be addressed to Karim Guida; guida.karim@gmail.com and Lahcen Ibnelazyz; ibnelazyzlahcen@gmail.com

This paper deals with the existence and uniqueness of solutions for a new class of coupled systems of Hilfer fractional pantograph differential equations with nonlocal integral boundary conditions. First of all, we are going to give some definitions that are necessary for the understanding of the manuscript; second of all, we are going to prove our main results using the fixed point theorems, namely, Banach's contraction principle and Krasnoselskii's fixed point theorem; in the end, we are giving two examples to illustrate our results.

1. Introduction

Differential equations play a very important role in the understanding of qualitative features of many phenomenon and processes in different areas and practical fields. A lot of works have been done concerning these equations in the recent years for their importance in applied sciences; for more details about differential equations and their applications, we refer the readers to [1–7].

A more general way to describe natural differential equations is through fractional calculus. Fractional calculus has attracted many researchers recently; this branch of mathematics is used in the modelling of many problems in various fields, like biology, physics, control theory, and economics; for more details, we give the following classical references [8–13].

There are many different definitions of fractional integrals and derivatives in the literature [12]; the most popular definitions are the Riemann-Liouville and the Caputo fractional derivatives. A generalization of these derivatives was introduced by Hilfer in [14], known by the Hilfer fractional derivative of order α and type $\beta \in [0, 1]$, and we can find the Riemann-Liouville fractional derivative when $\beta = 0$, and the Caputo fractional derivative when $\beta = 1$. Fractional differen-

tial equations involving the Hilfer fractional derivative have many applications, see [15–18] and the references therein.

On the other hand, another important class of differential equations are called pantograph equations, which are a special class of delay differential equations arising in deterministic situations and are of the form

$$\begin{cases} g'(t) = kg(t) + \lg(\lambda t), t \in [0, b], b > 0, 0 < \lambda < 1, \\ g(0) = g_0. \end{cases} \quad (1)$$

They are also called equations with proportional delays. This class of differential equations was not properly investigated under fractional derivatives. Pantograph is a device used in drawing and scaling. But, recently, this device is being used in electric trains [19, 20]. Many researchers studied the pantograph differential equations and their applications in many sciences such as biology, physics, economics, and electrodynamics. For more details, please see [21, 22].

In [23], the authors studied nonlocal boundary value problems for the Hilfer fractional derivative. Initial value problems involving Hilfer fractional derivatives were studied in [24–26]. Initial value problems for pantograph equations with the Hilfer fractional derivative were studied in [22, 27].

To the best of our knowledge, there is no work involving systems of integral boundary value problems for pantograph equations with the Hilfer fractional derivative. Thus, the objective of this work is to introduce a new class of coupled systems of Hilfer fractional differential pantograph equations with nonlocal integral boundary conditions of the form

$$\begin{cases} {}^H D^{\alpha_1,\beta_1} x(t) = f_1(t, x(t), x(\lambda_1 t), y(t)) & t \in [a, b], \\ {}^H D^{\alpha_2,\beta_2} y(t) = f_2(t, x(t), y(t), y(\lambda_2 t)) & t \in [a, b], \\ x(a) = 0, A_1 x(b) + B_1 I^{\delta_1} x(\mu_1) = C_1, & \mu_1 \in (a, b), \\ y(a) = 0, A_2 y(b) + B_2 I^{\delta_2} y(\mu_2) = C_2, & \mu_2 \in (a, b), \end{cases} \tag{2}$$

where ${}^H D^{\alpha_1,\beta_1}$, ${}^H D^{\alpha_2,\beta_2}$ are the Hilfer fractional derivatives of order α_1 and α_2, $1 < \alpha_1, \alpha_2 < 2$ and parameter β_1, β_2, $0 < \beta_1, \beta_2 < 1$, respectively, $f_1, f_2 : [a, b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are two continuous functions; $I^{\delta_1}, I^{\delta_2}$ are the Riemann-Liouville fractional integrals of order δ_1 and δ_2, respectively, $a \geq 0$, $A_1, A_2, B_1, B_2, C_1, C_2 \in \mathbb{R}$, and $0 < \lambda_1, \lambda_2 < 1$.

This paper is organized as follows: we first give some definitions and notions that will be used throughout the work, after that we will establish the existence and uniqueness results by means of the fixed point theorems, and last but not least, we will give some examples that illustrate the results.

2. Preliminaries and Notations

In this section, we introduce some notations and definitions related to fractional calculus that we will use throughout this paper.

We first define the following spaces:

$C([a, b], \mathbb{R})$ with $a \geq 0$ is the Banach space of all continuous functions from $[a, b]$ to $\mathbb{R}$, $L(a, b)$ is the space of Lebesgue integrable functions on a finite closed interval $[a, b](b > a)$ of the real line $\mathbb{R}$, and $AC^k[a, b]$ is the space of real-valued functions $f(t)$ which have continuous derivatives up to order $k - 1$ on $[a, b]$ such that $f^{(k-1)}(t)$ belongs to the space of absolutely continuous functions $AC[a, b]$.

Definition 1. (see [8, 11]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $f : [a, \infty) \to \mathbb{R}$, is defined by

$$I^\alpha = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} f(s) ds, \tag{3}$$

provided the right-hand side exists on (a, ∞).

Definition 2 (see [8, 11]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function f, is defined by

$$^{RL} D^\alpha f(t) := D^n I^{n-\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t - s)^{n-\alpha-1} f(s) ds, \tag{4}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the real number α, provided the right-hand side is pointwise defined on (a, ∞).

Definition 3 (see [8, 11]). The Caputo fractional derivative of order $\alpha > 0$ of a continuous function f, is defined by

$$^C D^\alpha f(t) := I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - s)^{n-\alpha-1} \\ \cdot \left(\frac{d}{dt}\right)^n f(s) ds, n - 1 < \alpha < n, \tag{5}$$

provided the right-hand side is pointwise defined on (a, ∞).

Definition 4 (see [14]). The Hilfer fractional derivative of order α and parameter β of a function f is given by

$$^H D^{\alpha,\beta} f(t) = I^{\beta(n-\alpha)} D^n I^{(1-\beta)(n-\alpha)} f(t), \tag{6}$$

where $n - 1 < \alpha < n$, $0 \leq \beta \leq 1$, $t > a$, and $D = d/dt$.

Remark 5. When $\beta = 0$, the Hilfer fractional derivative becomes the Riemann-Liouville fractional derivative, while when $\beta = 1$, the Hilfer fractional derivative becomes the Caputo fractional derivative.

The following lemma gives a composition between the Riemann-Liouville fractional integral operator and the Hilfer fractional derivative operator.

Lemma 6 (see [15]). *Let* $f \in L(a, b)$, $n - 1 < \alpha < n$, $n \in \mathbb{N}$, $0 < \beta < 1$, $I^{(n-\alpha)(1-\beta)} f \in AC^k[a, b]$; *then, we have*

$$\left(I^{\alpha H} D^{\alpha,\beta} f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t - a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k - (n - \alpha)(1 - \beta) + 1)} \lim_{t \to a^+} \\ \cdot \frac{d^k}{dt^k} \left(I^{(1-\beta)(n-\alpha)} f\right)(t). \tag{7}$$

Now, we give a lemma which is the solution of a variant of the integral boundary value coupled systems (2).

Lemma 7. *Let $a \geq 0$, for $i = 1, 2$, $1 < \alpha_i < 2$, $\gamma_i = \alpha_i + 2\beta_i - \alpha_i\beta_i$, $h_i \in C([a, b], \mathbb{R})$, and*

$$\Lambda_i = \frac{A_i(b-a)^{\gamma_i-1}}{\Gamma(\gamma_i)} + \frac{B_i(\mu_i-a)^{\gamma_i+\delta_i-1}}{\Gamma(\gamma_i+\delta_i)} \neq 0. \quad (8)$$

Then, the following problem

$$\begin{cases} {}^H D^{\alpha_1,\beta_1} x(t) = h_1(t), & t \in [a, b], \\ {}^H D^{\alpha_2,\beta_2} y(t) = h_2(t), & t \in [a, b], \\ x(a) = 0, A_1 x(b) + B_1 I^{\delta_1} x(\mu_1) = C_1, & \mu_1 \in (a, b), \\ y(a) = 0, A_2 y(b) + B_2 I^{\delta_2} y(\mu_2) = C_2, & \mu_2 \in (a, b), \end{cases} \quad (9)$$

is equivalent to the system of equations:

$$\begin{cases} x(t) = I^{\alpha_1} h_1(t) + \dfrac{(t-a)^{\gamma_1-1}}{\Lambda_1 \Gamma(\gamma_1)} \left[C_1 - A_1 I^{\alpha_1} h_1(b) - B_1 I^{\alpha_1+\delta_1} h_1(\mu_1) \right], \\ y(t) = I^{\alpha_2} h_2(t) + \dfrac{(t-a)^{\gamma_2-1}}{\Lambda_2 \Gamma(\gamma_2)} \left[C_2 - A_2 I^{\alpha_2} h_2(b) - B_2 I^{\alpha_2+\delta_2} h_2(\mu_2) \right]. \end{cases} \quad (10)$$

Proof. Let us assume that (x, y) is a solution of problem (9). Applying the fractional integrals I^{α_1} and I^{α_2} on both sides of the equations in (9) and using Lemma 6, we obtain

$$\begin{cases} x(t) = c_{01} \dfrac{(t-a)^{-(2-\alpha_1)(1-\beta_1)}}{\Gamma(1-(2-\alpha_1)(1-\beta_1))} + c_{11} \dfrac{(t-a)^{1-(2-\alpha_1)(1-\beta_1)}}{\Gamma(2-(2-\alpha_1)(1-\beta_1))} + I^{\alpha_1} h_1(t), \\ y(t) = c_{02} \dfrac{(t-a)^{-(2-\alpha_2)(1-\beta_2)}}{\Gamma(1-(2-\alpha_2)(1-\beta_2))} + c_{12} \dfrac{(t-a)^{1-(2-\alpha_2)(1-\beta_2)}}{\Gamma(2-(2-\alpha_2)(1-\beta_2))} + I^{\alpha_2} h_2(t). \end{cases} \quad (11)$$

Since for $i = 1, 2$, $(1 - \beta_i)(2 - \alpha_1) = 2 - \gamma_i$, we obtain

$$\begin{cases} x(t) = c_{01} \dfrac{(t-a)^{\gamma_1-2}}{\Gamma(\gamma_1-1)} + c_{11} \dfrac{(t-a)^{\gamma_1-1}}{\Gamma(\gamma_1)} + I^{\alpha_1} h_1(t), \\ y(t) = c_{02} \dfrac{(t-a)^{\gamma_2-2}}{\Gamma(\gamma_2-1)} + c_{12} \dfrac{(t-a)^{\gamma_2-1}}{\Gamma(\gamma_2)} + I^{\alpha_2} h_2(t), \end{cases} \quad (12)$$

where for $i = 1, 2$, c_{0i}, c_{1i} are real constants.

Since we have $x(0) = y(0) = 0$ and $\lim_{t\to a}(t-a)^{\gamma_1-2} = \lim_{t\to a}(t-a)^{\gamma_2-2} = \infty$, we can obtain that $c_{01} = c_{02} = 0$. Then, we get

$$\begin{cases} x(t) = c_{11} \dfrac{(t-a)^{\gamma_1-1}}{\Gamma(\gamma_1)} + I^{\alpha_1} h_1(t), \\ y(t) = c_{12} \dfrac{(t-a)^{\gamma_2-1}}{\Gamma(\gamma_2)} + I^{\alpha_2} h_2(t), \end{cases} \quad (13)$$

from the conditions: $A_1 x(b) + B_1 I^{\delta_1} x(\mu_1) = C_1$ and $A_2 y(b) + B_2 I^{\delta_2} y(\mu_2) = C_2$; we can find that

$$\begin{cases} c_{11} = \dfrac{1}{\Lambda_1} \left[C_1 - A_1 I^{\alpha_1} h_1(b) - B_1 I^{\alpha_1+\delta_1} h_1(\mu_1) \right], \\ c_{12} = \dfrac{1}{\Lambda_2} \left[C_2 - A_2 I^{\alpha_2} h_2(b) - B_2 I^{\alpha_2+\delta_2} h_2(\mu_2) \right]. \end{cases} \quad (14)$$

By substituting the values above in (13), we obtain

$$\begin{cases} x(t) = I^{\alpha_1} h_1(t) + \dfrac{(t-a)^{\gamma_1-1}}{\Lambda_1 \Gamma(\gamma_1)} \left[C_1 - A_1 I^{\alpha_1} h_1(b) - B_1 I^{\alpha_1+\delta_1} h_1(\mu_1) \right], \\ y(t) = I^{\alpha_2} h_2(t) + \dfrac{(t-a)^{\gamma_2-1}}{\Lambda_2 \Gamma(\gamma_2)} \left[C_2 - A_2 I^{\alpha_2} h_2(b) - B_2 I^{\alpha_2+\delta_2} h_2(\mu_2) \right], \end{cases} \quad (15)$$

which is the solution to problem (9).

We get the converse by direct computations. This ends the proof.

3. Main Results

The space $X = \{x : x(t) \in C([a, b], \mathbb{R})\}$ endowed with the norm $\|x\| = \sup \{|x(t)|, t \in [a, b]\}$ and the space $Y = \{y : y(t) \in C([a, b], \mathbb{R})\}$ endowed with the norm $\|y\| = \sup \{|y(t)|, t \in [a, b]\}$ are two Banach spaces. Moreover, the product space $(X \times Y, \|(x, y)\|)$ is a Banach space with the norm $\|(x, y)\| = \|x\| + \|y\|$.

In view of Lemma 7, we define the operator $T : X \times Y \to X \times Y$ by

$$T(x, y)(t) = (U(x, y)(t), V(x, y)(t)), \quad (16)$$

where

$$\begin{cases} U(x,y)(t) = \dfrac{(t-a)^{\gamma_1-1}}{\Lambda_1 \Gamma(\gamma_1)}\left[C_1 - A_1 I^{\alpha_1} f_1(s, x(s), x(\lambda_1 s), y(s))(b) - B_1 I^{\alpha_1+\delta_1} f_1(s, x(s), x(\lambda_1 s), y(s))(\mu_1) \right] + I^{\alpha_1} f_1(s, x(s), x(\lambda_1 s), y(s))(t), \\[2mm] V(x,y)(t) = \dfrac{(t-a)^{\gamma_2-1}}{\Lambda_2 \Gamma(\gamma_2)}\left[C_2 - A_2 I^{\alpha_2} f_2(s, x(s), y(s), y(\lambda_2 s))(b) - B_2 I^{\alpha_2+\delta_2} f_2(s, x(s), y(s), y(\lambda_2 s))(\mu_2) \right] + I^{\alpha_2} f_2(s, x(s), y(s), y(\lambda_2 s))(t). \end{cases}$$

$$(17)$$

We should note that problem (2) has a solution (x, y) if and only if the operator T has a fixed point.

In what is coming, for convenience, we set the following: For $i = 1, 2$,

$$\begin{aligned} \Omega_i = \frac{(b-a)^{\gamma_i-1}}{|\Lambda_i|\Gamma(\gamma_i)} & \left[|A_i| \frac{(b-a)^{\alpha_i}}{\Gamma(\alpha_i+1)} \right. \\ & \left. + |B_i| \frac{(\mu_i-a)^{\alpha_i+\delta_i}}{\Gamma(\alpha_i+\delta_i+1)} \right] + \frac{(b-a)^{\alpha_i}}{\Gamma(\alpha_i+1)}. \end{aligned} \tag{18}$$

We are going to prove the existence and uniqueness as well as the existence results for problem (2) by using the Banach contraction principle and Krasnoselskii's fixed point theorem.

The first result is based on Banach's fixed point theorem.

Theorem 8. *Assume that*

(H_1) *For all* $x, \bar{x}, y, \bar{y} \in \mathbb{R}, t \in [a, b]$ *there exist* $L_{f_1}, L_{f_2} > 0$ *such that*

$$\begin{aligned} & |f_1(t, x(t), x(\lambda_1 t), y(t)) - f_1(t, \bar{x}(t), \bar{x}(\lambda_1 t), \bar{y}(t))| \\ & \quad \le L_{f_1}[2|x-\bar{x}| + |y-\bar{y}|], \end{aligned}$$

$$\begin{aligned} & |f_2(t, x(t), y(t), y(\lambda_2 t)) - f_1(t, \bar{x}(t), \bar{y}(t), \bar{y}(\lambda_2 t))| \\ & \quad \le L_{f_2}[|x-\bar{x}| + 2|y-\bar{y}|], \end{aligned} \tag{19}$$

in addition, if we have

$$3\left(L_{f_1}\Omega_1 + L_{f_2}\Omega_2\right) < 1, \tag{20}$$

where Ω_1, Ω_2 *are defined by (18); then, the boundary coupled systems (2) has a unique solution* (x^*, y^*) *on* $[a, b]$.

Proof. We transform the boundary value coupled systems (2) into a fixed point problem. Applying the Banach contraction

mapping principle, we show that T defined by (16) and (17) has a unique fixed point. We let $\sup_{t\in(a,b)}|f_1(t, 0, 0, 0)| = M_1$ and $\sup_{t\in(a,b)}|f_2(t, 0, 0, 0)| = M_2$ and choose

$$r_0 \ge \frac{M_1\Omega_1 + M_2\Omega_2 + |C_1|(b-a)^{\gamma_1-1}/|\Lambda_1|\Gamma(\gamma_1) + |C_2|(b-a)^{\gamma_2-1}/|\Lambda_2|\Gamma(\gamma_2)}{1 - 3\left(L_{f_1}\Omega_1 + L_{f_2}\Omega_2\right)}. \tag{21}$$

We first show that $TB_{r_0} \subset B_{r_0}$, where $B_{r_0} = \{(x, y) \in X \times Y : \|(x, y)\| \le r_0\}$.

For any $(x, y) \in B_{r_0}$, we have

$$\begin{aligned} |U(x,y)(t)| &\le \sup_{t\in[a,b]}\left[\frac{(t-a)^{\gamma_1-1}}{\Lambda_1 \Gamma(\gamma_1)} \left(|C_1| + |A_1| I^{\alpha_1}|f_1(s, x(s), x(\lambda_1 s), y(s))|(b) \right.\right. \\ & \quad \left. + |B_1| I^{\alpha_1+\delta_1}|f_1(s, x(s), x(\lambda_1 s), y(s))|(\mu_1)\right) \\ & \quad \left. + I^{\alpha_1}|f_1(s, x(s), x(\lambda_1 s), y(s))|(t) \right] \\ &\le \left[\frac{(b-a)^{\gamma_1-1}}{\Lambda_1 \Gamma(\gamma_1)} \left(|C_1| + |A_1| I^{\alpha_1}(|f_1(s, x(s), x(\lambda_1 s), y(s)) \right.\right. \\ & \quad - f_1(s, 0, 0, 0)| + |f_1(s, 0, 0, 0)|)(b) \\ & \quad + |B_1| I^{\alpha_1+\delta_1}(|f_1(s, x(s), x(\lambda_1 s), y(s)) - f_1(s, 0, 0, 0)| \\ & \quad + |f_1(s, 0, 0, 0)|)(\mu_1)) + I^{\alpha_1}(|f_1(s, x(s), x(\lambda_1 s), y(s)) \\ & \quad \left.\left. - f_1(s, 0, 0, 0)| + |f_1(s, 0, 0, 0)|)(b) \right] \right. \\ &\le \left(L_{f_1}(2\|x\| + \|y\|) + M_1 \right)\left[\frac{(b-a)^{\gamma_1-1}}{\Lambda_1 \Gamma(\gamma_1)} \right. \\ & \quad \left. \cdot \left(|A_1| \frac{(b-a)^{\alpha_1}}{\Gamma(\alpha_1+1)} + |B_1| \frac{(\mu_1-a)^{\alpha_1+\delta_1}}{\Gamma(\alpha_1+\delta_1+1)} \right) + \frac{(b-a)^{\alpha_1}}{\Gamma(\alpha_1+1)} \right] \\ & \quad + |C_1| \frac{(b-a)^{\gamma_1-1}}{\Lambda_1 \Gamma(\gamma_1)} \le \left(3r_0 L_{f_1} + M_1 \right)\Omega_1 + |C_1| \frac{(b-a)^{\gamma_1-1}}{\Lambda_1 \Gamma(\gamma_1)}. \end{aligned}$$

$$(22)$$

Similarly, we get

$$|V(x,y)(t)| \le \left(3r_0 L_{f_2} + M_2 \right)\Omega_2 + |C_2| \frac{(b-a)^{\gamma_2-1}}{\Lambda_2 \Gamma(\gamma_2)}. \tag{23}$$

Finally,

$$|T(x, y)(t)| \leq \left(3r_0 L_{f_1} + M_1\right)\Omega_1 + \left(3r_0 L_{f_2} + M_2\right)\Omega_2$$
$$+ |C_1|\frac{(b-a)^{\gamma_1 - 1}}{\Lambda_1 \Gamma(\gamma_1)} + |C_2|\frac{(b-a)^{\gamma_2 - 1}}{\Lambda_2 \Gamma(\gamma_2)} \leq r_0,$$
(24)

which implies that $TB_{r_0} \subset B_{r_0}$.

Next, we show that the operator T is a contraction; we let $(x, y), (\bar{x}, \bar{y}) \in X \times Y$; then for $t \in [a, b]$, we have

$$|U(x, y)(t) - U(\bar{x}, \bar{y})(t)|$$
$$\leq \left[\frac{(b-a)^{\gamma_1 - 1}}{\Lambda_1 \Gamma(\gamma_1)} (|A_1| I^{\alpha_1} | f_1(s, x(s), x(\lambda_1 s), y(s)) \right.$$
$$- f_1(s, \bar{x}(s), \bar{x}(\lambda_1 s), \bar{y}(s))|(b)$$
$$+ |B_1| I^{\alpha_1 + \delta_1} |f_1(s, x(s), x(\lambda_1 s), y(s))$$
$$- f_1(s, \bar{x}(s), \bar{x}(\lambda_1 s), \bar{y}(s))|(\mu_1)$$
$$+ I^{\alpha_1} |f_1(s, x(s), x(\lambda_1 s), y(s))$$
$$\left. - f_1(s, \bar{x}(s), \bar{x}(\lambda_1 s), \bar{y}(s))|(b) \right]$$
$$\leq L_{f_1} \left[\frac{(b-a)^{\gamma_1 - 1}}{\Lambda_1 \Gamma(\gamma_1)} \left(|A_1| \frac{(b-a)^{\alpha_1}}{\Gamma(\alpha_1 + 1)} + |B_1| \frac{(\mu_1 - a)^{\alpha_1 + \delta_1}}{\Gamma(\alpha_1 + \delta_1 + 1)} \right) \right.$$
$$\left. + \frac{(b-a)^{\alpha_1}}{\Gamma(\alpha_1 + 1)} \right] (2\|x - \bar{x}\| + \|y - \bar{y}\|)$$
$$\leq L_{f_1} \Omega_1 (2\|x - \bar{x}\| + \|y - \bar{y}\|) \leq 3 L_{f_1} \Omega_1 (\|x - \bar{x}\| + \|y - \bar{y}\|),$$
(25)

with a similar method, we also get

$$|V(x, y)(t) - V(\bar{x}, \bar{y})(t)| \leq 3 L_{f_2} \Omega_2 (\|x - \bar{x}\| + \|y - \bar{y}\|). \quad (26)$$

Finally, we can obtain

$$|T(x, y)(t) - T(\bar{x}, \bar{y})(t)| \leq 3 \left(L_{f_1} \Omega_1 + L_{f_2} \Omega_2 \right) (\|x - \bar{x}\| + \|y - \bar{y}\|).$$
(27)

And since, $3(L_{f_1} \Omega_1 + L_{f_2} \Omega_2) < 1$, then the operator T is a contraction.

Therefore, we conclude by Banach's contraction mapping principle that T has a fixed point which is the unique solution (x^*, y^*) of problem (2). The proof is completed.

Next, we present a result based on Krasnoselskii's fixed point theorem.

Theorem 9. *Let $f : [a, b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies (H_1).*
In addition, we assume that
$(H_2)|f_1(t, x(t), x(\lambda_1 t), y(t))| \leq 2C_{f_1}|x| + D_{f_1}|y| + M_{f_1}$,

$$|f_2(t, x(t), y(t), y(\lambda_2 t))| \leq C_{f_2}|x| + 2D_{f_2}|y| + M_{f_2}, \quad (28)$$

for all $x, y \in \mathbb{R}$, $0 < \lambda_1, \lambda_2 < 1$ and $t \in [a, b]$, with $C_{f_1}, C_{f_2}, D_{f_1}, D_{f_2}, M_{f_1}, M_{f_2}$ are positive real numbers.
Then, problem (2) has at least a solution (x, y) on $[a, b]$, provided

$$\sum_{i=1}^{2} 3L_{f_i} \frac{(b-a)^{\gamma_i - 1}}{|\Lambda_i| \Gamma(\gamma_i)} \left[|A_i| \frac{(b-a)^{\alpha_i}}{\Gamma(\alpha_i + 1)} + |B_i| \frac{(\mu_i - a)^{\alpha_i + \delta_i}}{\Gamma(\alpha_i + \delta_i + 1)} \right] < 1.$$
(29)

Proof. We set $\varepsilon_0 \geq ((M_{f_1} \Omega_1 + |C_1|((b-a)^{\gamma_1 - 1}/|\Lambda_1| \Gamma(\gamma_1)))/(1 - 2C_{f_1} - D_{f_1})) + ((M_{f_2} \Omega_2 + |C_2|((b-a)^{\gamma_2 - 1}/|\Lambda_2| \Gamma(\gamma_2)))/(1 - C_{f_2} - 2D_{f_2}))$, where Ω_1, Ω_2 are defined in (18), and we consider $B_{\varepsilon_0} = \{(x, y) \in X \times Y : \|(x, y)\| \leq \varepsilon_0\}$.

We define the operator T by $T(x, y)(t) = (U(x, y)(t), V(x, y)(t))$, for any $(x, y) \in B_{\varepsilon_0}$, and $t \in [a, b]$ where

$$U(x, y)(t) = \frac{(t-a)^{\gamma_1 - 1}}{\Lambda_1 \Gamma(\gamma_1)} [C_1 - A_1 I^{\alpha_1} f_1(s, x(s), x(\lambda_1 s), y(s))(b)$$
$$- B_1 I^{\alpha_1 + \delta_1} f_1(s, x(s), x(\lambda_1 s), y(s))(\mu_1)]$$
$$+ I^{\alpha_1} f_1(s, x(s), x(\lambda_1 s), y(s))(t),$$

$$V(x, y)(t) = \frac{(t-a)^{\gamma_2 - 1}}{\Lambda_2 \Gamma(\gamma_2)} [C_2 - A_2 I^{\alpha_2} f_2(s, x(s), y(s), y(\lambda_2 s))(b)$$
$$- B_2 I^{\alpha_2 + \delta_2} f_2(s, x(s), y(s), y(\lambda_2 s))(\mu_2)]$$
$$+ I^{\alpha_2} f_2(s, x(s), y(s), y(\lambda_2 s))(t),$$
(30)

by splitting the two operators above, we have

$$U_1(x, y)(t) = I^{\alpha_1} f_1(s, x(s), x(\lambda_1 s), y(s))(t),$$

$$U_2(x, y)(t) = \frac{(t-a)^{\gamma_1 - 1}}{\Lambda_1 \Gamma(\gamma_1)} [C_1 - A_1 I^{\alpha_1} f_1(s, x(s), x(\lambda_1 s), y(s))(b)$$
$$- B_1 I^{\alpha_1 + \delta_1} f_1(s, x(s), x(\lambda_1 s), y(s))(\mu_1)],$$

$$V_1(x, y)(t) = I^{\alpha_2} f_2(s, x(s), y(s), y(\lambda_2 s))(t),$$

$$V_2(x, y)(t) = \frac{(t-a)^{\gamma_2-1}}{\Lambda_2 \Gamma(\gamma_2)} \left[C_2 - A_2 I^{\alpha_2} f_2(s, x(s), y(s), y(\lambda_2 s))(b) \right. \\ \left. - B_2 I^{\alpha_2+\delta_2} f_2(s, x(s), y(s), y(\lambda_2 s))(\mu_2) \right].$$

(31)

This upcoming part of the proof requires us to rewrite the operator T as

$$T(x, y)(t) = T_1(x, y)(t) + T_2(x, y)(t),$$

(32)

where

$$T_1(x, y)(t) = (U_1(x, y)(t), V_1(x, y)(t)),$$
$$T_2(x, y)(t) = (U_2(x, y)(t), V_2(x, y)(t)).$$

(33)

For any $(x, y) \in B_{\varepsilon_0}$, we have

$$|T_1(x, y)(t)| \le \sup_{t \in [a,b]} \left[\frac{(t-a)^{\gamma_1-1}}{\Lambda_1 \Gamma(\gamma_1)} \right. \\ \cdot \left(|C_1| + |A_1| I^{\alpha_1} |f_1(s, x(s), x(\lambda_1 s), y(s))|(b) \right. \\ \left. + |B_1| I^{\alpha_1+\delta_1} |f_1(s, x(s), x(\lambda_1 s), y(s))|(\mu_1) \right) \\ \left. + I^{\alpha_1} |f_1(s, x(s), x(\lambda_1 s), y(s))|(t) \right] \\ \le \left(2C_{f_1} \|x\| + D_{f_1} \|y\| + M_{f_1} \right) \Omega_1 + |C_1| \frac{(b-a)^{\gamma_1-1}}{|\Lambda_1| \Gamma(\gamma_1)},$$

(34)

Similarly, we have

$$|T_2(x, y)(t)| \le \left(C_{f_2} \|x\| + 2D_{f_2} \|y\| + M_{f_2} \right) \Omega_2 \\ + |C_2| \frac{(b-a)^{\gamma_2-1}}{|\Lambda_2| \Gamma(\gamma_2)},$$

(35)

Since $\varepsilon_0 \ge ((M_{f_1} \Omega_1 + |C_1|((b-a)^{\gamma_1-1}/|\Lambda_1| \Gamma(\gamma_1)))/(1 - 2C_{f_1} - D_{f_1})) + (M_{f_2} \Omega_2 + |C_2|((b-a)^{\gamma_2-1}/|\Lambda_2| \Gamma(\gamma_2)))/(1 - C_{f_2} - 2D_{f_2})$.

Then, this shows that $T(x, y) \in B_{\varepsilon_0}$.

Next, we show that T_2 is a contraction mapping.

For all $(x, y), (\bar{x}, \bar{y}) \in X \times Y$, and for $t \in [a, b]$, we have

$$|U_2(x, y)(t) - U_2(\bar{x}, \bar{y})(t)| \\ \le 3L_{f_1} \frac{(b-a)^{\gamma_1-1}}{|\Lambda_1| \Gamma(\gamma_1)} \left[|A_1| \frac{(b-a)^{\alpha_1}}{\Gamma(\alpha_1+1)} \right. \\ \left. + |B_1| \frac{(\mu_1-a)^{\alpha_1+\delta_1}}{\Gamma(\alpha_1+\delta_1+1)} \right] (\|x - \bar{x}\| + \|y - \bar{y}\|).$$

(36)

Similarly,

$$|V_2(x, y)(t) - V_2(\bar{x}, \bar{y})(t)| \\ \le 3L_{f_2} \frac{(b-a)^{\gamma_2-1}}{|\Lambda_2| \Gamma(\gamma_2)} \left[|A_2| \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} \right. \\ \left. + |B_2| \frac{(\mu_2-a)^{\alpha_2+\delta_2}}{\Gamma(\alpha_2+\delta_2+1)} \right] (\|x - \bar{x}\| + \|y - \bar{y}\|).$$

(37)

It is easy to see, using (29), that T_2 is a contraction mapping.

The continuity of the functions f_1 and f_2 implies the continuity of the operator T_1. In addition, T_1 is uniformly bounded on B_{ε_0} as

$$|T_1(x, y)(t)| \le |U_1(x, y)(t)| + |V_1(x, y)(t)| \\ \le \frac{(b-a)^{\alpha_1}}{\Gamma(\alpha_1+1)} \left(2C_{f_1} \|x\| + D_{f_1} \|y\| + M_{f_1} \right) \\ + \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} \left(C_{f_2} \|x\| + 2D_{f_2} \|y\| + M_{f_2} \right) \le \varepsilon_0.$$

(38)

Now we prove the compactness of the operator T_1.

For any $t_1, t_2 \in [a, b]$, with $t_1 < t_2$, we have

$$|U_1(x, y)(t_2) - U_1(x, y)(t_1)| \\ = \frac{1}{\Gamma(\alpha_1)} \left| \int_a^{t_1} \left[(t_2 - s)^{\alpha_1-1} - (t_1 - s)^{\alpha_1-1} \right] f_1 \right. \\ \times (t, x(s), x(\lambda_1 s), y(s)) ds \\ \left. + \int_{t_1}^{t_2} (t_2 - s)^{\alpha_1-1} f_1(t, x(s), x(\lambda_1 s), y(s)) ds \right| \\ \le \frac{\left(2C_{f_1} \|x\| + D_{f_1} \|y\| + M_{f_1} \right)}{\Gamma(\alpha_1+1)} \\ \times \left[2(t_2 - t_1)^{\alpha_1} + |(t_2 - a)^{\alpha_1} - (t_1 - a)^{\alpha_1}| \right],$$

(39)

which tends to zero as $t_2 - t_1 \to 0$ independently of (x, y).

Similarly, we have $|V_1(x, y)(t_2) - V_2(x, y)(t_1)| \to 0$ as $t_2 - t_1 \to 0$.

Finally, the operator T_1 is equicontinuous, which means that T_1 is relatively compact on B_{ε_0}. Hence, by Arzela-Ascoli theorem, T_1 is compact on B_{ε_0}. Thus, all the assumptions of Krasnoselskii's fixed point theorem are satisfied. So,

the boundary value coupled systems (2) have at least a solution (x, y) on $[a, b]$.

4. Examples

Example 1. We first consider the following problem:

$$\begin{cases} {}^H D^{3/2,2/5} x(t) = \dfrac{t^3 + \sin|x(t)| + \cos|y(t)|}{90} + \dfrac{\sin|x(t/3)|}{100}, & t \in \left[\dfrac{1}{2}, \dfrac{5}{2}\right], \\[2mm] {}^H D^{3/2,2/5} y(t) = \dfrac{\cos|x(t) + y(t)|}{50} + \dfrac{|y(t/3)|}{(t^3 + 5)^4}, & t \in \left[\dfrac{1}{2}, \dfrac{5}{2}\right], \\[2mm] x\left(\dfrac{1}{2}\right) = 0, & \dfrac{2}{3}x\left(\dfrac{5}{2}\right) + \dfrac{3}{4}I^{1/2}x\left(\dfrac{3}{2}\right) = \dfrac{4}{5}, \\[2mm] y\left(\dfrac{1}{2}\right) = 0, & \dfrac{2}{3}y\left(\dfrac{5}{2}\right) + \dfrac{3}{4}I^{1/2}y\left(\dfrac{3}{2}\right) = \dfrac{4}{5}, \end{cases} \tag{40}$$

where, $\alpha_1 = \alpha_2 = 3/2$, $\beta_1 = \beta_2 = 2/5$, $\lambda_1 = \lambda_2 = 1/3$, $a = 1/2$, $b = 5/2$, $A_1 = A_2 = 2/3$, $B_1 = B_2 = 3/4$, $\delta_1 = \delta_2 = 1/2$, $\mu_1 = \mu_2 = 3/2$, and $C_1 = C_2 = 4/5$.

The setting yields $\gamma_1 = \gamma_2 = 17/10$, $\Lambda_1 = \Lambda_2 = 1.872599119$, and $\Omega_1 = \Omega_2 = 4.129461300$.

Now, we put

$$f_1(t, x(t), x(\lambda_1 t), y(t)) = \dfrac{t^3 + \sin|x(t)| + \cos|y(t)|}{90} + \dfrac{\sin|x(t/3)|}{100},$$

$$f_2(t, x(t), y(t), y(\lambda_2 t)) = \dfrac{\cos|x(t) + y(t)|}{50} + \dfrac{|y(t/3)|}{(t^3 + 5)^4}, \tag{41}$$

which satisfies (H_1) as

$$|f_1(t, x(t), x(\lambda_1 t), y(t)) - f_1(t, \bar{x}(t), \bar{x}(\lambda_1 t), \bar{y}(t))| \\ \leq 0.0111[2|x - \bar{x}| + |y - \bar{y}|],$$

$$|f_2(t, x(t), y(t), y(\lambda_2 t)) - f_1(t, \bar{x}(t), \bar{y}, \bar{y}(\lambda_2 t))| \\ \leq 0.020[|x - \bar{x}| + 2|y - \bar{y}|], \tag{42}$$

Setting: $L_{f_1} = 0.0111$, $L_{f_2} = 0.020$, we obtain $3(L_{f_1}\Omega_1 + L_{f_2}\Omega_2) \simeq 0.3852787392 < 1$, which shows that inequality (20) is verified. Then, by Theorem 8, we can conclude that problem (40) has a unique solution (x^*, y^*) on $[1/2, 5/2]$.

We now consider the following problem:

$$\begin{cases} {}^H D^{5/3,1/2} x(t) = \dfrac{\sin|x(t)| + \sin|x(t/4)|}{100} + \dfrac{\sin|y(t)|}{90} + e^{1-3t}, & t \in \left[\dfrac{1}{3}, \dfrac{5}{3}\right], \\[2mm] {}^H D^{5/3,1/2} y(t) = \dfrac{\sin|x(t)| + \sin|y(t)|}{90} + \dfrac{\sin|y(t/4)|}{100} + e^{1-3t}, & t \in \left[\dfrac{1}{3}, \dfrac{5}{2}\right], \\[2mm] x\left(\dfrac{1}{3}\right) = 0, & \dfrac{3}{5}x\left(\dfrac{5}{3}\right) + \dfrac{1}{4}I^{3/2}x\left(\dfrac{2}{3}\right) = \dfrac{3}{4}, \\[2mm] y\left(\dfrac{1}{3}\right) = 0, & \dfrac{3}{5}y\left(\dfrac{5}{3}\right) + \dfrac{1}{4}I^{3/2}y\left(\dfrac{2}{3}\right) = \dfrac{3}{4}, \end{cases} \tag{43}$$

where, $\alpha_1 = \alpha_2 = 5/3$, $\beta_1 = \beta_2 = 1/2$, $\lambda_1 = \lambda_2 = 1/4$, $a = 1/3$, $b = 5/3$, $A_1 = A_2 = 3/5$, $B_1 = B_2 = 1/4$, $\delta_1 = \delta_2 = 3/2$, $\mu_1 = \mu_2 = 2/3$, and $C_1 = C_2 = 3/4$.

The setting yields $\gamma_1 = \gamma_2 = 11/6$, $\Lambda_1 = \Lambda_2 = 0.8175877260$.

The function f_1, f_2 verify (H_1) as

$$|f_1(t, x(t), x(\lambda_1 t), y(t)) - f_1(t, \bar{x}(t), \bar{x}(\lambda_1 t), \bar{y}(t))| \\ \leq \dfrac{1}{90}[2|x - \bar{x}| + |y - \bar{y}|],$$

$$|f_2(t, x(t), y(t), y(\lambda_2 t)) - f_1(t, \bar{x}(t), \bar{y}, \bar{y}(\lambda_2 t))|$$
$$\leq \frac{1}{90}[|x - \bar{x}| + 2|y - \bar{y}|]. \tag{44}$$

Hence, $L_{f_1} = L_{f_2} = 1/90$, by the definitions of f_1 and f_2 the conditon (H_2) is also satisfied and we have

$$\sum_{i=1}^{2} 3L_{f_i} \frac{(b-a)^{\gamma_i-1}}{|\Lambda_i| \Gamma(\gamma_i)} \left[|A_i| \frac{(b-a)^{\alpha_i}}{\Gamma(\alpha_i+1)} \right.$$
$$\left. + |B_i| \frac{(\mu_i-a)^{\alpha_i+\delta_i}}{\Gamma(\alpha_i+\delta_i+1)} \right] \simeq 0.07108034 < 1, \tag{45}$$

which shows that problem (43) has at least a solution (x, y) on $[1/3, 5/3]$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] Y. Benia, M. Ruggieri, and A. Scapellato, "Exact solutions for a modified Schrödinger equation," *Mathematics*, vol. 7, no. 10, p. 908, 2019.

[2] Y. Benia and A. Scapellato, "Existence of solution to Korteweg-de Vries equation in a non-parabolic domain," *Nonlinear Analysis*, vol. 195, article 111758, 2020.

[3] G. M. Coclite and L. di Ruvo, "Well-posedness of the classical solutions for a Kawahara–Korteweg–de Vries-type equation," *Journal of Evolution Equations*, 2020.

[4] G. M. Coclite and L. di Ruvo, "Convergence of the Solutions on the Generalized Korteweg–de Vries Equation," *Mathematical Modelling and Analysis*, vol. 21, no. 2, pp. 239–259, 2016.

[5] M. Ruggieri and A. Valenti, "Symmetries and reduction techniques for dissipative models," *Journal of Mathematical Physics*, vol. 50, no. 6, article 063506, 2009.

[6] M. Ruggieri and M. P. Speciale, "Approximate analysis of a nonlinear dissipative model," *Acta Applicandae Mathematicae*, vol. 132, no. 1, pp. 549–559, 2014.

[7] M. Ruggieri and A. Valenti, "Exact solutions for a nonlinear model of dissipative media," *Journal of Mathematical Physics*, vol. 52, no. 4, article 043520, 2011.

[8] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[9] V. Lakshmikantham, S. Leela, and J. V. Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, Cambridge, 2009.

[10] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, Wiley, New York, NY, USA, 1993.

[11] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, NY, USA, 1999.

[12] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives*, Gordon & Breach, Yverdon, 1993.

[13] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

[14] R. Hilfer, Ed., *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[15] R. Hilfer, Y. Luchko, and Z. Tomovski, "Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives," *Fractional Calculus & Applied Analysis*, vol. 12, pp. 299–318, 2009.

[16] T. T. Soong, *Random Differential Equations in Science and Engineering*, Academic Press, New York, NY, USA, 1973.

[17] R. Almeida, A. B. Malinowska, and M. T. T. Monteiro, "Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications," *Mathematicsl Methods in the Applied Sciences*, vol. 41, no. 1, pp. 336–352, 2018.

[18] R. Hilfer, "Experimental evidence for fractional time evolution in glass forming materials," *The Journal of Chemical Physics*, vol. 284, no. 1-2, pp. 399–408, 2002.

[19] S. Sedaghat, Y. Ordokhani, and M. Dehghan, "Numerical solution of the delay differential equations of pantograph type via Chebyshev polynomials," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 12, pp. 4815–4830, 2012.

[20] M. M. Bahşı, M. Çevik, and M. Sezer, "Orthoexponential polynomial solutions of delay pantograph differential equations with residual error estimation," *Mathematics of Computation*, vol. 271, pp. 11–21, 2015.

[21] K. Balachandran, S. Kiruthika, and J. J. Trujillo, "Existence of solutions of nonlinear fractional pantograph equations," *Acta Mathematica Scientia*, vol. 33, no. 3, pp. 712–720, 2013.

[22] D. Vivek, K. Kanagarajan, and S. Harikrishnan, "Existence and uniqueness results for pantograph equations with generalized fractional derivative," *Journal of Nonlinear Analysis and Application*, vol. 2, pp. 105–112, 2017.

[23] S. Asawasamrit, A. Kijjathanakorn, S. K. Ntouyas, and J. Tariboon, "Nonlocal boundary value problems for Hilfer fractional differential equations," *Bulletin of the Korean Mathematical Society*, vol. 55, pp. 1639–1657, 2018.

[24] K. M. Furati, N. D. Kassim, and N. Tatar, "Existence and uniqueness for a problem involving Hilfer fractional derivative," *Computers & Mathematcs with Applications*, vol. 64, no. 6, pp. 1616–1626, 2012.

[25] J. Wang and Y. Zhang, "Nonlocal initial value problems for differential equations with Hilfer fractional derivative," *Applied Mathematics and Computation*, vol. 266, pp. 850–859, 2015.

[26] H. Gu and J. J. Trujillo, "Existence of mild solution for evolution equation with Hilfer fractional derivative," *Applied Mathematics and Computation*, vol. 257, pp. 344–354, 2015.

[27] D. Vivek, E. M. Elsayed, and K. Kanagarajan, "Existence, uniqueness and stability of Hilfer type neutral pantograph differential equations with nonlocal conditions," *International Journal of Scientific and Innovative Mathematical Research*, vol. 6, pp. 42–53, 2018.