

## Research Article

# A Riemann-Hilbert Approach to the Multicomponent Kaup-Newell Equation

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A Riemann-Hilbert approach is developed to the multicomponent Kaup-Newell equation. The formula is presented of  $N$ -soliton solutions through an identity jump matrix related to the inverse scattering problems with reflectionless potential.

## 1. Introduction

Many nonlinear partial differential equations especially soliton equations have exact solutions [1–3]. There are a lot of methods to solve soliton equations such as the Hirota bilinear method [2–6], Wronskian technique (Casoratian technique) [5–9], and Darboux transformation [10, 11]. The inverse scattering transformation (IST) is one of the most powerful tools and closely connected with those methods mentioned above [1, 3]. It is also called nonlinear Fourier transform for its procedure to solve the nonlinear equations is similar to the linear Fourier transform. One advantage of the IST is that it can be applied to the whole soliton hierarchies [3]. Recently, researches show that the IST can solve not only classic soliton hierarchies but also soliton equations with self-consistent sources [12], nonisospectral soliton hierarchies [13], hierarchies mixed with isospectral and nonisospectral ones [14], and nonlocal soliton hierarchies [15]. Furthermore, it can generate both soliton and general matrix exponent solutions [16, 17].

The Riemann-Hilbert (RH) approach is another effective method to solve soliton equations. It actually shares a close relationship with the IST [18–20]. Both of them start from same matrix spectral problems which possess bounded eigenfunctions analytically extendable to the upper or lower half-plane. To get scattering data, we must consider the asymptotic conditions at infinity on real axis by the IST to solve soliton equations. In fact, the considered conditions

are used as the solutions to the corresponding RH problems. When the jump matrix is an identity matrix, the RH problem is equivalent to the IST with reflectionless potentials, and  $N$ -soliton solutions can be generated [21–23]. Recently, Ma has already used the method to solve multicomponent soliton equations such as multicomponent AKNS integrable hierarchies and a coupled mKdV equation [24–26].

It is known to us all that the three famous derivative nonlinear Schrödinger equations, the Chen-Lee-Liu (CLL) equation [17, 27], Kaup-Newell (KN) equation [28], and Gerdjikov-Ivanov (GI) equation [29, 30], can be reduced from the Kundu equation by choosing different value of the arbitrary parameter [31, 32]. Many properties of them have been researched such as exact solutions [30, 33], conservation laws [34], multi-Hamilton structure [31], and  $\tau$ -symmetry algebra [32, 34].

In this paper, we will present the multicomponent KN equation with its  $(n + 1) \times (n + 1)$  matrix Lax pairs. To formulate an RH problem of the equation, we consider a modify matrix Lax pairs. The formula of generating the  $N$ -soliton solutions to multicomponent KN equation will be obtained through taking the identify jump matrix.

The paper is organized as follows. In Section 2, we will introduce the multicomponent isospectral KN equation and its Lax pairs. In Section 3, we will construct a multicomponent RH problem to the equation introduced in the previous section. In Section 4, the expression of  $N$ -soliton solutions will be obtained. We conclude the paper in Section 5.

## 2. The Multicomponent KN Equation

In this section, we will present the isospectral multicomponent KN equation from a  $(n+1) \times (n+1)$  matrix spectral problem by the zero-curvature representation. To our knowledge, there is another powerful method to build soliton equation hierarchies through Kac-Moody algebra and principal gradation [35].

Suppose that  $p_j$  and  $q_j$  are smooth functions of variables  $x$  and  $t$  ( $j = 1, 2, \dots, n$ ),  $T$  denotes the transpose of matrix, and  $I_n$  is an  $n \times n$  identity matrix. Let us consider the following Lax pairs

$$-i\phi_x = U\phi, U = \begin{pmatrix} \alpha_1\lambda^2 & \lambda p \\ \lambda q & \alpha_2 I_n \lambda^2 \end{pmatrix} = \lambda^2 \Lambda + \lambda P, \quad (1a)$$

$$-i\phi_t = V\phi, V = \begin{pmatrix} \alpha_1\lambda^4 - \frac{1}{\alpha}pq\lambda^2 & p\lambda^3 - \left[\frac{i}{\alpha}p_x + \frac{2}{\alpha^2}pqp\right]\lambda \\ q\lambda^3 + \left[\frac{i}{\alpha}q_x - \frac{2}{\alpha^2}pqq\right]\lambda & \alpha_2 I_n \lambda^4 + \frac{1}{\alpha}qp\lambda^2 \end{pmatrix}, \quad (1b)$$

where  $\alpha_1$  and  $\alpha_2$  are two real constants;  $\alpha = \alpha_1 - \alpha_2$ ,  $p = (p_1, p_2, \dots, p_n)$ , and  $q = (q_1, q_2, \dots, q_n)^T$  are potential functions;  $\lambda$  is a spectral parameter; and

$$\Lambda = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 I_n \end{pmatrix}, P = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}. \quad (2)$$

Obviously,  $p$  and  $q$  are smooth component functions of variables  $t$  and  $x$ . Assume that  $p, q$ , and their derivatives of any order with respect to  $x$  vanish rapidly as  $x \rightarrow \infty$ .

The compatibility condition of (1), i.e., the zero curvature equation

$$U_t - V_x + i[U, V] = 0, \quad (3)$$

generates the multicomponent KN soliton equation

$$\begin{pmatrix} p^T \\ q \end{pmatrix}_t = \frac{1}{\alpha} \begin{pmatrix} -ip_{xx}^T - \frac{2}{\alpha}(pqp^T)_x \\ iq_{xx} - \frac{2}{\alpha}(pqq)_x \end{pmatrix}. \quad (4)$$

For example, when  $n=2$ , the spectral problem (1a) becomes

$$-i\hat{\phi}_x = \hat{U}\hat{\phi}, \hat{U} = \begin{pmatrix} \alpha_1\lambda^2 & \lambda p_1 & \lambda p_2 \\ \lambda q_1 & \alpha_2\lambda^2 & 0 \\ \lambda q_2 & 0 & \alpha_2\lambda^2 \end{pmatrix} = \lambda^2 \hat{\Lambda} + \lambda \hat{P}, \quad (5)$$

where

$$\hat{\Lambda} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}, \hat{P} = \begin{pmatrix} 0 & p_1 & p_2 \\ q_1 & 0 & 0 \\ q_2 & 0 & 0 \end{pmatrix}. \quad (6)$$

Its time evolution is

$$-i\hat{\phi}_t = \hat{V}\hat{\phi}, \hat{V} = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & \alpha_2\lambda^4 + \frac{1}{\alpha}q_1p_1\lambda^2 & \frac{1}{\alpha}q_1p_2\lambda^2 \\ V_{31} & \frac{1}{\alpha}q_2p_1\lambda^2 & \alpha_2\lambda^4 + \frac{1}{\alpha}q_2p_2\lambda^2 \end{pmatrix}, \quad (7)$$

with

$$\begin{aligned} V_{11} &= \alpha_1\lambda^4 - \frac{1}{\alpha} \langle \hat{p}, \hat{q} \rangle \lambda^2, \quad V_{12} = p_1\lambda^3 - \left[ \frac{i}{\alpha}p_{1x} + \frac{2}{\alpha^2} \langle \hat{p}, \hat{q} \rangle p_1 \right] \lambda, \\ V_{13} &= p_2\lambda^3 - \left[ \frac{i}{\alpha}p_{2x} + \frac{2}{\alpha^2} \langle \hat{p}, \hat{q} \rangle p_2 \right] \lambda, \quad V_{21} = q_1\lambda^3 + \left[ \frac{i}{\alpha}q_{1x} - \frac{2}{\alpha^2} \langle \hat{p}, \hat{q} \rangle q_1 \right] \lambda, \\ V_{31} &= q_2\lambda^3 + \left[ \frac{i}{\alpha}q_{2x} - \frac{2}{\alpha^2} \langle \hat{p}, \hat{q} \rangle q_2 \right] \lambda, \quad \langle \hat{p}, \hat{q} \rangle = p_1q_1 + p_2q_2. \end{aligned} \quad (8)$$

The 4-component KN equation is

$$\begin{cases} p_{1t} = -\frac{i}{\alpha}p_{1xx} - \frac{2}{\alpha^2}[(p_1q_1 + p_2q_2)p_1]_x, \\ p_{2t} = -\frac{i}{\alpha}p_{2xx} - \frac{2}{\alpha^2}[(p_1q_1 + p_2q_2)p_2]_x, \\ q_{1t} = \frac{i}{\alpha}q_{1xx} - \frac{2}{\alpha^2}[(p_1q_1 + p_2q_2)q_1]_x, \\ q_{2t} = \frac{i}{\alpha}q_{2xx} - \frac{2}{\alpha^2}[(p_1q_1 + p_2q_2)q_2]_x. \end{cases} \quad (9)$$

## 3. The RH Problem to the Multicomponent KN Equation

In this section, we will build the RH problem to the multicomponent KN equation (5). Here, we only focus on the positive flows. Constructing the RH problem from negative symmetry flows have already appeared in [36] for the homogeneous  $A_m$ -hierarchy and its  $\hat{\mathfrak{gl}}(m+1, \mathbb{C})$ .

Setting

$$V = \lambda^4 \Lambda + Q, \quad (10)$$

it is obvious that the trace of  $Q$  is zero, where

$$Q = \begin{pmatrix} -\frac{1}{\alpha}pq\lambda^2 & p\lambda^3 - \left[\frac{i}{\alpha}p_x + \frac{2}{\alpha^2}pqp\right]\lambda \\ q\lambda^3 + \left[\frac{i}{\alpha}q_x - \frac{2}{\alpha^2}pqq\right]\lambda & \frac{1}{\alpha}qp\lambda^2 \end{pmatrix}. \quad (11)$$

Thus, the equation (5) has the following Lax pairs

$$\begin{cases} -i\phi_x = \lambda^2 \Lambda \phi + \lambda P \phi, \\ -i\phi_t = \lambda^4 \Lambda \phi + Q \phi. \end{cases} \quad (12)$$

Next, we will present the scattering and inverse scattering methods for the multicomponent KN equation (5) by the RH approach. The resulting results will lay the groundwork for  $N$ -soliton solutions in the next section. Suppose that all the potentials rapidly vanish when  $x \rightarrow \pm\infty$  or  $t \rightarrow \pm\infty$  and satisfy

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x|^m |t|^n \sum_{j=1}^n (|p_j| + |q_j|) dx dt < \infty, m, n \geq 0. \quad (13)$$

In the RH approach, we treat  $\phi$  in the spectral problem (1a) as a fundamental matrix. From (12), we note, under (13), one has the asymptotic behavior:  $\phi \sim E_g = e^{i\lambda^2 \Lambda x + i\lambda^4 \Lambda t}$ . This motivate us to introduce the variable transformation

$$\psi = \phi e^{-i\lambda^2 \Lambda x - i\lambda^4 \Lambda t}, \quad (14)$$

to have the canonical normalization for the associated RH problem:

$$\psi \rightarrow I_{n+1}, \text{ when } x, t \rightarrow \pm\infty, \quad (15)$$

where  $I_{n+1}$  is the  $(n+1) \times (n+1)$  identity matrix. This way, the spectral problems in (12) equivalently lead to

$$\psi_x = i\lambda^2 [\Lambda, \psi] + \lambda P \psi, \quad (16a)$$

$$\psi_t = i\lambda^4 [\Lambda, \psi] + Q \psi, \quad (16b)$$

where  $P = iP$  and  $Q = iQ$ . Noticing  $\text{tr}(P) = \text{tr}(Q) = 0$ , we have

$$\det \psi = 1, \quad (17)$$

by the Abel's formula.

Let us now consider the formulation of an associated RH problem with the variable  $x$ . In the scattering problem, we first introduce the matrix solutions  $\psi^\pm(x, \lambda)$  of (16a) with the asymptotics conditions

$$\psi^\pm \rightarrow I_{n+1}, \text{ when } x \rightarrow \pm\infty, \quad (18)$$

respectively. The subscripts above refer to which end of the  $x$ -axis the boundary conditions are required. Then, by (17), we have the determinant  $\det \psi^\pm = 1$  for all  $x \in \mathbb{R}$ . Since  $\phi^\pm = \psi^\pm E$  are both solutions of (12), they must be linearly related, and so, we have

$$\psi^- E = \psi^+ E S(\lambda), \lambda^2 \in \mathbb{R}, \quad (19)$$

where

$$E = e^{i\lambda^2 \Lambda x}, S(\lambda) = \begin{pmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,n+1} \\ s_{2,1} & s_{2,2} & \cdots & s_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n+1,1} & s_{n+1,2} & \cdots & s_{n+1,n+1} \end{pmatrix}, \lambda^2 \in \mathbb{R}, \quad (20)$$

is the scattering matrix. Note that  $\det(S(\lambda)) = 1$  since  $\det(\psi^\pm) = 1$ . Using the method of variation in parameters as well as the boundary condition (19), we can turn the  $x$ -part of (12) into the following Volterra integral equation for  $\psi^\pm$ :

$$\psi^-(\lambda, x) = I_{n+1} + \int_{-\infty}^x e^{i\lambda^2 \Lambda(x-y)} P(y) \psi^-(\lambda, y) e^{i\lambda^2 \Lambda(y-x)} dy, \quad (21a)$$

$$\psi^+(\lambda, x) = I_{n+1} - \int_x^{\infty} e^{i\lambda^2 \Lambda(x-y)} P(y) \psi^+(\lambda, y) e^{i\lambda^2 \Lambda(y-x)} dy. \quad (21b)$$

Thus,  $\psi^\pm$  allows analytical continuations off the real axis  $\lambda^2 \in \mathbb{R}$  as long as the integrals on their right hand sides converge. Taking  $\alpha = \alpha_1 - \alpha_2$ , it is direct to see that the integral equation for the first column of  $\psi^-$  contains only the exponential factor  $e^{-i\alpha\lambda^2(x-y)}$ . When  $\lambda$  is in the first or third quadrant, i.e.,  $\text{Im}(\lambda^2) > 0$ , let  $\lambda^2 = r + is, s > 0$ . Then,  $e^{-i\alpha\lambda^2(x-y)} = e^{-i\alpha r(x-y) + \alpha s(x-y)}$  due to  $y < x$  in the integral decays as  $\alpha < 0$ , and the integral equation for the last  $n$  columns of  $\psi^+$  contains only the exponential factor  $e^{i\alpha\lambda^2(x-y)}$ , which due to  $y > x$  in the integral, also decays when  $\lambda^2$  is in the upper half-plane  $\mathbb{C}^+$ . Thus, these  $n+1$  columns can be analytically continued to the first or third quadrants. Similarly, we find that the last  $n$  columns of  $\psi^-$  and the first column of  $\psi^+$  can be analytically continued to the second and fourth quadrants. Let us express

$$\psi^\pm = (\psi_1^\pm, \psi_2^\pm, \dots, \psi_{n+1}^\pm), \quad (22)$$

where  $\psi_k^\pm$  stands for the  $k$ th column of  $\psi^\pm (1 \leq k \leq n+1)$ . Then, the matrix solution

$$P^+ = P^+(x, \lambda) = (\psi_1^-, \psi_2^+, \dots, \psi_{n+1}^+) = \psi^- H_1 + \psi^+ H_2, \quad (23)$$

is analytic in the first and third quadrants of  $\lambda$ , and the matrix solution

$$(\psi_1^+, \psi_2^-, \dots, \psi_{n+1}^-) = \psi^+ H_1 + \psi^- H_2, \quad (24)$$

is analytic in the second and fourth quadrants of  $\lambda$ , where

$H_1 = \text{diag}(1, 0, \dots, 0)$  and  $H_2 = \text{diag}(0, 1, \dots, 1)$ . In addition, from the Volterra integral equation (21), we know that

$$\begin{aligned} P^+(x, \lambda) &\longrightarrow I_{n+1}, \text{ when } \lambda^2 \in \mathbb{C}^+ \longrightarrow \infty, \\ (\Psi_1^+, \Psi_2^-, \dots, \Psi_{n+1}^-) &\longrightarrow I_{n+1}, \text{ when } \lambda^2 \in \mathbb{C}^- \longrightarrow \infty. \end{aligned} \quad (25)$$

Next, we construct the analytic counterpart of  $P^+$  in the second and fourth quadrants of  $\lambda$ . Note that the adjoint equation of the  $x$ -part of (12) and the adjoint equation of (16) read as

$$\begin{aligned} i\tilde{\phi}_x &= \tilde{\phi}(\lambda^2 \Lambda + \lambda P), \\ i\tilde{\psi}_x &= \lambda^2[\tilde{\psi}, \Lambda] - i\lambda\tilde{\psi}P. \end{aligned} \quad (26)$$

It is easy to see that the inverse matrices  $\tilde{\phi}^\pm = (\phi^\pm)^{-1}$  and  $\tilde{\psi}^\pm = (\psi^\pm)^{-1}$  solve these adjoint equations, respectively. If we express  $\tilde{\psi}^\pm$  as follows:

$$\tilde{\psi}^\pm = \left( \psi^{\sim \pm, 1} \psi^{\sim \pm, 2} : \psi^{\sim \pm, n+1} \right), \quad (27)$$

where  $\psi^{\sim \pm, k}$  is the  $k$ th row of  $\tilde{\psi}^\pm$  ( $1 \leq k \leq n+1$ ). Then, by similar arguments, we can show that the adjoint matrix solution

$$P^- = \left( \psi^{\sim -, 1} \psi^{\sim -, 2} : \psi^{\sim -, n+1} \right) = H_1 \tilde{\psi}^- + H_2 \tilde{\psi}^+ = H_1 (\psi^-)^{-1} + H_2 (\psi^+)^{-1}, \quad (28)$$

is analytic when  $\lambda$  is in the second or fourth quadrants, and the other matrix solution

$$\left( \psi^{\sim +, 1} \psi^{\sim +, 2} : \psi^{\sim +, n+1} \right) = H_1 \tilde{\psi}^+ + H_2 \tilde{\psi}^- = H_1 (\psi^+)^{-1} + H_2 (\psi^-)^{-1}, \quad (29)$$

is analytic for  $\lambda$  in first and third quadrants. In the same way, we see that

$$\begin{aligned} P^-(x, \lambda) &\longrightarrow I_{n+1}, \text{ when } \lambda^2 \in \mathbb{C}^- \longrightarrow \infty, \\ \left( \psi^{\sim +, 1} \psi^{\sim +, 2} : \psi^{\sim +, n+1} \right) &\longrightarrow I_{n+1}, \text{ when } \lambda^2 \in \mathbb{C}^+ \longrightarrow \infty. \end{aligned} \quad (30)$$

Now, we have constructed two matrix functions  $P^+$  and  $P^-$ , which are analytic in the first or third quadrants and second or fourth quadrants, respectively. Defining

$$\begin{aligned} G^+(x, \lambda) &= P^+(x, \lambda), \lambda \in \text{first or third quadrant}, \\ (G^-)^{-1}(x, \lambda) &= P^-(x, \lambda), \lambda \in \text{second or fourth quadrant}, \end{aligned} \quad (31)$$

we can easily find that if  $\lambda$  on the real axis or imaginary axis, the two matrix functions  $P^+$  and  $P^-$  are related by

$$P^-(x, \lambda)P^+(x, \lambda) = G(x, \lambda), \lambda^2 \in \mathbb{R}, \quad (32)$$

where

$$\begin{aligned} G(x, \lambda) &= (G^-)^{-1}(x, \lambda)G^+(x, \lambda) \\ &= E(H_1 + H_2 S(\lambda))(H_1 + S^{-1}(\lambda)H_2)E^{-1} \\ &= E \begin{pmatrix} 1 & \widehat{s}_{1,2} & \widehat{s}_{1,3} & \cdots & \widehat{s}_{1,n+1} \\ s_{2,1} & 1 & 0 & \cdots & 0 \\ s_{3,1} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n+1,1} & 0 & 0 & \cdots & 1 \end{pmatrix} E^{-1}, \end{aligned} \quad (33)$$

$$S^{-1}(\lambda) = (S(\lambda))^{-1} = (\widehat{s}_{j,l})_{n+1, n+1}^{-1}, j, l = 1, 2, \dots, n+1. \quad (34)$$

Eq. (32) and Eq. (33) are exactly the associated matrix RH problem we wanted to present. The asymptotic conditions

$$P^+(x, \lambda) \longrightarrow I_{n+1}, \text{ when } \lambda \text{ in the first or third quadrant}, \quad (35a)$$

$$P^-(x, \lambda) \longrightarrow I_{n+1}, \text{ when } \lambda \text{ in the second or fourth quadrant}, \quad (35b)$$

provide the canonical normalization condition for the established RH problem.

To finish the direct scattering transform, we take the derivative of (19) with time  $t$  and use the vanishing conditions of the potentials; we can show that  $S$  satisfies

$$S_t = i\lambda^4[\Lambda, S], \quad (36)$$

which gives the time evolution of the scattering coefficients:

$$s_{1,j} = s_{1,j}(0, \lambda)e^{ia\lambda^4 t}, s_{j,1} = s_{j,1}(0, \lambda)e^{-ia\lambda^4 t}, 0 \leq j \leq n+1, \quad (37)$$

and the other scattering data do not depend on time  $t$ .

#### 4. $N$ -Soliton Solutions

The RH problems with zeros can generate soliton solutions. The uniqueness of the associated RH problem (32) does not hold unless the zeros of  $\det P^+$  in the first or third quadrants and  $\det P^-$  in the second or fourth quadrants are specified and kernel structures of  $P^\pm$  at these zeros are determined. Following the definitions of  $P^\pm$  as well as the scattering relation between  $\psi^+$  and  $\psi^-$ , we find that

$$\begin{aligned} \det P^+(x, \lambda) &= \det(\psi^- H_1 + \psi^+ H_2), \\ &= s_{1,1}(\lambda), \end{aligned} \quad (38)$$

where we have used the fact  $\det \psi^+ = 1$ . Similarly, we have  $\det P^-(x, \lambda) = \widehat{s}_{1,1}(\lambda)$  and

$$\widehat{s}_{1,1}(\lambda) = (S^{-1})_{11} = \begin{vmatrix} s_{2,2} & s_{2,3} & \cdots & s_{2,n+1} \\ s_{3,2} & s_{3,3} & \cdots & s_{3,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n+1,2} & s_{n+1,3} & \cdots & s_{n+1,n+1} \end{vmatrix}. \quad (39)$$

Suppose that  $s_{1,1}(\lambda)$  has zeros  $\{\lambda_k, \lambda_k^2 \in \mathbb{C}^+, 1 \leq k \leq 2N\}$ , and  $\widehat{s}_{1,1}$  has zeros  $\{\widehat{\lambda}_k, \widehat{\lambda}_k^2 \in \mathbb{C}^-, 1 \leq k \leq 2N\}$ . For simplicity, we assume that all these zeros,  $\lambda_k$  and  $\widehat{\lambda}_k$ ,  $1 \leq k \leq 2N$ , are simple. Then, each of  $\ker P^+(\lambda_k)$  contains only a single column vector, denoted by  $v_k$ , and each of  $\ker P^+(\widehat{\lambda}_k)$  contains a row vector, denoted by  $\widehat{v}_k$ :

$$P^+(\lambda_k)v_k = 0, \widehat{v}_k P^-(\widehat{\lambda}_k) = 0, 1 \leq k \leq 2N. \quad (40)$$

The RH problem (32) with the canonical normalization condition (35) and the zero structure (40) can be solved explicitly, and thus, one can readily reconstruct the potential  $P$  as follows. Note that  $P^+$  is a solution to the spectral problem (16). Therefore, as long as we expand  $P^+$  at large  $\lambda$  as

$$P^+(x, \lambda) = I_{n+1} + \frac{1}{\lambda} P_1^+(x) + O\left(\frac{1}{\lambda^2}\right), \lambda \longrightarrow \infty, \quad (41)$$

inserting this expansion into (16) and comparing  $O(1)$  terms lead to

$$P = -i[\Lambda, P_1^+], \quad (42)$$

which implies that

$$P = -[\Lambda, P_1^+] = \begin{pmatrix} 0 & -\alpha(P_1^+)_{12} & -\alpha(P_1^+)_{13} & \cdots & -\alpha(P_1^+)_{1,n+1} \\ \alpha(P_1^+)_{21} & 0 & 0 & \cdots & 0 \\ \alpha(P_1^+)_{31} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha(P_1^+)_{n+1,1} & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (43)$$

where  $P_1^+ = ((P_1^+)_{kl})_{1 \leq k, l \leq n+1}$ . Further, the potentials  $p_k$  and  $q_k$ ,  $k = 1, 2, \dots, n$  can be computed as

$$p_j = -\alpha(P_1^+)_{1,j+1}, q_j = \alpha(P_1^+)_{j+1,1}, j = 1, 2, \dots, n. \quad (44)$$

To obtain soliton solutions, we set  $G = I_{n+1}$  in the RH problem (32). This can be achieved if we assume  $s_{1,j} = s_{j,1} = 0$ ,  $2 \leq j \leq n+1$ , which means that there is no reflection in

the scattering problem. The solutions to this specific RH problem can be given as follows [24, 25]

$$\begin{aligned} P^+(\lambda) &= I_{n+1} - \sum_{k,l=1}^N \frac{v_k (M^{-1})_{kl} \widehat{v}_l}{\lambda - \widehat{\lambda}_l}, P^-(\lambda) \\ &= I_{n+1} + \sum_{k,l=1}^N \frac{v_k (M^{-1})_{kl} \widehat{v}_l}{\lambda - \lambda_l}, \end{aligned} \quad (45)$$

where  $M = (M_{kl})_{N \times N}$  is a square matrix whose entries read

$$M_{kl} = \frac{\widehat{v}_k v_l}{\lambda_l - \widehat{\lambda}_k}, 1 \leq k, l \leq 2N. \quad (46)$$

Noting that the zeros  $\lambda_k$  and  $\widehat{\lambda}_k$  are constants, i.e., space and time independent, we can easily find the spatial and temporal evolutions for the vectors,  $v_k(x, t)$  and  $\widehat{v}_k(x, t)$ ,  $1 \leq k \leq N$ . For example, let us take the  $x$ -derivative of both sides of the equation  $P^+(\lambda_k)v_k = 0$ . By using (16) and  $P^+(\lambda_k)v_k = 0$ , we get

$$P^+(\lambda_k, x) \left( \frac{dv_k}{dx} - i\lambda_k^2 \Lambda v_k \right) = 0, 1 \leq k \leq 2N, \quad (47)$$

which implies

$$\frac{dv_k}{dx} = i\lambda_k^2 \Lambda v_k, 1 \leq k \leq 2N. \quad (48)$$

The time dependence of  $v_k$ :

$$\frac{dv_k}{dt} = i\lambda_k^4 \Lambda v_k, 1 \leq k \leq 2N, \quad (49)$$

can be determined similarly through an associated RH problem with the variable  $t$ . Summing up, we obtain

$$v_k(x, t) = e^{i\lambda_k^2 \Lambda x + i\lambda_k^4 \Lambda t} v_{k,0}, 1 \leq k \leq 2N, \quad (50a)$$

$$\widehat{v}_k(x, t) = \widehat{v}_{k,0} e^{-i\lambda_k^2 \Lambda x - i\lambda_k^4 \Lambda t}, 1 \leq k \leq 2N, \quad (50b)$$

where  $v_{k,0}$  and  $\widehat{v}_{k,0}$ ,  $1 \leq k \leq 2N$ , are arbitrary constant vectors. Finally, from (45), we get

$$P_1^+ = - \sum_{k,l=1}^N v_k (M^{-1})_{kl} \widehat{v}_l, \quad (51)$$

and thus by (44), the  $N$ -soliton solution to the system of multicomponent KN equations (4):

$$\begin{cases} p_j = \alpha \sum_{k,l=1}^N v_{k,1} (M^{-1})_{kl} \widehat{v}_{l,j+1}, \\ q_j = -\alpha \sum_{k,l=1}^N v_{k,j+1} (M^{-1})_{kl} \widehat{v}_{l,1}, \end{cases} \quad (52)$$

where  $v_k = (v_{k,1}, v_{k,2}, \dots, v_{k,n+1})^T$  and  $\widehat{v}_k = (\widehat{v}_{k,1}, \widehat{v}_{k,2}, \dots, \widehat{v}_{k,n+1})$ ,  $1 \leq k \leq 2N$ , are arbitrary.

## 5. Conclusions

In general, we construct the RH problem for the multicomponent KN equation in this paper. To build the special RH problem with the identity jump matrix, we introduced a variable transformation to canonical normalization spectral problem. By recombining the solutions of the canonical spectral problem and its adjoint spectral problem, a general jump matrix to the special RH problem was constructed. Letting the general jump matrix to be identity jump matrix, the RH problem was solved. Finally, we obtained the expression of the  $N$ -soliton solutions through power series expansion of the spectral parameter in the canonical normalization spectral problem.

In this method, the jump matrix is corresponding to the scattering matrix, and the identity jump matrix is equivalent to reflectionless coefficient of the IST. It is well known that there are not only soliton solutions to soliton equations but also rational solutions, Matveev solutions, complexiton solutions, and so on. Recently, there have been active studies on lumps and their interaction solutions with solitons [37, 38]. It would be very interesting to generalize this method to  $(2+1)$ -dimensional equations and consider their lumps and interaction solutions. These will be our future projects.

## Data Availability

The data that supports the findings of this study are available within the article.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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