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Research Article

Fixed-Point Results for Generalized α-Admissible Hardy-Rogers' Contractions in Cone $\mathfrak{b}_2$-Metric Spaces over Banach's Algebras with Application

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In the current manuscript, the notion of a cone $\mathfrak{b}_2$-metric space over Banach's algebra with parameter $\mathfrak{b} \succeq \mathfrak{e}$ is introduced. Furthermore, using α-admissible Hardy-Rogers' contractive conditions, we have proven fixed-point theorems for self-mappings, which generalize and strengthen many of the conclusions in existing literature. In order to verify our key result, a nontrivial example is given, and as an application, we proved a theorem that shows the existence of a solution of an infinite system of integral equations.

1. Introduction and Preliminaries

There are many generalizations in the literature about the concept of metric spaces like $\mathfrak{b}$-metric spaces [1], 2-metric spaces [2], N_b-metric spaces [3], and weak partial $\mathfrak{b}$-metric spaces [4]. Gähler incorporated the notion of a 2-metric space in [2]. Recall that a 2-metric is not a continuous function of its variables, whereas a standard metric is. This led Dhage to implement the D-metric notion in [5]. In [6, 7] Mustafa and Sims implemented the G-metric notion for overcoming D-metric flaws. Following that, several fixed-point theorems were proven on G-metric spaces (see [8]). The authors in [9] found that fixed-point theorems in G-metric spaces can potentially be deduced from metric or quasimetric spaces in a variety of cases. Different researchers have additionally indicated that the fixed-point results about cone metric spaces can be acquired in a few cases by diminishing them to their standard metric partners; see for instance [10–12]. It is worth noting that a 2-metric space was not considered to be topologically equivalent to an ordinary metric in the generalizations described above.

Bakhtin [1] analyzed the phenomenon of a $\mathfrak{b}$-metric space. After this theory, Czerwik [13] demonstrated the contraction mapping method in $\mathfrak{b}$-metric spaces which generalized the renowned Banach contraction principle in $\mathfrak{b}$-metric spaces.

Replacing the set of real numbers by an ordered Banach space, Huang and Zhang [14] generalized the concept of metric spaces and defined the cone metric space, where they studied certain fixed-point results for contractive mapping in the context of cone metric space. Later, Mustafa et al. [15] set the space structure $\mathfrak{b}_2$-metric as a generalization of $\mathfrak{b}$-metric and 2-metric spaces. They illustrated some fixed-point theorems in a partially ordered $\mathfrak{b}_2$-metric space under different contractive conditions and provided some smart examples and an application to integral equations for their main outcomes.

Recently, the equivalence of cone metric space and metric space has become an extremely fascinating topic after the work of several researchers discovered that the fixed-point results in a cone metric space are special cases of metric spaces in some cases. They found that $(\mathfrak{X}, \partial)$ is equivalent to any cone metric if the real-valued function ∂^* is replaced by a nonlinear scalariztion function ξ_e or by a Minkowski functional q_e. To address these shortcomings, Liu and Xu [16] presented the definition of cone metric space over Banach's algebra.

Fernandez et al.[17] presented the concept of cone $\mathfrak{b}_2$ -metric spaces over Banach's algebra with coefficient $\mathfrak{b} \geq 1$ as an extension of $\mathfrak{b}_2$-metric spaces and cone metric spaces over Banach's algebras. They also presented many fixed-point results under different contractive conditions in the said structure. As an application, they discussed the existence of solutions to the integral equation.

On the other hand, Hardy and Rogers [18] introduced a new concept of mapping called the Hardy-Rogers contraction which generalize the Banach contraction principle and Reich's [19] theorem in the setting of metric spaces. Samet et al. [20] initiated the α-admissibility of mappings and gave a result of α-ψ-contractive mapping which generalized the Banach contraction principle. After that, many researchers worked on the Hardy-Rogers contraction and α-admissibility of mapping in different settings; for examples, see [21–27] and the references therein.

Motivated by the work done in [17, 18, 20] we study some results for the generalized α-admissible Hardy-Roger contractions in cone $\mathfrak{b}_2$-metric spaces over Banach's algebras. We note that some well-known results in the literature can be deduced by using the presented work.

In the sequel, we need the following definitions and results from the existing literature.

Definition 1. (see [28]). Let $\mathfrak{B}$ be a real Banach algebra, and the multiplication operation is defined according to the following properties (for all $\mathfrak{s}, \mathfrak{m}, \mathfrak{z} \in \mathfrak{B}$ and $\lambda \in \mathbb{R}$):

($\mathfrak{a}_1$) $(\mathfrak{s}\mathfrak{m})\mathfrak{z} = \mathfrak{s}(\mathfrak{m}\mathfrak{z})$;
($\mathfrak{a}_2$) $\mathfrak{s}(\mathfrak{m} + \mathfrak{z}) = \mathfrak{s}\mathfrak{m} + \mathfrak{s}\mathfrak{z}$ and $(\mathfrak{s} + \mathfrak{m})\mathfrak{z} = \mathfrak{s}\mathfrak{z} + \mathfrak{m}\mathfrak{z}$;
($\mathfrak{a}_3$) $\lambda(\mathfrak{s}\mathfrak{m}) = (\lambda\mathfrak{s})\mathfrak{m} = \mathfrak{s}(\lambda\mathfrak{m})$;
($\mathfrak{a}_4$) $\|\mathfrak{s}\mathfrak{m}\| \leq \|\mathfrak{s}\|\|\mathfrak{m}\|$.

We will presume in the course of this article that $\mathfrak{B}$ is a real Banach algebra, unless otherwise specified. We call $\mathfrak{e}$ the unit of $\mathfrak{B}$, if there is $\mathfrak{s} \in \mathfrak{B}$, such that $\mathfrak{e}\mathfrak{s} = \mathfrak{s}\mathfrak{e} = \mathfrak{s}$. In this case, we call $\mathfrak{B}$ a unital. It is said that an element $\mathfrak{s} \in \mathfrak{B}$ is invertible if an inverse element $\mathfrak{m} \in \mathfrak{B}$ occurs, such that $\mathfrak{s}\mathfrak{m} = \mathfrak{m}\mathfrak{s} = \mathfrak{e}$. In such case, the inverse of $\mathfrak{s}$ is unique and is denoted by $\mathfrak{s}^{-1}$. In the sequel, we need the following propositions.

Proposition 2. (see [28]). *Let $\mathfrak{e}$ be the unit element of the Banach algebra $\mathfrak{B}$ and $\mathfrak{s} \in \mathfrak{B}$ be arbitrary. If the spectral radius $r(\mathfrak{s}) < 1$, that is*

$$r(\mathfrak{s}) = \lim_{n\to\infty} \|\mathfrak{s}^n\|^{1/n} = \inf \|\mathfrak{s}^n\|^{1/n} < 1, \quad (1)$$

then, $\mathfrak{e} - \mathfrak{s}$ is invertible. In fact

$$(\mathfrak{e} - \mathfrak{s})^{-1} = \sum_{k=1}^{\infty} \mathfrak{s}^k. \quad (2)$$

Remark 3. From [28] we see that, for all $\mathfrak{s}$ in the Banach algebra $\mathfrak{B}$ with unit $\mathfrak{e}$, we have $r(\mathfrak{s}) \leq \|\mathfrak{s}\|$.

Remark 4. (see [29]). In Proposition 2, if we replace "$r(\mathfrak{s}) < 1$" by $\|\mathfrak{s}\| \leq 1$, then the conclusion remains true.

Remark 5. (see [29]). If $r(\mathfrak{s}) < 1$, then $\|\mathfrak{s}^n\| \longrightarrow 0$ as $(n \longrightarrow \infty)$.

Definition 6. Let θ be the zero element of the unital Banach algebra $\mathfrak{B}$ and $\mathfrak{C}_{\mathfrak{B}} \neq \emptyset$. Then, $\mathfrak{C}_{\mathfrak{B}} \subset \mathfrak{B}$ is a cone in $\mathfrak{B}$ if

($\mathfrak{b}_1$) $\mathfrak{e} \in \mathfrak{C}_{\mathfrak{B}}$;
($\mathfrak{b}_2$) $\mathfrak{C}_{\mathfrak{B}} + \mathfrak{C}_{\mathfrak{B}} \subset \mathfrak{C}_{\mathfrak{B}}$;
($\mathfrak{b}_3$) $\lambda\mathfrak{C}_{\mathfrak{B}} \subset \mathfrak{C}_{\mathfrak{B}}$ for all $\lambda \geq 0$;
($\mathfrak{b}_4$) $\mathfrak{C}_{\mathfrak{B}} \cdot \mathfrak{C}_{\mathfrak{B}} \subset \mathfrak{C}_{\mathfrak{B}}$;
($\mathfrak{b}_5$) $\mathfrak{C}_{\mathfrak{B}} \cap (-\mathfrak{C}_{\mathfrak{B}}) = \{\theta\}$.

Define a partial order relation $\preceq$ in $\mathfrak{B}$ w.r.t. $\mathfrak{C}_{\mathfrak{B}}$ by $\mathfrak{s} \preceq \mathfrak{m}$ if and only if $\mathfrak{m} - \mathfrak{s} \in \mathfrak{C}_{\mathfrak{B}}$ and also $\mathfrak{s} \prec \mathfrak{m}$ if $\mathfrak{s} \preceq \mathfrak{m}$ but $\mathfrak{s} \neq \mathfrak{m}$ while $\mathfrak{s} \ll \mathfrak{m}$ stands for $\mathfrak{m} - \mathfrak{s} \in \operatorname{int} \mathfrak{C}_{\mathfrak{B}}$, where int $\mathfrak{C}_{\mathfrak{B}}$ is the interior of $\mathfrak{C}_{\mathfrak{B}}$. $\mathfrak{C}_{\mathfrak{B}}$ is solid if int $\mathfrak{C}_{\mathfrak{B}} \neq \emptyset$.

If there is $\mathfrak{M} > 0$ such that for all $\mathfrak{s}, \mathfrak{m} \in \mathfrak{C}_{\mathfrak{B}}$, we have

$$\theta \leq \mathfrak{s} \leq \mathfrak{m} \text{ implies } \|\mathfrak{s}\| \leq \mathfrak{M}\|\mathfrak{m}\|, \quad (3)$$

then, $\mathfrak{C}_{\mathfrak{B}}$ is normal. If$\mathfrak{M}$is the least and positive among those cited above, then it is a normal constant of $\mathfrak{C}_{\mathfrak{B}}$ [14].

Onward, we assume that $\mathfrak{C}_{\mathfrak{B}}$ is a cone in $\mathfrak{B}$ with int $\mathfrak{C}_{\mathfrak{B}} \neq \emptyset$, and $\preceq$ is a partial order with respect to the cone $\mathfrak{C}_{\mathfrak{B}}$.

Definition 7. (see [16–14]). Let $\mathfrak{X} \neq \emptyset$ and the mapping $\partial : \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathfrak{B}$ be

($\mathfrak{c}_1$) $\partial(\mathfrak{s}, \mathfrak{m}) \succeq \theta$ for all $\mathfrak{s}, \mathfrak{m} \in \mathfrak{X}$, and $\partial(\mathfrak{s}, \mathfrak{m}) = \theta$ if and only if $\mathfrak{s} = \mathfrak{m}$;
($\mathfrak{c}_2$) $\partial(\mathfrak{s}, \mathfrak{m}) = \partial(\mathfrak{m}, \mathfrak{s})$ for all $\mathfrak{s}, \mathfrak{m} \in \mathfrak{X}$;
($\mathfrak{c}_3$) $\partial(\mathfrak{s}, \mathfrak{z}) \preceq \partial(\mathfrak{s}, \mathfrak{m}) + \partial(\mathfrak{m}, \mathfrak{z})$ for all $\mathfrak{s}, \mathfrak{m}, \mathfrak{z} \in \mathfrak{X}$.

Then ∂ is a cone metric and $(\mathfrak{X}, \partial)$ is a cone metric space over the Banach algebra $\mathfrak{B}$.

Definition 8. (see [13]). Let $\mathfrak{X} \neq \emptyset$ and $\mathfrak{b} \geq 1$ be a real number. Then, the mapping $\partial : \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathbb{R}^+$ is a $\mathfrak{b}$ -metric if, for all $\mathfrak{s}, \mathfrak{m}, \mathfrak{z} \in \mathfrak{X}$, the following holds:

($\mathfrak{d}_1$) $\partial(\mathfrak{s}, \mathfrak{m}) = 0$ if and only if $\mathfrak{s} = \mathfrak{m}$;
($\mathfrak{d}_2$) $\partial(\mathfrak{s}, \mathfrak{m}) = \partial(\mathfrak{m}, \mathfrak{s})$;
($\mathfrak{d}_3$) $\partial(\mathfrak{s}, \mathfrak{z}) \leq \mathfrak{b}[\partial(\mathfrak{s}, \mathfrak{m}) + \partial(\mathfrak{m}, \mathfrak{z})]$.

Here, the pair $(\mathfrak{X}, \partial)$ is a $\mathfrak{b}$-metric space.

The cone $\mathfrak{b}$-metric space over a Banach algebra with constant $\mathfrak{b} \geq 1$ is introduced in [30]. Mitrovic and Hussain in [26] introduced the cone $\mathfrak{b}$-metric space over a Banach algebra with constant $\mathfrak{b} \succeq \mathfrak{e}$.

Definition 9. (see [26]). Let $\mathfrak{X} \neq \emptyset$. A function $\partial : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{B}$ is a cone $\mathfrak{b}$ -metric if

($\mathfrak{e}_1$) $\theta \preceq \partial(\mathfrak{s}, \mathfrak{m})$ for all $\mathfrak{s}, \mathfrak{m} \in \mathfrak{X}$, and $\partial(\mathfrak{s}, \mathfrak{m}) = \theta$ if and only if $\mathfrak{s} = \mathfrak{m}$;
($\mathfrak{e}_2$) $\partial(\mathfrak{s}, \mathfrak{m}) = \partial(\mathfrak{m}, \mathfrak{s})$ for all $\mathfrak{s}, \mathfrak{m} \in \mathfrak{X}$;
($\mathfrak{e}_3$) There exists $\mathfrak{b} \in \mathfrak{C}_{\mathfrak{B}}$, $\mathfrak{b} \succeq \mathfrak{e}$ such that $\partial(\mathfrak{s}, \mathfrak{z}) \preceq \mathfrak{b}[\partial(\mathfrak{s}, \mathfrak{m}) + \partial(\mathfrak{m}, \mathfrak{z})]$ for all $\mathfrak{s}, \mathfrak{m}, \mathfrak{z} \in \mathfrak{X}$.

Here, the pair $(\mathfrak{X}, \partial)$ is a cone $\mathfrak{b}$-metric space over $\mathfrak{B}$. If $\mathfrak{b} = \mathfrak{e}$, then $(\mathfrak{X}, \partial)$ becomes a cone metric space over $\mathfrak{B}$.

Definition 10. (see [2]). Let $\mathfrak{X} \neq \emptyset$, $\partial : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathbb{R}^+$ satisfy the following conditions:

($\mathfrak{f}_1$) For $\mathfrak{s}, \mathfrak{m} \in \mathfrak{X}$, there is a point $\mathfrak{z} \in \mathfrak{X}$ with at least two of $\mathfrak{s}, \mathfrak{m}, \mathfrak{z}$ which are not equal, then $\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) \neq 0$;

($\mathfrak{f}_2$) $\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) = 0$ if at least two of $\mathfrak{s}, \mathfrak{m}, \mathfrak{z}$ are equal;

($\mathfrak{f}_3$) For all $\mathfrak{s}, \mathfrak{m}, \mathfrak{z} \in \mathfrak{X}, \partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) = \partial(P(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}))$ where $P(\mathfrak{s}, \mathfrak{m}, \mathfrak{z})$ stands for all permutations of $\mathfrak{s}, \mathfrak{m}, \mathfrak{z}$;

($\mathfrak{f}_4$) For all $\mathfrak{s}, \mathfrak{m}, \mathfrak{z}, \mathfrak{t} \in \mathfrak{X}, \partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) \leq \partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{t}) + \partial(\mathfrak{s}, \mathfrak{z}, \mathfrak{t}) + \partial(\mathfrak{m}, \mathfrak{z}, \mathfrak{t})$.

Then, the function ∂ is a 2-metric and $(\mathfrak{X}, \partial)$ is 2-metric space.

Definition 11. (see [17]). Let $\mathfrak{X} \neq \emptyset$ and $\mathfrak{b} \geq 1$ be a real number. Let $\partial : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathfrak{B}$ satisfy the following:

($\mathfrak{g}_1$) For $\mathfrak{s}, \mathfrak{m} \in \mathfrak{X}$, there is a point $\mathfrak{z} \in \mathfrak{X}$ with at least two of $\mathfrak{s}, \mathfrak{m}, \mathfrak{z}$ which are not equal, then $\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) \neq \theta$;

($\mathfrak{g}_2$) $\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) = \theta$ if at least two of $\mathfrak{s}, \mathfrak{m}, \mathfrak{z}$ are equal;

($\mathfrak{g}_3$) For all $\mathfrak{s}, \mathfrak{m}, \mathfrak{z} \in \mathfrak{X}, \partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) = \partial(P(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}))$ where $P(\mathfrak{s}, \mathfrak{m}, \mathfrak{z})$ stands for all permutations of $\mathfrak{s}, \mathfrak{m}, \mathfrak{z}$;

($\mathfrak{g}_4$) For all $\mathfrak{s}, \mathfrak{m}, \mathfrak{z}, \mathfrak{t} \in \mathfrak{X}, \partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) \preceq \mathfrak{b}[\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{t}) + \partial(\mathfrak{s}, \mathfrak{z}, \mathfrak{t}) + \partial(\mathfrak{m}, \mathfrak{z}, \mathfrak{t})]$.

Then, the function ∂ is a cone $\mathfrak{b}_2$-metric and $(\mathfrak{X}, \partial)$ is a cone $\mathfrak{b}_2$-metric space over the Banach algebra $\mathfrak{B}$ with parameter $\mathfrak{b}$. It reduces to a cone 2-metric space if we take $\mathfrak{b} = 1$ mentioned above. For other details about the cone 2-metric space over the Banach algebra $\mathfrak{B}$, we refer the reader to [31].

Definition 12. (see [32]). Let $\{\mathfrak{s}_\mathfrak{n}\}$ be a sequence in $\mathfrak{B}$, then

($\mathfrak{j}_1$) $\{\mathfrak{s}_\mathfrak{n}\}$ is a $\mathfrak{c}$-sequence, if for each $\mathfrak{c} \gg \theta$ there exists a natural number $\mathfrak{N}$ such that $\mathfrak{s}_\mathfrak{n} \ll \mathfrak{c}$ for all $\mathfrak{n} > \mathfrak{N}$;

($\mathfrak{j}_2$) $\{\mathfrak{s}_\mathfrak{n}\}$ is a θ-sequence, if $\mathfrak{s}_\mathfrak{n} \longrightarrow \theta$ as $\mathfrak{n} \longrightarrow \infty$.

Lemma 13. (see [33]). *Let $\mathfrak{B}$ be Banach's algebra and* int $\mathfrak{C}_\mathfrak{B} \neq \emptyset$. *Also, let $\{\mathfrak{s}_\mathfrak{n}\}$ be $\mathfrak{c}$-sequences in $\mathfrak{B}$, then for arbitrary $\mathfrak{k} \in \mathfrak{C}_\mathfrak{B}$, $\{\mathfrak{k}\mathfrak{s}_\mathfrak{n}\}$ is a $\mathfrak{c}$ -sequence.*

Lemma 14. (see [33]). *Let $\mathfrak{B}$ be Banach's algebra and* int $\mathfrak{C}_\mathfrak{B} \neq \emptyset$. *Let $\{\mathfrak{s}_\mathfrak{n}\}$ and $\{\mathfrak{z}_\mathfrak{n}\}$ be $\mathfrak{c}$ -sequences in $\mathfrak{B}$. Let η and $\zeta \in \mathfrak{C}_\mathfrak{B}$ be arbitrarily given vectors, then $\{\eta\mathfrak{s}_\mathfrak{n} + \zeta\mathfrak{z}_\mathfrak{n}\}$ is a $\mathfrak{c}$ -sequence.*

Lemma 15. (see [33]). *Let $\mathfrak{B}$ be Banach's algebra and* int $\mathfrak{C}_\mathfrak{B} \neq \emptyset$. *Let $\{\mathfrak{s}_\mathfrak{n}\} \subset \mathfrak{C}_\mathfrak{B}$ such that $\|\mathfrak{s}_\mathfrak{n}\| \longrightarrow 0$ as $\mathfrak{n} \longrightarrow \infty$. Then, $\{\mathfrak{s}_\mathfrak{n}\}$ is a $\mathfrak{c}$ -sequence.*

Lemma 16. (see [28]). *Let $\mathfrak{e}$ be the unit element of $\mathfrak{B}$, and $\mathfrak{s} \in \mathfrak{B}$, then $\lim_{\mathfrak{n}\to\infty}\|\mathfrak{s}^\mathfrak{n}\|^{1/\mathfrak{n}}$ exists and the spectral radius $r(\mathfrak{s})$ satisfies*

$$r(\mathfrak{s}) = \lim_{\mathfrak{n}\to\infty}\|\mathfrak{s}^\mathfrak{n}\|^{1/\mathfrak{n}} = \inf \|\mathfrak{s}^\mathfrak{n}\|^{1/\mathfrak{n}}. \tag{4}$$

If $r(\mathfrak{s}) < |\lambda|$, then $(\lambda\mathfrak{e} - \mathfrak{s})$ is invertible in $\mathfrak{B}$, moreover, we have

$$(\lambda\mathfrak{e} - \mathfrak{s})^{-1} = \sum_{\mathfrak{k}=0}^{\infty} \frac{\mathfrak{s}^\mathfrak{k}}{\lambda^{\mathfrak{k}+1}}. \tag{5}$$

Lemma 17. (see [28]). *Let $\mathfrak{e}$ be the unit element of $\mathfrak{B}$, and $\mathfrak{s}, \mathfrak{m} \in \mathfrak{B}$. If $\mathfrak{s}, \mathfrak{m}$ commute, then*

($\mathfrak{k}_1$) $r(\mathfrak{s} + \mathfrak{m}) \leq r(\mathfrak{s}) + r(\mathfrak{m})$;

($\mathfrak{k}_2$) $r(\mathfrak{s}\mathfrak{m}) \leq r(\mathfrak{s})r(\mathfrak{m})$.

Lemma 18. (see [34]). *Let $\mathfrak{e}$ be the unit element of $\mathfrak{B}$, and $\mathfrak{C}_\mathfrak{B} \neq \emptyset$. Let $\mathfrak{L} \in \mathfrak{B}$, and $\mathfrak{s}_\mathfrak{n} = \mathfrak{L}^\mathfrak{n}$. If $r(\mathfrak{L}) < 1$, then $\{\mathfrak{s}_\mathfrak{n}\}$ is a $\mathfrak{c}$ -sequence.*

Lemma 19. (see [34]). *Let $\mathfrak{e}$ be the unit element of $\mathfrak{B}$, and $\mathfrak{s} \in \mathfrak{B}$. Let λ be a complex number, and $r(\mathfrak{s}) < |\lambda|$, then*

$$r\left((\lambda\mathfrak{e} - \mathfrak{s})^{-1}\right) \leq \frac{1}{|\lambda| - r(\mathfrak{s})}. \tag{6}$$

Lemma 20. (see [35]). *Let $\mathfrak{C}_\mathfrak{B} \subset \mathfrak{B}$ be a cone.*

($\mathfrak{l}_1$) If $\mathfrak{s}, \mathfrak{m} \in \mathfrak{B}$, $\mathfrak{k} \in \mathfrak{C}_\mathfrak{B}$, and $\mathfrak{s} \preceq \mathfrak{m}$, then $\mathfrak{k}\mathfrak{s} \preceq \mathfrak{k}\mathfrak{m}$;

($\mathfrak{l}_2$) If $\mathfrak{s}, \mathfrak{k} \in \mathfrak{C}_\mathfrak{B}$ are such that $r(\mathfrak{k}) < 1$ and $\mathfrak{s} \preceq \mathfrak{k}\mathfrak{s}$, then $\mathfrak{s} = 0$;

($\mathfrak{l}_3$) If $\mathfrak{k} \in \mathfrak{C}_\mathfrak{B}$ and $r(\mathfrak{k}) < 1$, then for any fixed $\mathfrak{n} \in \mathbb{N}$, we have $r(\mathfrak{k}^\mathfrak{n}) < 1$.

Lemma 21. (see [32]). *Let $\mathfrak{C}_\mathfrak{B} \neq \emptyset$ and $\mathfrak{C}_\mathfrak{B} \subset \mathfrak{B}$.*

($\mathfrak{m}_1$) Let $\mathfrak{k} \in \mathfrak{C}_\mathfrak{B}$. Then $\{\mathfrak{k}^\mathfrak{n}\}$ is a θ-sequence if and only if $r(\mathfrak{k}) < 1$;

($\mathfrak{m}_2$) Every θ-sequence in $\mathfrak{B}$ is a $\mathfrak{c}$-sequence;

($\mathfrak{m}_3$) $\mathfrak{C}_\mathfrak{B}$ is normal if and only if each $\mathfrak{c}$-sequence in $\mathfrak{C}_\mathfrak{B}$ is a θ-sequence.

Lemma 22. (see [36]). *Let $\mathfrak{B}$ be a Banach algebra and* int $\mathfrak{C}_\mathfrak{B} \neq \emptyset$. *Then the following are always true:*

($\mathfrak{n}_1$) If $\mathfrak{s}, \mathfrak{m}, \mathfrak{z} \in \mathfrak{B}$ and $\mathfrak{s} \preceq \mathfrak{m} \ll \mathfrak{z}$, then $\mathfrak{s} \ll \mathfrak{c}$;

($\mathfrak{n}_2$) If $\mathfrak{s} \in \mathfrak{C}_\mathfrak{B}$ and $\mathfrak{s} \ll \mathfrak{c}$ for each $\mathfrak{c} \gg \theta$, then $\mathfrak{s} = \theta$.

Definition 23. (see [37]). Let a cone $\mathfrak{b}_2$ -metric space be $(\mathfrak{X}, \partial)$ over the Banach algebra $\mathfrak{B}$ with parameter $\mathfrak{b}$, $(\mathfrak{b} \geq \mathfrak{e})$, $\mathfrak{C}_\mathfrak{B}$ be a solid cone, $\alpha : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathfrak{C}_\mathfrak{B}$, and $\mathfrak{d} : \mathfrak{X} \longrightarrow \mathfrak{X}$ be two mappings. If for any sequence $\{\mathfrak{s}_\mathfrak{n}\} \in \mathfrak{X}$, with $\alpha(\mathfrak{s}_\mathfrak{n}, \mathfrak{s}_{\mathfrak{n}+1}, \mathfrak{z}) \geq \mathfrak{e}$ for each $\mathfrak{n} \in \mathbb{N}$ and $\mathfrak{s}_\mathfrak{n} \longrightarrow \mathfrak{s}$ as $\mathfrak{n} \longrightarrow \infty$, it follows that $\alpha(\mathfrak{s}_\mathfrak{n}, \mathfrak{s}, \mathfrak{z}) \geq \mathfrak{e}$ for all $\mathfrak{n} \in \mathbb{N}$ and for all $\mathfrak{z} \in \mathfrak{X}$; then, we say that $(\mathfrak{X}, \partial)$ is α -regular.

2. Results and Discussion

We introduced here the notion of cone $\mathfrak{b}_2$-metric space over Banach's algebra with parameter $\mathfrak{b} \geq \mathfrak{e}$.

Definition 24. Let $\mathfrak{X} \neq \emptyset$ and $\partial : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathfrak{B}$ satisfy the following:

($\mathfrak{h}_1$) For $\mathfrak{s}, \mathfrak{m} \in \mathfrak{X}$, there is a point $\mathfrak{z} \in \mathfrak{X}$ with at least two of $\mathfrak{s}, \mathfrak{m}, \mathfrak{z}$ which are not equal, then $\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) \neq \theta$;

($\mathfrak{h}_2$) $\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) = \theta$ if at least two of $\mathfrak{s}, \mathfrak{m}, \mathfrak{z}$ are equal;

($\mathfrak{h}_3$) For all $\mathfrak{s}, \mathfrak{m}, \mathfrak{z} \in \mathfrak{X}, \partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) = \partial(P(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}))$ where $P(\mathfrak{s}, \mathfrak{m}, \mathfrak{z})$ stands for all permutations of $\mathfrak{s}, \mathfrak{m}, \mathfrak{z}$;

($\mathfrak{h}_4$) For all $\mathfrak{s}, \mathfrak{m}, \mathfrak{z}, \mathfrak{t} \in \mathfrak{X}$, there exists $\mathfrak{b} \in \mathfrak{C}_{\mathfrak{B}}$, $\mathfrak{b} \geq \mathfrak{e}$ such that $\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) \preceq \mathfrak{b}[\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{t}) + \partial(\mathfrak{s}, \mathfrak{z}, \mathfrak{t}) + \partial(\mathfrak{m}, \mathfrak{z}, \mathfrak{t})]$.

Then, the function ∂ is a cone $\mathfrak{b}_2$-metric and $(\mathfrak{X}, \partial)$ is a cone $\mathfrak{b}_2$-metric space over Banach's algebra with parameter $\mathfrak{b}$. It is reduced to a cone 2-metric space if we take $\mathfrak{b} = \mathfrak{e}$ mentioned above.

Remark 25. Note that every cone 2-metric space is a cone $\mathfrak{b}_2$ -metric space with parameter $\mathfrak{b} = \mathfrak{e}$ over Banach's algebra. But the converse is not true.

Example 26. Let $\mathfrak{B} = \mathbb{R}^2$. For each $(\mathfrak{s}_1, \mathfrak{s}_2) \in \mathfrak{B}$, $\|(\mathfrak{s}_1, \mathfrak{s}_2)\| = |\mathfrak{s}_1| + |\mathfrak{s}_2|$. The multiplication is defined by $\mathfrak{s}\mathfrak{m} = (\mathfrak{s}_1, \mathfrak{s}_2)(\mathfrak{m}_1, \mathfrak{m}_2) = (\mathfrak{s}_1\mathfrak{m}_1, \mathfrak{s}_1\mathfrak{m}_2 + \mathfrak{s}_2\mathfrak{m}_1)$. Then $\mathfrak{B}$ is a Banach algebra with unit $\mathfrak{e} = (1, 0)$. Let $\mathfrak{C}_{\mathfrak{B}} = \{(\mathfrak{s}_1, \mathfrak{s}_2) \in \mathbb{R}^2 \mid \mathfrak{s}_1, \mathfrak{s}_2 \geq 0\}$. Then $\mathfrak{C}_{\mathfrak{B}}$ is a cone in $\mathfrak{B}$.

Let $\mathfrak{X} = \{(\mathfrak{k}, 0) \in \mathbb{R}^2 \mid \mathfrak{k} \geq 0\} \cup \{(0, 2)\} \subset \mathbb{R}^2$. Define $\partial : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathfrak{B}$ as follows:

$$\partial(\mathfrak{S}, \mathfrak{M}, \mathfrak{Z}) = \begin{cases} (0, 0), & \text{if at least two of } \mathfrak{S}, \mathfrak{M}, \mathfrak{Z} \text{ are equal}, \\ \partial(P(\mathfrak{S}, \mathfrak{M}, \mathfrak{Z})), & P \text{ denotes permutations}, \\ (\Delta, \Delta), & \text{otherwise}, \end{cases} \tag{7}$$

where Δ = square of the area of triangle $\mathfrak{S}, \mathfrak{M}, \mathfrak{Z}$. We have

$$\begin{aligned} \partial((\mathfrak{s}, 0), (\mathfrak{m}, 0), (0, 2)) \preceq \partial((\mathfrak{s}, 0), (\mathfrak{m}, 0), (\mathfrak{z}, 0)) \\ + \partial((\mathfrak{s}, 0), (\mathfrak{z}, 0), (0, 2)) \\ + \partial((\mathfrak{z}, 0), (\mathfrak{m}, 0), (0, 2)). \end{aligned} \tag{8}$$

That is, $(\mathfrak{s} - \mathfrak{m})^2 \preceq (\mathfrak{s} - \mathfrak{z})^2 + (\mathfrak{z} - \mathfrak{m})^2$, which shows that ∂ is not a cone 2-metric, because $(-9/2, -9/2) \in \mathfrak{C}_{\mathfrak{B}}$ for $\mathfrak{s}, \mathfrak{m}, \mathfrak{z} \geq 0$ with $\mathfrak{s} = 5$, $\mathfrak{m} = 0$, and $\mathfrak{z} = 1/2$. But the parameter $\mathfrak{b} = (\mathfrak{p}, 0) \geq \mathfrak{e}$ with $\mathfrak{p} \geq 2$ is a cone $\mathfrak{b}_2$-metric space over the Banach algebra $\mathfrak{B}$.

Example 27. Let $\mathfrak{B} = C^1_{\mathbb{R}}[0, 1]$. For each $\mathfrak{f}(\mathfrak{t}) \in \mathfrak{B}$, $\|\mathfrak{f}(\mathfrak{t})\| = \|\mathfrak{f}(\mathfrak{t})\|_\infty + \|\mathfrak{f}'(\mathfrak{t})\|_\infty$. The multiplication is defined point wise. Then, $\mathfrak{B}$ is a Banach algebra with unit $\mathfrak{e} = 1$ a constant function. Let $\mathfrak{C}_{\mathfrak{B}} = \{\mathfrak{f}(\mathfrak{t}) \in \mathfrak{B} \mid \mathfrak{f}(\mathfrak{t}) \geq 0, \mathfrak{t} \in [0, 1]\}$. Then, $\mathfrak{C}_{\mathfrak{B}}$ is a cone in $\mathfrak{B}$.

Let $\mathfrak{X} = \{(\mathfrak{k}, 0) \in \mathbb{R}^2 \mid 0 \leq \mathfrak{k} \leq 1\} \cup \{(0, 1)\}$. Define $\partial : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathfrak{B}$ as follows:

$$\partial(\mathfrak{S}, \mathfrak{M}, \mathfrak{Z}) = \begin{cases} \partial(p(\mathfrak{S}, \mathfrak{M}, \mathfrak{Z})), & p \text{ denotes permutations}, \\ \Delta \cdot \mathfrak{f}(\mathfrak{t}), & \text{otherwise}, \end{cases} \tag{9}$$

for all $\mathfrak{S}, \mathfrak{M}, \mathfrak{Z} \in \mathfrak{X}$, where Δ = square of the area of triangle $\mathfrak{S}, \mathfrak{M}, \mathfrak{Z}$ and $\mathfrak{f} : [0, 1] \longrightarrow \mathbb{R}$ is such that $\mathfrak{f}(\mathfrak{t}) = \mathfrak{e}^{\mathfrak{t}}$. We have

$$\begin{aligned} \partial((\mathfrak{s}, 0), (\mathfrak{m}, 0), (0, 1)) \cdot \mathfrak{e}^{\mathfrak{t}} \preceq \partial((\mathfrak{s}, 0), (\mathfrak{m}, 0), (\mathfrak{z}, 0)) \cdot \mathfrak{e}^{\mathfrak{t}} \\ + \partial((\mathfrak{s}, 0), (\mathfrak{z}, 0), (0, 1)) \cdot \mathfrak{e}^{\mathfrak{t}} \\ + \partial((\mathfrak{z}, 0), (\mathfrak{m}, 0), (0, 1)) \cdot \mathfrak{e}^{\mathfrak{t}}. \end{aligned} \tag{10}$$

That is, $(1/4)(\mathfrak{s} - \mathfrak{m})^2 \cdot \mathfrak{e}^{\mathfrak{t}} \preceq (1/4)((\mathfrak{s} - \mathfrak{z})^2 + (\mathfrak{z} - \mathfrak{m})^2) \cdot \mathfrak{e}^{\mathfrak{t}}$, which shows that ∂ is not a cone 2-metric, because $-(3/16)\mathfrak{e}^{\mathfrak{t}} \notin \mathfrak{C}_{\mathfrak{B}}$ for $0 \leq \mathfrak{s}, \mathfrak{m}, \mathfrak{z} \leq 1$ with $\mathfrak{s} = 1$, $\mathfrak{m} = 0$, and $\mathfrak{z} = 1/2$. But the parameter $\mathfrak{b} \geq 2 \in \mathfrak{C}_{\mathfrak{B}}$ is a cone $\mathfrak{b}_2$-metric space over the Banach algebra $\mathfrak{B}$.

Definition 28. Let a cone $\mathfrak{b}_2$ -metric space be $(\mathfrak{X}, \partial)$ over the Banach algebra $\mathfrak{B}$ with parameter $\mathfrak{b}$, and let $\{\mathfrak{s}_{\mathfrak{n}}\}$ be a sequence in $(\mathfrak{X}, \partial)$, then

($\mathfrak{i}_1$) $\{\mathfrak{s}_{\mathfrak{n}}\}$ converges to $\mathfrak{s} \in \mathfrak{X}$ if for every $\mathfrak{c} \gg \theta$ there exists $\mathfrak{N} \in \mathbb{N}$ such that $\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}, \mathfrak{a}) \ll \mathfrak{c}$ for all $\mathfrak{n} \geq \mathfrak{N}$. We denote it by

$$\lim_{\mathfrak{n} \to \infty} \mathfrak{s}_{\mathfrak{n}} = \mathfrak{s}, \tag{11}$$

or

$$\mathfrak{s}_{\mathfrak{n}} \longrightarrow \mathfrak{s}(\mathfrak{n} \longrightarrow \infty). \tag{12}$$

($\mathfrak{i}_2$) If for $\mathfrak{c} \gg \theta$ there is $\mathfrak{N} \in \mathbb{N}$ such $\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{m}}, \mathfrak{a}) \ll \mathfrak{c}$ for all $\mathfrak{n}, \mathfrak{m} \geq \mathfrak{N}$, then $\{\mathfrak{s}_{\mathfrak{n}}\}$ is a Cauchy sequence.

($\mathfrak{i}_3$) If every Cauchy sequence is convergent in $\mathfrak{X}$, then $(\mathfrak{X}, \partial)$ is complete.

Next in the framework of cone $\mathfrak{b}_2$-metric space over Banach's algebra, we introduce the notion of α-admissibility of mappings [20] and give the consequence of Hardy and Rogers [18] through α-admissibility in cone $\mathfrak{b}_2$-metric spaces over Banach's algebras.

Definition 29. Let $\mathfrak{X} \neq \emptyset$ and $\mathfrak{C}_{\mathfrak{B}}$ be a cone in a Banach algebra $\mathfrak{B}$. We say $\mathfrak{d}$ is α-admissible if $\mathfrak{d} : \mathfrak{X} \longrightarrow \mathfrak{X}$ and $\alpha : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathfrak{C}_{\mathfrak{B}}$, such that

$$\begin{gathered} \mathfrak{s}, \mathfrak{m} \in \mathfrak{X}, \\ \alpha(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) \geq \mathfrak{e} \forall \mathfrak{z} \in \mathfrak{X} \Longrightarrow \alpha(\mathfrak{d}\mathfrak{s}, \mathfrak{d}\mathfrak{m}, \mathfrak{z}) \geq \mathfrak{e} \forall \mathfrak{z} \in \mathfrak{X}. \end{gathered} \tag{13}$$

Definition 30. Let $\mathfrak{d} : \mathfrak{X} \longrightarrow \mathfrak{X}$ and $(\mathfrak{X}, \partial)$ is a cone $\mathfrak{b}_2$-metric space over the Banach algebra $\mathfrak{B}$. We say $\mathfrak{d}$ is continuous at point $\mathfrak{s}_0 \in \mathfrak{X}$, if for every sequence $\mathfrak{s}_{\mathfrak{n}} \in \mathfrak{X}$ we have $\mathfrak{d}\mathfrak{s}_{\mathfrak{n}} \longrightarrow \mathfrak{d}\mathfrak{s}_0$ as $\mathfrak{n} \longrightarrow \infty$, whenever $\mathfrak{s}_{\mathfrak{n}} \longrightarrow \mathfrak{s}_0$ as $\mathfrak{n} \longrightarrow \infty$. $\mathfrak{d}$ is continuous if it is continuous at every point of $\mathfrak{X}$.

Definition 31. Let a cone $\mathfrak{b}_2$ -metric space be $(\mathfrak{X}, \partial)$ over a Banach algebra $\mathfrak{B}$ with parameter $\mathfrak{b}$, $(\mathfrak{b} \geq \mathfrak{e})$, let $\mathfrak{C}_{\mathfrak{B}}$ be a solid cone, $\alpha : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathfrak{C}_{\mathfrak{B}}$, and let $\mathfrak{d} : \mathfrak{X} \longrightarrow \mathfrak{X}$ be two mappings. Then $\mathfrak{d}$ is the α -admissible Hardy-Rogers contraction with vectors $\mathfrak{A}_{\mathfrak{k}} \in \mathfrak{C}_{\mathfrak{B}}$, $\mathfrak{k} \in \{1, \cdots, 5\}$ such that $\sum_{\mathfrak{k}=1}^{5} r(\mathfrak{A}_{\mathfrak{k}}) < 1$. If

$$\alpha(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) \succeq \mathfrak{e} \forall \mathfrak{z} \in \mathfrak{X} \Longrightarrow \alpha(\mathfrak{d}\mathfrak{s}, \mathfrak{d}\mathfrak{m}, \mathfrak{z}) \succeq \mathfrak{e} \forall \mathfrak{z} \in \mathfrak{X},$$

$$\begin{aligned}\alpha(\mathfrak{s}, \mathfrak{m}, \mathfrak{z})\partial(\mathfrak{d}\mathfrak{s}, \mathfrak{d}\mathfrak{m}, \mathfrak{z}) \preceq \mathfrak{A}_1\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) &+ \mathfrak{A}_2\partial(\mathfrak{s}, \mathfrak{d}\mathfrak{s}, \mathfrak{z}) \\ &+ \mathfrak{A}_3\partial(\mathfrak{m}, \mathfrak{d}\mathfrak{m}, \mathfrak{z}) + \mathfrak{A}_4\partial(\mathfrak{s}, \mathfrak{d}\mathfrak{m}, \mathfrak{z}) \\ &+ \mathfrak{A}_5\partial(\mathfrak{m}, \mathfrak{d}\mathfrak{s}, \mathfrak{z}),\end{aligned} \tag{14}$$

for all $\mathfrak{s}, \mathfrak{m}, \mathfrak{z} \in \mathfrak{X}$ with $\alpha(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) \succeq \mathfrak{e}$.

Next, we ensure the existence of a fixed point for a continuous generalized α-admissible Hardy-Rogers contraction mapping in the context of a cone $\mathfrak{b}_2$-metric space over Banach's algebra.

Theorem 32. *Let a complete cone $\mathfrak{b}_2$-metric space be $(\mathfrak{X}, \partial)$ over the Banach algebra $\mathfrak{B}$ with parameter $\mathfrak{b}$, $(\mathfrak{b} \succeq \mathfrak{e})$, and* int $\mathfrak{C}_{\mathfrak{B}} \neq \varnothing$. *Let $\{\mathfrak{d}_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\infty}$ be a family of self-maps from $\mathfrak{X}$ to itself and vectors $\mathfrak{A}_{\mathfrak{k}} \in \mathfrak{C}_{\mathfrak{B}}$, $\mathfrak{k} \in \{1, \cdots, 5\}$ such that*

$$\begin{aligned}\alpha(\mathfrak{s}, \mathfrak{m}, \mathfrak{z})\partial\left(\mathfrak{d}_{\mathfrak{i}}(\mathfrak{s}), \mathfrak{d}_{\mathfrak{j}}(\mathfrak{m}), \mathfrak{z}\right) \preceq \mathfrak{A}_1\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) &+ \mathfrak{A}_2\partial(\mathfrak{s}, \mathfrak{d}_{\mathfrak{i}}(\mathfrak{s}), \mathfrak{z}) \\ &+ \mathfrak{A}_3\partial\left(\mathfrak{m}, \mathfrak{d}_{\mathfrak{j}}(\mathfrak{m}), \mathfrak{z}\right) \\ &+ \mathfrak{A}_4\partial\left(\mathfrak{s}, \mathfrak{d}_{\mathfrak{j}}(\mathfrak{m}), \mathfrak{z}\right) \\ &+ \mathfrak{A}_5\partial(\mathfrak{m}, \mathfrak{d}_{\mathfrak{i}}(\mathfrak{s}), \mathfrak{z}),\end{aligned} \tag{15}$$

for $\mathfrak{i}, \mathfrak{j} \geq 1$ and for all $\mathfrak{s}, \mathfrak{m}, \mathfrak{z} \in \mathfrak{X}$ together with the following:

$\mathfrak{o}_1$ *There is $\mathfrak{s}_0 \in \mathfrak{X}$ such that $\alpha(\mathfrak{s}_0, \mathfrak{d}_{\mathfrak{i}}(\mathfrak{s}_0), \mathfrak{z}) \succeq \mathfrak{e}$ for all $\mathfrak{z} \in \mathfrak{X}$;*

$\mathfrak{o}_2\{\mathfrak{d}_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\infty}$ *are continuous for all $\mathfrak{i} \geq 1$;*

$\mathfrak{o}_3\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{A}_4, \mathfrak{A}_5, \mathfrak{b}$ *commute with each other;*

$\mathfrak{o}_4 r(\mathfrak{A}_3 + \mathfrak{A}_4\mathfrak{b}) < 1$ *and $r(\mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_4\mathfrak{b}) < 1$.*

Then $\{\mathfrak{d}_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\infty}$ share a common fixed point in $\mathfrak{X}$.

Proof. Choose $\mathfrak{s}_0 \in \mathfrak{X}$ in such a way that

$$\alpha(\mathfrak{s}_0, \mathfrak{d}_{\mathfrak{i}}(\mathfrak{s}_0), \mathfrak{z}) \succeq \mathfrak{e} \text{ for all } \mathfrak{z} \in \mathfrak{X}. \tag{16}$$

Now, let $\mathfrak{s}_1 = \mathfrak{d}_1(\mathfrak{s}_0)$. Then, $\alpha(\mathfrak{s}_0, \mathfrak{s}_1, \mathfrak{z}) \succeq \mathfrak{e}$ for all $\mathfrak{z} \in \mathfrak{X}$. Again, we put $\mathfrak{s}_2 = \mathfrak{d}_2(\mathfrak{s}_1)$ and using α-admissibility of $\mathfrak{d}_{\mathfrak{i}}$, we have

$$\alpha(\mathfrak{d}_1(\mathfrak{s}_0), \mathfrak{d}_2(\mathfrak{s}_1), \mathfrak{z}) = \alpha(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{z}) \succeq \mathfrak{e}. \tag{17}$$

Putting $\mathfrak{s}_3 = \mathfrak{d}_3(\mathfrak{s}_2)$ and using α-admissibility of $\mathfrak{d}_{\mathfrak{i}}$, we have

$$\alpha(\mathfrak{d}_2(\mathfrak{s}_1), \mathfrak{d}_3(\mathfrak{s}_2), \mathfrak{z}) = \alpha(\mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{z}) \succeq \mathfrak{e}. \tag{18}$$

By induction, we construct a sequence $\{\mathfrak{s}_{\mathfrak{n}}\}$ in $\mathfrak{X}$ by $\mathfrak{s}_{\mathfrak{n}+1} = \mathfrak{d}_{\mathfrak{n}+1}(\mathfrak{s}_{\mathfrak{n}})$ for $\mathfrak{n} \in \mathbb{N}$ such that

$$\alpha(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{n}+1}, \mathfrak{z}) \succeq \mathfrak{e} \text{ for all } \mathfrak{z} \in \mathfrak{X}. \tag{19}$$

From condition (3) we obtain

$$\begin{aligned}\alpha(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{n}}, \mathfrak{z})\partial(\mathfrak{d}_{\mathfrak{n}}(\mathfrak{s}_{\mathfrak{n}-1}), \mathfrak{d}_{\mathfrak{n}+1}(\mathfrak{s}_{\mathfrak{n}}), \mathfrak{z}) &\preceq \mathfrak{A}_1\partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{n}}, \mathfrak{z}) \\ &+ \mathfrak{A}_2\partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{d}_{\mathfrak{s}_{\mathfrak{n}}}(\mathfrak{s}_{\mathfrak{n}-1}), \mathfrak{z}) + \mathfrak{A}_3\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{d}_{\mathfrak{n}+1}(\mathfrak{s}_{\mathfrak{n}}), \mathfrak{z}) \\ &+ \mathfrak{A}_4\partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{d}_{\mathfrak{n}+1}(\mathfrak{s}_{\mathfrak{n}}), \mathfrak{z}) + \mathfrak{A}_5\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{d}_{\mathfrak{n}}(\mathfrak{s}_{\mathfrak{n}-1}), \mathfrak{z}),\end{aligned} \tag{20}$$

that is

$$\begin{aligned}\alpha(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{n}}, \mathfrak{z})\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{n}+1}, \mathfrak{z}) \preceq \mathfrak{A}_1\partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{n}}, \mathfrak{z}) &+ \mathfrak{A}_2\partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{n}}, \mathfrak{z}) \\ &+ \mathfrak{A}_3\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{n}+1}, \mathfrak{z}) \\ &+ \mathfrak{A}_4\partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{n}+1}, \mathfrak{z}).\end{aligned} \tag{21}$$

Since, $\alpha(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{n}}, \mathfrak{z}) \succeq \mathfrak{e}$, then we obtain for all $\mathfrak{z} \in \mathfrak{X}$ and for all $\mathfrak{n} \in \mathbb{N}$

$$\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{n}+1}, \mathfrak{z}) \leq \alpha(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{n}}, \mathfrak{z})\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{n}+1}, \mathfrak{z}). \tag{22}$$

Therefore, (21) becomes

$$\begin{aligned}(\mathfrak{e} - \mathfrak{A}_3)\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{n}+1}, \mathfrak{z}) &\preceq (\mathfrak{A}_1 + \mathfrak{A}_2)\partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{n}}, \mathfrak{z}) \\ &\quad + \mathfrak{A}_4\partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{n}+1}, \mathfrak{z}) \\ &\preceq (\mathfrak{A}_1 + \mathfrak{A}_2)\partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{n}}, \mathfrak{z}) \\ &\quad + \mathfrak{A}_4\mathfrak{b}[\partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{n}+1}, \mathfrak{s}_{\mathfrak{n}}) \\ &\quad + \partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{n}}, \mathfrak{z}) + \partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{n}+1}, \mathfrak{z})].\end{aligned} \tag{23}$$

Assume that for any $\mathfrak{t} \in \mathbb{N}$, we have

$$\begin{aligned}\alpha(\mathfrak{s}_{\mathfrak{t}-1}, \mathfrak{s}_{\mathfrak{t}}, \mathfrak{s}_{\mathfrak{t}-1})\partial(\mathfrak{d}_{\mathfrak{t}}(\mathfrak{s}_{\mathfrak{t}-1}), \mathfrak{d}_{\mathfrak{t}+1}(\mathfrak{s}_{\mathfrak{t}}), \mathfrak{s}_{\mathfrak{t}-1}) &\preceq \mathfrak{A}_1\partial(\mathfrak{s}_{\mathfrak{t}-1}, \mathfrak{s}_{\mathfrak{t}}, \mathfrak{s}_{\mathfrak{t}-1}) \\ &+ \mathfrak{A}_2\partial(\mathfrak{s}_{\mathfrak{t}-1}, \mathfrak{d}_{\mathfrak{t}}(\mathfrak{s}_{\mathfrak{t}-1}), \mathfrak{s}_{\mathfrak{t}-1}) + \mathfrak{A}_3\partial(\mathfrak{s}_{\mathfrak{t}}, \mathfrak{d}_{\mathfrak{t}+1}(\mathfrak{s}_{\mathfrak{t}}), \mathfrak{s}_{\mathfrak{t}-1}) \\ &+ \mathfrak{A}_4\partial(\mathfrak{s}_{\mathfrak{t}-1}, \mathfrak{d}_{\mathfrak{t}+1}(\mathfrak{s}_{\mathfrak{t}}), \mathfrak{s}_{\mathfrak{t}-1}) + \mathfrak{A}_5\partial(\mathfrak{s}_{\mathfrak{t}}, \mathfrak{d}_{\mathfrak{t}}(\mathfrak{s}_{\mathfrak{t}-1}), \mathfrak{s}_{\mathfrak{t}-1}),\end{aligned} \tag{24}$$

that is

$$\alpha(\mathfrak{s}_{\mathfrak{t}-1}, \mathfrak{s}_{\mathfrak{t}}, \mathfrak{s}_{\mathfrak{t}-1})\partial(\mathfrak{s}_{\mathfrak{t}}, \mathfrak{s}_{\mathfrak{t}+1}, \mathfrak{s}_{\mathfrak{t}-1}) \preceq \mathfrak{A}_3\partial(\mathfrak{s}_{\mathfrak{t}}, \mathfrak{s}_{\mathfrak{t}+1}, \mathfrak{s}_{\mathfrak{t}-1}). \tag{25}$$

Since, $\alpha(\mathfrak{s}_{\mathfrak{t}-1}, \mathfrak{s}_{\mathfrak{t}}, \mathfrak{z}) \succeq \mathfrak{e}$ for all $\mathfrak{z} \in \mathfrak{X}$, particularly, if $\mathfrak{z} = \mathfrak{s}_{\mathfrak{t}-1}$ for $\mathfrak{t} \in \mathbb{N}$, then we have $\alpha(\mathfrak{s}_{\mathfrak{t}-1}, \mathfrak{s}_{\mathfrak{t}}, \mathfrak{z}) \succeq \mathfrak{e}$, and hence

$$\partial(\mathfrak{s}_{\mathfrak{t}}, \mathfrak{s}_{\mathfrak{t}+1}, \mathfrak{s}_{\mathfrak{t}-1}) \preceq \mathfrak{A}_3\partial(\mathfrak{s}_{\mathfrak{t}}, \mathfrak{s}_{\mathfrak{t}+1}, \mathfrak{s}_{\mathfrak{t}-1}), \tag{26}$$

which is possible only when $\partial(\mathfrak{s}_{\mathfrak{t}}, \mathfrak{s}_{\mathfrak{t}+1}, \mathfrak{s}_{\mathfrak{t}-1}) = \theta$ by Lemma 20. Therefore, (5) becomes

$$(\mathfrak{e} - \mathfrak{A}_3 - \mathfrak{A}_4\mathfrak{b})\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{n}+1}, \mathfrak{z}) \preceq (\mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_4\mathfrak{b})\partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{n}}, \mathfrak{z}). \tag{27}$$

Since $r(\mathfrak{A}_3 + \mathfrak{A}_4\mathfrak{b}) \leq r(\mathfrak{A}_3) + r(\mathfrak{A}_4)r(\mathfrak{b}) < 1$, from Proposition 2, we have

$$\partial(\mathfrak{s}_{\mathfrak{n}+1}, \mathfrak{s}_{\mathfrak{n}}, \mathfrak{z}) \preceq \mathfrak{L}\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{z}), \tag{28}$$

where $\mathfrak{L} = (\mathfrak{e} - \mathfrak{A}_3 - \mathfrak{A}_4\mathfrak{b})^{-1}(\mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_4\mathfrak{b})$.

Similarly, $\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{z}) \preceq \mathfrak{L}\partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{n}-2}, \mathfrak{z})$, and hence we have for all $\mathfrak{z} \in \mathfrak{X}$

$$\partial(\mathfrak{s}_{\mathfrak{n}+1}, \mathfrak{s}_{\mathfrak{n}}, \mathfrak{z}) \leq \mathfrak{L}^{\mathfrak{n}}\partial(\mathfrak{s}_1, \mathfrak{s}_0, \mathfrak{z}). \tag{29}$$

In this case, for all $\mathfrak{l} < \mathfrak{k}$, proceeding in a similar way as above, we have

$$\partial(\mathfrak{s}_{\mathfrak{k}}, \mathfrak{s}_{\mathfrak{k}-1}, \mathfrak{s}_{\mathfrak{l}}) \preceq \mathfrak{L}^{\mathfrak{k}-\mathfrak{l}-1}\partial(\mathfrak{s}_{\mathfrak{l}+1}, \mathfrak{s}_{\mathfrak{l}}, \mathfrak{s}_{\mathfrak{l}}) = \theta, \tag{30}$$

that is

$$\partial(\mathfrak{s}_{\mathfrak{k}}, \mathfrak{s}_{\mathfrak{k}-1}, \mathfrak{s}_{\mathfrak{l}}) = \theta. \tag{31}$$

Therefore, for all $\mathfrak{m} < \mathfrak{n}$, we have

$$\begin{aligned}
\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{m}}, \mathfrak{z}) &\preceq \mathfrak{b}[\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{m}}, \mathfrak{s}_{\mathfrak{n}-1}) + \partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{z}) + \partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{m}}, \mathfrak{z})] \\
&= \mathfrak{b}\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{m}}) + \mathfrak{b}\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{z}) + \mathfrak{b}\partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{m}}, \mathfrak{z}) \\
&\preceq \mathfrak{b}\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{z}) + \mathfrak{b}[\mathfrak{b}\partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{m}}, \mathfrak{s}_{\mathfrak{n}-2}) \\
&\quad + \mathfrak{b}\partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{n}-2}, \mathfrak{z}) + \mathfrak{b}\partial(\mathfrak{s}_{\mathfrak{n}-2}, \mathfrak{s}_{\mathfrak{m}}, \mathfrak{z})] \\
&\preceq \mathfrak{b}\partial(\mathfrak{s}_{\mathfrak{n}}, \mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{z}) + \mathfrak{b}^2\partial(\mathfrak{s}_{\mathfrak{n}-1}, \mathfrak{s}_{\mathfrak{n}-2}, \mathfrak{z}) + \cdots + \mathfrak{b}^{\mathfrak{n}-\mathfrak{m}}\partial(\mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{s}_{\mathfrak{m}}, \mathfrak{z}) \\
&\preceq \left(\mathfrak{b}\mathfrak{L}^{\mathfrak{n}-1} + \mathfrak{b}^2\mathfrak{L}^{\mathfrak{n}-2} + \mathfrak{b}^3\mathfrak{L}^{\mathfrak{n}-3} + \cdots + \mathfrak{b}^{\mathfrak{n}-\mathfrak{m}}\mathfrak{L}^{\mathfrak{m}}\right)\partial(\mathfrak{s}_1, \mathfrak{s}_0, \mathfrak{z}) \\
&= \mathfrak{b}^{\mathfrak{n}-\mathfrak{m}}\mathfrak{L}^{\mathfrak{m}}\left[\mathfrak{e} + \mathfrak{b}^{-1}\mathfrak{L} + \left(\mathfrak{b}^{-1}\mathfrak{L}\right)^2 + \cdots + \left(\mathfrak{b}^{-1}\mathfrak{L}\right)^{\mathfrak{n}-\mathfrak{m}-1}\right]\partial(\mathfrak{s}_1, \mathfrak{s}_0, \mathfrak{z}) \\
&= \mathfrak{b}^{\mathfrak{n}-\mathfrak{m}}\mathfrak{L}^{\mathfrak{m}}\left[\sum_{\mathfrak{k}=0}^{\infty}\left(\mathfrak{b}^{-1}\mathfrak{L}\right)^{\mathfrak{k}}\right]\partial(\mathfrak{s}_1, \mathfrak{s}_0, \mathfrak{z}) \\
&= \left(\mathfrak{e} - \mathfrak{b}^{-1}\mathfrak{L}\right)^{-1}\mathfrak{b}^{\mathfrak{n}-\mathfrak{m}}\mathfrak{L}^{\mathfrak{m}}\partial(\mathfrak{s}_1, \mathfrak{s}_0, \mathfrak{z}).
\end{aligned} \tag{32}$$

From Lemma 17 and Lemma 19, we have

$$\begin{aligned}
r(\mathfrak{L}) &= r\left((\mathfrak{e} - \mathfrak{A}_3 - \mathfrak{A}_4\mathfrak{b})^{-1}(\mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_4\mathfrak{b})\right) \\
&\leq \frac{r(\mathfrak{A}_1) + r(\mathfrak{A}_2) + r(\mathfrak{A}_4)r(\mathfrak{b})}{1 - r(\mathfrak{A}_3) - r(\mathfrak{A}_4)r(\mathfrak{b})} < 1.
\end{aligned} \tag{33}$$

As $r(\mathfrak{L}) < 1$, so that in the light of Remark 5, we can get to know $\|\mathfrak{b}^{\mathfrak{n}-\mathfrak{m}}\mathfrak{L}^{\mathfrak{m}}\partial(\mathfrak{s}_1, \mathfrak{s}_0, \mathfrak{z})\| \leq \|\mathfrak{b}^{\mathfrak{n}-\mathfrak{m}}\mathfrak{L}^{\mathfrak{m}}\|\|\partial(\mathfrak{s}_1, \mathfrak{s}_0, \mathfrak{z})\| \longrightarrow 0$ as $(n \longrightarrow \infty)$, by Lemma 15 we have $\{\mathfrak{b}^{\mathfrak{n}-\mathfrak{m}}\mathfrak{L}^{\mathfrak{m}}\partial(\mathfrak{s}_1, \mathfrak{s}_0, \mathfrak{z})\}$, a $\mathfrak{c}$-sequence in $\mathfrak{X}$. At last, by using Lemmas 13 and 22, we get that $\{\mathfrak{s}_{\mathfrak{n}}\}$ is a Cauchy sequence in $\mathfrak{X}$. In addition, $(\mathfrak{X}, \partial)$ is complete; therefore, there exists some $\mathfrak{s}^* \in \mathfrak{X}$ such that

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{s}_{\mathfrak{n}} = \mathfrak{s}^*. \tag{34}$$

Since $\mathfrak{d}_{\mathfrak{i}}{}'^{\mathfrak{s}}$ are continuous for $\mathfrak{i} = 1, 2, 3, \cdots$.

Therefore, for $\mathfrak{s}_{\mathfrak{n}} \longrightarrow \mathfrak{s}^*$, we have $\mathfrak{d}_{\mathfrak{n}+1}(\mathfrak{s}_{\mathfrak{n}}) \longrightarrow \mathfrak{d}_{\mathfrak{n}+1}(\mathfrak{s}^*)$ as $\mathfrak{n} \longrightarrow \infty$. But as $\mathfrak{s}_{\mathfrak{n}+1} = \mathfrak{d}_{\mathfrak{n}+1}(\mathfrak{s}_{\mathfrak{n}}) \longrightarrow \mathfrak{d}_{\mathfrak{n}+1}(\mathfrak{s}^*)$ as $\mathfrak{n} \longrightarrow \infty$, therefore, from the uniqueness of the limit, we get $\mathfrak{d}_{\mathfrak{n}+1}(\mathfrak{s}^*) = \mathfrak{s}^*$, that is, $\mathfrak{s}^*$ is a common fixed point of $\{\mathfrak{d}_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\infty}$.

Remark 33. Our Theorem 32 generalizes Theorem 1 in [38] from a cone $\mathfrak{b}$ -metric space over a Banach algebra to a cone $\mathfrak{b}_2$ -metric space over a Banach algebra.

Theorem 34. *Let a complete cone $\mathfrak{b}_2$ -metric space be $(\mathfrak{X}, \partial)$ over a Banach algebra $\mathfrak{B}$ with parameter $\mathfrak{b}$, $(\mathfrak{b} \geq \mathfrak{e})$, and* int $\mathfrak{C}_{\mathfrak{B}} \neq \emptyset$*. Let $\{\mathfrak{d}\}_{\mathfrak{i}=1}^{\infty}$ be a family of self-maps from $\mathfrak{X}$ to itself. Assume that $\{\mathfrak{m}_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\infty}$ is a nonnegative integer sequence and vectors $\mathfrak{A}_{\mathfrak{k}} \in \mathfrak{C}_{\mathfrak{B}}$, $\mathfrak{k} \in \{1, \cdots, 5\}$ such that*

$$\begin{aligned}
\partial\left(\mathfrak{d}_{\mathfrak{i}}^{\mathfrak{m}_{\mathfrak{i}}}(\mathfrak{s}), \mathfrak{d}_{\mathfrak{j}}^{\mathfrak{m}_{\mathfrak{j}}}(\mathfrak{m}), \mathfrak{z}\right) \preceq\, &\mathfrak{A}_1\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) + \mathfrak{A}_2\partial\left(\mathfrak{s}, \mathfrak{d}_{\mathfrak{i}}^{\mathfrak{m}_{\mathfrak{i}}}(\mathfrak{s}), \mathfrak{z}\right) \\
&+ \mathfrak{A}_3\partial\left(\mathfrak{m}, \mathfrak{d}_{\mathfrak{j}}^{\mathfrak{m}_{\mathfrak{j}}}(\mathfrak{m}), \mathfrak{z}\right) \\
&+ \mathfrak{A}_4\partial\left(\mathfrak{s}, \mathfrak{d}_{\mathfrak{j}}^{\mathfrak{m}_{\mathfrak{j}}}(\mathfrak{m}), \mathfrak{z}\right) \\
&+ \mathfrak{A}_5\partial\left(\mathfrak{m}, \mathfrak{d}_{\mathfrak{i}}^{\mathfrak{m}_{\mathfrak{i}}}(\mathfrak{s}), \mathfrak{z}\right),
\end{aligned} \tag{35}$$

or

$$\begin{aligned}
\partial\left(\mathfrak{d}_{\mathfrak{i}}(\mathfrak{s}), \mathfrak{d}_{\mathfrak{j}}(\mathfrak{m}), \mathfrak{z}\right) \preceq\, &\mathfrak{A}_1\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) + \mathfrak{A}_2\partial(\mathfrak{s}, \mathfrak{d}_{\mathfrak{i}}(\mathfrak{s}), \mathfrak{z}) \\
&+ \mathfrak{A}_3\partial\left(\mathfrak{m}, \mathfrak{d}_{\mathfrak{j}}(\mathfrak{m}), \mathfrak{z}\right) + \mathfrak{A}_4\partial\left(\mathfrak{s}, \mathfrak{d}_{\mathfrak{j}}(\mathfrak{m}), \mathfrak{z}\right) \\
&+ \mathfrak{A}_5\partial(\mathfrak{m}, \mathfrak{d}_{\mathfrak{i}}(\mathfrak{s}), \mathfrak{z}),
\end{aligned} \tag{36}$$

for $\mathfrak{i}, \mathfrak{j} \geq 1$ and for all $\mathfrak{s}, \mathfrak{m}, \mathfrak{z} \in \mathfrak{X}$ together with

$\mathfrak{p}_1$ $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, $\mathfrak{A}_4$, $\mathfrak{A}_5$, and $\mathfrak{b}$ commute with each other;

$\mathfrak{p}_2$ $r(\mathfrak{A}_3 + \mathfrak{A}_4\mathfrak{b}) < 1$, $r(\mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_4\mathfrak{b}) < 1$, $r(\mathfrak{b}\mathfrak{A}_2) < 1$, and $r(\mathfrak{A}_1 + \mathfrak{A}_4 + \mathfrak{A}_5) < 1$.

Then $\{\mathfrak{d}_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\infty}$ share a unique common fixed point in $\mathfrak{X}$.

Proof. On taking $\alpha = \mathfrak{e}$ in Theorem 32, set $\Theta_{\mathfrak{i}} = \mathfrak{d}_{\mathfrak{i}}^{\mathfrak{m}_{\mathfrak{i}}}$ for $\mathfrak{i} = 1, 2, 3, \cdots$. Then (35) becomes

$$\begin{aligned}
\partial\left(\Theta_{\mathfrak{i}}(\mathfrak{s}), \Theta_{\mathfrak{j}}(\mathfrak{m}), \mathfrak{z}\right) \preceq\, &\mathfrak{A}_1\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) + \mathfrak{A}_2\partial(\mathfrak{s}, \Theta_{\mathfrak{i}}(\mathfrak{s}), \mathfrak{z}) \\
&+ \mathfrak{A}_3\partial\left(\mathfrak{m}, \Theta_{\mathfrak{j}}(\mathfrak{m}), \mathfrak{z}\right) + \mathfrak{A}_4\partial\left(\mathfrak{s}, \Theta_{\mathfrak{j}}(\mathfrak{m}), \mathfrak{z}\right) \\
&+ \mathfrak{A}_5\partial(\mathfrak{m}, \Theta_{\mathfrak{i}}(\mathfrak{s}), \mathfrak{z}).
\end{aligned} \tag{37}$$

Choose $\mathfrak{s}_0 \in \mathfrak{X}$ arbitrarily and construct a sequence $\{\mathfrak{s}_{\mathfrak{n}}\}$ by $\mathfrak{s}_{\mathfrak{n}+1} = \Theta_{\mathfrak{s}_{\mathfrak{n}+1}}(\mathfrak{s}_{\mathfrak{n}})$ for $\mathfrak{n} \in \mathbb{N}$, then using the same method as the proof of Theorem 32, one can easily show that $\{\mathfrak{s}_{\mathfrak{n}}\}$ is a Cauchy sequence, and hence from the completeness of $(\mathfrak{X}, \partial)$, there exists $\mathfrak{s}^* \in \mathfrak{X}$ such that

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{s}_{\mathfrak{n}} = \mathfrak{s}^*. \tag{38}$$

Now, we show that $\mathfrak{s}^*$ is a fixed point for a family of self-maps $\{\Theta_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\infty}$:

$$\begin{aligned}\partial(\Theta_{\mathfrak{n}}(\mathfrak{s}^*), \mathfrak{s}^*, \mathfrak{z}) &\preceq \mathfrak{b}[\partial(\Theta_{\mathfrak{n}}(\mathfrak{s}^*), \mathfrak{s}^*, \mathfrak{s}_{\mathfrak{m}+1}) \\ &\quad + \partial(\Theta_{\mathfrak{n}}(\mathfrak{s}^*), \mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{z}) + \partial(\mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{s}^*, \mathfrak{z})] \\ &= \mathfrak{b}\partial(\mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{s}^*, \mathfrak{z}) + \mathfrak{b}\partial(\Theta_{\mathfrak{n}}(\mathfrak{s}^*), \Theta_{\mathfrak{m}+1}(\mathfrak{s}_{\mathfrak{m}}), \mathfrak{z}) \\ &\quad + \mathfrak{b}\partial(\Theta_{\mathfrak{n}}(\mathfrak{s}^*), \Theta_{\mathfrak{m}+1}(\mathfrak{s}_{\mathfrak{m}}), \mathfrak{s}^*) \preceq \mathfrak{b}\partial(\mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{s}^*, \mathfrak{z}) \\ &\quad + [\mathfrak{b}\mathfrak{A}_1\partial(\mathfrak{s}^*, \mathfrak{s}_{\mathfrak{m}}, \mathfrak{z}) + \mathfrak{b}\mathfrak{A}_2\partial(\mathfrak{s}^*, \Theta_{\mathfrak{n}}(\mathfrak{s}^*), \mathfrak{z}) \\ &\quad + \mathfrak{b}\mathfrak{A}_3\partial(\mathfrak{s}_{\mathfrak{m}}, \mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{z}) + \mathfrak{b}\mathfrak{A}_4\partial(\mathfrak{s}^*, \mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{z}) \\ &\quad + \mathfrak{b}\mathfrak{A}_5\partial(\mathfrak{s}_{\mathfrak{m}}, \Theta_{\mathfrak{n}}(\mathfrak{s}^*), \mathfrak{z})] + [\mathfrak{b}\mathfrak{A}_1\partial(\mathfrak{s}^*, \mathfrak{s}_{\mathfrak{m}}, \mathfrak{s}^*) \\ &\quad + \mathfrak{b}\mathfrak{A}_2\partial(\mathfrak{s}^*, \Theta_{\mathfrak{n}}(\mathfrak{s}^*), \mathfrak{s}^*) + \mathfrak{b}\mathfrak{A}_3\partial(\mathfrak{s}_{\mathfrak{m}}, \mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{s}^*) \\ &\quad + \mathfrak{b}\mathfrak{A}_4\partial(\mathfrak{s}^*, \mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{s}^*) + \mathfrak{b}\mathfrak{A}_5\partial(\mathfrak{s}_{\mathfrak{m}}, \Theta_{\mathfrak{n}}(\mathfrak{s}^*), \mathfrak{s}^*)] \\ &\preceq \mathfrak{b}\partial(\mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{s}^*, \mathfrak{z}) + \mathfrak{b}\mathfrak{A}_1\partial(\mathfrak{s}^*, \mathfrak{s}_{\mathfrak{m}}, \mathfrak{z}) \\ &\quad + \mathfrak{b}\mathfrak{A}_2\partial(\mathfrak{s}^*, \Theta_{\mathfrak{n}}(\mathfrak{s}^*), \mathfrak{z}) + \mathfrak{b}\mathfrak{A}_3\partial(\mathfrak{s}_{\mathfrak{m}}, \mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{z}) \\ &\quad + \mathfrak{b}\mathfrak{A}_4\partial(\mathfrak{s}^*, \mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{z}) + \mathfrak{b}\mathfrak{A}_5\partial(\mathfrak{s}_{\mathfrak{m}}, \Theta_{\mathfrak{n}}(\mathfrak{s}^*), \mathfrak{z}) \\ &\quad + \mathfrak{b}\mathfrak{A}_3\partial(\mathfrak{s}_{\mathfrak{m}}, \mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{s}^*) + \mathfrak{b}\mathfrak{A}_5\partial(\mathfrak{s}_{\mathfrak{m}}, \Theta_{\mathfrak{n}}(\mathfrak{s}^*), \mathfrak{s}^*).\end{aligned} \tag{39}$$

That is,

$$\begin{aligned}(\mathfrak{e} - \mathfrak{b}\mathfrak{A}_2)\partial(\Theta_{\mathfrak{n}}(\mathfrak{s}^*), \mathfrak{s}^*, \mathfrak{z}) &\preceq (\mathfrak{b} + \mathfrak{b}\mathfrak{A}_4)\partial(\mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{s}^*, \mathfrak{z}) \\ &\quad + \mathfrak{b}\mathfrak{A}_1\partial(\mathfrak{s}^*, \mathfrak{s}_{\mathfrak{m}}, \mathfrak{z}) \\ &\quad + \mathfrak{b}\mathfrak{A}_3\partial(\mathfrak{s}_{\mathfrak{m}}, \mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{z}) \\ &\quad + \mathfrak{b}\mathfrak{A}_5\partial(\mathfrak{s}_{\mathfrak{m}}, \Theta_{\mathfrak{n}}(\mathfrak{s}^*), \mathfrak{z}) \\ &\quad + \mathfrak{b}\mathfrak{A}_5\partial(\mathfrak{s}_{\mathfrak{m}}, \Theta_{\mathfrak{n}}(\mathfrak{s}^*), \mathfrak{s}^*) \\ &\quad + \mathfrak{b}\mathfrak{A}_3\partial(\mathfrak{s}_{\mathfrak{m}}, \mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{s}^*).\end{aligned} \tag{40}$$

Since $r(\mathfrak{b}\mathfrak{A}_2) \leq r(\mathfrak{b})r(\mathfrak{A}_2) < 1$, so by Proposition 2, we have $(\mathfrak{e} - \mathfrak{b}\mathfrak{A}_2)$ which is invertible:

$$\begin{aligned}\partial(\Theta_{\mathfrak{n}}(\mathfrak{s}^*), \mathfrak{s}^*, \mathfrak{z}) &\preceq (\mathfrak{e} - \mathfrak{b}\mathfrak{A}_2)^{-1}[(\mathfrak{b} + \mathfrak{b}\mathfrak{A}_4)\partial(\mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{s}^*, \mathfrak{z}) \\ &\quad + \mathfrak{b}\mathfrak{A}_1\partial(\mathfrak{s}^*, \mathfrak{s}_{\mathfrak{m}}, \mathfrak{z}) + \mathfrak{b}\mathfrak{A}_3\partial(\mathfrak{s}_{\mathfrak{m}}, \mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{z}) \\ &\quad + \mathfrak{b}\mathfrak{A}_5\partial(\mathfrak{s}_{\mathfrak{m}}, \Theta_{\mathfrak{n}}(\mathfrak{s}^*), \mathfrak{z}) + \mathfrak{b}\mathfrak{A}_5\partial(\mathfrak{s}_{\mathfrak{m}}, \Theta_{\mathfrak{n}}(\mathfrak{s}^*), \mathfrak{s}^*) \\ &\quad + \mathfrak{b}\mathfrak{A}_3\partial(\mathfrak{s}_{\mathfrak{m}}, \mathfrak{s}_{\mathfrak{m}+1}, \mathfrak{s}^*)].\end{aligned} \tag{41}$$

Keeping $\mathfrak{n}$ fixed and using Lemma 13 and Lemma 14, the right-hand side of the above inequality is a $\mathfrak{c}$-sequence.

Therefore, for any $\mathfrak{c} \in \mathfrak{X}$ with $\mathfrak{c} \gg \theta$ and using Lemma 22, we have $\partial(\Theta_{\mathfrak{n}}(\mathfrak{s}^*), \mathfrak{s}^*, \mathfrak{z}) = \theta$ for all $\mathfrak{z} \in \mathfrak{X}$. Hence, $\Theta_{\mathfrak{n}}(\mathfrak{s}^*) = \mathfrak{s}^*$ for all $\mathfrak{n} = 1, 2, 3, \cdots$, that is, $\mathfrak{s}^*$ is a fixed point of $\Theta_{\mathfrak{n}}$.

Assume that $\mathfrak{o}^*$ be another fixed point of $\Theta_{\mathfrak{n}}$, that is, $\Theta_{\mathfrak{n}}(\mathfrak{o}^*) = \mathfrak{o}^*$. Then using (37), we have

$$\begin{aligned}\partial(\mathfrak{s}^*, \mathfrak{o}^*, \mathfrak{z}) &= \partial\left(\Theta_{\mathfrak{n}}(\mathfrak{s}^*), \Theta_{(\mathfrak{o}^*)}, \mathfrak{z}\right) \preceq \mathfrak{A}_1\partial(\mathfrak{s}^*, \mathfrak{o}^*, \mathfrak{z}) \\ &\quad + \mathfrak{A}_2\partial(\mathfrak{s}^*, \mathfrak{s}^*, \mathfrak{z}) + \mathfrak{A}_3\partial(\mathfrak{o}^*, \mathfrak{o}^*, \mathfrak{z}) \\ &\quad + \mathfrak{A}_4\partial(\mathfrak{s}^*, \mathfrak{o}^*, \mathfrak{z}) + \mathfrak{A}_5\partial(\mathfrak{o}^*, \mathfrak{s}^*, \mathfrak{z}) \\ &= (\mathfrak{A}_1 + \mathfrak{A}_4 + \mathfrak{A}_5)\partial(\mathfrak{s}^*, \mathfrak{o}^*, \mathfrak{z}),\end{aligned} \tag{42}$$

that is, $\partial(\mathfrak{s}^*, \mathfrak{o}^*, \mathfrak{z}) = \theta$ for all $\mathfrak{z} \in \mathfrak{X}$.

Therefore, $\mathfrak{s}^* = \mathfrak{o}^*$ is the unique fixed point of $\{\Theta_{\mathfrak{n}}\}_{\mathfrak{n}=1}^{\infty}$.

Thus, we have $\Theta_{\mathfrak{n}}(\mathfrak{s}^*) = \mathfrak{d}_{\mathfrak{n}}^{\mathfrak{m}_{\mathfrak{n}}}(\mathfrak{s}^*) = \mathfrak{s}^*$.

Also, $\mathfrak{d}_{\mathfrak{n}}(\mathfrak{s}^*) = \mathfrak{d}_{\mathfrak{n}}(\mathfrak{d}_{\mathfrak{n}}^{\mathfrak{m}_{\mathfrak{n}}}(\mathfrak{s}^*)) = \mathfrak{d}_{\mathfrak{n}}^{\mathfrak{m}_{\mathfrak{n}}}(\mathfrak{d}_{\mathfrak{n}}(\mathfrak{s}^*)) = \Theta_{\mathfrak{n}}(\mathfrak{d}_{\mathfrak{n}}(\mathfrak{s}^*))$.

That is, $\mathfrak{d}_{\mathfrak{n}}(\mathfrak{s}^*) = \Theta_{\mathfrak{n}}(\mathfrak{d}_{\mathfrak{n}}(\mathfrak{s}^*))$, which implies that $\mathfrak{d}_{\mathfrak{n}}(\mathfrak{s}^*)$ is also a fixed point of $\Theta_{\mathfrak{n}}$. But the fixed point of $\Theta_{\mathfrak{n}}$ is unique which is $\mathfrak{s}^*$; therefore, we must accept that $\mathfrak{d}_{\mathfrak{n}}(\mathfrak{s}^*) = \mathfrak{s}^*$.

For uniqueness, let $\mathfrak{d}_{\mathfrak{n}}(\mathfrak{z}^*) = \mathfrak{z}^*$. That is, $\mathfrak{d}_{\mathfrak{n}}^{\mathfrak{m}_{\mathfrak{n}}}(\mathfrak{z}^*) = \mathfrak{z}^* = \Theta_{\mathfrak{n}}(\mathfrak{z}^*)$.

Since the fixed point of $\Theta_{\mathfrak{n}}$ is unique and is $\mathfrak{s}^*$, therefore, $\mathfrak{s}^* = \mathfrak{z}^*$.

Remark 35. Our Theorem 34 generalizes Theorem 3.2 in [27] from a cone 2-metric space over a Banach algebra to a cone $\mathfrak{b}_2$ -metric space over Banach's algebra.

From Theorem 34, we obtain the following corollaries.

Corollary 36. *Let a complete cone $\mathfrak{b}_2$ -metric space be $(\mathfrak{X}, \partial)$ over the Banach algebra $\mathfrak{B}$ with parameter $\mathfrak{b}$, $(\mathfrak{b} \succeq \mathfrak{e})$, and* int $\mathfrak{C}_{\mathfrak{B}} \neq \emptyset$. *Let $\{\mathfrak{d}_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\infty}$ be a family of self-maps from $\mathfrak{X}$ to itself. Assume that $\{\mathfrak{m}_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\infty}$ is a nonnegative integer sequence and vectors $\mathfrak{A}_{\mathfrak{k}} \in \mathfrak{C}_{\mathfrak{B}}$, $\mathfrak{k} \in \{1, 2, 3\}$ such that*

$$\begin{aligned}\partial\left(\mathfrak{d}_{\mathfrak{i}}^{\mathfrak{m}_{\mathfrak{i}}}(\mathfrak{s}), \mathfrak{d}_{\mathfrak{j}}^{\mathfrak{m}_{\mathfrak{j}}}(\mathfrak{m}), \mathfrak{z}\right) &\preceq \mathfrak{A}_1\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) + \mathfrak{A}_2\partial\left(\mathfrak{s}, \mathfrak{d}_{\mathfrak{i}}^{\mathfrak{m}_{\mathfrak{i}}}(\mathfrak{s}), \mathfrak{z}\right) \\ &\quad + \mathfrak{A}_3\partial\left(\mathfrak{m}, \mathfrak{d}_{\mathfrak{j}}^{\mathfrak{m}_{\mathfrak{j}}}(\mathfrak{m}), \mathfrak{z}\right),\end{aligned} \tag{43}$$

or

$$\begin{aligned}\partial\left(\mathfrak{d}_{\mathfrak{i}}(\mathfrak{s}), \mathfrak{d}_{\mathfrak{j}}(\mathfrak{m}), \mathfrak{z}\right) &\preceq \mathfrak{A}_1\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) + \mathfrak{A}_2\partial(\mathfrak{s}, \mathfrak{d}_{\mathfrak{i}}(\mathfrak{s}), \mathfrak{z}) \\ &\quad + \mathfrak{A}_3\partial\left(\mathfrak{m}, \mathfrak{d}_{\mathfrak{j}}(\mathfrak{m}), \mathfrak{z}\right),\end{aligned} \tag{44}$$

for all positive integers $\mathfrak{i}$, $\mathfrak{j}$ and for all $\mathfrak{s}, \mathfrak{m}, \mathfrak{z} \in \mathfrak{X}$ with the following conditions:

($\mathfrak{q}_1$) $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, and $\mathfrak{b}$ commute with each other;

($\mathfrak{q}_2$) $\sum_{\mathfrak{k}=1}^{3} r(\mathfrak{A}_{\mathfrak{k}}) + 2r(\mathfrak{A}_2)r(\mathfrak{b}) < 1$.

Then $\{\mathfrak{d}_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\infty}$ shares a unique common fixed point in $\mathfrak{X}$.

Proof. By taking $\mathfrak{A}_4 = \mathfrak{A}_5 = \theta$ in Theorem 34, we can get the required unique fixed point for $\{\mathfrak{d}_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\infty}$.

Remark 37. Our Corollary 36 generalizes Theorem 6.1 in [17] and Theorem 3.1 in [31]. It also extends Corollary 3.1 in [27] from a cone 2-metric space to a cone $\mathfrak{b}_2$ -metric space over a Banach algebra.

Corollary 38. *Let a complete cone $\mathfrak{b}_2$ -metric space be $(\mathfrak{X}, \partial)$ over a Banach algebra $\mathfrak{B}$ with parameter $\mathfrak{b}$, $(\mathfrak{b} \succeq \mathfrak{e})$, and* int $\mathfrak{C}_{\mathfrak{B}} \neq \emptyset$. *Let $\{\mathfrak{d}_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\infty}$ be a family of self-maps from $\mathfrak{X}$ to itself. Assume that $\{\mathfrak{m}_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\infty}$ is a nonnegative integer sequence and vectors $\mathfrak{A}_1 \in \mathfrak{C}_{\mathfrak{B}}$ such that*

$$\partial\left(\mathfrak{d}_{\mathfrak{i}}^{\mathfrak{m}_{\mathfrak{i}}}(\mathfrak{s}), \mathfrak{d}_{\mathfrak{j}}^{\mathfrak{m}_{\mathfrak{j}}}(\mathfrak{m}), \mathfrak{z}\right) \preceq \mathfrak{A}_1\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}), \tag{45}$$

or

$$\partial\left(\mathfrak{d}_{\mathfrak{i}}(\mathfrak{s}), \mathfrak{d}_{\mathfrak{j}}(\mathfrak{m}), \mathfrak{z}\right) \preceq \mathfrak{A}_1\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}), \tag{46}$$

for all $\mathfrak{s}, \mathfrak{m}, \mathfrak{z} \in \mathfrak{X}$ with $r(\mathfrak{A}_1) < 1$. Then $\{\mathfrak{d}_i\}_{i=1}^{\infty}$ shares a unique common fixed point in $\mathfrak{X}$.

Proof. By taking $\mathfrak{A}_2 = \mathfrak{A}_3 = \theta$ in Corollary 36, we can get the required unique fixed point for $\{\mathfrak{d}_i\}_{i=1}^{\infty}$.

Remark 39. Corollary 38 extends Corollary 3.4 in [27] from a cone 2-metric space to a cone $\mathfrak{b}_2$ -metric space over a Banach algebra and Corollary 6.2 in [17].

In the next theorem, the continuity assumption is relaxed.

Theorem 40. *Let a complete cone $\mathfrak{b}_2$ -metric space be $(\mathfrak{X}, \partial)$ over a Banach algebra $\mathfrak{B}$ with parameter $\mathfrak{b}$, $(\mathfrak{b} \succeq \mathfrak{e})$, and* int $\mathfrak{C}_{\mathfrak{B}} \neq \varnothing$. *Let $\{\mathfrak{d}_i\}_{i=1}^{\infty}$ be a family of self-maps from $\mathfrak{X}$ to itself and vectors $\mathfrak{A}_{\mathfrak{k}} \in \mathfrak{C}_{\mathfrak{B}}$, $\mathfrak{k} \in \{1, \cdots, 5\}$ such that*

$$\begin{aligned}\alpha(\mathfrak{s}, \mathfrak{m}, \mathfrak{z})\partial\left(\mathfrak{d}_i(\mathfrak{s}), \mathfrak{d}_j(\mathfrak{m}), \mathfrak{z}\right) \preceq {} & \mathfrak{A}_1\partial(\mathfrak{s}, \mathfrak{m}, \mathfrak{z}) + \mathfrak{A}_2\partial(\mathfrak{s}, \mathfrak{d}_i(\mathfrak{s}), \mathfrak{z}) \\ & + \mathfrak{A}_3\partial\left(\mathfrak{m}, \mathfrak{d}_j(\mathfrak{m}), \mathfrak{z}\right) \\ & + \mathfrak{A}_4\partial\left(\mathfrak{s}, \mathfrak{d}_j(\mathfrak{m}), \mathfrak{z}\right) \\ & + \mathfrak{A}_5\partial(\mathfrak{m}, \mathfrak{d}_i(\mathfrak{s}), \mathfrak{z}),\end{aligned} \tag{47}$$

for $\mathfrak{i}, \mathfrak{j} \geq 1$ and for all $\mathfrak{s}, \mathfrak{m}, \mathfrak{z} \in \mathfrak{X}$ together with

$\mathfrak{r}_1$ There is $\mathfrak{s}_0 \in \mathfrak{X}$ such that $\alpha(\mathfrak{s}_0, \mathfrak{d}_i(\mathfrak{s}_0), \mathfrak{z}) \succeq \mathfrak{e}$ for all $\mathfrak{z} \in \mathfrak{X}$;

$\mathfrak{r}_2(\mathfrak{X}, \partial)$ is α-regular;

$\mathfrak{r}_3\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, $\mathfrak{A}_4$, $\mathfrak{A}_5$, and $\mathfrak{b}$ commute with each other;

$\mathfrak{r}_4 r(\mathfrak{b}\mathfrak{A}_3 + \mathfrak{b}^2\mathfrak{A}_4) < 1$;

Then, $\{\mathfrak{d}_i\}_{i=1}^{\infty}$ shares a common fixed point in $\mathfrak{X}$.

Proof. Choose $\mathfrak{s}_0 \in \mathfrak{X}$ in such a way that $\alpha(\mathfrak{s}_0, \mathfrak{d}_i(\mathfrak{s}_0), \mathfrak{z}) \succeq \mathfrak{e}$ for all $\mathfrak{z} \in \mathfrak{X}$, and construct a sequence $\{\mathfrak{s}_n\}$ in $\mathfrak{X}$ by $\mathfrak{s}_{n+1} = \mathfrak{d}_{n+1}(\mathfrak{s}_n)$ such that $\alpha(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{z}) \succeq \mathfrak{e}$ for all $\mathfrak{z} \in \mathfrak{X}$ and $\mathfrak{n} \in \mathbb{N}$. Then, by using the same method as the proof of Theorem 32, one can get that $\{\mathfrak{x}_n\}$ is a Cauchy sequence in $(\mathfrak{X}, \partial)$. But, as $(\mathfrak{X}, \partial)$ is complete, there exists $\mathfrak{s}^* \in \mathfrak{X}$ such that

$$\lim_{n\to\infty} \mathfrak{s}_n = \mathfrak{s}^*. \tag{48}$$

Since $\alpha(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{z}) \succeq \mathfrak{e}$ and $(\mathfrak{X}, \partial)$ is α-regular such that $\mathfrak{s}_n \longrightarrow \mathfrak{s}^*$ as $\mathfrak{n} \longrightarrow \infty$; therefore, $\alpha(\mathfrak{s}_n, \mathfrak{s}^*, \mathfrak{z}) \succeq \mathfrak{e}$ for all $\mathfrak{z} \in \mathfrak{X}$ and $\mathfrak{n} \in \mathbb{N}$.

Now, we obtain that $\mathfrak{s}^*$ is a fixed point of $\mathfrak{d}_i$. Namely, we have

$$\begin{aligned}\partial(\mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{s}^*, \mathfrak{z}) \preceq {} & \mathfrak{b}\partial(\mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{s}^*, \mathfrak{d}_{n+1}(\mathfrak{s}_n)) \\ & + \mathfrak{b}\partial(\mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{d}_{n+1}(\mathfrak{s}_n), \mathfrak{z}) \\ & + \mathfrak{b}\partial(\mathfrak{d}_{n+1}(\mathfrak{s}_n), \mathfrak{s}^*, \mathfrak{z}).\end{aligned} \tag{49}$$

As

$$\begin{aligned}&\partial(\mathfrak{d}_{n+1}(\mathfrak{s}_n), \mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{s}^*) \preceq \alpha(\mathfrak{s}_n, \mathfrak{s}^*, \mathfrak{s}^*)\partial(\mathfrak{d}_{n+1}(\mathfrak{s}_n), \mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{s}^*), \\ &\partial(\mathfrak{d}_{n+1}(\mathfrak{s}_n), \mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{z}) \preceq \alpha(\mathfrak{s}_n, \mathfrak{s}^*, \mathfrak{z})\partial(\mathfrak{d}_{n+1}(\mathfrak{s}_n), \mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{z}),\end{aligned} \tag{50}$$

and $\mathfrak{d}_i{}'^{\mathfrak{s}}$ are the α-admissible Hardy-Rogers contraction; therefore, (49) becomes

$$\begin{aligned}&\partial(\mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{s}^*, \mathfrak{z}) \\ &\preceq \mathfrak{b}\alpha(\mathfrak{s}_n, \mathfrak{s}^*, \mathfrak{s}^*)\partial(\mathfrak{d}_{n+1}(\mathfrak{s}_n), \mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{s}^*) \\ &\quad + \mathfrak{b}\alpha(\mathfrak{s}_n, \mathfrak{s}^*, \mathfrak{z})\partial(\mathfrak{d}_{n+1}(\mathfrak{s}_n), \mathfrak{d}_{\mathfrak{s}}(\mathfrak{x}^*), \mathfrak{z}) + \mathfrak{b}\partial(\mathfrak{s}_{n+1}, \mathfrak{s}^*, \mathfrak{z}) \\ &\preceq [\mathfrak{b}\mathfrak{A}_1\partial(\mathfrak{s}_n, \mathfrak{s}^*, \mathfrak{s}^*) + \mathfrak{b}\mathfrak{A}_2\partial(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{s}^*) \\ &\qquad + \mathfrak{b}\mathfrak{A}_3\partial(\mathfrak{s}^*, \mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{s}^*) + \mathfrak{b}\mathfrak{A}_4\partial(\mathfrak{s}_n, \mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{s}^*) \\ &\qquad + \mathfrak{b}\mathfrak{A}_5\partial(\mathfrak{s}^*, \mathfrak{s}_{n+1}, \mathfrak{s}^*)] \\ &+ [\mathfrak{b}\mathfrak{A}_1\partial(\mathfrak{s}_n, \mathfrak{s}^*, \mathfrak{z}) + \mathfrak{b}\mathfrak{A}_5\partial(\mathfrak{s}^*, \mathfrak{s}_{n+1}, \mathfrak{z})] + \mathfrak{b}\partial(\mathfrak{s}_{n+1}, \mathfrak{s}^*, \mathfrak{z}) \\ &\preceq \mathfrak{b}\mathfrak{A}_2\partial(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{s}^*) + \mathfrak{b}\mathfrak{A}_4\partial(\mathfrak{s}_n, \mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{s}^*) \\ &\quad + \mathfrak{b}\mathfrak{A}_1\partial(\mathfrak{s}_n, \mathfrak{s}^*, \mathfrak{z}) + \mathfrak{b}\mathfrak{A}_2\partial(\mathfrak{s}_n, \mathfrak{s}_{n+1}, \mathfrak{z}) \\ &\quad + \mathfrak{b}\mathfrak{A}_3\partial(\mathfrak{s}^*, \mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{z}) + \mathfrak{b}^2\mathfrak{A}_4\partial(\mathfrak{s}_n, \mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{s}^*) \\ &\quad + \mathfrak{b}^2\mathfrak{A}_4\partial(\mathfrak{s}_n, \mathfrak{s}^*\mathfrak{z}) + \mathfrak{b}^2\mathfrak{A}_4\partial(\mathfrak{s}^*, \mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{z}) \\ &\quad + \mathfrak{b}\mathfrak{A}_5\partial(\mathfrak{s}^*, \mathfrak{s}_{n+1}, \mathfrak{z}) + \mathfrak{b}\partial(\mathfrak{s}_{n+1}, \mathfrak{s}^*, \mathfrak{z}).\end{aligned} \tag{51}$$

Because $\lim_{n\to\infty}\partial(\mathfrak{s}_n, \mathfrak{s}^*, \mathfrak{z}) = \theta$ and $\lim_{n\to\infty}\partial(\mathfrak{s}_{n+1}, \mathfrak{s}_n, \mathfrak{z}) = \theta$ for all $\mathfrak{z} \in \mathfrak{X}$, we obtain

$$\partial(\mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{s}^*, \mathfrak{z}) \leq \left(\mathfrak{b}\mathfrak{A}_3 + \mathfrak{b}^2\mathfrak{A}_4\right)\partial(\mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{s}^*, \mathfrak{z}). \tag{52}$$

Because, $r(\mathfrak{b}\mathfrak{A}_3 + \mathfrak{b}^2\mathfrak{A}_4) < 1$, from Lemma 20, we claim that $\partial(\mathfrak{d}_n(\mathfrak{s}^*), \mathfrak{s}^*, \mathfrak{z}) = \theta$, that is, $\mathfrak{d}_n(\mathfrak{s}^*) = \mathfrak{s}^*$.

Example 41. Consider Example 26 which is what we claim a complete cone $\mathfrak{b}_2$ -metric space over the Banach algebra $\mathfrak{B} = \mathbb{R}^2$ with parameter $\mathfrak{b} = (2, 0) \geq \mathfrak{e}$. Define $\Gamma_i : \mathfrak{X} \longrightarrow \mathfrak{X}$ by

$$\begin{aligned}&\mathfrak{d}_i(\mathfrak{s}, 0) = \begin{cases} \left(\dfrac{\mathfrak{s}}{\mathfrak{i}+4}, 0\right), & \text{if } \mathfrak{s} \in [0, 1], \\ (0, 0), & \text{otherwise}, \end{cases} \\ &\mathfrak{d}_i(0, 2) = (0, 0) \text{ for all } (\mathfrak{s}, 0) \in \mathfrak{X}, \\ &\qquad \mathfrak{i} \in \mathbb{N}.\end{aligned} \tag{53}$$

Also, define $\alpha : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathfrak{C}_{\mathfrak{B}}$ by

$$\alpha((\mathfrak{s}, 0), (\mathfrak{m}, 0), (\mathfrak{z}_1, \mathfrak{z}_2)) = \begin{cases} (1, 0), & \text{if } \mathfrak{s}, \mathfrak{m} \in [0, 1], \\ (0, 0), & \text{otherwise}, \end{cases} \tag{54}$$

for all $\mathfrak{z} = (\mathfrak{a}_1, \mathfrak{a}_2) \in \mathfrak{X}$.

Choose $\mathfrak{A}_1 = ((1/2), 0)$, $\mathfrak{A}_2 = \mathfrak{A}_3 = ((1/4), 0)$, and $\mathfrak{A}_4 = \mathfrak{A}_5 = ((1/10), 0)$. Clearly $r(\mathfrak{A}_3 + \mathfrak{b}\mathfrak{A}_4) = (9/20) < 1$ and $r(\mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{b}\mathfrak{A}_4) = (19/20) < 1$.

Considering the contractive condition

$$\begin{aligned}\alpha(\mathfrak{S}, \mathfrak{M}, \mathfrak{Z})\mathfrak{d}(\mathfrak{d}_i(\mathfrak{S}), \mathfrak{d}_j(\mathfrak{M}), \mathfrak{Z}) &\preceq \mathfrak{A}_1\mathfrak{d}(\mathfrak{S}, \mathfrak{M}, \mathfrak{Z}) + \mathfrak{A}_2\mathfrak{d}(\mathfrak{S}, \mathfrak{d}_i(\mathfrak{S}), \mathfrak{Z}) + \mathfrak{A}_3\mathfrak{d}(\mathfrak{M}, \mathfrak{d}_j(\mathfrak{M}), \mathfrak{Z}) \\ &\quad + \mathfrak{A}_4\mathfrak{d}(\mathfrak{S}, \mathfrak{d}_j(\mathfrak{M}), \mathfrak{Z}) + \mathfrak{A}_5\mathfrak{d}(\mathfrak{M}, \mathfrak{d}_i(\mathfrak{S}), \mathfrak{Z}),\end{aligned} \tag{55}$$

with $\mathfrak{A}_1 = ((1/2), 0)$, $\mathfrak{A}_2 = \mathfrak{A}_3 = ((1/4), 0)$, and $\mathfrak{A}_4 = \mathfrak{A}_5 = ((1/10), 0)$, we have the following eight cases:

(i) $\mathfrak{S} \in \mathfrak{A}$, $\mathfrak{M} \in \mathfrak{A}$, and $\mathfrak{Z} \in \mathfrak{A}$

(ii) $\mathfrak{S} \in \mathfrak{B}$, $\mathfrak{M} \in \mathfrak{B}$, and $\mathfrak{Z} \in \mathfrak{A}$

(iii) $\mathfrak{S} \in \mathfrak{A}$, $\mathfrak{M} \in \mathfrak{B}$, and $\mathfrak{Z} \in \mathfrak{A}$

(iv) $\mathfrak{S} \in \mathfrak{B}$, $\mathfrak{M} \in \mathfrak{A}$, and $\mathfrak{Z} \in \mathfrak{A}$

(v) $\mathfrak{S} \in \mathfrak{B}$, $\mathfrak{M} \in \mathfrak{B}$, and $\mathfrak{Z} \in \mathfrak{B}$

(vi) $\mathfrak{S} \in \mathfrak{A}$, $\mathfrak{M} \in \mathfrak{A}$, and $\mathfrak{Z} \in \mathfrak{B}$

(vii) $\mathfrak{S} \in \mathfrak{A}$, $\mathfrak{M} \in \mathfrak{B}$, and $\mathfrak{Z} \in \mathfrak{B}$

(viii) $\mathfrak{S} \in \mathfrak{B}$, $\mathfrak{M} \in \mathfrak{A}$, and $\mathfrak{Z} \in \mathfrak{B}$

All the cases are trivial, except case (vi), in which case we have

$$\begin{aligned}&\alpha((\mathfrak{s}, 0), (\mathfrak{m}, 0), (0, 2))\partial(\mathfrak{d}_i(\mathfrak{s}, 0), \mathfrak{d}_j(\mathfrak{m}, 0), (0, 2)) \\ &\preceq \left(\frac{1}{2}, 0\right)\partial((\mathfrak{s}, 0), (\mathfrak{m}, 0), (0, 2)) \\ &+ \left(\frac{1}{4}, 0\right)\left[\partial\left((\mathfrak{s}, 0), \left(\frac{\mathfrak{s}}{\mathfrak{i}+4}, 0\right), (0, 2)\right) + \partial\left((\mathfrak{m}, 0), \left(\frac{\mathfrak{m}}{\mathfrak{j}+4}, 0\right), (0, 2)\right)\right] \\ &+ \left(\frac{1}{10}, 0\right)\left[\partial\left((\mathfrak{s}, 0), \left(\frac{\mathfrak{m}}{\mathfrak{j}+4}, 0\right), (0, 2)\right) + \partial\left((\mathfrak{m}, 0), \left(\frac{\mathfrak{s}}{\mathfrak{i}+4}, 0\right), (0, 2)\right)\right].\end{aligned} \tag{56}$$

Since

$$\begin{aligned}\left(\frac{\mathfrak{s}}{\mathfrak{i}+4} - \frac{\mathfrak{m}}{\mathfrak{j}+4}\right)^2 &\preceq \frac{1}{2}(\mathfrak{s} - \mathfrak{m})^2 + \frac{1}{4}\left[\left(\frac{3\mathfrak{s} + \mathfrak{i}\mathfrak{s}}{\mathfrak{i}+4}\right)^2 + \left(\frac{3\mathfrak{m} + \mathfrak{j}\mathfrak{m}}{\mathfrak{j}+4}\right)^2\right] \\ &\quad + \frac{1}{10}\left[\left(\frac{\mathfrak{j}\mathfrak{s} + 4\mathfrak{s} - \mathfrak{m}}{\mathfrak{j}+4}\right)^2 + \left(\frac{\mathfrak{i}\mathfrak{m} + 4\mathfrak{m} - \mathfrak{s}}{\mathfrak{i}+4}\right)^2\right],\end{aligned} \tag{57}$$

is always true for all $\mathfrak{i}, \mathfrak{j} \in \mathbb{N}$ and $\mathfrak{s}, \mathfrak{m} \geq 0$, case (vi) is also satisfied.

The mappings $\mathfrak{d}_i{}'^{\mathfrak{s}}$ are α-admissible. In fact, let $\mathfrak{S}, \mathfrak{M} \in \mathfrak{X}$ such that $\alpha((\mathfrak{s}, 0), (\mathfrak{m}, 0), \mathfrak{Z}) \succeq \mathfrak{e}$ for all $\mathfrak{Z} \in \mathfrak{X}$. By definition of α, it implies that $\mathfrak{s}, \mathfrak{m} \in [0, 1]$. Therefore, for $\mathfrak{i}, \mathfrak{j} \in \mathbb{N}$ and $\mathfrak{s}, \mathfrak{m} \in [0, 1]$, we have $\mathfrak{d}_i(\mathfrak{s}, 0) = \mathfrak{s}/\mathfrak{i} + 4$, $\mathfrak{d}_j(\mathfrak{m}, 0) = \mathfrak{m}/\mathfrak{j} + 4 \in [0, 1]$, and so that $\alpha(\mathfrak{d}_i(\mathfrak{s}, 0), \mathfrak{d}_j(\mathfrak{m}, 0), \mathfrak{Z}) \succeq \mathfrak{e}$ for all $\mathfrak{Z} \in \mathfrak{X}$.

Further, there is $\mathfrak{s}_0 \in \mathfrak{X}$ such that $\alpha(\mathfrak{s}_0, \mathfrak{d}_i(\mathfrak{s}_0), \mathfrak{Z}) \succeq \mathfrak{e}$ for all $\mathfrak{Z} \in \mathfrak{X}$. Indeed, for $\mathfrak{s}_0 = (1, 0)$, we have

$$\begin{aligned}\alpha((1, 0), \mathfrak{d}_i(1, 0), (\mathfrak{z}_1, \mathfrak{z}_2)) &= \alpha\left((1, 0), \left(\frac{1}{\mathfrak{i}+4}, 0\right), (\mathfrak{z}_1, \mathfrak{z}_2)\right) \\ &\succeq \mathfrak{e} \text{ for all } (\mathfrak{z}_1, \mathfrak{z}_2) \in \mathfrak{X}.\end{aligned} \tag{58}$$

Thus, all the assumptions of Theorem 32 are fulfilled, and we conclude the existence of at least one fixed point for each $\mathfrak{d}_i{}'^{\mathfrak{s}}$. Indeed, $(0, 0)$ is the common fixed point of the family of mapping $\{\mathfrak{d}_i\}_{i=1}^{\infty}$.

Next, we use the following property [20] to guarantee the uniqueness of the fixed point of $\mathfrak{d}_i{}'^{\mathfrak{s}}$.

(H). Denote Fix$(\mathfrak{d}_i)$ to be the set of all fixed points of $\{\mathfrak{d}_i\}_{i=1}^{\infty}$. Assume for all $\mathfrak{s}^*, \mathfrak{o}^* \in \text{Fix}(\mathfrak{d}_i)$, there exists $\mathfrak{m} \in \mathfrak{X}$ such that $\alpha(\mathfrak{s}^*, \mathfrak{m}, \mathfrak{z}) \succeq \mathfrak{e}$ and $\alpha(\mathfrak{o}^*, \mathfrak{m}, \mathfrak{z}) \succeq \mathfrak{e}$ for all $\mathfrak{z} \in \mathfrak{X}$.

Theorem 42. *To add condition (H) in Theorem 32 (resp., Theorem 40) we obtain uniqueness of the fixed point of each* $\{\Gamma_i\}_{i=1}^{\infty}$.

Proof. Using related claims to those in the proof of Theorem 32 (resp., Theorem 40), we achieve fixed-point existence. Let (H) be satisfied and $\mathfrak{s}^*, \mathfrak{o}^* \in \text{Fix}(\mathfrak{d}_{i'\mathfrak{s}})$ and $\mathfrak{s}^* \neq \mathfrak{o}^*$. By condition (H), there exists $\mathfrak{m} \in \mathfrak{X}$ such that

$$\begin{aligned}\alpha(\mathfrak{s}^*, \mathfrak{m}, \mathfrak{z}) &\succeq \mathfrak{e}, \\ \alpha(\mathfrak{o}^*, \mathfrak{m}, \mathfrak{z}) &\succeq \mathfrak{e},\end{aligned} \tag{59}$$

$$\text{for all } \mathfrak{z} \in \mathfrak{X}. \tag{60}$$

Since $\mathfrak{d}_i{}'^{\mathfrak{s}}$ are α-admissible mappings and $\mathfrak{s}^*, \mathfrak{o}^* \in$ Fix $(\mathfrak{d}_{i'\mathfrak{s}})$. From (59), we have

$$\begin{aligned}\alpha(\mathfrak{s}^*, \mathfrak{d}_i^{\mathfrak{n}}(\mathfrak{m}), \mathfrak{z}) &\succeq \mathfrak{e}, \\ \alpha(\mathfrak{o}^*, \mathfrak{d}_i^{\mathfrak{n}}(\mathfrak{m}), \mathfrak{z}) &\succeq \mathfrak{e}, \\ \text{for all } \mathfrak{z} \in \mathfrak{X} &\text{ and } \mathfrak{i}, \mathfrak{n} \in \mathbb{N}.\end{aligned} \tag{61}$$

As, $\alpha(\mathfrak{s}^*, \mathfrak{d}_i^{\mathfrak{n}}(\mathfrak{m}), \mathfrak{z}) \succeq \mathfrak{e}$ for all $\mathfrak{z} \in \mathfrak{X}$, therefore, we have

$$\begin{aligned}\partial(\mathfrak{s}^*, \mathfrak{d}_i^{\mathfrak{n}}(\mathfrak{m}), \mathfrak{z}) &\preceq \alpha(\mathfrak{s}^*, \mathfrak{m}, \mathfrak{z})\partial\left(\mathfrak{d}_i(\mathfrak{s}^*), \mathfrak{d}_i\left(\mathfrak{d}_i^{\mathfrak{n}-1}\mathfrak{m}\right), \mathfrak{z}\right) \\ &\preceq \mathfrak{A}_1\partial\left(\mathfrak{s}^*, \mathfrak{d}_i^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{z}\right) + \mathfrak{A}_2\partial(\mathfrak{s}^*, \mathfrak{d}_i(\mathfrak{s}^*), \mathfrak{z}) \\ &\quad + \mathfrak{A}_3\partial\left(\mathfrak{d}_i^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{d}_i\left(\mathfrak{d}_i^{\mathfrak{n}-1}\mathfrak{m}\right), \mathfrak{z}\right) \\ &\quad + \mathfrak{A}_4\partial\left(\mathfrak{s}^*, \mathfrak{d}_i\left(\mathfrak{d}_i^{\mathfrak{n}-1}\mathfrak{m}\right), \mathfrak{z}\right) \\ &\quad + \mathfrak{A}_5\partial\left(\mathfrak{d}_i^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{d}_i(\mathfrak{s}^*), \mathfrak{z}\right).\end{aligned} \tag{62}$$

That is, we have

$$\begin{aligned}\partial(\mathfrak{s}^*, \mathfrak{d}_i^{\mathfrak{n}}(\mathfrak{m}), \mathfrak{z}) &\preceq \mathfrak{A}_1\partial\left(\mathfrak{s}^*, \mathfrak{d}_i^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{z}\right) + \mathfrak{A}_3\partial\left(\mathfrak{d}_i^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{d}_i^{\mathfrak{n}}(\mathfrak{m}), \mathfrak{a}\right) \\ &\quad + \mathfrak{A}_4\partial(\mathfrak{s}^*, \mathfrak{d}_i^{\mathfrak{n}}(\mathfrak{m}), \mathfrak{z}) + \mathfrak{A}_5\partial\left(\mathfrak{d}_i^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{s}^*, \mathfrak{z}\right).\end{aligned} \tag{63}$$

Hence, we have

$$(\mathfrak{e}-\mathfrak{A}_4)\partial(\mathfrak{s}^*, \mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}), \mathfrak{z}) \preceq (\mathfrak{A}_1+\mathfrak{A}_5)\partial(\mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{s}^*, \mathfrak{z}) + \mathfrak{A}_3\partial(\mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}), \mathfrak{z}). \quad (64)$$

Similarly, we have

$$\partial(\mathfrak{s}^*, \mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}), \mathfrak{z}) \leq \mathfrak{A}_1\partial(\mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{s}^*, \mathfrak{z}) + \mathfrak{A}_2\partial(\mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}), \mathfrak{z}) + \mathfrak{A}_4\partial(\mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{s}^*, \mathfrak{z}) + \mathfrak{A}_5\partial(\mathfrak{s}^*, \mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}), \mathfrak{z}). \quad (65)$$

That is, we have

$$(\mathfrak{e}-\mathfrak{A}_5)\partial(\mathfrak{s}^*, \mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}), \mathfrak{z}) \preceq (\mathfrak{A}_1+\mathfrak{A}_4)\partial(\mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{s}^*, \mathfrak{z}) + \mathfrak{A}_2\partial(\mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}), \mathfrak{z}). \quad (66)$$

Adding up (64) and (66), we have

$$(2\mathfrak{e}-\mathfrak{A}_4-\mathfrak{A}_5)\partial(\mathfrak{s}^*, \partial_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}), \mathfrak{z}) \preceq (2\mathfrak{A}_1+\mathfrak{A}_4+\mathfrak{A}_5)\partial(\partial_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{s}^*, \mathfrak{z}) + (\mathfrak{A}_2+\mathfrak{A}_3)\partial(\partial_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}), \mathfrak{z}). \quad (67)$$

Since, $r(\mathfrak{A}_4+\mathfrak{A}_5)<1$, therefore, we have

$$\partial(\mathfrak{s}^*, \mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}), \mathfrak{z}) \preceq (2\mathfrak{e}-\mathfrak{A}_4-\mathfrak{A}_5)^{-1}(2\mathfrak{A}_1+\mathfrak{A}_4+\mathfrak{A}_5)\partial(\mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{s}^*, \mathfrak{z}) + (2\mathfrak{e}-\mathfrak{A}_4-\mathfrak{A}_5)^{-1}(\mathfrak{A}_2+\mathfrak{A}_3)\partial(\mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}), \mathfrak{z}). \quad (68)$$

That is, we have

$$\partial(\mathfrak{s}^*, \mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}), \mathfrak{z}) \preceq \mathfrak{L}_1\partial(\mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{s}^*, \mathfrak{z}) + \mathfrak{L}_2\partial(\mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}), \mathfrak{z}), \quad (69)$$

where $\mathfrak{L}_1=(2\mathfrak{e}-\mathfrak{A}_4-\mathfrak{A}_5)^{-1}(2\mathfrak{A}_1+\mathfrak{A}_4+\mathfrak{A}_5)$ and $\mathfrak{L}_2=(2\mathfrak{e}-\mathfrak{A}_4-\mathfrak{A}_5)^{-1}(\mathfrak{A}_2+\mathfrak{A}_3)$. Hence, we have

$$\begin{aligned}\partial(\mathfrak{s}^*, \mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}), \mathfrak{z}) &\preceq \mathfrak{L}_1\partial(\mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{s}^*, \mathfrak{z}) + \mathfrak{L}_2\partial(\mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}), \mathfrak{z})\\ &\preceq {\mathfrak{L}_1}^2\partial(\mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-2}(\mathfrak{m}), \mathfrak{s}^*, \mathfrak{z}) + {\mathfrak{L}_2}^2\partial(\mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-2}(\mathfrak{m}), \mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{z}),\\ &\vdots \quad \vdots\\ &\preceq {\mathfrak{L}_1}^\mathfrak{n}\partial(\mathfrak{m}, \mathfrak{s}^*, \mathfrak{z}) + {\mathfrak{L}_2}^\mathfrak{n}\partial(\mathfrak{m}, \mathfrak{d}_\mathfrak{i}(\mathfrak{m}), \mathfrak{z}).\end{aligned} \quad (70)$$

Since, $r({\mathfrak{L}_1}^\mathfrak{n})<1$ and $r({\mathfrak{L}_2}^\mathfrak{n})<1$, by Remark 5, it follows that $\|{\mathfrak{L}_1}^\mathfrak{n}\| \longrightarrow 0(\mathfrak{n}\longrightarrow\infty)$ and $\|{\mathfrak{L}_2}^\mathfrak{n}\| \longrightarrow 0(\mathfrak{n}\to\infty)$, and so

$$\begin{aligned}&\|{\mathfrak{L}_1}^\mathfrak{n}\partial(\mathfrak{m}, \mathfrak{s}^*, \mathfrak{z}) + {\mathfrak{L}_2}^\mathfrak{n}\partial(\mathfrak{m}, \mathfrak{d}_\mathfrak{i}(\mathfrak{m}), \mathfrak{z})\|\\ &\quad\leq \|{\mathfrak{L}_1}^\mathfrak{n}\partial(\mathfrak{m}, \mathfrak{s}^*, \mathfrak{z})\| + \|{\mathfrak{L}_2}^\mathfrak{n}\partial(\mathfrak{m}, \mathfrak{d}_\mathfrak{i}(\mathfrak{m}), \mathfrak{z})\|\\ &\quad\leq \|{\mathfrak{L}_1}^\mathfrak{n}\|\|\partial(\mathfrak{m}, \mathfrak{s}^*, \mathfrak{z})\| + \|{\mathfrak{L}_2}^\mathfrak{n}\|\|\partial(\mathfrak{m}, \mathfrak{d}_\mathfrak{i}(\mathfrak{m}), \mathfrak{z})\|\\ &\quad\longrightarrow 0(\mathfrak{n}\to\infty).\end{aligned} \quad (71)$$

Therefore, based on Lemma 15, we conclude that for any $\mathfrak{c}\in\mathfrak{B}$ with $\mathfrak{c}\gg\theta$, there exists $\mathfrak{N}\in\mathbb{N}$ such that

$$\partial(\mathfrak{s}^*, \mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}), \mathfrak{z}) \preceq \mathfrak{L}_1\partial(\mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{s}^*, \mathfrak{z}) + \mathfrak{L}_2\partial(\mathfrak{d}_\mathfrak{i}^{\mathfrak{n}-1}(\mathfrak{m}), \mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}), \mathfrak{z}) \ll \mathfrak{c} \quad \text{for all } \mathfrak{i}\in\mathbb{N} \text{ and } \mathfrak{z}\in\mathfrak{X}. \quad (72)$$

Hence, $\mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}) \longrightarrow \mathfrak{s}^*(\mathfrak{n}\longrightarrow\infty)$. Similarly, we get that $\mathfrak{d}_\mathfrak{i}^\mathfrak{n}(\mathfrak{m}) \longrightarrow \mathfrak{o}^*(\mathfrak{n}\longrightarrow\infty)$. Then, by the uniqueness of the limit, we have $\mathfrak{s}^*=\mathfrak{o}^*$.

3. Applications

We give here a couple of auxiliary facts that will be used in our further considerations.

Let $\mathfrak{B}$ with norm $\|\cdot\|_\mathfrak{B}$ be a real infinite-dimensional Banach's algebra. Let $\mathfrak{J}=[0,\mathfrak{J}]$ and denote $C=C(\mathfrak{J},\mathfrak{B})$ the space consisting of all continuous functions defined on interval $\mathfrak{J}$ with values in the Banach algebra $\mathfrak{B}=\mathbb{R}^\infty$ (the collection of all real sequences).

The space C will be equipped with $\|\mathfrak{s}\|_C=\max\{\|\mathfrak{s}(\mathfrak{a})\|_\mathfrak{B} : \mathfrak{a}\in\mathfrak{J}\}$.

The purpose of this section is to establish and demonstrate a result on the existence of solutions of a class of an infinite system of integral equations of the form (74).

Let $\mathfrak{X}=C$, and $\partial:\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X}\longrightarrow\mathfrak{B}$ be defined by

$$\partial(\mathfrak{s}_\mathfrak{i}(\mathfrak{a}), \mathfrak{m}_\mathfrak{i}(\mathfrak{a}), \mathfrak{z}_\mathfrak{i}(\mathfrak{a})) = [\min\{|\mathfrak{s}_\mathfrak{i}(\mathfrak{a})-\mathfrak{m}_\mathfrak{i}(\mathfrak{a})|, |\mathfrak{m}_\mathfrak{i}(\mathfrak{a})-\mathfrak{z}_\mathfrak{i}(\mathfrak{a})|, |\mathfrak{s}_\mathfrak{i}(\mathfrak{a})-\mathfrak{z}_\mathfrak{i}(\mathfrak{a})|\}]^\mathfrak{p}, \quad (73)$$

where $\mathfrak{i}=1,2,3,\cdots$ and for all $\mathfrak{s}_\mathfrak{i}(\mathfrak{a})$, $\mathfrak{m}_\mathfrak{i}(\mathfrak{a})$, and $\mathfrak{z}_\mathfrak{i}(\mathfrak{a})\in\mathfrak{X}$. Then $(\mathfrak{X},\partial)$ is a complete cone $\mathfrak{b}_2$-metric space over Banach's algebra. Consider the infinite system of integral equations

$$\mathfrak{s}_\mathfrak{i}(\mathfrak{a}) = \mathfrak{g}_\mathfrak{i}(\mathfrak{a}) + \int_0^\mathfrak{J} \mathfrak{H}_\mathfrak{i}(\mathfrak{t},\mathfrak{w})\mathfrak{f}_\mathfrak{i}(\mathfrak{w}, \mathfrak{s}_1(\mathfrak{w}), \mathfrak{s}_2(\mathfrak{w}), \cdots)d\mathfrak{w}, \quad (74)$$

where $\mathfrak{i}=1,2,3,\cdots$ and let $\mathfrak{d}_\mathfrak{i}:\mathfrak{X}\longrightarrow\mathfrak{X}$ be defined by

$$\mathfrak{d}_\mathfrak{i}(\mathfrak{s}_\mathfrak{i}(\mathfrak{a})) = \mathfrak{g}_\mathfrak{i}(\mathfrak{a}) + \int_0^\mathfrak{J} \mathfrak{H}_\mathfrak{i}(\mathfrak{a},\mathfrak{w})\mathfrak{f}_\mathfrak{i}(\mathfrak{w}, \mathfrak{s}_1(\mathfrak{w}), \mathfrak{s}_2(\mathfrak{w}), \cdots)d\mathfrak{s} \quad \text{for all } \mathfrak{i}=1,2,3,\cdots. \quad (75)$$

We assume that $\mathfrak{d}$

(1) $\mathfrak{g}_\mathfrak{i}\sim\mathfrak{J}\longrightarrow\mathbb{R}$ are continuous

(2) $\mathfrak{H}_\mathfrak{i}\sim\mathfrak{J}\times\mathbb{R}\longrightarrow[0,+\infty)$ are continuous and $\int_0^\mathfrak{J}\mathfrak{H}_\mathfrak{i}(\mathfrak{t},\mathfrak{w})\leq 1$

(3) $\mathfrak{f}_\mathfrak{i}\sim\mathfrak{J}\times\mathbb{R}^\infty\longrightarrow\mathbb{R}$ are continuous such that

$$\begin{aligned}&|\mathfrak{f}_\mathfrak{i}(\mathfrak{w}, \mathfrak{s}_1(\mathfrak{w}), \mathfrak{s}_2(\mathfrak{w}), \cdots) - \mathfrak{f}_\mathfrak{i}(\mathfrak{w}, \mathfrak{m}_1(\mathfrak{w}), \mathfrak{m}_2(\mathfrak{w}), \cdots)|\\ &\quad\leq \mathfrak{r}^{1/\mathfrak{p}}[\min\{|\mathfrak{s}_\mathfrak{i}(\mathfrak{w})-\mathfrak{m}_\mathfrak{i}(\mathfrak{w})|, |\mathfrak{m}_\mathfrak{i}(\mathfrak{w})-\mathfrak{z}_\mathfrak{i}(\mathfrak{w})|, |\mathfrak{s}_\mathfrak{i}(\mathfrak{w})-\mathfrak{z}_\mathfrak{i}(\mathfrak{w})|\}],\end{aligned} \quad (76)$$

for all $\mathfrak{z}_\mathfrak{i}(\mathfrak{w})\in\mathfrak{X}$, where $0\leq\mathfrak{r}<1$.

Theorem 43. *Under the assumptions (1)–(3), the infinite system of integral equation (74) has a solution in C.*

Proof. Take $\mathfrak{B} = \mathbb{R}^{\infty}$ with norm $\|\mathfrak{o}\| = \|(\mathfrak{o}_1, \mathfrak{o}_2, \cdots)\| = \sum_{i=1}^{\infty} |\mathfrak{o}_i|$ and multiplication defined by

$$\mathfrak{o}\mathfrak{q} = ((\mathfrak{o}_1, \mathfrak{o}_2, \cdots)(\mathfrak{q}_1, \mathfrak{q}_2, \cdots)) = (\mathfrak{o}_1\mathfrak{q}_1, \mathfrak{o}_2\mathfrak{q}_2, \cdots). \tag{77}$$

Let $\mathfrak{C}_{\mathfrak{B}} = \{\mathfrak{o} = (\mathfrak{o}_1, \mathfrak{o}_2, \cdots) \in \mathfrak{B} : \mathfrak{o}_1, \mathfrak{o}_2, \cdots \geq 0\}$. It is clear that $\mathfrak{C}_{\mathfrak{B}}$ is a normal cone, and $\mathfrak{B}$ is a Banach algebra with unit $\mathfrak{e} = (1, 0, \cdots)$.

Consider the family of mapping $\mathfrak{d}_i : \mathfrak{X} \longrightarrow \mathfrak{X}$ defined by (75). Let $\mathfrak{s}_i(\mathfrak{a})$, $\mathfrak{m}_i(\mathfrak{a})$, and $\mathfrak{z}_i(\mathfrak{a}) \in \mathfrak{X}$.

From (15), we deduce that

$$\begin{aligned}
&\partial(\mathfrak{d}_i(\mathfrak{s}_i(\mathfrak{a})), \mathfrak{d}_i(\mathfrak{m}_i(\mathfrak{a})), \mathfrak{z}_i(\mathfrak{a})) \\
&= \max_{\mathfrak{a}\in[0,\mathfrak{I}]} [\min \{|\mathfrak{d}_i(\mathfrak{s}_i(\mathfrak{a})) - \mathfrak{d}_i(\mathfrak{m}_i(\mathfrak{a}))|, |\mathfrak{d}_i(\mathfrak{m}_i(\mathfrak{a})) \\
&\qquad - \mathfrak{z}_i(\mathfrak{a})|, |\mathfrak{d}_i(\mathfrak{s}_i(\mathfrak{a})) - \mathfrak{z}_i(\mathfrak{a})|\}]^{\mathfrak{p}} \\
&\preceq \left(\max_{\mathfrak{a}\in[0,\mathfrak{I}]} |\mathfrak{d}_i(\mathfrak{s}_i(\mathfrak{a})) - \mathfrak{d}_i(\mathfrak{m}_i(\mathfrak{a}))| \right)^{\mathfrak{p}} \\
&= \left(\max_{\mathfrak{a}\in[0,\mathfrak{I}]} \left| \int_0^{\mathfrak{I}} \mathfrak{H}_i(\mathfrak{a}, \mathfrak{w}) \mathfrak{f}_i(\mathfrak{w}, \mathfrak{s}_1(\mathfrak{w}), \mathfrak{s}_2(\mathfrak{w}), \cdots) d\mathfrak{w} \right.\right. \\
&\qquad \left.\left. - \int_0^{\mathfrak{I}} \mathfrak{H}_i(\mathfrak{a}, \mathfrak{w}) \mathfrak{f}_i(\mathfrak{w}, \mathfrak{m}_1(\mathfrak{w}), \mathfrak{m}_2(\mathfrak{w}), \cdots) d\mathfrak{w} \right| \right)^{\mathfrak{p}} \\
&= \left(\max_{\mathfrak{a}\in[0,\mathfrak{I}]} \left| \int_0^{\mathfrak{I}} \mathfrak{H}_i(\mathfrak{a}, \mathfrak{w}) [\mathfrak{f}_i(\mathfrak{w}, \mathfrak{s}_1(\mathfrak{w}), \mathfrak{s}_2(\mathfrak{w}), \cdots) \right.\right. \\
&\qquad \left.\left. - \mathfrak{f}_i(\mathfrak{w}, \mathfrak{m}_1(\mathfrak{s}), \mathfrak{m}_2(\mathfrak{w}), \cdots)] d\mathfrak{w} \right| \right)^{\mathfrak{p}} \\
&\preceq \left(\max_{\mathfrak{a}\in[0,\mathfrak{I}]} \int_0^{\mathfrak{I}} \mathfrak{H}_i(\mathfrak{a}, \mathfrak{w}) |\mathfrak{f}_i(\mathfrak{w}, \mathfrak{s}_1(\mathfrak{w}), \mathfrak{s}_2(\mathfrak{w}), \cdots) \right. \\
&\qquad \left. - \mathfrak{f}_i(\mathfrak{w}, \mathfrak{m}_1(\mathfrak{w}), \mathfrak{m}_2(\mathfrak{w}), \cdots)| d\mathfrak{w} \right)^{\mathfrak{p}} \\
&\preceq \left(\max_{\mathfrak{a}\in[0,\mathfrak{I}]} \int_0^{\mathfrak{I}} \mathfrak{H}_i(\mathfrak{a}, \mathfrak{w}) \mathfrak{r}^{1/\mathfrak{p}} [\min \{|\mathfrak{s}_i(\mathfrak{w}) - \mathfrak{m}_i(\mathfrak{w})|, |\mathfrak{m}_i(\mathfrak{w}) \right. \\
&\qquad \left. - \mathfrak{z}_i(\mathfrak{w})|, |\mathfrak{s}_i(\mathfrak{w}) - \mathfrak{z}_i(\mathfrak{w})|\}] d\mathfrak{w} \right)^{\mathfrak{p}} \\
&\preceq \left(\int_0^{\mathfrak{I}} \left(\max_{\mathfrak{a}\in[0,\mathfrak{I}]} \mathfrak{H}_i(\mathfrak{a}, \mathfrak{w}) \mathfrak{r}^{1/\mathfrak{p}} \right) \left(\max_{\mathfrak{a}\in[0,\mathfrak{T}]} [\min \{|\mathfrak{s}_i(\mathfrak{w}) - \mathfrak{m}_i(\mathfrak{w})|, |\mathfrak{m}_i(\mathfrak{w}) \right.\right. \\
&\qquad \left.\left. - \mathfrak{z}_i(\mathfrak{w})|, |\mathfrak{s}_i(\mathfrak{w}) - \mathfrak{z}_i(\mathfrak{w})|\}]^{\mathfrak{p}} \right)^{1/\mathfrak{p}} d\mathfrak{w} \right)^{\mathfrak{p}} \\
&\preceq \left(\int_0^{\mathfrak{I}} \left(\max_{\mathfrak{a}\in[0,\mathfrak{I}]} \mathfrak{H}_i(\mathfrak{a}, \mathfrak{w}) \mathfrak{r}^{1/\mathfrak{p}} \right) [\partial(\mathfrak{s}_i(\mathfrak{a}), \mathfrak{m}_i(\mathfrak{a}), \mathfrak{z}_i(\mathfrak{a}))]^{1/\mathfrak{p}} \right)^{\mathfrak{p}} \\
&= \mathfrak{r}\partial(\mathfrak{s}_i(\mathfrak{a}), \mathfrak{m}_i(\mathfrak{a}), \mathfrak{z}_i(\mathfrak{a})) \left(\sup_{\mathfrak{a}\in[0,\mathfrak{I}]} \int_0^{\mathfrak{I}} \mathfrak{H}_i(\mathfrak{a}, \mathfrak{w}) \right)^{\mathfrak{p}} \\
&\preceq \mathfrak{r}\partial(\mathfrak{s}_i(\mathfrak{a}), \mathfrak{m}_i(\mathfrak{a}), \mathfrak{z}_i(\mathfrak{a})).
\end{aligned} \tag{78}$$

Therefore, we have

$$\partial(\mathfrak{d}_i(\mathfrak{s}_i(\mathfrak{a})), \mathfrak{d}_i(\mathfrak{m}_i(\mathfrak{a})), \mathfrak{z}_i(\mathfrak{a})) \preceq \mathfrak{r}\partial(\mathfrak{s}_i(\mathfrak{a}), \mathfrak{m}_i(\mathfrak{a}), \mathfrak{z}_i(\mathfrak{a})). \tag{79}$$

Now, all the hypotheses of Corollary 38 are satisfied, and the family of mapping $\{\mathfrak{d}_i\}_{i=0}^{\infty}$ has a unique fixed point in $\mathfrak{X}$, which means that the infinite system of integral equations (74) has a solution.

Data Availability

No data were used.

Conflicts of Interest

None of the authors have any conflicts of interest.

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