Hindawi
Advances in Mathematical Physics
Volume 2020, Article ID 8319584, 7 pages
https://doi.org/10.1155/2020/8319584

Research Article

A New Extension of the Rectangular b-Metric Spaces

Nabil Mlaiki[1] **Mohamed Hajji**[2] **and Thabet Abdeljawad**[3,4,5]

[1] *Department of Mathematics and General Sciences, Prince Sultan University Riyadh, Saudi Arabia 11586*
[2] *Institut Superieur des Sciences Appliques et de Technologie de Kasserine, Kairouan University, BP 471, Kasserine 1200, Tunisia*
[3] *Department of Mathematics and General Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia*
[4] *Department of Medical Research, China Medical University, Taichung 40402, Taiwan*
[5] *Department of Computer Sciences and Information Engineering, Asia University, Taichung, Taiwan*

Correspondence should be addressed to Thabet Abdeljawad; tabdeljawad@psu.edu.sa

Received 7 April 2020; Accepted 18 May 2020; Published 1 June 2020

Academic Editor: Soheil Salahshour

In this paper, we introduce a generalization of rectangular b-metric spaces, by changing the rectangular inequality as follows: $D_\zeta(a, b) \leq \zeta(a, b, u, v)[D_\zeta(a, u) + D_\zeta(u, v) + D_\zeta(v, b)]$, for all distinct $a, b, u, v \in X$. We prove some fixed point theorems, and we use our results to present a nice application in the last section of this paper.

1. Introduction

It will not be an exaggeration if we say that Banach [1] in 1922 introduced in some way a new area in mathematics, which is called fixed point theory, and that is due to the fact that he proved the existence and uniqueness of a fixed point for self-contractive mappings in metric spaces. Since 1922, mathematicians around the world start to generalize his result either by changing the type of contractions or by generalizing the type of metric spaces (see [2–19]). The question here is what is the point of all these generalizations? Well, in fact, the answer to that is quite simple and that is the larger the class of functions or metrics, the more fields that results can be applied to, such as computer sciences and engineering.

In this paper, and inspired by the work done in [20–27], we introduce the notion of controlled rectangular b-metric spaces as a generalization of the rectangular metric spaces and rectangular b-metric spaces. In the second section, we present some preliminaries; in the third section, we prove our main result; in the fourth section, we present an application of our result to polynomial equations; and in the closing section, we give a conclusion with some open questions.

2. Preliminaries

The concept of rectangular metric spaces was introduced by Branciari in [28] as follows.

Definition 1 [28] (rectangular (or Branciari) metric spaces). Let X be a nonempty set. A mapping $\Delta : X^2 \longrightarrow [0,\infty)$ is called a rectangular metric on X if for any $x, y \in X$ and all distinct points $u, v \in X \setminus \{x, y\}$; it satisfies the following conditions:

(R_1) $x = y$ if and only if $\Delta(x, y) = 0$
(R_2) $\Delta(x, y) = \Delta(y, x)$
(R_3) $\Delta(x, y) \leq \Delta(x, u) + \Delta(u, v) + \Delta(v, y)$

In this case, the pair (X, Δ) is called a rectangular metric space.

In [29], George et al. introduced the concept of b-rectangular metric spaces as follows.

Definition 2 [29] (rectangular b-metric spaces). Let X be a nonempty set. A mapping $B : X^2 \longrightarrow [0,\infty)$ is called a rectangular b-metric on X if there exists a constant $a \geq 1$ such that for any $x, y \in X$ and all distinct points $u, v \in X \setminus \{x, y\}$; it satisfies the following conditions:

(R_{b1}) $x = y$ if and only if $B(x, y) = 0$

(R_{b2}) $B(x, y) = B(y, x)$

(R_{b3}) $B(x, y) \leq a[B(x, u) + B(u, v) + B(v, y)]$

In this case, the pair (X, B) is called a rectangular metric space.

As a generalization of rectangular b-metric spaces, Abdeljawad et al. in [30] introduced the concept of extended Branciari b-distance spaces as follows.

Definition 3 [30]. For a nonempty set S and a mapping $\omega : S \times S \longrightarrow [1,\infty)$, we say that a function $B_{\text{dist}} : S \times S \longrightarrow [0,\infty)$ is called an extended Branciari b-distance if it satisfies

(i) $B_{\text{dist}}(x, y) = 0$ if and only if $x = y$

(ii) $B_{\text{dist}}(x, y) = B_{\text{dist}}(y, x)$

(iii) $B_{\text{dist}}(x, y) \leq \omega(x, y)[B_{\text{dist}}(x, u) + B_{\text{dist}}(u, v) + B_{\text{dist}}(v, y)]$

for all $x, y \in \mathcal{S}$ and all distinct $u, v \in \mathcal{S} \setminus \{x, y\}$. The couple of the symbols (S, B_{dist}) denotes an extended Branciari b-distance space (shortly, B_{dist}-metric space).

Now, we present the definition of controlled rectangular b-metric spaces.

Definition 4. Let X be a nonempty set, a function $\zeta : X^4 \longrightarrow [1,\infty)$, and $D_\zeta : X^2 \longrightarrow [0,\infty)$. We say that (X, D_ζ) is a controlled rectangular b-metric space if all distinct $a, b, u, v \in X$; we have

(1) $D_\zeta(a, b) = 0$ if and only if $a = b$

(2) $D_\zeta(a, b) = D_\zeta(b, a)$

(3) $D_\zeta(a, b) \leq \zeta(a, b, u, v)[D_\zeta(a, u) + D_\zeta(u, v) + D_\zeta(v, b)]$

Next, we present the topology of controlled rectangular b-metric spaces.

Definition 5. Let (X, D_ζ) be a controlled rectangular b-metric space.

(1) A sequence $\{a_n\}$ is called D_ζ-convergent in a controlled rectangular b-metric space (X, D_ζ), if there exists $a \in X$ such that $\lim_{n\to\infty} D_\zeta(a_n, a) = D_\zeta(a, a)$

(2) A sequence $\{a_n\}$ is called D_ζ-Cauchy if and only if $\lim_{n,m\to\infty} \rho(a_n, a_m)$ exists and is finite

(3) A controlled rectangular b-metric space (X, D_ζ) is called D_ζ-complete if for every D_ζ-Cauchy sequence $\{a_n\}$ in X, there exists $v \in X$, such that $\lim_{n\to\infty} D_\zeta(a_n, v) = \lim_{n,m\to\infty} \rho_r(a_n, a_m) = D_\zeta(v, v)$

(4) Let $a \in X$ define an open ball in a controlled rectangular b-metric space (X, D_ζ) by $B_\zeta(a, \eta) = \{b \in X \mid D_\zeta(a, b) < \eta\}$

Notice that rectangular metric spaces and rectangular b-metric spaces are controlled rectangular b-metric spaces, but the converse is not always true. In the following example, we present a controlled rectangular b-metric space which is not a rectangular metric space.

Example 1. Let $X = Y \cup Z$, where $Y = \{1/m \mid m \text{ is a natural number}\}$ and Z be the set of positive integers. We define $D_\zeta : X^2 \longrightarrow [0,\infty)$ by

$$D_\zeta(a, b) = \begin{cases} 0, & \text{if and only if } a = b, \\ 2\beta, & \text{if } a, b \in Y, \\ \dfrac{\beta}{2}, & \text{otherwise,} \end{cases} \tag{1}$$

where β is a constant bigger than 0. Now, define $\zeta : X^4 \longrightarrow [1,\infty)$ by $\zeta(a, b, u, v) = \max\{a, b, u, v\} + 2\beta$. It is not difficult to check that (X, D_ζ) is a controlled rectangular b-metric space. However, (X, D_ζ) is not a rectangular metric space; for instance, notice that $D_\zeta(1/2, 1/3) = 2\beta > D_\zeta(1/2, 2) + D_\zeta(2, 3) + D_\zeta(3, 1/3) = 3\beta/2$.

3. Main Results

Theorem 6. *Let (X, D_ζ)be a controlled rectangular b-metric space and T a self-mapping on X. If there exists $0 < \delta < 1$, such that $D_\zeta(Tx, Ty) \leq \delta\rho(x, y)$and $\sup_{m>1}\lim_{n\to\infty}\zeta(x_n, x_{n+1}, x_{n+2}, x_m) \leq 1/\delta$, then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$ and define the sequence $\{x_n\}$ as follows: $x_1 = Tx_0, x_2 = T^2x_0, \cdots, x_n = T^n x_0, \cdots$. Now, by the hypothesis of the theorem, we have

$$\begin{aligned} D_\zeta(x_n, x_{n+1}) &\leq \delta D_\zeta(x_{n-1}, x_n) \\ &\leq \delta^2 D_\zeta(x_{n-2}, x_{n-1}) \leq \cdots \leq \delta^n D_\zeta(x_0, x_1). \end{aligned} \tag{2}$$

Note that if we take the limit of the above inequality as $n \longrightarrow \infty$, we deduce that $D_\zeta(x_n, x_{n+1}) \longrightarrow 0$ as $n \longrightarrow \infty$. Now, consider $D_i = D_\zeta(x_{n+i}, x_{n+i+1})$. Thus, for all $n \geq 1$, we have two cases.

Case 1. Let $x_n = x_m$ for some integers $n \neq m$. So, if for $m > n$, we have $T^{m-n}(x_n) = x_n$. Choose $y = x_n$ and $p = m - n$. Then, $T^p y = y$; that is, y is a periodic point of T. Thus, $D_\zeta(y, Ty) = D_\zeta(T^p y, T^{p+1} y) \leq k^p D_\zeta(y, Ty)$. Since $\delta \in (0, 1)$, we get $D_\zeta(y, Ty) = 0$, so $y = Ty$; that is, y is a fixed point of T.

Case 2. Suppose that $T^n x \neq T^m x$ for all integers $n \neq m$. Let $n < m$ be two natural numbers; to show that $\{x_n\}$ is a D_ζ-Cauchy sequence, we need to consider two subcases:

Subcase 1. Assume that $m = n + 2p + 1$. By property (R_3) of the controlled rectangular b-metric spaces, we have

$$\begin{aligned}
D_\zeta(x_n, x_{n+2p+1}) \le{} & \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \\
& \cdot [D_\zeta(x_n, x_{n+1}) + D_\zeta(x_{n+1}, x_{n+2}) \\
& + D_\zeta(x_{n+2}, x_{n+2p+1})] \\
\le{} & \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) D_\zeta(x_n, x_{n+1}) \\
& + D_\zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) D_\zeta(x_{n+1}, x_{n+2}) \\
& + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \\
& \cdot \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p+1}) \\
& \cdot [D_\zeta(x_{n+2}, x_{n+3}) + D_\zeta(x_{n+3}, x_{n+4}) \\
& + D_\zeta(x_{n+4}, x_{n+2p+1})] \\
\le{} & \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) D_\zeta(x_n, x_{n+1}) \\
& + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) D_\zeta(x_{n+1}, x_{n+2}) \\
& + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \\
& \cdot \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p+1}) D_\zeta(x_{n+2}, x_{n+3}) \\
& + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \\
& \cdot \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p+1}) \\
& \cdot D_\zeta(x_{n+3}, x_{n+4}) + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \\
& \cdot \zeta(x_{n+2}, x_{n+3}, \\
& \cdot x_{n+4}, x_{n+2p+1}) D_\zeta(x_{n+4}, x_{n+2p+1}) \le \cdots.
\end{aligned} \tag{3}$$

Thus,

$$\begin{aligned}
D_\zeta(x_n, x_{n+2p+1}) \le{} & \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) D_\zeta(x_n, x_{n+1}) \\
& + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) D_\zeta(x_{n+1}, x_{n+2}) \\
& + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \zeta(x_{n+2}, x_{n+3}, \\
& \cdot x_{n+4}, x_{n+2p+1}) D_\zeta(x_{n+2}, x_{n+3}) + \zeta(x_n, x_{n+1}, \\
& \cdot x_{n+2}, x_{n+2p+1}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p+1}) \\
& \cdot D_\zeta(x_{n+3}, x_{n+4}) + \cdots + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \\
& \cdot \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p+1}) \cdots \zeta(x_{n+2p-2}, x_{n+2p-1}, \\
& \cdot x_{n+2p}, x_{n+2p+1}) D_\zeta(x_{n+2p}, x_{n+2p+1}) \\
\le{} & \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) D_0 + \zeta(x_n, x_{n+1}, x_{n+2}, \\
& \cdot x_{n+2p+1}) D_1 + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \zeta(x_{n+2}, \\
& \cdot x_{n+3}, x_{n+4}, x_{n+2p+1}) D_2 + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \\
& \cdot \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p+1}) D_3 + \cdots + \zeta(x_n, x_{n+1}, \\
& \cdot x_{n+2}, x_{n+2p+1}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p+1}) \times \cdots \times \cdots \\
& \cdot \zeta(x_{n+2p-2}, x_{n+2p-1}, x_{n+2p}, x_{n+2p+1}) D_{2p} \\
={} & \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) [D_0 + D_1] + \zeta(x_n, x_{n+1}, \\
& \cdot x_{n+2}, x_{n+2p+1}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p+1}) [D_2 + D_3] \\
& + \cdots + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, \\
& \cdot x_{n+2p+1}) \times \cdots \times \cdots \zeta(x_{n+2p-2}, x_{n+2p-1}, x_{n+2p}, x_{n+2p+1}) \\
& \cdot [D_{2p-1} + D_{2p}].
\end{aligned} \tag{4}$$

Therefore,

$$\begin{aligned}
D_\zeta(x_n, x_{n+2p+1}) \le{} & \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) [(\delta^n + \delta^{n+1}) \\
& \cdot D_\zeta(x_0, x_1)] + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \\
& \cdot \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p+1}) [(\delta^{n+2} + \delta^{n+3}) \\
& \cdot D_\zeta(x_0, x_1)] + \cdots + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \\
& \cdot \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p+1}) \times \cdots \times \cdots \\
& \cdot \zeta(x_{n+2p-2}, x_{n+2p-1}, x_{n+2p}, x_{n+2p+1}) \\
& \cdot [(\delta^{n+2p-2} + \delta^{n+2p-1}) D_\zeta(x_0, x_1)] \\
\le{} & [\zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) (\delta^n + \delta^{n+1}) \\
& + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \zeta(x_{n+2}, x_{n+3}, \\
& \cdot x_{n+4}, x_{n+2p+1}) (\delta^{n+2} + \delta^{n+3}) + \cdots + \zeta(x_n, \\
& \cdot x_{n+1}, x_{n+2}, x_{n+2p+1}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, \\
& \cdot x_{n+2p+1}) \times \cdots \times \cdots \zeta(x_{n+2p-2}, x_{n+2p-1}, \\
& \cdot x_{n+2p}, x_{n+2p+1}) (\delta^{n+2p-2} + \delta^{n+2p-1})] D_\zeta(x_0, x_1) \\
={} & \sum_{l=0}^{p-1} \prod_{i=0}^{l} \zeta(x_{n+2i}, x_{n+2i+1}, x_{n+2i+2}, x_{n+2p+1}) \\
& \cdot \left[\delta^{n+2l} + \delta^{n+2l+1}\right] D_\zeta(x_0, x_1) \\
={} & \sum_{l=0}^{p-1} \prod_{i=0}^{l} \zeta(x_{n+2i}, x_{n+2i+1}, x_{n+2i+2}, x_{n+2p+1}) \\
& \cdot [1 + \delta] \delta^{n+2l} D_\zeta(x_0, x_1).
\end{aligned} \tag{5}$$

Now, using the fact that $\delta < 1$, the above inequalities imply the following:

$$\begin{aligned}
D_\zeta(x_n, x_{n+2p+1}) < {} & \sum_{l=0}^{p-1} \prod_{i=0}^{l} \zeta(x_{n+2i}, x_{n+2i+1}, x_{n+2i+2}, x_{n+2p+1}) \\
& \cdot 2\delta^{n+2l} D_\zeta(x_0, x_1).
\end{aligned} \tag{6}$$

Since $\sup_{m>1} \lim_{n\to\infty} \zeta(x_n, x_{n+1}, x_{n+2}, x_m) \le 1/\delta$, we deduce

$$\begin{aligned}
\lim_{n,p\to\infty} D_\zeta(x_n, x_{n+2p+1}) < {} & \sum_{l=0}^{\infty} \prod_{i=0}^{l} \zeta(x_{n+2i}, x_{n+2i+1}, x_{n+2i+2}, x_{n+2p+1}) \\
& \cdot 2\delta^{n+2l} D_\zeta(x_0, x_1) \le \sum_{l=0}^{\infty} \frac{1}{\delta^{l+1}} 2\delta^{n+2l} D_\zeta(x_0, x_1) \\
& \le \sum_{l=0}^{\infty} 2\delta^{n+l-1} D_\zeta(x_0, x_1).
\end{aligned} \tag{7}$$

Note that the series $\sum_{l=0}^{\infty} 2\delta^{n+l-1} D_\zeta(x_0, x_1)$ converges by the ratio test, which implies that $D_\zeta(x_n, x_{n+2p+1})$ converges as $n, p \longrightarrow \infty$.

Subcase 2. $m=n+2p$. First of all, note that

$$D_\zeta(x_n,x_{n+2}) \le \delta D_\zeta(x_{n-1},x_{n+1}) \le \delta^2 D_\zeta(x_{n-2},x_n) \le \cdots \le \delta^n D_\zeta(x_0,x_2), \tag{8}$$

which leads us to conclude that $D_\zeta(x_n,x_{n+2}) \longrightarrow 0$ as $n \longrightarrow \infty$. Similar to Subcase 1, we have

$$\begin{aligned}
D_\zeta(x_n,x_{n+2p}) &\le \zeta(x_n,x_{n+1},x_{n+2},x_{n+2p}) \cdot [D_\zeta(x_n,x_{n+1}) + D_\zeta(x_{n+1},x_{n+2}) + D_\zeta(x_{n+2},x_{n+2p})] \\
&\le \zeta(x_n,x_{n+1},x_{n+2},x_{n+2p}) D_\zeta(x_n,x_{n+1}) + \zeta(x_n,x_{n+1},x_{n+2},x_{n+2p}) \cdot D_\zeta(x_{n+1},x_{n+2}) \\
&\quad + \zeta(x_n,x_{n+1},x_{n+2},x_{n+2p}) \cdot \zeta(x_{n+2},x_{n+3},x_{n+4},x_{n+2p}) \cdot [D_\zeta(x_{n+2},x_{n+3}) + D_\zeta(x_{n+3},x_{n+4}) + D_\zeta(x_{n+4},x_{n+2p})] \\
&\le \zeta(x_n,x_{n+1},x_{n+2},x_{n+2p}) \cdot D_\zeta(x_n,x_{n+1}) + \zeta(x_n,x_{n+1},x_{n+2},x_{n+2p}) \cdot D_\zeta(x_{n+1},x_{n+2}) \\
&\quad + \zeta(x_n,x_{n+1},x_{n+2},x_{n+2p}) \cdot \zeta(x_{n+2},x_{n+3},x_{n+4},x_{n+2p}) \cdot D_\zeta(x_{n+2},x_{n+3}) \\
&\quad + \zeta(x_n,x_{n+1},x_{n+2},x_{n+2}) \cdot \zeta(x_{n+2},x_{n+3},x_{n+4},x_{n+2p}) \cdot D_\zeta(x_{n+3},x_{n+4}) \\
&\quad + \zeta(x_n,x_{n+1},x_{n+2},x_{n+2p}) \cdot \zeta(x_{n+2},x_{n+3},x_{n+4},x_{n+2p}) \cdot D_\zeta(x_{n+4},x_{n+2p}) \\
&\le \zeta(x_n,x_{n+1},x_{n+2},x_{n+2p}) D_0 + \zeta(x_n,x_{n+1},x_{n+2},x_{n+2p}) D_1 \\
&\quad + \zeta(x_n,x_{n+1},x_{n+2},x_{n+2p}) \cdot \zeta(x_{n+2},x_{n+3},x_{n+4},x_{n+2p}) D_2 \\
&\quad + \zeta(x_n,x_{n+1},x_{n+2},x_{n+2p}) \cdot \zeta(x_{n+2},x_{n+3},x_{n+4},x_{n+2p}) D_3 + \cdots \\
&\quad + \zeta(x_n,x_{n+1},x_{n+2},x_{n+2p}) \cdot \zeta(x_{n+2},x_{n+3},x_{n+4},x_{n+2p}) \times \cdots \times \cdots \zeta(x_{n+2p-3},x_{n+2p-2},x_{n+2p-1},x_{n+2p}) D_{2p} \\
&\quad + \prod_{i=0}^{2p-2} \zeta(x_{n+2i},x_{n+2i+1},x_{n+2i+1},x_{n+2p}) \cdot D_\zeta(x_{n+2p-2},x_{n+2p}).
\end{aligned} \tag{9}$$

Hence,

$$\begin{aligned}
D_\zeta(x_n,x_{n+2p}) &= \sum_{l=0}^{p-1} \prod_{i=0}^{l} \zeta(x_{n+2i},x_{n+2i+1},x_{n+2i+2},x_{n+2p+1}) \cdot \left[\delta^{n+2l} + \delta^{n+2l+1}\right] D_\zeta(x_0,x_1) \\
&\quad + \prod_{i=0}^{2p-2} \zeta(x_{n+2i},x_{n+2i+1},x_{n+2i+1},x_{n+2p}) \cdot D_\zeta(x_{n+2p-2},x_{n+2p}) = \sum_{l=0}^{p-1} \prod_{i=0}^{l} \\
&\quad \cdot \zeta(x_{n+2i},x_{n+2i+1},x_{n+2i+2},x_{n+2p+1}) \cdot [1+\delta]\delta^{n+2l} D_\zeta(x_0,x_1) + \prod_{i=0}^{2p-2} \\
&\quad \cdot \zeta(x_{n+2i},x_{n+2i+1},x_{n+2i+1},x_{n+2p}) \cdot D_\zeta(x_{n+2p-2},x_{n+2p}) \le \sum_{l=0}^{p-1} \prod_{i=0}^{l} \\
&\quad \cdot \zeta(x_{n+2i},x_{n+2i+1},x_{n+2i+2},x_{n+2p+1}) \cdot [1+\delta]\delta^{n+2l} D_\zeta(x_0,x_1) + \prod_{i=0}^{2p-2} \\
&\quad \cdot \zeta(x_{n+2i},x_{n+2i+1},x_{n+2i+1},x_{n+2p}) \delta^{n+2p-2} \cdot D_\zeta(x_0,x_2).
\end{aligned} \tag{10}$$

Since $\sup_{m>1}\lim_{n\to\infty}\zeta(x_n,x_{n+1},x_{n+2},x_m) \le 1/\delta$, we deduce

$$\begin{aligned}
\lim_{n,p\to\infty} D_\zeta(x_n,x_{n+2p}) &\le \lim_{n,p\to\infty} \sum_{l=0}^{p-1} \frac{1}{\delta^{l+1}} [1+\delta]\delta^{n+2l} D_\zeta(x_0,x_1) + \delta^{2p-1}\delta^{n+2p-2} D_\zeta(x_0,x_2) \\
&= \lim_{n,p\to\infty} \sum_{l=0}^{p-1} [1+\delta]\delta^{n+l-1} D_\zeta(x_0,x_1) + \delta^{n-1} D_\zeta(x_0,x_2) \\
&\le \sum_{m=0}^{\infty} [1+\delta]\delta^m D_\zeta(x_0,x_1) + \delta^m D_\zeta(x_0,x_2).
\end{aligned} \tag{11}$$

By using the ratio test, it is not difficult to see that the series

$$\sum_{m=0}^{\infty} [1+\delta]\delta^m D_\zeta(x_0,x_1) + \delta^m D_\zeta(x_0,x_2), \tag{12}$$

converges. Hence, $D_\zeta(x_n,x_{n+2p})$ converges as n and p go toward ∞ Thus, by Subcases 1 and 2, we deduce that the sequence $\{x_n\}$ is a D_ζ-Cauchy sequence. Since (X,D_ζ) is a D_ζ-complete extended rectangular b-metric space, we deduce that $\{x_n\}$ converges to some $\nu \in X$. We claim that ν is a fixed point of T. Note that there exists an integer N such that $x_N=\nu$. Due to Case 2, $T^n x \ne \nu$ for all $n>N$. Similarly,

$T^n x \neq T\nu$ for all $n > N$. Hence, we are in Case 1, so ν is a fixed point of T.

Also, there exists an integer N such that $T^N x = T\nu$. Again, necessarily, $T^n x \neq \nu$ and $T^n x \neq T\nu$ for all $n > N$. Thus, $T \nu = \nu$. Therefore, we may assume that for all n, we have $x_n \in \{\nu, T\nu\}$.

$$\begin{aligned} D_\zeta(\nu, T\nu) &\leq \zeta(\nu, T\nu, x_n, x_{n+1})\left[D_\zeta(\nu, x_n) \right.\\ &\quad \left. + D_\zeta(x_n, x_{n+1}) + D_\zeta(x_{n+1}, T\nu)\right] \\ &\leq D_\zeta(\nu, T\nu, x_n, x_{n+1}) \\ &\quad \cdot \left[D_\zeta(\nu, x_n) + D_\zeta(x_n, x_{n+1}) + D_\zeta(Tx_n, T\nu)\right] \\ &\leq \zeta(\nu, T\nu, x_n, x_{n+1}) \\ &\quad \cdot \left[D_\zeta(\nu, x_n) + D_\zeta(x_n, x_{n+1}) + \delta D_\zeta(x_n, \nu)\right]. \end{aligned} \tag{13}$$

Now, taking the limit as $n \longrightarrow \infty$, we deduce that $D_\zeta (\nu, T\nu) = 0$; that is, $T\nu = \nu$ and ν is a fixed point of T as desired.

Finally, to show uniqueness assume, there exist two fixed points of T say ν and μ such that $\nu \neq \mu$. By the contractive property of T, we have

$$D_\zeta(\nu, \mu) = D_\zeta(T\nu, T\mu) \leq \delta D_\zeta(\nu, \mu) < D_\zeta(\nu, \mu), \tag{14}$$

which leads us to a contradiction. Thus, T has a unique fixed point as required.

Theorem 7. *Let*(X, D_ζ)*be a complete extended rectangular b -metric space and T a self-mapping on X satisfying the following condition; for all*$a, b \in X$*, there exists* $0 < \delta < 1/2$ *such that*

$$D_\zeta(Ta, Tb) \leq \delta\left[D_\zeta(a, Ta) + D_\zeta(b, Tb)\right]. \tag{15}$$

Also, if

$$\sup_{m>1} \lim_{n\to\infty} \zeta(x_n, x_{n+1}, x_{n+2}, x_m) \leq 1/\delta, \tag{16}$$

and for all$u, v \in X$*, we have*

$$\lim_{n\to\infty} \zeta(u, v, x_n, x_{n+1}) \leq 1\,; \tag{17}$$

then, T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and define the sequence $\{x_n\}$ as follows:

$$x_1 = Tx_0, x_2 = Tx_1 = T^2 x_0, \cdots, x_n = Tx_{n-1} = T^n x_0, \cdots. \tag{18}$$

First of all, note that for all $n \geq 1$, we have

$$\begin{aligned} D_\zeta(x_n, x_{n+1}) &\leq \delta\left[D_\zeta(x_{n-1}, x_n) + D_\zeta(x_n, x_{n+1})\right] \\ &\Rightarrow (1-k) D_\zeta(x_n, x_{n+1}) \leq \delta \rho(x_{n-1}, x_n) \\ &\Rightarrow D_\zeta(x_n, x_{n+1}) \leq \frac{\delta}{1-\delta} D_\zeta(x_{n-1}, x_n). \end{aligned} \tag{19}$$

Since $0 < \delta < 1/2$, one can easily deduce that $0 < \delta/(1-\delta) < 1$. So, let $\mu = \delta/(1-\delta)$. Hence,

$$\begin{aligned} D_\zeta(x_n, x_{n+1}) &\leq \mu D_\zeta(x_{n-1}, x_n) \\ &\leq \mu^2 D_\zeta(x_{n-2}, x_{n-1}) \leq \cdots \leq \mu^n D_\zeta(x_0, x_1). \end{aligned} \tag{20}$$

Therefore,

$$D_\zeta(x_n, x_{n+1}) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{21}$$

Also, for all $n \geq 1$, we have

$$D_\zeta(x_n, x_{n+2}) \leq \delta\left[D_\zeta(x_{n-1}, x_n) + D_\zeta(x_{n+1}, x_{n+2})\right]. \tag{22}$$

Thus, by using the fact that $D_\zeta(x_n, x_{n+1}) \longrightarrow 0$ as $n \longrightarrow \infty$, we deduce that

$$D_\zeta(x_n, x_{n+2}) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{23}$$

Now, similar to the proof of Cases 1 and 2 of Theorem 6, we deduce that the sequence $\{x_n\}$ is a D_ζ-Cauchy sequence. Since (X, D_ζ) is a D_ζ-complete extended rectangular b-metric space, we conclude that $\{x_n\}$ converges to some $\nu \in X$. Using the argument in the proof of Theorem 6, we may assume that for all $n \geq 1$, we have $x_n \in \{\nu, T\nu\}$. Thus,

$$\begin{aligned} D_\zeta(\nu, T\nu) &\leq \zeta(\nu, T\nu, x_n, x_{n+1}) \\ &\quad \cdot \left[D_\zeta(\nu, x_n) + D_\zeta(x_n, x_{n+1}) + D_\zeta(x_{n+1}, T\nu)\right] \\ &\leq \zeta(\nu, T\nu, x_n, x_{n+1}) \\ &\quad \cdot \left[D_\zeta(\nu, x_n) + D_\zeta(x_n, x_{n+1}) + D_\zeta(Tx_n, T\nu)\right] \\ &\leq \zeta(\nu, T\nu, x_n, x_{n+1}) \\ &\quad \cdot \left[D_\zeta(\nu, x_n) + D_\zeta(x_n, x_{n+1}) + \delta D_\zeta(x_n, Tx_n) + \delta D_\zeta(\nu, T\nu)\right]. \end{aligned} \tag{24}$$

Taking the limit of the above inequalities, we get

$$D_\zeta(\nu, T\nu) \leq \left[0 + 0 + 0 + \delta D_\zeta(\nu, T\nu)\right] < D_\zeta(\nu, T\nu). \tag{25}$$

Thus, $D_\zeta(\nu, T\nu) = 0$ which implies that $T\nu = \nu$, and hence, ν is a fixed point of T. Finally, to show uniqueness, assume there exist two fixed points of T say ν and μ such that $\nu \neq \mu$. By the contractive property of T, we have

$$D_\zeta(\nu, \mu) = D_\zeta(T\nu, T\mu) \leq \delta D_\zeta(\nu, \mu) < D_\zeta(\nu, \mu), \tag{26}$$

which leads us to a contradiction. Thus, T has a unique fixed point as required.

4. Application

In closing, we present the following application for our results.

Theorem 8. *For any natural number$\kappa \geq 3$, the equation*

$$s^{\kappa} + 1 = \left(\kappa^4 - 1\right)s^{\kappa+1} + \kappa^4 s, \tag{27}$$

has a unique real solution.

Proof. First of all, note that if $|s| > 1$, Equation (3.1) does not have a solution. So, let $X = [-1, 1]$, and for all $s, r \in X$, let $D_\zeta(s, r) = |s - r|$ and $\zeta(s, r, u, v) = \max\{s, r, u, v\} + 2$. It is not difficult to see that (X, D_ζ) is a D_ζ-complete controlled rectangular b-metric space. Now, let

$$Ts = \frac{s^\kappa + 1}{(\kappa^4 - 1)s^\kappa + \kappa^4}. \tag{28}$$

Notice that since $\kappa \geq 2$, we can deduce that $\kappa^4 \geq 6$. Thus,

$$\begin{aligned} D_\zeta(Ts, Tr) &= \left| \frac{s^\kappa + 1}{(\kappa^4 - 1)s^\kappa + \kappa^4} - \frac{r^\kappa + 1}{(\kappa^4 - 1)r^\kappa + \kappa^4} \right| \\ &= \left| \frac{s^\kappa - r^\kappa}{((\kappa^4 - 1)s^\kappa + \kappa^4)((\kappa^4 - 1)r^\kappa + \kappa^4)} \right| \\ &\leq \frac{|s - r|}{\kappa^4} \leq \frac{|s - r|}{6} = \frac{1}{6} D_\zeta(s, r). \end{aligned} \tag{29}$$

Hence,

$$D_\zeta(Ts, Tr) \leq \delta D_\zeta(s, r), \quad \delta = \frac{1}{6}. \tag{30}$$

On the other hand, notice that for all $s_0 \in X$, we have

$$s_n = T^n s_0 \leq \frac{2}{\kappa^4}. \tag{31}$$

Thus,

$$\sup_{n\geq 1} \lim_{i\to\infty} \zeta(s_i, s_{i+1}, s_{i+2}, s_n) = \frac{2}{\kappa^4} \leq 2 < 6 = \frac{1}{\delta}. \tag{32}$$

Finally, note that T satisfies all the hypothesis of Theorem 6. Therefore, T has a unique fixed point in X, which implies that Equation (3.1) has a unique real solution as desired.

Example 2. The following equation

$$s^{100} + 1 = 99999999 s^{101} + 100000000 s, \tag{33}$$

has a unique real solution.

Proof. The proof is a direct consequence of Theorem 6, by taking $\kappa = 100$.

5. Conclusion

In closing, we would like to bring to the readers' attention to the following open questions:

Question 1. Let (X, D_ζ) be a controlled rectangular b-metric space and T a self-mapping on X. Also, assume that for all distinct $s, r, Ts, Tr \in X$, there exists $\delta \in (0, 1)$ such that

$$D_\zeta(Ts, Tr) \leq \delta\zeta(s, r, Ts, Tr) D_\zeta(s, r). \tag{34}$$

What are the other hypotheses we should add so that T has a unique fixed point in the whole space X?

Question 2. Let (X, D_ζ) be a controlled rectangular b-metric space and T a self-mapping on X. Also, assume that for all distinct $s, r, Ts, Tr \in X$, there exists $\delta \in (0, 1)$ such that

$$D_\zeta(Ts, Tr) \leq \zeta(s, r, Ts, Tr)\left[D_\zeta(s, Ts) + D_\zeta(r, Tr)\right]. \tag{35}$$

What are the other hypotheses we should add so that T has a unique fixed point in the whole space X?

Data Availability

Data availability is not applicable.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All three authors contributed equally to this manuscript.

Acknowledgments

The first and third authors would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM), group number RG-DES-2017-01-17.

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, no. 1, pp. 133–181, 1922.

[2] T. A. Maraaba, F. Jarad, and D. Baleanu, "On the existence and the uniqueness theorem for fractional differential equations with bounded delay within Caputo derivatives," *Science in China Series A: Mathematics*, vol. 51, no. 10, pp. 1775–1786, 2008.

[3] T. Maraaba (Abdeljawad), D. Baleanu, and F. Jarad, "Existence and uniqueness theorem for a class of delay differential equations with left and right Caputo fractional derivatives," *Journal of Mathematical Physics*, vol. 49, no. 8, article 083507, 2008.

[4] T. Abdeljawad, "Meir-Keeler α-contractive fixed and common fixed point theorems," *Fixed Point Theory and Applications*, vol. 2013, no. 1, 2013.

[5] K. Abodayeh, N. Mlaiki, T. Abdeljawad, and W. Shatanawi, "Relations between partial metric spaces and M-metric spaces, Caristi Kirk's theorem in M–metric type spaces," *Journal of Mathematical Analysis*, vol. 7, no. 3, pp. 1–12, 2016.

[6] T. Abdeljawad, K. Abodayeh, and N. Mlaiki, "On fixed point generalizations to partial b-metric spaces," *Journal of

Computational Analysis and Applications, vol. 19, no. 5, pp. 883–891, 2015.

[7] H. Afshari, H. Aydi, and E. Karapınar, "On generalized α-ψ-Geraghty contractions on b-metric spaces," *Georgian Mathematical Journal*, vol. 27, no. 1, pp. 9–21, 2020.

[8] E. Ameer, H. Aydi, M. Arshad, and M. de la Sen, "Hybrid Ćirić type graphic Υ,Λ-contraction mappings with applications to electric circuit and fractional differential equations," *Symmetry*, vol. 12, no. 3, p. 467, 2020.

[9] H. Aydi, M. F. Bota, E. Karapinar, and S. Moradi, "A common fixed point for weak ϕ-contractions on b-metric spaces," *Fixed Point Theory*, vol. 13, no. 2, pp. 337–346, 2012.

[10] H. Aydi, R. Banković, I. Mitrović, and M. Nazam, "Nemytzki-Edelstein-Meir-Keeler type results in b-metric spaces," *Discrete Dynamics in Nature and Society*, vol. 2018, Article ID 4745764, 7 pages, 2018.

[11] I. A. Bakhtin, "The contraction mapping principle in almost metric spaces," *Functional Analysis*, vol. 30, pp. 26–37, 1989.

[12] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.

[13] E. Karapinar, S. Czerwik, and H. Aydi, "(α,ψ)-Meir-Keeler contraction mappings in generalized b-metric spaces," *Journal of Function Spaces*, vol. 2018, Article ID 3264620, 4 pages, 2018.

[14] A. A. Kilbas, M. H. Srivastava, and J. J. Trujillo, "Theory and application of fractional differential equations," *North Holland Mathematics Studies*, vol. 204, 2006.

[15] M. A. Khamsi and W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, John Willey and Sons, INC., 1996.

[16] N. Mlaiki, A. Mukheimer, Y. Rohen, N. Souayah, and T. Abdeljawad, "Fixed point theorems for $\alpha-\psi-$contractive mapping in Υ_b-metric spaces," *Journal of Mathematical Analysis*, vol. 8, no. 5, pp. 40–46, 2017.

[17] D. K. Patel, T. Abdeljawad, and D. Gopal, "Common fixed points of generalized Meir-Keeler α-contractions," *Fixed Point Theory and Applications*, vol. 2013, no. 1, 2013.

[18] N. Souayah, N. Mlaiki, and M. Mrad, "The G_M -contraction principle for mappings on an M-metric spaces endowed with a graph and fixed point theorems," *IEEE Access*, vol. 6, pp. 25178–25184, 2018.

[19] N. Souayaha and N. Mlaikib, "A fixed point theorem in S_b-metric spaces," *Journal of Mathematics and Computer Science*, vol. 16, no. 2, pp. 131–139, 2016.

[20] T. Abdeljawad, N. Mlaiki, H. Aydi, and N. Souayah, "Double controlled metric type spaces and some fixed point results," *Mathematics*, vol. 6, no. 12, p. 320, 2018.

[21] N. Mlaiki, H. Aydi, N. Souayah, and T. Abdeljawad, "Controlled metric type spaces and the related contraction principle," *Mathematics*, vol. 6, no. 10, p. 194, 2018.

[22] A. Mukheimer, N. Mlaiki, K. Abodayeh, and W. Shatanawi, "New theorems on extended b-metric spaces under new contractions," *Nonlinear Analysis: Modelling and Control*, vol. 24, no. 6, 2019.

[23] H. Qawaqneh, M. S. M. Noorani, and W. Shatanawi, "Fixed point theorems for (α, k, θ)-contractive multi-valued mapping in b-metric space and applications," *International Journal of Mathematics and Computer Science*, vol. 14, no. 1, pp. 263–283, 2019.

[24] W. Shatanawi, K. Abodayeh, and A. Mukhemer, "Some fixed point theorems in extended b-metric spaces," *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, vol. 80, no. 4, pp. 71–78, 2018.

[25] W. Shatanawi, A. Pitea, and R. Lazovic, "Contraction conditions using comparison functions on b-metric spaces," *Fixed Point Theory and Applications*, vol. 2014, no. 1, 2014.

[26] J. R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei, and W. Shatanawi, "Common fixed points of almost generalized $(\psi,\varphi)_s$-contractive mappings in ordered b-metric spaces," *Fixed Point Theory and Applications*, vol. 2013, no. 1, 2013.

[27] H. Qawaqneh, M. Md Noorani, W. Shatanawi, H. Aydi, and H. Alsamir, "Fixed point results for multi-valued contractions in b-metric spaces and an application," *Mathematics*, vol. 7, no. 2, p. 132, 2019.

[28] A. Branciari, "A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces," *Publicationes Mathematicae*, vol. 57, no. 1-2, pp. 31–37, 2000.

[29] R. George, S. Radenović, K. P. Reshma, and S. Shukla, "Rectangular b-metric space and contraction principles," *Journal of Nonlinear Sciences and Applications*, vol. 8, no. 6, pp. 1005–1013, 2015.

[30] T. Abdeljawad, E. Karapınar, S. K. Panda, and N. Mlaiki, "Solutions of boundary value problems on extended-Branciari b-distance," *Journal of Inequalities and Applications*, vol. 2020, no. 1, 2020.