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Research Article

Existence and Multiplicity of Solutions for a Class of Anisotropic Double Phase Problems

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We consider the following double phase problem with variable exponents: $\begin{cases} -\text{div}\,(|\nabla u|^{p(x)-2}\nabla u + a(x)|\nabla u|^{q(x)-2}\nabla u) = \lambda f(x,u) \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega \end{cases}$. By using the mountain pass theorem, we get the existence results of weak solutions for the aforementioned problem under some assumptions. Moreover, infinitely many pairs of solutions are provided by applying the Fountain Theorem, Dual Fountain Theorem, and Krasnoselskii's genus theory.

1. Introduction and Statement of Results

In this paper, we deal with the existence and multiplicity of solutions for the following double phase problem

$$\begin{cases} -\text{div}\left(|\nabla u|^{p(x)-2}\nabla u + a(x)|\nabla u|^{q(x)-2}\nabla u\right) = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (P_\lambda) \tag{1}$$

where $\lambda > 0$ is a real parameter, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with smooth boundary, $p^*(\cdot) = Np(\cdot)/(N - p(\cdot))$, $p(\cdot)$, and $q(\cdot)$ are Lipschitz continuous in $\mathbb{R}^N$. Moreover,

$$\frac{q(\cdot)}{p(\cdot)} < 1 + \frac{1}{N}, \; a : \bar{\Omega} \longrightarrow [0, +\infty), \text{ is Lipschitz continuous} \tag{2}$$

and we also assume that the nonlinearity f satisfies the following conditions:

(f_1) $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function and there exists $C_1 > 0$ such that

$$|f(x,t)| \leq C_1\left(1 + |t|^{\alpha(x)-1}\right), \tag{3}$$

for all $(x,t) \in \Omega \times \mathbb{R}$, where $\alpha \in C(\bar{\Omega})$, $1 < q^+ < \alpha^- \leq \alpha^+ < p^*(\cdot)$.$(f_2)$ $\lim_{t\to 0}(f(x,t)/|t|^{q^+-1}) = 0$, uniformly for a.e. $x \in \Omega$. (f_3) $\lim_{t\to+\infty} F(x,t)/|t|^{q^+} = +\infty$, uniformly for a.e. $x \in \Omega$, where $F(x,t) = \int_0^t f(x,s)ds$.$(f_4)$ There exists a constant $C_0 > 0$ such that

$$G(x,t) \leq G(x,s) + C_0, \tag{4}$$

for any $x \in \Omega, 0 < t < s$ or $s < t < 0$, where $G(x,t) = tf(x,t) - q^+F(x,t)$.$(f_4^*)$ There exists $T_0 > 0$ such that $f(x,t)/|t|^{q^+-2}t$ is nondecreasing in t when $t \geq T_0$ and nonincreasing in $t \leq -T_0$ for all $x \in \Omega$.(f_5) $f(x,-t) = -f(x,t)$, for all $x \in \Omega$ and $t \in \mathbb{R}$.

Remark 1. We point out that the condition (f_4) is weaker than (f_4^*). It is not difficult to check that the condition (f_4^*) is equivalent to the following condition (see [1]): (f_4^{**})

$G(x,t)$ is increasing in $t \geq T_0$ and decreasing in $t \leq -T_0$ for all $x \in \Omega$.

Hence, (f_4^*) implies (f_4).

Similar problems have been investigated and it is well known they have a strong physical meaning because they appear in the models of strongly anisotropic materials, see, e.g., [2, 3]. The energy functionals of the form

$$u \mapsto \int_\Omega \mathscr{H}(x, |\nabla u(x)|)dx, \quad \mathscr{H}(x,t) = t^{p(x)} + a(x)t^{q(x)}, \quad q(x) > p(x) > 1, \quad a(\cdot) > 0, \tag{5}$$

where the integrand $\mathscr{H}$ switches between two different elliptic behaviors have been intensively studied in recent years, see [2–11]. Recently, Mingione et al. have obtained the regularity theory for minimizers of (5), see, e.g., [7].

When $a(x) = 1$ and $\lambda = 1$, problem (P_λ) becomes a $(p(x), q(x))$-Laplacian problem of the form

$$\begin{cases} -\Delta_{p(x)}u(x) - \Delta_{q(x)}u(x) = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $-\Delta_{p(x)}u := -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. In particular, we refer to [9] where the authors proved the existence of one and three nontrivial weak solutions of (6), by the mountain pass theory and Morse theory.

If $p(x) = q(x)$, then $a(x) = 1$. Vetro [12] studied the following Dirichlet boundary value problem involving the $p(x)$-Laplacian-like operator:

$$\begin{cases} -\Delta^l_{p(x)}u(x) + |u(x)|^{p(x)-2}u(x) = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where

$$-\Delta^l_{p(x)}u := \operatorname{div}\left(\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right), \tag{8}$$

is the $p(x)$-Laplacian-like. They have established the existence and multiplicity results for the problem (7) when λ is sufficiently small.

In the particular case of $p(x) \equiv p, q(x) \equiv q$, such problems have been recently studied in, e.g., [13–16]. The existence and multiplicity of weak solutions of problem (P_λ) with $\lambda = 1$ has been established in Liu and Dai [13]. In [15], by using the Morse theory, Perera and Squassina obtained a nontrivial weak solution of problem (P_λ). In [14], by utilizing the Nehari method, Liu and Dai obtained three ground state solutions. Usually, the authors in those references considered the nonlinearities $f(x,t)$ satisfying the Ambrosetti-Rabinowitz type condition ((AR) in short): i.e., there exist $L > 0$, $\theta > q$, such that for $|t| \geq L$ and a.e. $x \in \Omega$,

$$0 < \theta F(x,t) \leq tf(x,t). \tag{9}$$

Under some appropriate assumptions, one can consider a much weaker condition on $f(x,t)$

$$\lim_{|u|\to+\infty} \frac{F(x,u)}{|u|^q} = +\infty \text{uniformly in } x. \tag{10}$$

This means that F is q-superlinear at infinity. But the (AR) condition is useful and natural to ensure the mountain pass geometry and the Palais-Smale condition ((PS) in short). So it have attracted much interest in recent literature, see for example [13, 15, 17–19] and the references therein. However, in this paper, we consider the problem (P_λ) in the case when the nonlinearity F is q^+-superlinear at both infinity and origin (see conditions (f_2) and (f_3)). These conditions are weaker than the (AR) condition. For example, Papageorgiou, Vetro, and Vetro [16] investigated the following $(p,2)$-equation with combined nonlinearities:

$$\begin{cases} -\Delta_p u(x) - \Delta u(x) = \lambda f(x,u) + g(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where $\lambda > 0, 2 < p < +\infty, \Omega \subset \mathbb{R}^N$, be a bounded domain with a C^2-boundary $\partial\Omega$. Using the critical point theory, critical groups, and flow invariance arguments, the authors obtained at least five nontrivial smooth solutions of (11) when f is $(p-1)$-superlinear near $\pm\infty$ but does not satisfy the (AR) condition.

Now, a natural question is whether the results contained in [13] can be generalized to the variable exponents $(p(x), q(x))$ case. Moreover, can we assume that the nonlinearity f satisfies a more natural and weaker $(q^+ - 1)$-superlinear condition near $\pm\infty$ instead of the (AR) condition?

Inspired by the above works, we will answer these questions. For a detailed motivation of our context and additional references, we refer to the introduction of [8, 20]. To the best of our knowledge, there are very few papers related to the existence of solutions of problem (P_λ) with variable exponents. This paper was motivated by the interest in applications of the variable exponent Orlicz-Sobolev spaces. Before stating our main results, we introduce some notations.

1.1. Notations and definitions. Throughout this paper, we define the class

$$C_+(\bar{\Omega}) = \{p \in C(\bar{\Omega}),\ p(x) > 1 \text{ for all } x \in \bar{\Omega}\}. \tag{12}$$

For any $p \in C_+(\bar{\Omega})$, we denote

$$p^+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} p(x), \quad p^- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^N} p(x), \tag{13}$$

and we denote by $p_1 \ll p_2$ the fact that

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}^N}(p_2(x)-p_1(x))>0. \tag{14}$$

The letters $C, C_i, i=1,2,\cdots$, denote positive constants which may vary from line to line but are independent of the terms which will take part in any limit process. The notion of weak solution for problem (P_λ) is that $u \in W_0^{1,\mathcal{H}}(\Omega)$ is a solution of (P_λ) if

$$\int_\Omega \left(|\nabla u|^{p(x)-2}+a(x)|\nabla u|^{q(x)-2}\right)\nabla u\cdot\nabla v dx=\int_\Omega f(x,u)v dx, \quad \forall v\in W_0^{1,\mathcal{H}}(\Omega). \tag{15}$$

It is formulated in a suitable Orlicz-Sobolev space $W_0^{1,\mathcal{H}}(\Omega)$ that will be introduced in Section 2. It is easy to see that solutions of (P_λ) correspond to the critical points of the energy functional I_λ defined by

$$I_\lambda(u)=\int_\Omega\left(\frac{1}{p(x)}|\nabla u|^{p(x)}+\frac{a(x)}{q(x)}|\nabla u|^{q(x)}\right)\cdot dx-\lambda\int_\Omega F(x,u)dx, \forall u\in W_0^{1,\mathcal{H}}(\Omega), \tag{16}$$

where $F(x,t)=\int_0^t f(x,s)dx$.

Now, we present the main results of this paper as follows:

Theorem 2. *Suppose $(f_1)-(f_4)$ are satisfied. Then problem (P_λ) has at least one nontrivial weak solution in $W_0^{1,\mathcal{H}}(\Omega)$ for all $\lambda>0$.*

Theorem 3. *Suppose $(f_1),(f_3)-(f_4)$ are satisfied. Then there exists $\lambda_0>0$ such that for all $\lambda\in(0,\lambda_0)$, problem (P_λ) has at least one solution u_λ and*

$$\lim_{\lambda\to0^+}\|u_\lambda\|=+\infty. \tag{17}$$

Theorem 4. *Suppose $(f_1)-(f_5)$ are satisfied. Then problem (P_λ) has infinitely many solutions in $W_0^{1,\mathcal{H}}(\Omega)$ for all $\lambda>0$.*

Theorem 5. *Suppose (f_5) and the following condition $(f_6) f:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ is a Carathéodory function, and there exist positive constants d_0, d_1 such that*

$$d_0|t|^{\beta(x)-1}\le f(x,t)\le d_1|t|^{\beta(x)-1}, \tag{18}$$

for all $x\in\bar{\Omega}$and $t\ge0$, where $\beta\in C(\bar{\Omega})$ such that $1<\beta(x)<p^*(x)$ with $\beta^+<p^-$. Then problem (P_λ) has infinitely many solutions in $W_0^{1,\mathcal{H}}(\Omega)$ for all $\lambda>0$.

Theorem 6. *Suppose $(f_1),(f_3)-(f_5)$ are satisfied. Then for all $\lambda>0$, problem (P_λ) has infinitely many solutions $\{u_n\}_{n\in\mathbb{N}}W_0^{1,\mathcal{H}}(\Omega)$ such that $\lim_{n\to\infty}I_\lambda(u_n)=\infty$.*

Theorem 7. *Suppose $(f_1),(f_3)-(f_5)$ are satisfied. Then for all $0<\lambda<\alpha/q^+$, problem (P_λ) has infinitely many solutions $\{v_n\}_{n\in\mathbb{N}}W_0^{1,\mathcal{H}}(\Omega)$ such that $I_\lambda(v_n)<0, \lim_{n\to\infty}I_\lambda(v_n)=0$.*

Remark 8. Note that our Theorems 2–7 answer the above questions. To be precise, Theorems 2, 4, 6, and 7 extend the main results of [13] to the variable exponents $(p(x),q(x))$ case. Compared with [13], the main difficulty is that since both $p(x)$ and $q(x)$ are nonconstant functions, then (P_λ) has a more complicated structure, due to its nonhomogeneities and to the presence of the nonlinear term.

Remark 9. In Theorem 5, we obtain infinitely many solutions by using Krasnoselskii's genus theory. Moreover, we consider continuous functions $f=f(x,u)$ satisfying the growth condition

$$d_0|u|^{\beta(x)-1}\le f(x,u)\le d_1|u|^{\beta(x)-1}. \tag{19}$$

The rest of this paper is organized as follows. In Section 2, we state some preliminary notations and the main lemmas. In Section 3, we prove the Theorems 2 and 3. The proofs of Theorems 4–5 are given in Section 4. By using the Fountain Theorem and the Dual Fountain Theorem, infinitely many pairs of solutions are provided in Section 5.

2. Preliminaries

In order to discuss the problem (P_λ), we need some theories on generalized Orlicz spaces and Sobolev spaces. For more details, we refer to the references [20–23]. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega)=\left\{u \text{ is a measurable real valued function}\left|\int_\Omega|u(x)|^{p(x)}dx<+\infty\right|\right\}, \tag{20}$$

endowed with the Luxemburg norm

$$\|u\|_{p(\cdot)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}dx\le1\right\}. \tag{21}$$

Note that, if p is a constant function, the Luxemburg norm $\|u\|_{p(\cdot)}$ coincide with the standard norm $\|u\|_p$ of the Lebesgue space $L^p(\Omega)$. Then, $(L^{p(x)}(\Omega),\|u\|_{p(\cdot)})$ becomes a Banach space, and we call it the variable exponent Lebesgue space. It is easy to check that the embedding $L^{p_2(x)}(\Omega)\hookrightarrow L^{p_1(x)}(\Omega)$ is continuous, where $0<|\Omega|<\infty$ and p_1,p_2 are variable exponents such that $p_1\le p_2$ in Ω.

The following property of spaces with variable exponent is essentially due to Fan and Zhao [24].

Lemma 10. *The space $(L^{p(\cdot)}(\Omega),\|\cdot\|_{p(\cdot)})$ is a separable, uniformly convex Banach space, and its dual space is $L^{p'(\cdot)}(\Omega)$ where $(1/p(x))+(1/p'(x))=1$. For any $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$, we have*

$$\left|\int_{\Omega} uvdx\right| \le \left(\frac{1}{p^-} + \frac{1}{\left(p'\right)^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}. \tag{22}$$

The Musielak-Orlicz space $L^{\mathscr{H}}(\Omega)$ is defined by

$$L^{\mathscr{H}}(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable} : \int_{\Omega} \mathscr{H}(x, |u|)dx < +\infty\right\}, \tag{23}$$

endowed with the norm

$$\|u\|_{\mathscr{H}} = \inf\left\{\lambda > 0 : \rho\mathscr{H}\left(\frac{u}{\lambda}\right) \le 1\right\}, \tag{24}$$

where $\mathscr{H}$ is defined in (5). The space $L^{\mathscr{H}}(\Omega)$ is a separable, uniformly convex, and reflexive Banach space. We denote by $L_a^{q(\cdot)}(\Omega)$ the space of all measurable functions $u : \Omega \longrightarrow \mathbb{R}$ with the seminorm

$$\|u\|_{p(\cdot),a} \coloneqq \left(\int_{\Omega} a(x)|u|^{q(x)}dx\right)^{1/q(x)} < \infty. \tag{25}$$

It is easy to check that the embeddings

$$L^{q(\bullet)}(\Omega) \hookrightarrow L^{\mathscr{H}}(\Omega) \hookrightarrow L^{p(\bullet)}(\Omega) \cap L_a^{q(\bullet)}(\Omega), \tag{26}$$

are continuous. Since $\rho\mathscr{H}(u/\|u\|\mathscr{H}) = 1$ whenever $u \neq 0$, we have

$$\min\left\{\|u\|_{\mathscr{H}}^{p(x)}, \|u\|_{\mathscr{H}}^{q(x)}\right\} \le \|u\|_{p(\cdot)}^{p(x)} + \|u\|_{p(\cdot),a}^{p(x)} \le \max\left\{\|u\|_{\mathscr{H}}^{p(x)}, \|u\|_{\mathscr{H}}^{q(x)}\right\}, \quad \forall u \in L^{\mathscr{H}}(\Omega). \tag{27}$$

The related Sobolev space $W^{1,\mathscr{H}}(\Omega)$ is defined by

$$W^{1,\mathscr{H}}(\Omega) \coloneqq \left\{u \in L^{\mathscr{H}}(\Omega) : |\nabla u| \in L^{\mathscr{H}}(\Omega)\right\}, \tag{28}$$

equipped with the norm

$$\|u\| = \|u\|_{\mathscr{H}} + \|\nabla u\|_{\mathscr{H}}, \tag{29}$$

where $\|\nabla u\|_{\mathscr{H}} = \||\nabla u|\|_{\mathscr{H}}$. The completion of $C_0^{\infty}(\Omega)$ in $W^{1,\mathscr{H}}(\Omega)$ is denoted by $W_0^{1,\mathscr{H}}(\Omega)$ and it can be equivalently renormed by

$$\|u\| \coloneqq \|\nabla u\|_{\mathscr{H}}, \tag{30}$$

via a Poincaré-type inequality, cf ([6], Proposition 2.18(iv)), under assumption (2). The spaces $W^{1,\mathscr{H}}(\Omega)$ and $W_0^{1,\mathscr{H}}(\Omega)$ are uniformly convex, and hence reflexive, Banach space. By (27), We have

$$\min\left\{\|u\|^{p(x)}, \|u\|^{q(x)}\right\} \le \|\nabla u\|_{p(\cdot)}^{p(x)} + \|\nabla u\|_{p(\cdot),a}^{p(x)} \le \max\left\{\|u\|^{p(x)}, \|u\|^{q(x)}\right\}, \quad \forall u \in W_0^{1,\mathscr{H}}(\Omega). \tag{31}$$

We point out that if $r \in C_+(\bar{\Omega})$ and $r(x) \le p^*(x)$ for all $x \in \bar{\Omega}$, then $W^{1,\mathscr{H}}(\Omega) \hookrightarrow L^{r(\bullet)}(\Omega)$ is continuous. This embedding is compact if

$$\inf_{x\in\Omega}\{p^*(x) - r(x)\} > 0. \tag{32}$$

Let us now define $J(\cdot)$: $W_0^{1,\mathscr{H}}(\Omega) \longrightarrow \mathbb{R}$ as

$$J(u) = \int_{\Omega}\left(\frac{1}{p|(x)|}|\nabla u|^{p(x)} + \frac{a(x)}{q(x)}|\nabla u|^{q(x)}\right)dx, \tag{33}$$

and we denote the derivative operator by A, that is $A = J' : W_0^{1,\mathscr{H}}(\Omega) \longrightarrow \left(W_0^{1,\mathscr{H}}(\Omega)\right)^*$, with

$$\langle A(u), v\rangle = \int_{\Omega}\left(|\nabla u|^{p(x)-2} + a(x)|\nabla u|^{q(x)-2}\right)\nabla u \cdot \nabla v dx, \quad u, v \in W_0^{1,\mathscr{H}}(\Omega). \tag{34}$$

Here, $\left(W_0^{1,\mathscr{H}}(\Omega)\right)^*$ denotes the dual space of $W_0^{1,\mathscr{H}}(\Omega)$, and $\langle\cdot,\cdot\rangle$ denotes the paring between $W_0^{1,\mathscr{H}}(\Omega)$ and $\left(W_0^{1,\mathscr{H}}(\Omega)\right)^*$. In the following lemma, we summarize some properties of A, useful to study our problem. When $p(x) \equiv p, q(x) \equiv q$, we refer to ([13], Proposition 3.1).

Lemma 11 (see [19], Lemma 3.4). *Under the condition (2), A is a mapping of type (S_+), that is, if $u_n \rightharpoonup u$ in $W_0^{1,\mathscr{H}}(\Omega)$ and $\limsup_{n\to+\infty}\langle A(u_n) - A(u), u_n - u\rangle \le 0$,, then $u_n \longrightarrow u$ in $W_0^{1,\mathscr{H}}(\Omega)$.*

Lemma 12 (see [19], Lemma 3.2). *Under the condition (f_1), I_λ is well defined on $W_0^{1,\mathscr{H}}(\Omega)$, and $I_\lambda \in C^1(W_0^{1,\mathscr{H}}(\Omega), \mathbb{R})$ with Fréchet derivate given by*

$$\left\langle I_\lambda'(u), v\right\rangle = \int_{\Omega}\left(|\nabla u|^{p(x)-2} + a(x)|\nabla u|^{q(x)-2}\right)\nabla u \cdot \nabla v dx - \lambda\int_{\Omega} f(x, u)v dx, \ u, v \in W_0^{1,\mathscr{H}}(\Omega). \tag{35}$$

Firstly, we show the functional I_λ satisfies the $(C)_c$ condition.

Lemma 13. *If hypotheses (f_1),(f_3), and (f_4) hold, then I_λ satisfies the $(C)_c$ condition.*

Proof. For every $c \in \mathbb{R}$, let $\{u_n\} \subset W_0^{1,\mathscr{H}}(\Omega)$ be a $(C)_c$ − sequence, that is,

$$I_\lambda(u_n) \longrightarrow c, \text{and } \| I'_\lambda(u_n)\|_{(W_0^{1,\mathscr{H}}(\Omega))^*}(1+\|u_n\|) \longrightarrow 0, \text{as } n \longrightarrow \infty. \tag{36}$$

We claim that $\{u_n\}$ is bounded in $W_0^{1,\mathscr{H}}(\Omega)$. In fact, suppose by contradiction that $\|u_n\| \longrightarrow +\infty$, as $n \longrightarrow \infty$. Let $v_n = u_n/\|u_n\|$, $n \geq 1$. Up to a subsequence, we may assume that

$$\begin{cases} v_n \longrightarrow v, \text{ a.e. in } \Omega, \\ v_n \rightharpoonup v, \text{ weakly in } W_0^{1,\mathscr{H}}(\Omega), \\ v_n \longrightarrow v, \text{ strongly in } L^{q^+}(\Omega), \\ v_n \longrightarrow v, \text{ strongly in } L^{\alpha(x)}(\Omega). \end{cases} \tag{37}$$

We know that v satisfies the following alternative: $v = 0$ or $v \neq 0$. In what follows, we will show that under the condition $\|u_n\| \longrightarrow +\infty$, v satisfies neither $v = 0$ nor $v \neq 0$. This is a contradiction. Thus, $\{u_n\}$ is bounded.

If $v = 0$, then $v_n \longrightarrow 0$ a.e. $x \in \Omega$, as $n \longrightarrow \infty$. Since $I_\lambda(tu_n)$ is continuous in $t \in [0, 1]$, for each n, there exists $t_n \in [0, 1] (n = 1, 2, \cdots)$ such that

$$I_\lambda(t_n u_n) = \max_{t\in[0,1]} I_\lambda(tu_n). \tag{38}$$

It is easily seen that $t_n > 0$ and $I_\lambda(t_n u_n) \geq c > 0 = I_\lambda(0) = I_\lambda(0u_n)$. If $t_n < 1$, then $(d/dt) I_\lambda(tu_n)|_{t=t_n} = 0$, which implies

$$\left\langle {I_\lambda}'(t_n u_n), t_n u_n \right\rangle = 0. \tag{39}$$

Moreover, if $t_n = 1$, then, from(36) we have $\langle {I_\lambda}'(u_n), u_n \rangle = o_n(1)$. So, we always have

$$\left\langle {I_\lambda}'(t_n u_n), t_n u_n \right\rangle = o_n(1). \tag{40}$$

Let γ_k be a sequence of positive real numbers such that $\gamma_k > 1$ for any k and $\lim_{k\to+\infty} \gamma_k = +\infty$. Then $\|\gamma_k v_n\| = \gamma_k > 1$ for any k and n. Fix k, using (f_1), (37), and the Lebesgue dominated convergence theorem we deduce that

$$\lim_{n\to\infty} \int_\Omega F(x, \gamma_k v_n) dx = 0. \tag{41}$$

Recall that $\|u_n\| \longrightarrow +\infty$ as $n \longrightarrow \infty$. So, we have $\|u_n\| > \gamma_k$ or $0 < \gamma_k/\|u_n\| < 1$ for n large enough. Hence, from (31) and (38), we deduce that

$$\begin{aligned} I_\lambda(t_n u_n) &\geq I_\lambda\left(\frac{\gamma_k}{\|u_n\|} u_n\right) = I_\lambda(\gamma_k v_n) \\ &= \int_\Omega \left(\frac{\gamma_k^{p(x)}}{p(x)} |\nabla v_n|^{p(x)} + \frac{\gamma_k^{q(x)}}{q(x)} a(x) |\nabla v_n|^{q(x)} - \lambda F(x, \gamma_k v_n) \right) \\ &\quad \times dx \geq \frac{\gamma_k^{p^-}}{q^+} - \lambda \int_\Omega F(x, \gamma_k v_n) dx, \end{aligned} \tag{42}$$

for any n large enough. By combing this inequality with (41), as $n, k \longrightarrow +\infty$, we have

$$\limsup_{n\to\infty} I_\lambda(t_n u_n) = +\infty. \tag{43}$$

On the other hand, using condition (f_4) and (40), for all n large enough, we obtain

$$\begin{aligned} I_\lambda(t_n u_n) &= I_\lambda(t_n u_n) - \frac{1}{q^+} \left\langle I'_\lambda(t_n u_n), t_n u_n \right\rangle + o(1) \\ &= \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{q^+} \right) \left| \nabla(t_n u_n) \right|^{p(x)} dx \\ &\quad + \int_\Omega \left(\frac{1}{q(x)} - \frac{1}{q^+} \right) a(x) \left| \nabla(t_n u_n) \right|^{q(x)} dx \\ &\quad + \lambda \int_\Omega \left[\frac{1}{q^+} f(x, t_n u_n) t_n u_n - F(x, t_n u_n) \right] dx \\ &\leq I_\lambda(u_n) - \frac{1}{q^+} \left\langle I'_\lambda(u_n), u_n \right\rangle \\ &\quad + \frac{\lambda C_0 |\Omega|}{q^+} \longrightarrow c + \frac{\lambda C_0 |\Omega|}{q^+}, \text{as } n \longrightarrow \infty. \end{aligned} \tag{44}$$

From (43) and (44), we obtain a contradiction. This shows that $v \neq 0$, and thus,

$$v_n(x) \longrightarrow v(x) \neq 0 \text{ a.e. in } \Omega. \tag{45}$$

Let $\Omega_{\neq} \coloneqq \{x \in \Omega : v(x) \neq 0\}$. It implies that

$$|u_n(x)| \longrightarrow +\infty, \quad \text{in } \Omega_{\neq}, \text{ as } n \longrightarrow \infty. \tag{46}$$

Using condition (f_3), we obtain

$$\begin{aligned} \lim_{n\to+\infty} \frac{F(x, u_n(x))}{\|u_n(x)\|^{q^+}} &= \lim_{n\to+\infty} \frac{F(x, u_n(x))}{|u_n(x)|^{q^+}} \frac{|u_n(x)|^{q^+}}{\|u_n(x)\|^{q^+}} \\ &= \lim_{n\to+\infty} \frac{F(x, u_n(x))}{|u_n(x)|^{q^+}} |v_n(x)|^{q^+} = +\infty, \quad x \in \Omega_{\neq}. \end{aligned} \tag{47}$$

Also by (f_1) and (f_3), we can get a constant $C_2 > 0$ such that

$$F(x, t) \geq -C_2, \quad \forall (x, t) \in \bar{\Omega} \times \mathbb{R}. \tag{48}$$

Thus, we get

$$\frac{F(x,u_n)+C_2}{\|u_n\|^{q^+}} \ge 0. \tag{49}$$

From (31), we see that

$$\begin{aligned} c &= I_\lambda(u_n(x)) + o_n(1) \\ &= \int_\Omega \left(\frac{1}{p(x)} |\nabla u_n|^{p(x)} + \frac{a(x)}{q(x)} |\nabla u_n|^{q(x)} - \lambda F(x,u_n) \right) \\ &\quad \cdot dx + o_n(1) \ge \frac{1}{q^+} \|u_n\|^{p^-} - \int_\Omega \lambda F(x,u_n) dx + o_n(1), \end{aligned} \tag{50}$$

which implies

$$\int_\Omega F(x,u_n)dx \ge \frac{1}{\lambda q^+}\|u_n\|^{p^-} - \frac{c}{\lambda} + o_n(1) \longrightarrow +\infty, \quad \text{as } n \longrightarrow \infty. \tag{51}$$

Similarly, from (31), we also get

$$c = I_\lambda(u_n(x)) + o_n(1) \le \frac{1}{p^-}\|u_n\|^{q^+} - \int_\Omega \lambda F(x,u_n)dx + o_n(1), \tag{52}$$

which implies

$$\|u_n\|^{q^+} \ge p^- c + \lambda p^- \int_\Omega F(x,u_n)dx - o_n(1) > 0, \tag{53}$$

for n large enough.

We claim that $|\Omega_{\neq}| = 0$. Indeed, suppose by contradiction $|\Omega_{\neq}| \ne 0$, then by (47)–(53) and Fatou's lemma, we obtain

$$\begin{aligned} +\infty &= \int_{\Omega_{\neq}} \liminf_{n\to+\infty} \left(\frac{F(x,u_n(x))}{|u_n(x)|^{q^+}} |v_n(x)|^{q^+} + \frac{C_2}{\|u_n\|^{q^+}} \right) \\ &\le \liminf_{n\to+\infty} \int_{\Omega_{\neq}} \left(\frac{F(x,u_n(x))}{|u_n(x)|^{q^+}} |v_n(x)|^{q^+} + \frac{C_2}{\|u_n\|^{q^+}} \right) \\ &= \liminf_{n\to+\infty} \int_{\Omega_{\neq}} \frac{F(x,u_n(x))}{\|u_n(x)\|^{q^+}} \\ &\le \liminf_{n\to+\infty} \frac{\int_\Omega F(x,u_n(x))dx}{p^- c + \lambda p^- \int_\Omega F(x,u_n)dx - o_n(1)} = \frac{1}{\lambda p^-}, \end{aligned} \tag{54}$$

which yields a contradiction. Therefore the sequence $\{u_n\}$ is bounded in $W_0^{1,\mathscr{H}}(\Omega)$. Thus, there is a subsequence (which we still denote by $\{u_n\}$) that converges weakly to some $u \in W_0^{1,\mathscr{H}}(\Omega)$ and strongly in $L^{\alpha(\cdot)}(\Omega)$. It is easy to check from (f_1) and Hölder's inequality that

$$\begin{aligned} &\left| \int_\Omega f(x,u_n)(u_n - u)dx \right| \\ &\quad \le C(\|1 + |u_n|^{\alpha(x)-1}\|_{\alpha'(\cdot)} \|u_n - u\|_{\alpha(\cdot)} \longrightarrow 0. \end{aligned} \tag{55}$$

Then

$$\begin{aligned} \langle A(u_n), u_n - u \rangle &= \left\langle I'(u_n), u_n - u \right\rangle \\ &\quad + \lambda \int_\Omega f(x,u_n)(u_n - u)dx \longrightarrow 0. \end{aligned} \tag{56}$$

So $u_n \longrightarrow u$ follows from Lemma 11.

3. Proofs of Theorems 2 and 3

First, we will show the functional I_λ satisfies the mountain pass geometry [25].

Lemma 14. *Assume hypotheses $(f_1)-(f_3)$ hold. Then the functional I_λ satisfies the following properties:*

(i) There exist $\rho, \delta > 0$ such that $I_\lambda(u) \ge \delta$ for any $u \in W_0^{1,\mathscr{H}}(\Omega)$ with $\|u\| = \rho$

(ii) There exists a $\eta \in W_0^{1,\mathscr{H}}(\Omega) \setminus B_\rho$ such that $I_\lambda(\eta) \le 0$.

Proof. Let us check (i). For any $u \in W_0^{1,\mathscr{H}}(\Omega) \setminus \{0\}$ and $\varepsilon > 0$ small, it follows from $(f_1)-(f_2)$ that there exists $C_\varepsilon > 0$ such that

$$F(x,t) \le \varepsilon |t|^{q^+} + C_\varepsilon |t|^{\alpha(x)}, \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}^N. \tag{57}$$

Thus, for $u \in W_0^{1,\mathscr{H}}(\Omega)$ and $\|u\| \le 1$, we have

$$\begin{aligned} I_\lambda(u) &= \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{a(x)}{q(x)} |\nabla u|^{q(x)} - \lambda F(x,u) \right) dx, \\ &\ge \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{a(x)}{q(x)} |\nabla u|^{q(x)} \right) \\ &\quad \cdot dx - \lambda \int_\Omega \left(\varepsilon |u|^{q^+} + C_\varepsilon |u|^{\alpha(x)} \right) dx, \\ &\ge \frac{1}{q^+}\|u\|^{q^+} - \lambda C_3 \varepsilon \|u\|^{q^+} - \lambda C_\varepsilon C_4 \|u\|^{\alpha^-}, \end{aligned} \tag{58}$$

by the Sobolev embedding $W_0^{1,\mathscr{H}}(\Omega) \hookrightarrow L^{q^+}(\Omega)$ and $W_0^{1,\mathscr{H}}(\Omega) \hookrightarrow L^{\alpha(\cdot)}(\Omega)$. Since $q^+ < \alpha^-$ and ε arbitrarily small, there exist $\rho > 0$ and $\delta > 0$ such that $I_\lambda(u) \ge \delta > 0$ for $\|u\| = \rho$. Hence item (i) holds.

Let us check (ii). From (f_3), for any $M > 0$, we can choose a constant $C_5 > 0$ such that

$$F(x,t) \ge M|t|^{q^+} - C_5, \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{59}$$

Then, for $\omega \in W_0^{1,\mathscr{H}}(\Omega)$ and $t > 0$, we deduce that

$$\begin{aligned}\lim_{t\to+\infty}\frac{I(tw)}{t^{q^+}} &\le \lim_{t\to+\infty}\frac{\int_\Omega\left(1/p(x)|\nabla(tw)|^{p(x)}+(a(x)/q(x))|\nabla(tw)|^{q(x)}\right)dx-\lambda\int_\Omega\left(M|tw|^{q^+}-C_5\right)dx}{t^{q^+}},\\ &\le \lim_{t\to+\infty}\frac{1}{t^{q^+}}\int_\Omega\left(\frac{t^{p(x)}}{p(x)}|\nabla\omega|^{p(x)}+\frac{a(x)t^{q(x)}}{q(x)}|\nabla\omega|^{q(x)}-t^{q^+}\lambda M|\omega|^{q^+}+\lambda C_5\right)\\ &\cdot dx \le \int_\Omega\left(\frac{1}{p(x)}|\nabla\omega|^{p(x)}+\frac{a(x)}{q(x)}|\nabla\omega|^{q(x)}-\lambda M|\omega|^{q^+}\right)dx.\end{aligned} \tag{60}$$

If M is large enough such that

$$\int_\Omega\left(\frac{1}{p(x)}|\nabla\omega|^{p(x)}+\frac{a(x)}{q(x)}|\nabla\omega|^{q(x)}-\lambda M|\omega|^{q^+}\right)dx<0, \tag{61}$$

conclusion (ii) follows.

Proof of Theorem 2. Since the functional I_λ has the mountain pass geometry and satisfies the $(C)_c$ condition, the mountain pass theorem [25] gives that there exists a critical point $u \in W_0^{1,\mathscr{H}}(\Omega)$. Moreover, $I(u) = c \ge \alpha > 0 = I(0)$, so u is a nontrivial solution.

Lemma 15. *Assume (f_1) holds. Then there exist positive constants m_λ and ρ_λ such that $\lim_{\lambda\to0^+} m_\lambda = +\infty$ and $I_\lambda \ge m_\lambda > 0$ when $\|u\| = \rho_\lambda$.*

Proof. Let $u \in W_0^{1,\mathscr{H}}(\Omega)$ with $\|u\|>1$. It follows from (f_1) that there exists $C_6 > 0$ such that

$$|F(x,t)| \le C_6\left(|t|^{\alpha(x)}+1\right), \tag{62}$$

for all $(x,t) \in \Omega\times\mathbb{R}$, $q^+ < \alpha(x) < (p^*)^-$. Hence, we obtain

$$\begin{aligned}I_\lambda(u) &\ge \int_\Omega\left(\frac{1}{p(x)}|\nabla u|^{p(x)}+\frac{a(x)}{q(x)}|\nabla u|^{q(x)}\right)\\ &\cdot dx-\lambda C_6\int_\Omega\left(|u|^{\alpha(x)}+1\right)\\ &\cdot dx \ge \frac{1}{q^+}\|u\|^{p^-}-\lambda C_7\|u\|^{\alpha^+}-\lambda C_6|\Omega|.\end{aligned} \tag{63}$$

Let $\rho_\lambda = \lambda^{-s}$ where $s \in (0, 1/(\alpha^+ - p^+))$. Hence, we get $\rho_\lambda > 1$ for λ small enough. Therefore, substituting $\|u\| = \rho_\lambda = \lambda^{-s}$ in (63), we see that

$$I_\lambda(u) \ge \frac{1}{q^+}\lambda^{-sp^-}-C_7\lambda^{1-s\alpha^+}-\lambda C_6|\Omega|. \tag{64}$$

Let us define $m_\lambda = (1/q^+)\lambda^{-sp^-} - C_7\lambda^{1-s\alpha^+} - \lambda C_6|\Omega|$. From $s \in (0, 1/(\alpha^+ - p^+))$, we get that there exist λ_0 small enough such that $m_\lambda > 0$ for all $\lambda \in (0, \lambda_0)$ and $m_\lambda \longrightarrow +\infty$ as $\lambda \longrightarrow 0^+$.

Proof of Theorem 3. By Lemma 13, I_λ satisfies the $(C)_c$ condition. Now in view of Lemma 13 and Lemma 15 and Lemma 14(ii) we can apply the mountain pass theorem to obtain a nontrivial critical point u_λ for I_λ such that

$$I_\lambda(u_\lambda) = c \ge m_\lambda. \tag{65}$$

On the other hand, from (62), we have

$$\begin{aligned}I_\lambda(u) &\le \int_\Omega\left(\frac{1}{p(x)}|\nabla u|^{p(x)}+\frac{a(x)}{q(x)}|\nabla u|^{q(x)}\right)\\ &\cdot dx+\lambda C_6\int_\Omega\left(|u|^{\alpha^+}+1\right)dx, \le \frac{1}{p^-}\max\left\{\|u_\lambda\|^{p^-},\|u_\lambda\|^{q^+}\right\}\\ &+\lambda C_8\max\left\{\|u_\lambda\|^{\alpha^+}u_\lambda{}^{\alpha^-}\right\}+\lambda C_6|\Omega|.\end{aligned} \tag{66}$$

Taking the limit $\lambda \longrightarrow 0^+$ in (66) and using Lemma 15, one has $\lim_{\lambda\to0^+}\|u_\lambda\| = +\infty$.

4. Proofs of Theorems 4 and 5

Lemma 16. *Assume the hypotheses $(f_1)-(f_3)$ hold. Then the functional I_λ satisfies the following properties:*

(i) There exist constants $\rho, \delta > 0$, such that $I_\lambda(u) \ge \delta$ for any $u \in W_0^{1,\mathscr{H}}(\Omega)$ with $\|u\| = \rho$

(ii) For each finite dimensional subspace $\tilde{X} \subset W_0^{1,\mathscr{H}}(\Omega)$, there exists an $R = R(\tilde{X})$ such that $I_\lambda \le 0$, on $\tilde{X} \setminus B_R(\tilde{X})$.

Proof. As in the proof of Lemma 14, it is immediate to see that the case (i) is true. Let $e \in \tilde{X}$ and $\|e\| = 1$ be fixed. From (59), we obtain

$$I_\lambda(te) = \int_\Omega \left(\frac{1}{p(x)} |\nabla(te)|^{p(x)} + \frac{a(x)}{q(x)} |\nabla(te)|^{q(x)} - \lambda F(x, te) \right) \cdot dx \le \frac{t^{q^+}}{p^-} - \lambda M C_9 t^{q^+} + \lambda C_5 |\Omega|, \quad (67)$$

for all norms on $\tilde{X}$ are equivalent. Then, we can choose M large enough such that $1/p^- - \lambda M C_9 < 0$. Therefore, we see that $I_\lambda(te) \longrightarrow -\infty$, as $n \longrightarrow \infty$, and the step is proved by taking $v_0 = t_0 e$ with $t_0 > R$ large enough.

Proof of Theorem 4. According to our assumption (f_5), I_λ is an even functional. By the Lemma 13, I_λ satisfies the $(C)_c$ condition. Together with the Lemma 16, we can apply a Z_2 version of the mountain pass theorem (see [25], Theorem 9.12) to obtain an unbounded sequence of weak solutions of problem (P_λ).

We finalize the section presenting a relation between the genus of K and the number of solutions of the problem (P_λ), where K is a k-dimensional linear subspace $K \subset C_0^\infty(\Omega)$ of $W_0^{1,\mathscr{H}}(\Omega)$. We invoke Clark's Theorem in [25], Theorem 9.1. The next result is a compactness result on problem (P_λ) which we will use later.

Lemma 17. *Assume that condition (f_6) holds, then*

(i) I_λ is bounded from below

(ii) I_λ satisfies the (PS) condition.

Proof. (i) Using (f_6), and for $\|u\|>1$, $\lambda > 0$, we obtain

$$I_\lambda(u) \ge \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{a(x)}{q(x)} |\nabla u|^{q(x)} \right) \cdot dx - \frac{\lambda d_1}{\beta^-} \int_\Omega |u|^{\beta(x)} dx, \ge \frac{1}{q^+} \|u\|^{p^-} - C_{10} \|u\|^{\beta^+}. \quad (68)$$

Hence, I_λ is coercive following immediately from the above expression and $\beta^+ < p^-$. Therefore, I_λ is bounded from below.

(ii) Suppose $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence for I_λ. Thus $I_\lambda(u_n) \longrightarrow c$ and $I_\lambda'(u_n) \longrightarrow 0$ in $(W_0^{1,\mathscr{H}}(\Omega))^*$ as $n \longrightarrow +\infty$. It follows from (i) that $\{u_n\}$ is bounded in $W_0^{1,\mathscr{H}}(\Omega)$. Up to a subsequence, we may assume that

$$\begin{cases} u_n \longrightarrow u, & \text{a.e. in } \Omega, \\ u_n \rightharpoonup u, & \text{weakly in } W_0^{1,\mathscr{H}}(\Omega), \\ u_n \longrightarrow u, & \text{strongly in } L^{\beta(\cdot)}(\Omega). \end{cases} \quad (69)$$

Since $I_\lambda'(u_n) \longrightarrow 0$ and $u_n - u \rightharpoonup 0$ in $W_0^{1,\mathscr{H}}(\Omega)$, (see [26], Proposition 3.5), we get that

$$\lim_{n\to+\infty} \left\langle I_\lambda'(u_n), u_n - u \right\rangle = 0. \quad (70)$$

It is easy to check from (f_6) and Hölder's inequality that

$$\left| \int_\Omega f(x, u_n)(u_n - u) dx \right| \le C_{11} \|u_n\Big|^{\beta(x)-1} \cdot \|_{\beta}{}'(\cdot) \|u_n - u\|_{\beta(\cdot)} \longrightarrow 0, \text{ as } n \longrightarrow \infty, \quad (71)$$

where $\beta'(\cdot) = \beta(\cdot)/\beta(\cdot) - 1$. Then

$$\langle A(u_n), u_n - u \rangle = \left\langle I'(u_n), u_n - u \right\rangle + \lambda \int_\Omega f(x, u_n)(u_n - u) dx \longrightarrow 0, \text{ as } n \longrightarrow \infty. \quad (72)$$

So $u_n \to u$ follows from Lemma 11.

Proof of Theorem 5. Consider K is a k-dimensional linear subspace $K \subset C_0^\infty(\Omega)$ of $W_0^{1,\mathscr{H}}(\Omega)$. We claim $I_\lambda|_K < 0$ if $\|u\| \le r < 1$ is sufficiently small. Indeed, by the equivalence of norms on K, there exists a constant $C_{12} > 0$ such that $C_{12}\|u\|^{\beta^+} \le \int_\Omega |u|^{\beta(x)} dx$ for $u \in K$ with $\|u\| \le 1$. Therefore, by (f_6),

$$I_\lambda(u) \le \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{a(x)}{q(x)} |\nabla u|^{q(x)} \right) \cdot dx - \frac{\lambda d_0}{\beta^-} \int_\Omega |u|^{\beta(x)} dx \le \frac{1}{p^-} \|u\|^{p^-} - \lambda C_{13} \|u\|^{\beta^+} \le \|u\|^{\beta^+} \cdot \left(\frac{1}{p^-} \|u\|^{p^- - \beta^+} - \lambda C_{13} \right), \quad (73)$$

for $u \in K$ with $\|u\| < 1$. If $r \in (0, 1)$ is small enough, we have that

$$\frac{1}{p^-} r^{p^- - \beta^+} - \lambda C_{13} < 0. \quad (74)$$

The last inequality shows $I_\lambda|_K < 0$ for all $u \in S_r^k = \{u \in K : \|u\| = r\}$. It is clear that K is isomorphic to $\mathbb{R}^k$ and S_r^k is homeomorphic to $\mathbb{S}^{k-1}$ in $\mathbb{R}^k$. Hence, we obtain $\gamma(S_r^k) = k$. In the proof of Lemma 17, it was already established that $I_\lambda \in C^1(X, \mathbb{R})$ is bounded from below, satisfies the (PS) condition, and $I_\lambda(0) = 0$. Clearly, (f_5) implies I_λ is even. Consequently, by Clark's Theorem in [25] (Theorem 9.1), I_λ possesses at least k distinct pairs of nontrivial solutions. Since k is arbitrary, we obtain infinitely many nontrivial solutions.

5. Proofs of Theorems 6 and 7

In this section, we will show that (P_λ) has infinitely many pairs of solutions by using the Fountain Theorem and Dual Fountain Theorem. Firstly, we need to recall some preliminary results. Since $W_0^{1,\mathcal{H}}(\Omega)$ is a reflexive and separable Banach space, there are $e_j \subset W_0^{1,\mathcal{H}}(\Omega)$ and $e_j^* \subset (W_0^{1,\mathcal{H}}(\Omega))^*$ such that

$$\begin{aligned} W_0^{1,\mathcal{H}}(\Omega) &= \overline{\operatorname{span}\left\{e_j : j = 1, 2 \cdots\right\}}, \\ \left(W_0^{1,\mathcal{H}}(\Omega)\right)^* &= \overline{\operatorname{span}\left\{e_j^* : j = 1, 2 \cdots\right\}}, \\ \left\langle e_j^*, e_j \right\rangle &= \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \end{aligned} \tag{75}$$

Then, we define

$$X_j = \operatorname{span}\left\{e_j\right\}, Y_k = \oplus_{j=1}^k X_j, Z_k = \overline{\oplus_{j=k}^\infty X_j}. \tag{76}$$

We will apply the following Fountain Theorem ([25], Theorem 3.6).

Lemma 18. *Assume that X is a Banach space, and let $\varphi \in C^1(X, R)$ be an even functional. If, for every $k \in \mathbb{N}$, there exists $\rho_k > r_k > 0$ such that*

$$\begin{aligned} &(A_1) \quad b_k \coloneqq \inf_{\substack{u \in Z_k \\ \|u\| = r_k}} \varphi(u) \longrightarrow +\infty, k \longrightarrow +\infty, \\ &(A_2) \quad a_k \coloneqq \max_{\substack{u \in Y_k \\ \|u\| = \rho_k}} \varphi(u) \leq 0, \end{aligned} \tag{77}$$

(A_3) *φ satisfies the $(C)_c$ condition for every $c > 0$.*

Then φ has an unbounded sequence of critical values.

To prove Theorems 6 and 7, the following lemma is needed.

Lemma 19. *Assume that $\alpha(x) \in C_+(\bar{\Omega})$, $q^+ < \alpha(x) < (p^*)^-$, for any $x \in \bar{\Omega}$. Let*

$$\beta_k = \sup_{\substack{\|u\| = 1 \\ u \in Z_k}} \|u\|_{L^{\alpha(\cdot)}}, \tag{78}$$

then $\lim_{k \to +\infty} \beta_k = 0$.

Proof. Obviously, $0 < \beta_{k+1} \leq \beta_k$ and so $\beta_k \longrightarrow \beta \geq 0$. Let $u_k \in Z_k$ satisfy

$$\|u_k\| = 1, \ 0 \leq \beta_k - \|u_k\|_{L^{\alpha(x)}} < \frac{1}{k}. \tag{79}$$

Then, there exists a subsequence of $\{u_k\}$ (which we still denote by $\{u_k\}$) such that $u_k \rightharpoonup u$, and

$$\left\langle e_j^*, u \right\rangle = \lim_{k \to +\infty} \left\langle e_j^*, u_k \right\rangle = 0, j = 1, 2, \cdots, \tag{80}$$

which implies $u = 0$, and thus, $u_k \rightharpoonup 0$. Since $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow\hookrightarrow L^{\alpha(\cdot)}(\Omega)$, then $u_k \longrightarrow 0$ in $L^{\alpha(\cdot)}(\Omega)$. Hence, we get $\lim_{k \to +\infty} \beta_k = 0$.

Proof of Theorem 6. Let $X = W_0^{1,\mathcal{H}}(\Omega)$. According to $f(x, -t) = -f(x, t)$, I_λ is an even functional. As the proof of Lemma 13, it follows from (f_1), (f_3), and (f_4) that I_λ satisfies the $(C)_c$ condition. For every $k \in \mathbb{N}$, we shall prove that there exist $\rho_k > r_k > 0$ such that

$$\begin{aligned} &(A_1)\, b_k \coloneqq \inf_{\substack{u \in Z_k \\ \|u\| = r_k}} I_\lambda(u) \longrightarrow +\infty, k \longrightarrow +\infty, \\ &(A_2) a_k \coloneqq \max_{\substack{u \in Y_k \\ \|u\| = \rho_k}} I_\lambda(u) \leq 0, \end{aligned} \tag{81}$$

We first show that (A_1) holds. For any $u \in Z_k$, we choose $\|u\| = r_k = \left(2q^+ C_6 \lambda \beta_k^{\alpha^+}\right)^{1/(p^- - \alpha^+)}$. From Lemma 19 and $p^- < \alpha^+$, we see that $r_k \longrightarrow +\infty$ as $k \longrightarrow +\infty$. As before, we also have from (62) that

$$I_\lambda(u) \geq \begin{cases} \frac{1}{q^+}\|u\|^{p^-} - C_6\lambda - \lambda C_6|\Omega|, & \|u\|_{\alpha(\cdot)} \leq 1, \\ \frac{1}{q^+}\|u\|^{p^-} - C_6\lambda\beta_k^{\alpha^+}\|u\|^{\alpha^+} - \lambda C_6|\Omega|, & \|u\|_{\alpha(\cdot)} \geq 1, \end{cases} \geq \frac{1}{2q^+} r_k^{p^-} - \lambda C_{14}|\Omega|, \tag{82}$$

which implies that $b_k \longrightarrow +\infty$, $k \longrightarrow +\infty$.

Afterwards, we demonstrate that (A_2) holds. Let $\phi \in Y_k$ and $\|\phi\| = 1$, $t > 1$. From (59), we obtain

$$\begin{aligned} I_\lambda(t\phi) &= \int_\Omega \left(\frac{1}{p(x)} \nabla(t\phi)^{p(x)} + \frac{a(x)}{q(x)} |\nabla(t\phi)|^{q(x)} - F(x, t\phi) \right) \\ &\cdot dx \leq \frac{t^{q^+}}{p^-} - \lambda M C_{15} t^{q^+} + \lambda C_5 |\Omega|, \end{aligned} \tag{83}$$

for all norms on Y_k are equivalent. Then, we can choose M large enough such that $1/p^- - \lambda M C_{15} < 0$. Therefore, we see that $I_\lambda(t\phi) \longrightarrow -\infty$, as $t \longrightarrow +\infty$. Hence, there exists $t_1 > r_k > 1$ large enough such that $I_\lambda(t_1\phi) \leq 0$. Therefore, let $\rho_k = t_1$, we obtain that $a_k \coloneqq \max_{\substack{u \in Y_k \\ \|u\| = \rho_k}} I_\lambda(u) \leq 0$.

For the proof of Theorem 7, we need the following definitions and results.

Definition 20. Let X be a separable and reflexive Banach space, $I \in C^1(X, \mathbb{R})$, $c \in \mathbb{R}$. We say that I satisfies the $(C)_c^*$ condition (with respect to (Y_n)), if any sequence $\{u_n\}_{n \in \mathbb{N}} \subset X$ for which $u_n \in Y_n$, for any $n \in \mathbb{N}$, $I(u_n) \longrightarrow c$ and $\|(I_{|Y_n})'(u_n)\|_{X^*}(1 + \|u_n\|) \longrightarrow 0$, as $n \longrightarrow \infty$, contains a subsequence converging to a critical point of I.

We are now ready to prove the Theorem 7.

Proof of Theorem 7. According to the Dual Fountain Theorem ([25], Theorem 3.18), it suffices to prove that for every $k \geq k_0$, there exist $\rho_k > r_k > 0$ such that

(B_1) $a_k := \max_{u \in Y_k, \|u\|=r_k} I_\lambda(u) < 0,$

(B_2) $b_k := \inf_{u \in Z_k, \|u\|=\rho_k} I_\lambda(u) \geq 0,$

(B_3) $d_k := \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \longrightarrow 0, k \longrightarrow +\infty.$

(B_4) I_λ satisfies the$(C)_c^*$ condition for every $c \in \mathbb{R}$.

(84)

Firstly, we show that (B_1) holds. Let $\phi \in Y_k$ and $\|\phi\| = 1$, $t > 1$. Then similar to the proof of (A_2), we see that

$$I_\lambda(t\phi) = \int_\Omega \left(\frac{1}{p(x)}|\nabla(t\phi)|^{p(x)} + \frac{a(x)}{q(x)}|\nabla(t\phi)|^{q(x)} - F(x, t\phi)\right) \cdot dx \leq \frac{t^{q^+}}{p^-} - \lambda M C_{15} t^{q^+} + \lambda C_5 |\Omega|, \tag{85}$$

for all norms on Y_k are equivalent. Then, we can choose M large enough such that $1/p^- - \lambda M C_{15} < 0$. Therefore, we see that $I_\lambda(t\phi) \longrightarrow -\infty$, as $t \longrightarrow +\infty$. Hence, there exists $t_2 > 1$ large enough such that $I_\lambda(t_2\phi) < 0$. Therefore, let $r_k = t_2$, we obtain that

$$a_k := \max_{\substack{u \in Y_k \\ \|u\|=r_k}} I_\lambda(u) < 0. \tag{86}$$

We show that (B_2) holds. As we have done in the proof of Theorem 6, For any $u \in Z_k$, choosing $\|u\| = \rho_k = (2q^+ C_6 \lambda \beta_k^{\alpha^+})^{1/(p^- - \alpha^+)}$. From Lemma 19 and $p^- < \alpha^+$, we see that $\rho_k \longrightarrow +\infty$ as $k \longrightarrow +\infty$. As before, we also have from (88) that

$$I_\lambda(u) \geq \begin{cases} \dfrac{1}{q^+}\|u\|^{p^-} - C_6\lambda - \lambda C_6|\Omega|, & \|u\|_{\alpha(\cdot)} \leq 1, \\ \dfrac{1}{q^+}\|u\|^{p^-} - C_6\lambda\beta_k^{\alpha^+}\|u\|^{\alpha^+} - \lambda C_6|\Omega|, & \|u\|_{\alpha(\cdot)} \geq 1, \\ \geq \dfrac{1}{2q^+}\rho_k^{p^-} - \lambda C_{16}|\Omega|, & \end{cases} \tag{87}$$

which implies that there exists $k_0 \in \mathbb{N}$, for all $k \geq k_0$ choosing $\rho_k > r_k > 0$ such that $b_k \geq 0.$$(B_3)$ First from $Y_k \cap Z_k \neq \varnothing$ and $0 < r_k < \rho_k$, we observe that

$$d_k := \inf_{\substack{u \in Z_k \\ \|u\| \leq \rho_k}} I_\lambda(u) \leq a_k := \max_{\substack{u \in Y_k \\ \|u\|=r_k}} I_\lambda(u) < 0. \tag{88}$$

By (f_1), there exists $C_{17} > 0$ such that

$$|F(x,t)| \leq C_{17}\left(|t| + |t|^{\alpha(x)}\right) \tag{89}$$

for all $(x, t) \in \Omega \times \mathbb{R}$, $q^+ < \alpha(x) < (p^*)^-$. Now we define the function $\Psi_1, \Psi_2 : X \longrightarrow \mathbb{R}$ by

$$\Psi_1(u) = \int_\Omega \lambda C_{17}|u|^{\alpha(x)}dx, \quad \Psi_2(u) = \int_\Omega \lambda C_{17}|u|dx. \tag{90}$$

By the definition of Ψ_1, Ψ_2, we have $\Psi_i(0) = 0$, $i = 1, 2$, and they are weakly-strongly continuous. Consider

$$\xi_k = \sup_{\substack{u \in Z_k, \\ \|u\| \leq 1}} |\Psi_1(u)|, \zeta_k = \sup_{\substack{u \in Z_k, \\ \|u\| \leq 1}} |\Psi_2(u)|. \tag{91}$$

From the compact embedding $W_0^{1,\mathscr{H}}(\Omega) \hookrightarrow L^{\alpha(\cdot)}(\Omega)$ and Lemma 19, we have

$$\lim_{k \to +\infty} \xi_k = \lim_{k \to +\infty} \zeta_k = 0. \tag{92}$$

Let $\omega \in Z_k$ and $\|\omega\| = 1$, $0 < t < \rho_k$. Then, from (89) and (90), we obtain

$$\begin{aligned} I_\lambda(t\omega) &= \int_\Omega \left(\frac{1}{p(x)}|\nabla(t\omega)|^{p(x)} + \frac{a(x)}{q(x)}|\nabla(t\omega)|^{q(x)} - \lambda F(x, t\omega)\right) \\ &\quad \cdot dx \geq -\lambda \int_\Omega F(x, t\omega)dx \geq -\Psi_1(t\omega) - \Psi_2(t\omega) \\ &\geq -\rho_k^{\alpha^+}\Psi_1(\omega) - \rho_k\Psi_2(\omega) \geq -\rho_k^{\alpha^+}\xi_k - \rho_k\zeta_k. \end{aligned} \tag{93}$$

Passing the limit in the above inequality, as $k \longrightarrow +\infty$, we achieve that

$$\lim_{k \to +\infty} d_k \geq 0, \tag{94}$$

which, together with (88), implies that $\lim_{k \to +\infty} d_k = 0$.

(B_4) Let $\{u_n\}$ be any sequence in $W_0^{1,\mathscr{H}}(\Omega)$ such that

$$\begin{gathered} u_n \in Y_n, \\ I_\lambda(u_n) \longrightarrow c > 0, \\ \|I'_{\lambda|Y_n}(u_n)\|(1 + \|u_n\|) \longrightarrow 0, \\ \text{as } n \longrightarrow \infty \end{gathered} \tag{95}$$

Then similar to the proof of Lemma 13, we see that $\{u_n\}$ is bounded in $W_0^{1,\mathscr{H}}(\Omega)$. Thus, there is a subsequence (which we denote by $\{u_{n_k}\}$) that converges weakly to some $u \in W_0^{1,\mathscr{H}}(\Omega)$ and strongly in $L^{\alpha(\cdot)}(\Omega)$. It is easy to check from (f_1) and Hölder's inequality that

$$\left|\int_\Omega f(x, u_{n_k})(u_{n_k} - u)dx\right| \le C\|1 + |u_{n_k}|^{\alpha(x)-1}\|_{\alpha'(\cdot)}\|u_{n_k} - u\|_{\alpha(\cdot)} \longrightarrow 0. \tag{96}$$

Claim 21. $\lim_{k\to+\infty}\langle I_\lambda'(u_{n_k}), u_{n_k} - u\rangle = 0.$

If Claim 21 holds true, then

$$\langle A(u_{n_k}), u_{n_k} - u\rangle = \left\langle I'(u_{n_k}), u_{n_k} - u\right\rangle + \lambda\int_\Omega f(x, u_{n_k})(u_{n_k} - u)dx \longrightarrow 0. \tag{97}$$

So $u_{n_k} \longrightarrow u$ follows from Lemma 11. Hence, I_λ satisfies the $(C)_c^*$ condition. In order to prove Claim 21, we invoke $W_0^{1,\mathscr{H}}(\Omega) = \overline{\cup_n Y_n} = \overline{\text{span}\{e_n : n = 1, 2, \cdots\}}$ to choose $v_n \in Y_n$ such that $v_n \longrightarrow u$ strongly in $W_0^{1,\mathscr{H}}(\Omega)$. Since $I'_{\lambda|_{Y_{n_k}}}(u_{n_k}) \longrightarrow 0$ and $u_{n_k} - v_{n_k} \rightharpoonup 0$ in Y_{n_k}, (see [26], Proposition 3.5), we get that

$$\lim_{k\to+\infty}\left\langle I_\lambda'(u_{n_k}), u_{n_k} - v_{n_k}\right\rangle = 0. \tag{98}$$

Hence, we obtain

$$\lim_{k\to+\infty}\left\langle I_\lambda'(u_{n_k}), u_{n_k} - u\right\rangle = \lim_{k\to+\infty}\left\langle I_\lambda'(u_{n_k}), u_{n_k} - v_{n_k}\right\rangle + \lim_{k\to+\infty}\left\langle I_\lambda'(u_{n_k}), v_{n_k} - u\right\rangle = 0. \tag{99}$$

Therefore, the Claim holds true and we conclude that $I_\lambda'(u_{n_k}) \longrightarrow I_\lambda'(u)$ as $k \longrightarrow +\infty$. We next show that $I_\lambda'(u) = 0$. To see this, taking $\omega_j \in Y_j$, we have

$$\begin{aligned}\left\langle I_\lambda'(u), \omega_j\right\rangle &= \left\langle I_\lambda'(u) - I_\lambda'(u_{n_k}), \omega_j\right\rangle + \left\langle I_\lambda'(u_{n_k}), \omega_j\right\rangle \\ &= \left\langle I_\lambda'(u) - I_\lambda'(u_{n_k}), \omega_j\right\rangle + \left\langle I'_{\lambda|_{Y_{n_k}}}(u_{n_k}), \omega_j\right\rangle.\end{aligned} \tag{100}$$

We pass limit in the right side of (100) as $k \longrightarrow +\infty$ to obtain

$$\left\langle I_\lambda'(u), \omega_j\right\rangle = 0, \quad \text{for all } \omega_j \in Y_j. \tag{101}$$

Therefore, I_λ satisfies the $(C)_c^*$ condition for every $c \in \mathbb{R}$. The proof is complete.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] O. H. Miyagaki and M. A. S. Souto, "Superlinear problems without Ambrosetti and Rabinowitz growth condition," *Journal of Differential Equations*, vol. 245, no. 12, pp. 3628–3638, 2008.

[2] V. V. Zhikov, "On Lavrentiev's phenomenon," *Russian Journal of Mathematical Physics*, vol. 3, pp. 249–269, 1995.

[3] V. V. Zhikov, "On some variational problems," *Russian Journal of Mathematical Physics*, vol. 5, pp. 105–116, 1997.

[4] P. Baroni, M. Colombo, and G. Mingione, "Harnack inequalities for double phase functionals," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 121, pp. 206–222, 2015.

[5] P. Baroni, M. Colombo, and G. Mingione, "Regularity for general functionals with double phase," *Calculus of Variations and Partial Differential Equations*, vol. 57, no. 2, p. 62, 2018.

[6] F. Colasuonno and M. Squassina, "Eigenvalues for double phase variational integrals," *Annali di Matematica Pura ed Applicata (1923 -)*, vol. 195, no. 6, pp. 1917–1959, 2016.

[7] M. Colombo and G. Mingione, "Regularity for double phase variational problems," *Archive for Rational Mechanics and Analysis*, vol. 215, no. 2, pp. 443–496, 2015.

[8] P. Rabinowitz, "Minimax methods in critical point theory with applications to differential equations," *CBMS Regional Conference Series in Mathematics*, vol. 65, 1986.

[9] N. S. Papageorgiou and C. Vetro, "Superlinear $(p(z), q(z))$-equations," *Complex Variables and Elliptic Equations*, vol. 64, no. 1, pp. 8–25, 2019.

[10] C. Vetro and F. Vetro, "On problems driven by the $(p(\bullet), q(\bullet))$ -Laplace operator," *Mediterranean Journal of Mathematics*, vol. 17, no. 1, p. 24, 2020.

[11] N. S. Papageorgiou, V. D. Rădulescu, and D. D. Repovš, "Double-phase problems with reaction of arbitrary growth," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 69, no. 4, p. 108, 2018.

[12] C. Vetro, "Weak solutions to Dirichlet boundary value problem driven by p(x)-Laplacian-like operator," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 2017, no. 98, pp. 1–10, 2017.

[13] W. Liu and G. Dai, "Existence and multiplicity results for double phase problem," *Journal of Differential Equations*, vol. 265, no. 9, pp. 4311–4334, 2018.

[14] W. Liu and G. Dai, "Three ground state solutions for double phase problem," *Journal of Mathematical Physics*, vol. 59, no. 12, article 121503, 2018.

[15] K. Perera and M. Squassina, "Existence results for double-phase problems via Morse theory," *Communications in Contemporary Mathematics*, vol. 20, no. 2, article 1750023, 2018.

[16] N. S. Papageorgiou, C. Vetro, and F. Vetro, "Multiple solutions with sign information for a $(p, 2)$-equation with combined nonlinearities," *Nonlinear Analysis*, vol. 192, article 111716, 2020.

[17] A. Azzollini, P. d'Avenia, and A. Pomponio, "Quasilinear elliptic equations in $\mathbb{R}^N$ via variational methods and Orlicz-Sobolev embeddings," *Calculus of Variations and Partial Differential Equations*, vol. 49, no. 1-2, pp. 197–213, 2014.

[18] Y. Deng and W. Huang, "Ground state solutions for generalized quasilinear Schrödinger equations without (AR) condition," *Journal of Mathematical Analysis and Applications*, vol. 456, no. 2, pp. 927–945, 2017.

[19] I. H. Kim and Y. H. Kim, "Mountain pass type solutions and positivity of the infimum eigenvalue for quasilinear elliptic equations with variable exponents," *Manuscripta Mathematica*, vol. 147, no. 1-2, pp. 169–191, 2015.

[20] P. Harjulehto, P. Hästö, and R. Klén, "Generalized Orlicz spaces and related PDE," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 143, pp. 155–173, 2016.

[21] V. D. Radulescu and D. D. Repovs, *Partial differential equations with variable exponents: variational methods and qualitative analysis*, CRC Press, Boca Raton, 2015.

[22] W. Xie and H. Chen, "Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$," *Mathematische Nachrichten*, vol. 291, no. 16, pp. 2476–2488, 2018.

[23] L. Xu and H. Chen, "Ground state solutions for quasilinear Schrödinger equations via Pohožaev manifold in Orlicz space," *Journal of Differential Equations*, vol. 265, no. 9, pp. 4417–4441, 2018.

[24] X. Fan and D. Zhao, "On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$," *Journal of Mathematical Analysis and Applications*, vol. 263, no. 2, pp. 424–446, 2001.

[25] M. Williem, "Minimax theorems," in *Progress in nonlinear differential equations and their applications*, vol. 24, Birkhäuser, Boston, 1996.

[26] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer, New York, 2011.