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Research Article

Double Hopf Bifurcation in Microbubble Oscillators with Delay Coupling

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Using center manifold reduction methodswe investigate the double Hopf bifurcation in the dynamics of microbubble with delay couplingwith main attention focused on nonresonant double Hopf bifurcation. We obtain the normal form of the system in the vicinity of the double Hopf point and classify the bifurcations in a two-dimensional parameter space near the critical point. Some numerical simulations support the applicability of the theoretical results. In particularwe give the explanation for some physical phenomena of the system using the obtained mathematical results.

1. Introduction

At present, owing to the modeling of many technological and biological problems the dynamical systems with delay, delayed differential equations (DDEs), or functional differential equations (FDEs) have been an active area of research [1–6]. Double Hopf refers to same critical value, and there exist two different pure imaginary eigenvalues. We can observe much interesting phenomena through double Hopf bifurcation, such as periodic and quasi-periodic motions, three dimensional invariant torus [7], period doubling [8, 9], homoclinic and heteroclinic connections, and chaos. Recently, the center manifold reduction (CMR) and multiple time scales (MTS) are widely useful techniques which are used by researchers to study bifurcation of DDEs.

Bubble dynamics is an important branch of fluid mechanics which can be applied to some research fields. For example, the microbubbles filled with a drug as a carrier for local drug delivery can be improved drug utilization in medical field. The fact is that the dynamics of bubbles in a liquid are influencing each other via acoustic waves, which is a complex process described by time-delay system.

Plesset [10], Plesset and Prosperetti [11] set up the famous Rayleigh–Plesset equation (1) which used to describe the microbubble system:

$$(\dot{a}-c)\left(a\ddot{a}+\frac{3}{2}\dot{a}^2-a^{-3r}+\Delta\right)-\dot{a}^3-a^{-1}\left(a^2\Delta\right)=0, \quad (1)$$

here, $\Delta=\rho^{-1}(p(a)-p_0)$, where ρ is the density of the liquid, and $p(a)=\kappa\left((4/3)\pi a^3\right)^{-r}$ is the pressure inside the bubble, and r is the adiabatic exponent of the gas. However, Equation (1) merely describes the dynamics of a single micro-bubble system. In the general cases, there are a lot of bubbles (microbubbles) with the coupling effect in the liquid. Thus, to better analyze of micro-bubble nature of the system, we need to study an optimum system for describing further. In 2010, Heckman and Rand [12] set up a bubble coupled system with time delay:

$$\begin{cases}(\dot{a}-c)\left(a\ddot{a}+\frac{3}{2}\dot{a}^2-a^{-3r}+1\right)-\dot{a}^3-(3r-2)a^{-3r}\dot{a}-2\dot{a}=P\dot{b}(t-\tau),\\ \left(\dot{b}-c\right)\left(b\ddot{b}+\frac{3}{2}\dot{b}^2-b^{-3r}+1\right)-\dot{b}^3-(3r-2)b^{-3r}\dot{b}-2\dot{b}=P\dot{a}(t-\tau),\end{cases} \quad (2)$$

where r is positive-valued parameters, c is sound speed, and τ is the delay and P is a coupling coefficient. a and b are the radius of the microbubbles, respectively. a and b also denote, respectively, the two bubbles. Heckman et al. studied bifurcations of the in-phase manifold given by $a=b, \dot{a}=\dot{b}$ [13–16]. Recently, Heckman has studied Equation (2) and developed explanations for bifurcation structure of the in-phase manifold, However, these studies were limited in that he investigated only given by $a = b, \dot{a} = \dot{b}$. This work will extend the previous, by making use of the theory of center manifold [17] and the normal form method [18] to consider double Hopf bifurcation, which occur with more general cases.

This article is organized as follows: In Section 2, we discuss the occurring conditions for the existence of double Hopf bifurcation on Equation (2). In Section 3, we analyze the nonresonant double Hopf bifurcation for the system using the normal form method and the center manifold theory. We classify the bifurcations in a two-dimensional parameter space near the critical point and some numerical simulations support the applicability of the theoretical results in Section 4. Mathematical results explain the mechanical background of the model in Section 5. The concluding section contains a brief conclusion of this work.

2. Stability Analysis

After calculating $(a_e, b_e) = (1, 1)$ is an equilibrium point of Equation (2), we set $a = x(t) + 1$ and $b = y(t) + 1$, then Equation (2) can be written as

$$\begin{cases} P\dot{y}(t-\tau) = (\dot{x}(t) - c)((x(t)+1)\ddot{x}(t) \\ \qquad + \frac{3}{2}(\dot{x}(t))^2 - (x(t)+1)^{-3r} + 1) - (\dot{x}(t))^3 \\ \qquad - (3r-2)(x(t)+1)^{-3r}\dot{x}(t) - 2\dot{x}(t), \\ P\dot{x}(t-\tau) = (\dot{y}(t) - c)((y(t)+1)\ddot{y}(t) \\ \qquad + \frac{3}{2}(\dot{y}(t))^2 - (y(t)+1)^{-3r} + 1) - (\dot{y}(t))^3 \\ \qquad - (3r-2)(y(t)+1)^{-3r}\dot{y}(t) - 2\dot{y}(t). \end{cases} \quad (3)$$

Further, we let

$$\dot{x}(t) = h(t), \dot{y}(t) = g(t), \quad (4)$$

then Equation (3) can be written as

$$\begin{cases} \dot{x}(t) = h(t), \\ \dot{h}(t) = \frac{Pg(t-\tau) + 2h(t) + (h(t))^3 + (3r-2)(x(t)+1)^{-3r}h(t)}{(h(t)-c)(1+x(t))} + \frac{(-3/2)(h(t))^2 + (x(t)+1)^{-3r} - 1)}{1+x(t)}, \\ \dot{y}(t) = g(t), \\ \dot{g}(t) = \frac{Ph(t-\tau) + 2g(t) + (g(t))^3 + (3r-2)(y(t)+1)^{-3r}g(t)}{(g(t)-c)(1+y(t))} + \frac{(-3/2)(g(t))^2 + (y(t)+1)^{-3r} - 1)}{1+y(t)}. \end{cases} \quad (5)$$

Clearly, $(0, 0, 0, 0)$ is an equilibrium point of Equation (5). The characteristic equation of its corresponding linear system around the origin $(0, 0, 0, 0)$ is

$$\left(c\lambda^2 + 3rc + 3r\lambda + P\lambda e^{-\lambda\tau}\right)\left(c\lambda^2 + 3rc + 3r\lambda - P\lambda e^{-\lambda\tau}\right) = 0. \quad (6)$$

When $\tau = 0$, the roots for Equation (6) are

$$\lambda_{1,2} = \frac{1}{2c}\left(-(3r-P) \pm \sqrt{(3r-P)^2 - 12c^2 r}\right), \quad (7)$$

$$\lambda_{3,4} = \frac{1}{2c}\left(-(3r+P) \pm \sqrt{(3r+P)^2 - 12c^2 r}\right). \quad (8)$$

Clearly

$$\begin{cases} Re\lambda_{1,2} < 0, Re\lambda_{3,4} < 0, & 3r > P, \\ \lambda_{1,2} = \pm\sqrt{3r}i, Re\lambda_{3,4} < 0, & 3r = P, . \\ Re\lambda_{1,2} > 0, Re\lambda_{3,4} < 0, & 3r < P, \end{cases} \quad (9)$$

When $\tau \neq 0$, it is essential to discuss the properties of the roots into two cases.

Case 1. Let $\pm i\omega_1 (\omega_1 > 0)$ be a root of $c\lambda^2 + 3rc + 3r\lambda + P\lambda e^{-\lambda\tau} = 0$, then ω_1 satisfies

$$\begin{cases} P\omega_1 \sin \omega_1 \tau = c\left((\omega_1)^2 - 3r\right), \\ P\omega_1 \cos \omega_1 \tau = -3r\omega_1, \end{cases} \quad (10)$$

and the solutions to Equation (10) are then found to be

$$\omega_1^{\pm} = \frac{\sqrt{P^2 - 9r^2 + 12c^2 r} \pm \sqrt{P^2 - 9r^2}}{2c} > 0, \quad (11)$$

under the assumption $P > 3r$.

$$\tau_{k1}^{\pm} = \frac{1}{\omega_1^{\pm}}(\theta_1 + 2k\pi), \quad k = 0, 1, 2, \ldots, \quad (12)$$

where $\theta_1 = \pi - \arcsin\left(c\left((\omega_1^{\pm})^2 - 3r\right)/P\omega_1^{\pm}\right)$. By defining $\tau_{k1}^{+} (k = 0, 1, 2, 3, \ldots)$ and $\tau_{k1}^{-} (k = 0, 1, 2, 3, \ldots)$, we can obtain $\tau_{k1}^{+} < \tau_{(k+1)1}^{+}$ and $\tau_{k1}^{-} < \tau_{(k+1)1}^{-}$. Furthermore, we get $\tau_{k1}^{+} < \tau_{k1}^{-}$ and $\tau_{(k+1)1}^{+} - \tau_{k1}^{+} < \tau_{(k+1)1}^{-} - \tau_{k1}^{-}$ from $\omega_1^{+} < \omega_1^{-}$. Thus, there exists a positive integer $j > 0$, yielding

$$0 < \tau_{01}^{+} < \tau_{01}^{-} < \tau_{11}^{+} < \tau_{11}^{-} < \cdots < \tau_{(j-1)1}^{-} < \tau_{j1}^{+} < \tau_{(j+1)1}^{+} < \tau_{j1}^{-}. \quad (13)$$

Case 2. Let $\pm i\omega_2 (\omega_2 > 0)$ be a root of $c\lambda^2 + 3rc + 3r\lambda - P\lambda e^{-\lambda\tau} = 0$. Likewise available as a conclusion

$$\omega_1^{\pm} = \omega_2^{\pm}, \quad (14)$$

$$\tau_{k2}^{\pm} = \frac{1}{\omega_2^{\pm}}(\theta_2 + 2k\pi), \quad k = 0, 1, 2 \ldots, \quad (15)$$

$$\begin{gathered} 0 < \tau_{02}^{+} < \tau_{02}^{-} < \tau_{12}^{+} < \tau_{12}^{-} < \cdots < \tau_{(i-1)2}^{-} \\ < \tau_{i2}^{+} < \tau_{(i+1)2}^{+} < \tau_{i2}^{-}, \end{gathered} \quad (16)$$

here, $\theta_2 = \arcsin\left(-c\left((\omega_2^{\pm})^2 - 3r\right)/P\omega_2^{\pm}\right), P > 3r$.

After calculating, the corresponding transverse condition is the following:

$$sign\left(\frac{\mathrm{d}a(\tau)}{\mathrm{d}\tau}\right)|_{\tau=\tau 0} = \begin{cases} 1, & \tau_0 = \tau_{k1}^{+}, \ k = 0, 1, 2, 3, \ldots, \\ -1, & \tau_0 = \tau_{k1}^{-}, \ k = 0, 1, 2, 3, \ldots, \\ -1, & \tau_0 = \tau_{k2}^{+}, \ k = 0, 1, 2, 3, \ldots, \\ 1, & \tau_0 = \tau_{k2}^{-}, \ k = 0, 1, 2, 3, \ldots, \end{cases} \quad (17)$$

From [18, 19] and above analysis, we can obtain the following theorem about the existence of Hopf bifurcation. If $P > 3r$, system Equation (5) undergoes a Hopf bifurcation at its zero equilibrium when $\tau = \tau_{k_1}^{\pm}$ or $\tau = \tau_{k_2}^{\pm}$, where $k = 0, 1, 2, 3, \ldots$

In the following, we will discuss the occurring conditions for the existence of double Hopf bifurcation on Equation (5), and choose the (P, τ) as the parameters plane. Further, we can draw a (P, τ) diagram as shown in Figure 1. In fact, double Hopf bifurcation occurs at these points. (P_c, τ_c) is called a double Hopf bifurcation point, when $\tau_{k_1}^{+} = \tau_{k_2}^{-} = \tau_c$ is satisfied for some $k = 0, 1, 2, \ldots$

From

$$\tau_{k_1}^{+} = \frac{1}{\omega_1^{+}}(\theta_1 + 2k\pi) = \tau_{k_2}^{-} = \frac{1}{\omega_2^{-}}(\theta_2 + 2k\pi), \quad (18)$$

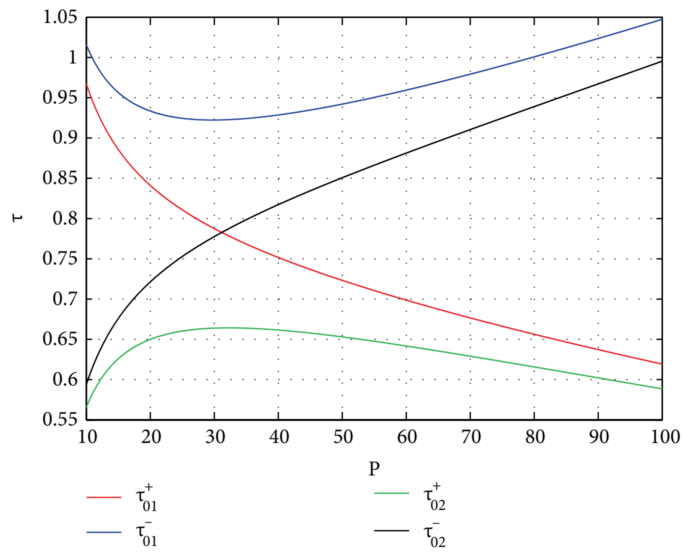

FIGURE 1: The Hopf bifurcation curves for Equation (5) at the trivial equilibrium (0, 0, 0, 0) when $r = 4/3, c = 94$, and $k = 0$.

we can obtain

$$\begin{aligned} \omega_1^+ : \omega_2^- &= \left(\sqrt{P^2 - 9r^2 + 12c^2 r} + \sqrt{P^2 - 9r^2} \right) \\ &\quad : \left(\sqrt{P^2 - 9r^2 + 12c^2 r} - \sqrt{P^2 - 9r^2} \right) \\ &\triangleq b_1 : b_2, \end{aligned} \tag{19}$$

the double Hopf bifurcation is called resonance when $b_1 : b_2 \in z_+$ and it is called nonresonance if $b_1 : b_2$ is an irrational number [16, 17].

3. Normal Form for Double Hopf Bifurcation of in Coupled Microbubble with Non-Resonance

In previous section, we have established some results of Hopf and double Hopf bifurcation and drew this conclusion by analyzing the stability of the origin after bifurcation. However, there is a direct way to approach the dynamic of near the non-resonance double Hopf bifurcation critical point by using the center manifold theory and the normal form method.

To begin, we will try to transform the infinite dimensional dynamic system into a finite dimensional system with the method of center manifold theory. Rescaling the time by $t \to t/\tau$ to normalize the delay so that system Equation (5) becomes

$$\begin{cases} \dot{x}(t) = \tau h(t), \\ \dot{h}(t) = \tau \left(\dfrac{Pg(t-1) + 2h(t) + (h(t))^3 + (3r-2)(x(t)+1)^{-3r} h(t)}{(h(t)-c)(1+x(t))} + \dfrac{(-3/2)(h(t))^2 + (x(t)+1)^{-3r} - 1)}{1+x(t)} \right), \\ \dot{y}(t) = \tau g(t), \\ \dot{g}(t) = \tau \left(\dfrac{Ph(t-1) + 2g(t) + (g(t))^3 + (3r-2)(y(t)+1)^{-3r} g(t)}{(g(t)-c)(1+y(t))} + \dfrac{(-3/2)(g(t))^2 + (y(t)+1)^{-3r} - 1)}{1+y(t)} \right). \end{cases} \tag{20}$$

Suppose that the system Equation (20) undergoes a double Hopf bifurcation at the critical point (P_c, τ_c) with two pairs of eigenvalues $\pm i\omega_1^+$ and $\pm i\omega_2^-$, we can choose

$$\eta(\theta) = \begin{cases} \tau_c A, & \theta = 0, \\ 0, & \theta \in (-1, 0), \\ -\tau_c B, & \theta = -1, \end{cases} \tag{21}$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -3r & -\frac{3r}{c} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3r & -\frac{3r}{c} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{P}{c} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{P}{c} & 0 & 0 \end{pmatrix}. \tag{22}$$

Then, the linearized equation of Equation (20) at the trivial equilibrium is

$$\dot{X}(t) = L_0 X_t, \tag{23}$$

where $L_0\phi = \int_{-1}^{0} d\eta(\theta)\phi(\theta)$, $\phi \in C([-1, 0], R^4)$, and the bilinear form on $C^* \times C$ (*stands for adjoint) is

$$(\psi, \varphi) = \psi(0)\varphi(0) - \int_{-r}^{0} \int_{\xi=0}^{\theta} \psi(\xi - \theta) d\eta(\theta) \varphi(\xi) d\xi, \tag{24}$$

where

$$\begin{aligned} \varphi(\theta) &= (\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta), \varphi_4(\theta)) \in C, \\ \psi(s) &= (\psi_1(s), \psi_2(s), \psi_3(s), \psi_4(s))^T \in C^*. \end{aligned} \tag{25}$$

Designate $\Lambda = \{\pm i\tau_c\omega_1^+, \pm i\tau_c\omega_2^-\}$, then as discussed above, we know that the phase space C can be decomposed as $C = P_\Lambda \oplus Q_\Lambda$, where $P_\Lambda \subset C$ is the 4-dimensional center subspace spanned by the basis vectors of the linear operator L_0 associated with the imaginary characteristic roots, and Q_Λ is the complement subspace of P_Λ. Further, Equation (20) can be rewritten as an ordinary differential equation in the Banach space BC (see [10]) of functions bounded; therefore, we can suppose that the bases for P_Λ and its adjoint P_Λ^* are given, respectively, by

$$\Phi(\theta) = (q_1(\theta), \overline{q_1}(\theta), q_2(\theta), \overline{q_2}(\theta)), \tag{26}$$

$$\psi(s) = \left(q_1^*(s), \overline{q_1^*}(s), q_2^*(s), \overline{q_2^*}(s)\right)^T, \tag{27}$$

where

$$q_1(\theta) = \left(1, i\omega_1^+\tau_c, -\frac{3r + (i\omega_1^+\tau_c + 3r/c)i\omega_1^+\tau_c}{(P/c)e^{-i\omega_1^+\tau_c}}, -\frac{3r + (i\omega_1^+\tau_c + 3r/c)i\omega_1^+\tau_c}{(P/c)e^{-i\omega_1^+\tau_c}i\omega_1^+}\right)^T e^{i\omega_1^+\tau_c\theta} \triangleq (q_{11}(0), q_{12}(0), q_{13}(0), q_{14}(0))^T e^{i\omega_1^+\tau_c\theta}, \tag{28}$$

$$q_2(\theta) = \left(1, i\omega_2^-\tau_c, -\frac{3r + (i\omega_2^-\tau_c + 3r/c)i\omega_2^-\tau_c}{(P/c)e^{-i\omega_2^-\tau_c}}, -\frac{3r + (i\omega_2^-\tau_c + 3r/c)i\omega_2^-\tau_c}{(P/c)e^{-i\omega_2^-\tau_c}i\omega_2^-}\right)^T e^{i\omega_2^-\tau_c\theta} \triangleq (q_{21}(0), q_{22}(0), q_{23}(0), q_{24}(0))^T e^{i\omega_2^-\tau_c\theta}, \tag{29}$$

$$q_1^*(s) = D_1\left(1, \frac{i\omega_1^+\tau_c}{3r}, \frac{9r^2 - 3ir\omega_1^+\tau_c(i\omega_1^+\tau_c - 3r/c)}{3iPr\omega_1^+\tau_c e^{i\omega_1^+\tau_c}/c}, \frac{3rc - i\omega_1^+\tau_c(ci\omega_1^+\tau_c - 3r)}{3Pr}e^{i\omega_1^+\tau_c}\right)e^{-i\omega_1^+\tau_c s} \triangleq D_1(q_{11}^*(0), q_{12}^*(0), q_{13}^*(0), q_{14}^*(0))e^{-i\omega_1^+\tau_c s}, \tag{30}$$

$$q_1^*(s) = D_2\left(1, \frac{i\omega_2^-\tau_c}{3r}, \frac{9r^2 - 3ir\omega_2^-\tau_c(i\omega_2^-\tau_c - 3r/c)}{3iPr\omega_2^-\tau_c e^{i\omega_2^-\tau_c}/c}, \frac{3rc - i\omega_2^-\tau_c(ci\omega_2^-\tau_c - 3r)}{3Pr}e^{i\omega_2^-\tau_c}\right)e^{-i\omega_2^-\tau_c s} \triangleq D_2(q_{21}^*(0), q_{22}^*(0), q_{23}^*(0), q_{24}^*(0))e^{-i\omega_2^-\tau_c s}, \tag{31}$$

$$D_1 = \left[\left(\overline{q_{11}(0)q_{11}^*(0)} + \overline{q_{12}(0)q_{12}^*(0)} + \overline{q_{13}(0)q_{13}^*(0)} + \overline{q_{14}(0)q_{14}^*(0)} + \frac{P}{c}\left(\overline{q_{12}(0)q_{14}^*(0)} + \overline{q_{14}(0)q_{12}^*(0)}\right)\right)\tau_c e^{i\omega_1^+\tau_c}\right]^{-1}, \tag{32}$$

$$D_2 = \left[\left(\overline{q_{21}(0)q_{21}^*(0)} + \overline{q_{22}(0)q_{22}^*(0)} + \overline{q_{23}(0)q_{23}^*(0)} + \overline{q_{24}(0)q_{24}^*(0)} + \frac{P}{c}\left(\overline{q_{22}(0)q_{24}^*(0)} + \overline{q_{24}(0)q_{22}^*(0)}\right)\right)\tau_c e^{i\omega_2^-\tau_c}\right]^{-2}. \tag{33}$$

We let $P = P_c + \varepsilon_1$ and $\tau = \tau_c + \varepsilon_2$ in Equation (20), where ε_1 and ε_2 are perturbation parameters, and denote $\varepsilon = (\varepsilon_1, \varepsilon_2)$. Then Equation (20) can be rewritten as

$$\dot{X}(t) = L(\varepsilon)X_t + F(X_t, \varepsilon), \tag{34}$$

where

$$L(\varepsilon)X_t = (\tau_c + \varepsilon_2)\begin{pmatrix} 0 & 1 & 0 & 0 \\ -3r & -\frac{3r}{c} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3r & -\frac{3r}{c} \end{pmatrix}\begin{pmatrix} x(0) \\ h(0) \\ y(0) \\ g(0) \end{pmatrix} + (\tau_c + \varepsilon_2)\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{P+\varepsilon_1}{c} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{P+\varepsilon_1}{c} & 0 & 0 \end{pmatrix}\begin{pmatrix} x(-1) \\ h(-1) \\ y(-1) \\ g(-1) \end{pmatrix} \tag{35}$$

and

$$F(X_t, \varepsilon) = (\tau_c + \varepsilon_2)\begin{pmatrix} 0 \\ B_1x(0)^3 + B_2x(0)^2h(0) + B_3h(0)^3 + B_4h(t)^2g(-1) \\ 0 \\ B_1y(0)^3 + B_2y(0)^2g(0) + B_3g(0)^3 + B_4g(t)^2h(-1) \end{pmatrix},$$
$$B_1 = -\frac{P}{2c^2}, \quad B_2 = -\frac{3r+2}{6c}, \quad B_3 = -\left(\frac{1}{c} + \frac{3r}{c^3} + \frac{1}{2}\right), \quad B_4 = -\frac{3P}{c}. \tag{36}$$

Thus, we can easily get Equation (20) that becomes an abstract ODE in the space BC.

$$\frac{d}{dt}\mu = A\mu + X_0\overline{F}, \tag{37}$$

where, $\mu \in C$, and A is defined by

$$A : C^1 \to BC, \quad A\mu = \dot{\mu} + X_0[L_0\mu - \dot{\mu}(0)] \tag{38}$$

and

$$\overline{F}(\mu, \varepsilon) = [L(\varepsilon) - L_0]\mu + F(\mu, \varepsilon). \tag{39}$$

By the continuous projection π, we can decompose the enlarged phase space by $\wedge = \{\pm\tau_c\omega_1^+, \pm\tau_c\omega_2^-\}$ as $BC = P_\Lambda \oplus ker\pi$, where $ker\pi = \{\phi + X_0c := \pi(\phi + X_0c) = 0\}$. Let $\mu_t = \Phi z(t) + \nu(t)$, Then, Equation (20) is decomposed as

$$\begin{cases} \dot{z} = Bz + \psi(0)\overline{F}(\Phi z + \nu, \varepsilon), \\ \dot{\nu} = A_{Q_1}\nu + (Id - \pi)X_0\overline{F}(\Phi z + \nu, \varepsilon), \end{cases} \tag{40}$$

where, we denote $\nu \in Q^1 = Q \cap C^1 \subset ker\pi$, and A_{Q_1} is the restriction of A as an operator from Q^1 to the Banach space $ker\pi$.

For the convenience of computing the normal form, Equation (40) can be written by neglecting higher order terms with respect to parameters, with

$$\begin{cases} \dot{z}_1 = i\tau_c\omega_1^+z_1 + \sum_{i=1}^4 D_1q_{1i}^*(0)f_i + h.o.t, \\ \dot{z}_2 = -i\tau_c\omega_1^+z_2 + \sum_{i=1}^4 \overline{D}_1\overline{q_{1i}^*(0)}f_i + h.o.t, \\ \dot{z}_3 = i\tau_c\omega_1^-z_3 + \sum_{i=1}^4 D_2q_{2i}^*(0)f_i + h.o.t, \\ \dot{z}_4 = -i\tau_c\omega_1^-z_4 + \sum_{i=1}^4 \overline{D}_2\overline{q_{2i}^*(0)}f_i + h.o.t, \end{cases} \tag{41}$$

where

$$f_1 = \varepsilon_2h(0), f_2 = -3r\varepsilon_2x(0) - \frac{3r\varepsilon_2}{c}h(0) - \frac{\tau_c\varepsilon_1 + \varepsilon_2P_c}{c}g(-1) + B_1x^3(0) + B_2x^2(0)h(0) + B_3h^3(0) + B_4h^2(0)g(-1),$$
$$f_3 = \varepsilon_2g(0), f_4 = -3r\varepsilon_2y(0) - \frac{3r\varepsilon_2}{c}g(0) - \frac{\tau_c\varepsilon_1 + \varepsilon_2P_c}{c}h(-1) + B_1x^3(0) + B_2y^2(0)h(0) + B_3g^3(0) + B_4g^2(0)h(-1) \tag{42}$$

and

$$\begin{pmatrix} x(0) \\ h(0) \\ y(0) \\ g(0) \end{pmatrix} = \Phi(0)z + \begin{pmatrix} \nu_1(0) \\ \nu_2(0) \\ \nu_3(0) \\ \nu_4(0) \end{pmatrix}, \quad \begin{pmatrix} x(-1) \\ h(-1) \\ y(-1) \\ g(-1) \end{pmatrix} = \Phi(-1)z + \begin{pmatrix} \nu_1(-1) \\ \nu_2(-1) \\ \nu_3(-1) \\ \nu_4(-1) \end{pmatrix}. \tag{43}$$

Next, let M_2^1 denote the operator defined in $V_2^6(C^4 \times ker\pi)$, with

$$M_2^1 : V_2^6(C^4) \mapsto V_2^6(C^4), \tag{44}$$

$$\left(M_2^1 P\right)(z,\varepsilon) = D_z P(z,\varepsilon)Bz - BP(z,\varepsilon), \quad (45)$$

where $V_2^6\left(C^4\right)$ represents the linear space of the second-order homogeneous polynomials in six variables $(z_1, z_2, z_3, z_4, \varepsilon_1, \varepsilon_2)$ with coefficients in C^4. It is easy to verify that one may choose the decomposition without considering the strong resonant

$$V_2^6\left(C^4\right) = Im\left(M_2^1\left(C^4\right)\right) \oplus Im\left(M_2^1\left(C^4\right)\right)^c \quad (46)$$

with complementary space

$$Im\left(M_2^1\left(C^4\right)\right)^c = span\left\{\begin{pmatrix} z_1\varepsilon_i \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2\varepsilon_i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_3\varepsilon_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_4\varepsilon_i \end{pmatrix}\right\} \quad i = 1, 2. \quad (47)$$

Then, the normal form of Equation (37) on the center manifold associated with the equilibrium near $\varepsilon = 0$ has the form

$$\dot{z} = Bz + \frac{1}{2} g_2^1(z, 0, \varepsilon) + h.o.t, \quad (48)$$

where g_2^1 is the function giving the quadratic terms in (z, ε) for $V = 0$, and is determined by $g_2^1(z, 0, \varepsilon) = Proj_{Im}\left(M_2^1\left(C^4\right)\right)^c \times f_2^1(z, 0, \varepsilon)$, where $f_2^1(z, 0, \varepsilon)$ is the function giving the quadratic terms in (z, ε) for $V = 0$ defined by the first equation of Equation (41). Thus, the normal form truncated at the quadratic order terms is given by

$$\begin{cases} \dot{z}_1 = i\tau_c\omega_1^+ z_1 + \lambda_1 z_1 + h.o.t, \\ \dot{z}_2 = -i\tau_c\omega_1^+ z_2 + \overline{\lambda}_1 z_2 + h.o.t, \\ \dot{z}_3 = i\tau_c\omega_1^- z_3 + \lambda_2 z_3 + h.o.t, \\ \dot{z}_4 = -i\tau_c\omega_1^- z_4 + \overline{\lambda}_2 z_4 + h.o.t, \end{cases} \quad (49)$$

where

$$\begin{aligned} \lambda_1 = 2D_1 \Bigg\{ & q_{11}^* \varepsilon_2 q_{12} + q_{12}^* \left(-3r\varepsilon_2 q_{11} - \frac{3r}{c}\varepsilon_2 q_{12} - \frac{\tau_c\varepsilon_1 + \varepsilon_2 P_c}{c} q_{14} e^{i\tau_c\omega_2^-}\right) \\ & + q_{13}^* \varepsilon_2 q_{14} + q_{14}^* \left(-3r\varepsilon_2 q_{13} - \frac{3r}{c}\varepsilon_2 q_{14} - \frac{\tau_c\varepsilon_1 + \varepsilon_2 P_c}{c} q_{12} e^{i\tau_c\omega_1^+}\right)\Bigg\}, \end{aligned} \quad (50)$$

$$\begin{aligned} \lambda_2 = 2D_2 \Bigg\{ & q_{21}^* \varepsilon_2 q_{32} + q_{22}^* \left(-3r\varepsilon_2 q_{31} - \frac{3r}{c}\varepsilon_2 q_{32} - \frac{\tau_c\varepsilon_1 + \varepsilon_2 P_c}{c} q_{34} e^{i\tau_c\omega_2^-}\right) \\ & + q_{23}^* \varepsilon_2 q_{34} + q_{24}^* \left(-3r\varepsilon_2 q_{33} - \frac{3r}{c}\varepsilon_2 q_{34} - \frac{\tau_c\varepsilon_1 + \varepsilon_2 P_c}{c} q_{32} e^{i\tau_c\omega_1^+}\right)\Bigg\}. \end{aligned} \quad (51)$$

To find the normal form up to third order, similarly, let M_3^1 denote the operator defined in $V_3^6\left(C^4 \times ker\pi\right)$, with

$$M_3^1 : V_2^6\left(C^4\right) \mapsto V_3^6\left(C^4\right), \quad (52)$$

$$\left(M_3^1 P\right)(z,\varepsilon) = D_z P(z,\varepsilon)Bz - BP(z,\varepsilon), \quad (53)$$

where $V_3^6\left(C^4\right)$ denotes the linear space of the third order homogeneous polynomials in four variables (z_1, z_2, z_3, z_4) with coefficients in C^4.

Decompose the space $V_3^6\left(C^4\right)$ as follows

$$V_3^6\left(C^4\right) = Im\left(M_3^1\left(C^4\right)\right) \oplus Im\left(M_3^1\left(C^4\right)\right)^c, \quad (54)$$

where the complementary space $Im\left(M_3^1\left(C^4\right)\right)^c$ spanned by the elements

$$\begin{aligned} Im\left(M_3^1\left(C^4\right)\right)^c = span\Bigg\{ & \begin{pmatrix} z_1^2 z_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} z_1 z_3 z_4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2^2 z_1 \\ 0 \\ 0 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ z_2 z_3 z_4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_1 z_2 z_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_3^2 z_4 \\ 0 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_1 z_2 z_4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_4^2 z_3 \end{pmatrix}\Bigg\}. \end{aligned} \quad (55)$$

Therefore, the normal form up to third-order terms is given by

$$\dot{z} = Bz + \frac{1}{2} g_2^1(z, 0, \varepsilon) + \frac{1}{3!} g_3^1(z, 0, 0) + h.o.t, \quad (56)$$

where

$$\frac{1}{3!} g_3^1(z, 0, 0) = \frac{1}{3!}\left(I - P_{1,3}^1\right) f_3^1(z, 0, 0), \quad (57)$$

and $f_3^1(z, 0, 0)$ is the function giving the cubic terms in (z, ε, ν) for $V = 0$, and $\varepsilon = 0$ is defined by the first equation of Equation (40).

Finally, the normal form on the center manifold arising from Equation (40) becomes

$$\begin{cases} \dot{z}_1 = i\tau_c\omega_1^+ z_1 + \lambda_1 z_1 + a_{11} z_1^2 z_2 + a_{12} z_1 z_3 z_4 + h.o.t, \\ \dot{z}_2 = -i\tau_c\omega_1^+ z_2 + \overline{\lambda}_1 z_2 + \overline{a}_{11} z_2^2 z_1 + \overline{a}_{12} z_1 z_3 z_4 + h.o.t, \\ \dot{z}_3 = i\tau_c\omega_2^- z_3 + \lambda_2 z_3 + a_{21} z_3^2 z_4 + a_{22} z_1 z_2 z_3 + h.o.t, \\ \dot{z}_4 = -i\tau_c\omega_2^- z_4 + \overline{\lambda}_2 z_4 + \overline{a}_{21} z_4^2 z_3 + \overline{a}_{22} z_1 z_2 z_4 + h.o.t, \end{cases} \quad (58)$$

where

$$\begin{aligned} a_{11} = {} & D_1 q_{12}^* \Big\{ 3B_1 q_{11}^2 q_{21} + B_2\left(q_{11}^2 q_{22} + 2q_{21} q_{12} q_{11}\right) + 3B_3 q_{12}^2 q_{22} \\ & + B_4\left(q_{12}^2 q_{24} e^{i\tau_c\omega_2^-} + 2q_{12} q_{22} q_{14} e^{i\tau_c\omega_2^-}\right)\Big\} \\ & + D_1 q_{14}^* \Big\{ 3B_1 q_{13}^2 q_{23} + B_2\left(q_{13}^2 q_{24} + 2q_{23} q_{14} q_{13}\right) \\ & + 3B_3 q_{14}^2 q_{24} + B_4\left(q_{14}^2 q_{22} e^{i\tau_c\omega_1^+} + 2q_{14} q_{24} q_{12} e^{i\tau_c\omega_1^+}\right)\Big\}, \end{aligned} \quad (59)$$

$$\begin{aligned} a_{12} = {} & D_1 q_{12}^* \Big\{ 6B_1 q_{11} q_{31} q_{41} + B_2(2q_{11} q_{31} q_{42} + 2q_{11} q_{41} q_{32} + 2q_{31} q_{41} q_{12}) \\ & + 6B_3 q_{12} q_{32} q_{42} + B_4\big(2q_{12} q_{32} q_{44} e^{i\tau_c\omega_2^-} \\ & + 2q_{12} q_{42} q_{34} e^{i\tau_c\omega_2^-} + 2q_{32} q_{42} q_{14} e^{i\tau_c\omega_2^-}\big)\Big\} \\ & + D_1 q_{14}^* \Big\{ 6B_1 q_{13} q_{33} q_{43} + B_2(2q_{13} q_{33} q_{44} + 2q_{13} q_{43} q_{34} + 2q_{33} q_{43} q_{14}) \\ & + 6B_3 q_{14} q_{34} q_{42} + B_4\big(2q_{14} q_{34} q_{42} e^{i\tau_c\omega_1^+} \\ & + 2q_{14} q_{44} q_{32} e^{i\tau_c\omega_1^+} + 2q_{34} q_{44} q_{12} e^{i\tau_c\omega_1^+}\big)\Big\}, \end{aligned} \quad (60)$$

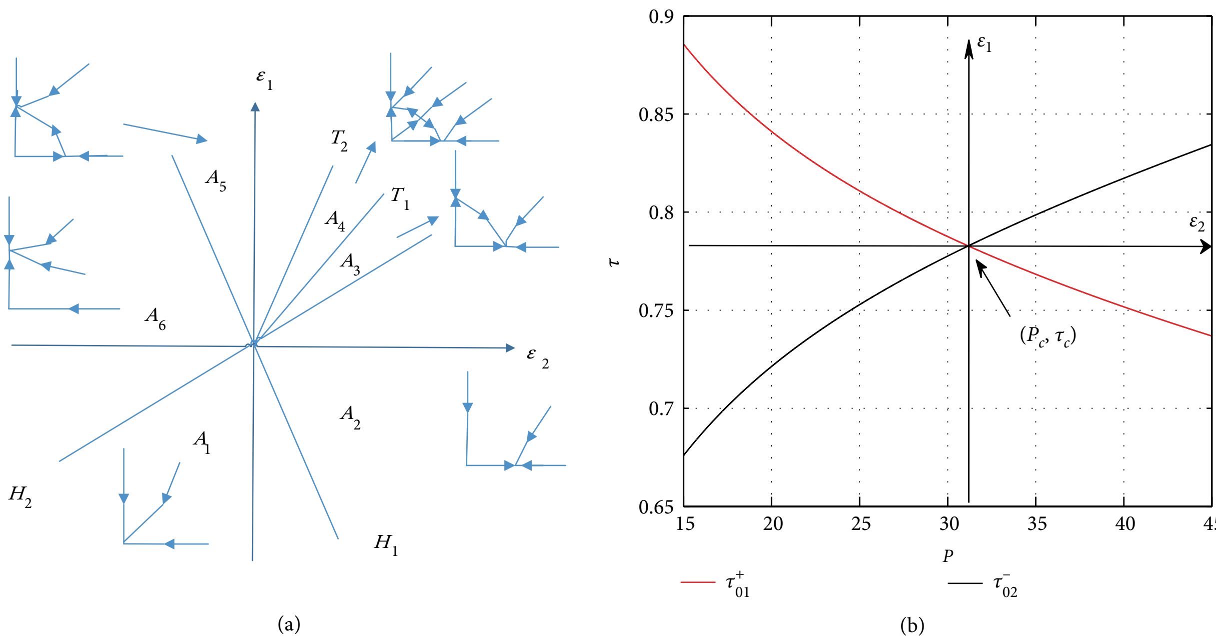

FIGURE 2: (a) Critical bifurcation lines in the $(\varepsilon_1, \varepsilon_1)$ parameter space near (P_c, τ_c); and the corresponding phase portraits in the (R_1, R_1) plane when $P_c = 30.21, \tau_c = 0.78$, (b) to embed Figure 2(a) into Figure 1, and yielding complete bifurcation sets near (P_c, τ_c).

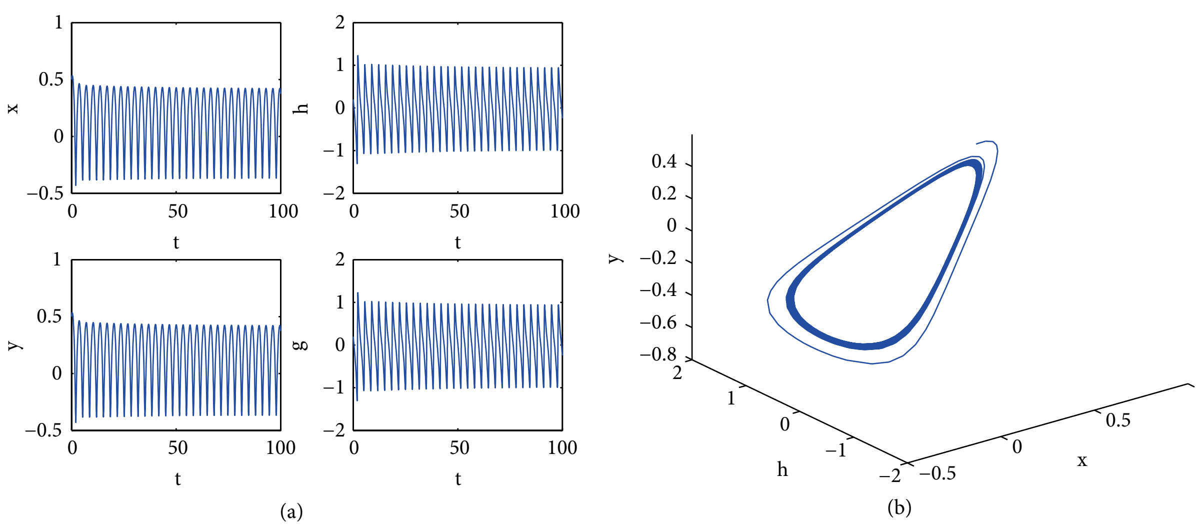

FIGURE 3: Simulated solution of Equation (20) for $(\varepsilon_1, \varepsilon_2) = (0.4, -0.08)$, showing a stable periodic solution.

$$\begin{aligned} a_{21} = {} & D_2 q_{22}^* \left\{ 3B_1 q_{31}^2 q_{41} + B_2\left(q_{31}^2 q_{42} + 2q_{41}q_{32}q_{31}\right) + 3B_3 q_{32}^2 q_{42} \right. \\ & \left. + B_4\left(q_{32}^2 q_{44} e^{i\tau_c\omega_2^-} + 2q_{32}q_{42}q_{34}e^{i\tau_c\omega_2^-}\right)\right\} \\ & + D_2 q_{24}^* \left\{ 3B_1 q_{33}^2 q_{43} + B_2\left(q_{33}^2 q_{44} + 2q_{43}q_{34}q_{33}\right) + 3B_3 q_{34}^2 q_{44} \right. \\ & \left. + B_4\left(q_{34}^2 q_{42} e^{i\tau_c\omega_1^+} + 2q_{34}q_{44}q_{32}e^{i\tau_c\omega_1^+}\right)\right\}, \end{aligned} \tag{61}$$

$$\begin{aligned} a_{22} = {} & D_2 q_{22}^* \left\{ 6B_1 q_{11}q_{21}q_{31} + B_2\left(2q_{11}q_{31}q_{22} \right.\right. \\ & \left. + 2q_{11}q_{21}q_{12} + 2q_{31}q_{21}q_{12}\right) + 6B_3 q_{12}q_{32}q_{22} \\ & \left. + B_4\left(2q_{12}q_{32}q_{24}e^{i\tau_c\omega_2^-} + 2q_{12}q_{22}q_{34}e^{i\tau_c\omega_2^-} + 2q_{32}q_{22}q_{14}e^{i\tau_c\omega_2^-}\right)\right\} \\ & + D_2 q_{24}^* \left\{ 6B_1 q_{13}q_{33}q_{23} + B_2\left(2q_{13}q_{33}q_{24} \right.\right. \\ & \left. + 2q_{13}q_{23}q_{34} + 2q_{33}q_{23}q_{14}\right) + 6B_3 q_{14}q_{34}q_{22} \\ & \left. + B_4\left(2q_{14}q_{34}q_{22}e^{i\tau_c\omega_1^+} + 2q_{14}q_{24}q_{32}e^{i\tau_c\omega_1^+} + 2q_{34}q_{24}q_{12}e^{i\tau_c\omega_1^+}\right)\right\}. \end{aligned} \tag{62}$$

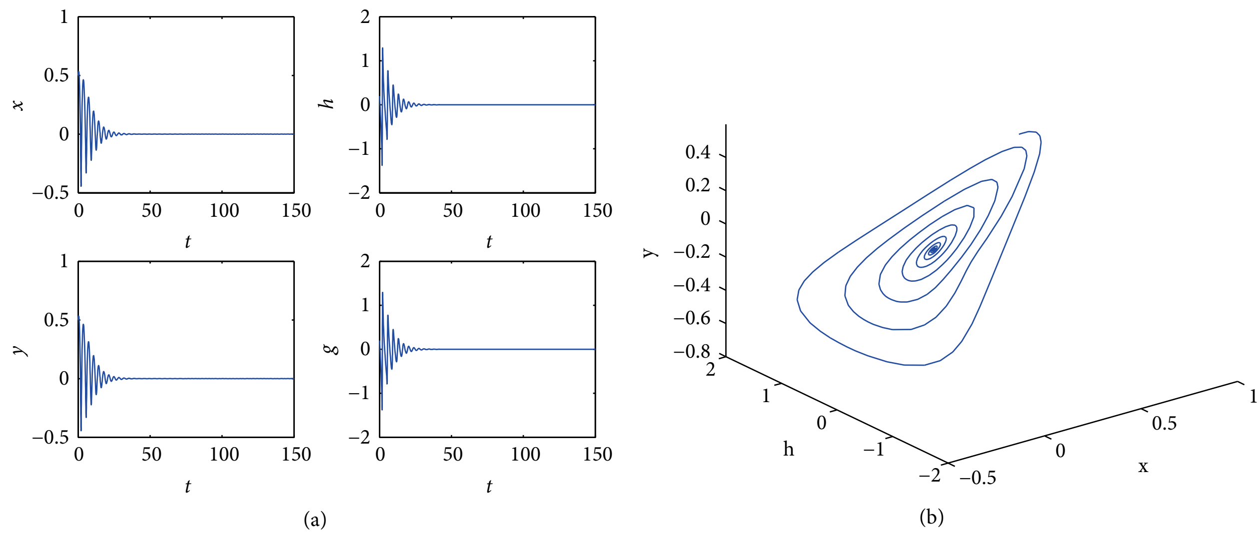

FIGURE 4: Simulated solution of Equation (20) for $(\varepsilon_1, \varepsilon_2) = (-0.4, -0.08)$, showing the equilibrium(0 0 0 0) is asymptotically stable.

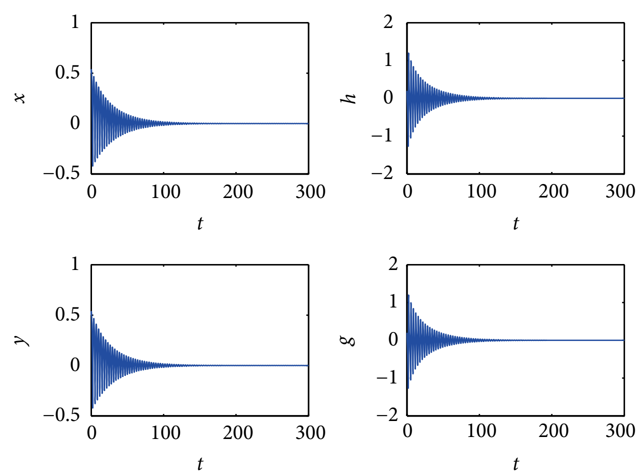

FIGURE 5: Simulated solution of Equation (2) for $P = 0.001$, $r = 4/3, c = 94, \tau = 1000$: the equilibrium (0 0 0 0) is asymptotically stable.

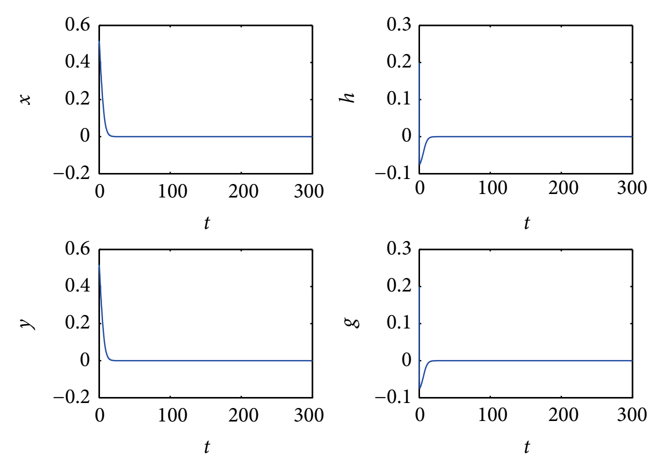

FIGURE 6: Simulated solution of Equation (2) for $P = 1000, r = 4/3$, $c = 94, \tau = 0.001$: the equilibrium (0 0 0 0) is asymptotically stable.

With the polar coordinates

$$\begin{cases} z_1 = r_1 e^{-i\rho_1} \\ z_2 = r_1 e^{i\rho_1} \\ z_3 = r_2 e^{-i\rho_2} \\ z_1 = r_2 e^{i\rho_2} \end{cases} \tag{63}$$

then, the amplitude equation resulted from Equation (58) is

$$\begin{cases} \dot{r}_1 = \left(\alpha_1 + \bar{a} r_1^2 + \bar{b} r_2^2\right) r_1 \\ \dot{r}_2 = \left(\alpha_2 + \bar{c} r_1^2 + \bar{d} r_2^2\right) r_1 \end{cases} \tag{64}$$

and the nature of bifurcation of Equation (64) can refer to [20]. We introduce new phase variables and rescale time according to $R_1 = -\bar{a} r_1, R_2 = -\bar{d} r_1, \tau = 2t$, and obtain

$$\begin{cases} \dot{R}_1 = (\alpha_1 - R_1 - \theta R_2) R_1, \\ \dot{R}_2 = (\alpha_2 - \delta R_1 - R_2) R_2, \end{cases} \tag{65}$$

where $\alpha_1 = Re\lambda_1, \alpha_2 = Re\lambda_2, \bar{a} = Rea_{11}, \bar{b} = Rea_{12}, \bar{c} = Rea_{21}$, $\bar{d} = Rea_{22}, \theta = \bar{b}/\bar{d}, \delta = \bar{c}/\bar{a}$.

4. Illustrations and Numerical Simulations

To give a more clear bifurcation picture, we choose $r = 4/3, c = 94, P_c = 30.21, \tau_c = 0.78$ by referring [12], and yield $\omega_1^+ = 2.166, \omega_2^- = 1.847, \omega_1^+ : \omega_2^- = 5\sqrt{2} : 6$. By calculating, we can obtain $\alpha_1 = 0.410\varepsilon_2 + 0.605\varepsilon_1$, $\alpha_2 = -30.514\varepsilon_2 + 24.351\varepsilon_1, \bar{a} = -4.729 < 0, \bar{b} = -10.031, \bar{c} = -.5.554, \bar{d} = -2.003 < 0, \theta = 5.007$, and $\delta = 1.117$. For the bifurcation behaviors of the original system Equation (20) in the neighborhood of the trivial equilibrium, the above critical bifurcation boundaries

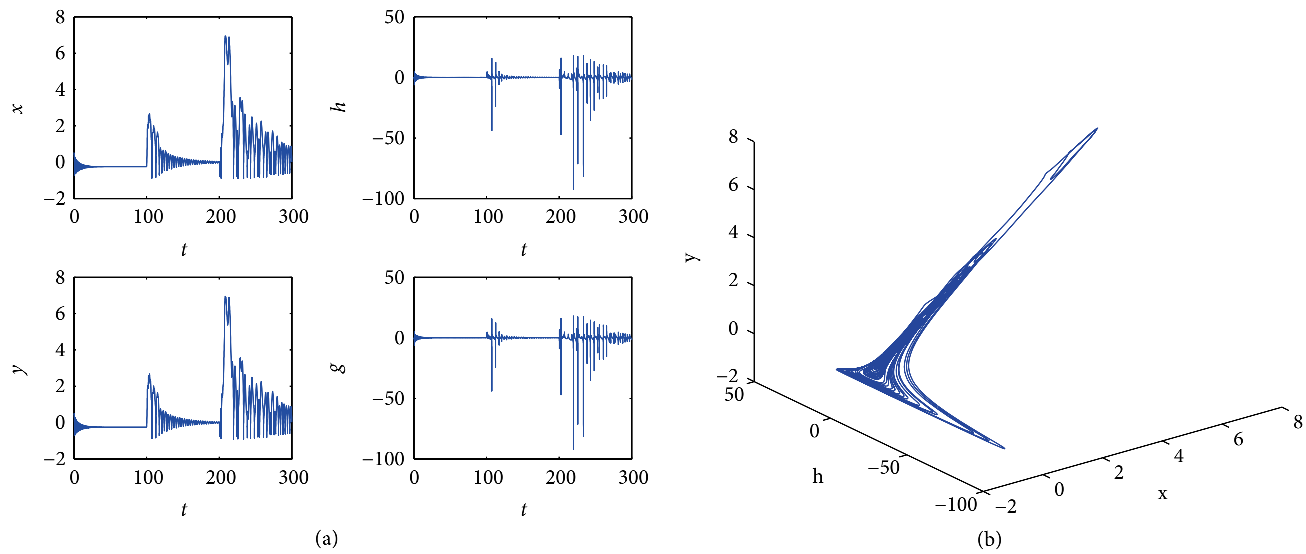

FIGURE 7: Simulated solution of Equation (2) for $P = 1000, r = 4/3, c = 94, \tau = 100$: showing the chaotic motion.

divide the parameter plane (P_c, τ_c) into six regions (see Figure 2(a)). In region A_1, Equation (20) exists one trivial equilibrium, when the parameters are changed from region A_1 to A_6, the stable trivial equilibrium becomes a saddle and a stable periodic solution is bifurcated. In region A_5, another stable periodic solution appears. When the parameters vary across line T_2 from region A_5 to A_4, an unstable periodic solution bifurcates an unstable quasi-periodic solution which is a nobble, and there coexists with two stable periodic solutions and an unstable quasi-periodic solution. Further, in region A_3, an unstable quasi-periodic solution disappears when parameters change. Finally, when the parameters are varied in the region A_2, the periodic solution collides with the trivial solution and the trivial solution ranges from a source to a saddle. Here, in Figure 2, the bifurcation critical lines are, respectively

$$H_1 = \{(\alpha_1, \alpha_2), \alpha_1 = 0\} : \varepsilon_1 = -0.698\varepsilon_2, \quad (66)$$

$$H_2 = \{(\alpha_1, \alpha_2), \alpha_2 = 0\} : \varepsilon_1 = 1.253\varepsilon_2, \quad (67)$$

which stand for the Hopf bifuration lines,

$$T_1 = \{(\alpha_1, \alpha_2), \alpha_2 > 0\} : \varepsilon_1 = 1.262\varepsilon_2 \quad (68)$$

$$T_2 = \{(\alpha_1, \alpha_2), \alpha_1 > 0\} : \varepsilon_1 = 1.308\varepsilon_2 \quad (69)$$

and which stand for the periodic solution of pitchfork bifurcation lines.

We will present some numerical simulation results and choose the values of the parameters, respectively, $(\varepsilon_1, \varepsilon_2) = (0.4, -0.08) \in A_6$ and $(\varepsilon_1, \varepsilon_2) = (-0.4, -0.08) \in A_1$. Corresponding to a stable periodic solution shown in Figure 3, the equilibrium (0 0 0 0) is asymptotically stable, shown in Figure 4. Thus, the result shows that the numerical simulations are coincident with the analytical predictions.

5. Explanation for the Mechanical Background of the Model Using Mathematical Results

As $P = 0$, then Equation (2) can be written as

$$(\dot{a} - c)\left(a\ddot{a} + \frac{3}{2}\dot{a}^2 - a^{-3r} + 1\right) - \dot{a}^3 - (3r-2)a^{-3r}\dot{a} - 2\dot{a} = 0, \quad (70)$$

which shows the behavior of systems range from double bubbles coupling to single bubble. The zero solution of Equation (70) is asymptotically stable (see [12]).

As noted in [12], $\tau = d/c$ shows that the distance of two bubbles d is proportional to the time delay τ when sound speed is constant. That coupling coefficient P decreases as the distance between bubbles increases (see [13]), and meaning the coupling coefficient P as a decreasing function of the delay τ. Clearly, if P is sufficiently small and τ is sufficiently big (τ is sufficiently small and P is sufficiently big), the behavior of Equation (2) at the equilibrium (0 0 0 0) will be close to Equation (70) (see Figures 5 and 6). And if neglecting relationship between P and τ and choosing them sufficiently big, Equation (2) will have a chaotic motion (see Figure 7). Therefore, the double bubble oscillators Equation (2) can show the possibility of more complicated dynamics.

Bubble dancing [21] is an interesting phenomenon in bubble dynamics. It creates a pull or push under the sound pressure gradient between the bubbles, which makes them undergo a periodic motion in turn, and the periodic motion is called bubble dancing. As noted in Figure 3, it shows a stable periodic solution by taking value of parameters, which can depict periodic motion of bubble' radius, and the change of bubble' radius can describe the motion of bubble in the liquid. Thus, the stable periodic solution of system can be used to better explain this physical phenomenon (bubble dancing).

6. Conclusions and Discussion

In this work, we mainly have discussed the nonresonant double Hopf bifurcation in dynamics of microbubble with delay coupling and have used center manifold reduction methods to compute the normal form near the double Hopf bifurcation point. Moreover, with bifurcation analysis near the double Hopf critical point given, we confirm the existence of a stable periodic solution and an unstable quasi-periodic solution. We discover the system coexists with two stable periodic solutions and an unstable quasi-periodic solution, which is a new phenomenon with studying the double Hopf bifurcation.

However, we only get the normal forms for nonresonant cases owing to the limitations of our knowledge, so analyzing the microbubble dynamics of the resonance case will be our next research.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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