Research Article

Investigating One-, Two-, and Triple-Wave Solutions via Multiple Exp-Function Method Arising in Engineering Sciences

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The multiple Exp-function method is employed for searching the multiple soliton solutions for the new extended $(3+1)$-dimensional Jimbo-Miwa-like (JM) equation, the extended $(2+1)$-dimensional Calogero-Bogoyavlenskii-Schiff (eCBS) equation, the generalization of the $(2+1)$-dimensional Bogoyavlensky-Konopelchenko (BK) equation, and a variable-coefficient extension of the DJKM (vDJKM) equation, which contain one-soliton-, two-soliton-, and triple-soliton-kind solutions. The physical phenomena of these gained multiple soliton solutions are analyzed and indicated in figures by selecting suitable values.

1. Introduction

Many nonlinear physical phenomena arise in various fields of engineering and science such as fluid dynamics, nuclear reactor dynamics, hydrodynamics, molecular biology, quantum mechanics, plasma physics, optical fibres, and solid state physics. To describe these complex physical phenomena, nonlinear differential equations play a significant role. Therefore, obtaining the solutions of these nonlinear equations is a topic of great interest in the study of many fields of science. To better understand the working of a physical problem, a mathematical model was brought into the picture in the form of nonlinear PDEs. The solutions of partial differential equations give the detailed summary about the nature of phenomena involved. Many numerical and analytical methods have been derived to deal with these kinds of scientific problems. We need to adopt an effective and powerful method to investigate such type of mathematical model which gives the solutions upholding physical reality. In most of the analytic

techniques, linearization of the system is the main topic to focus on, and also, it is assumed that the nonlinearities are relatively insignificant. Sometimes, these assumptions make a strong effect on the solutions in respect to the real physics of the phenomena involved. Thus, finding the solutions of nonlinear ODEs and PDEs is still a significant problem. For this, we need new techniques to develop approximate and exact solutions. Several analytical and numerical techniques have been formulated for tackling these types of nonlinear models, including the Exp-function method [1, 2], the homotopy analysis method [3], the homotopy perturbation method [4], the (G'/G)-expansion method [5], the improved tan $(\phi/2)$-expansion method ([6–8]), the Hirota bilinear method ([9–13]), the He variational principle [14, 15], the binary Darboux transformation [16], the Lie group analysis [17, 18], and the Bäcklund transformation method [19]. Moreover, many powerful methods have been used to investigate the new properties of mathematical models which are symbolizing serious real world problems ([20–24]). This

study is aimed at investigating the following four forms of nonlinear PDEs:

(i) Extended $(3 + 1)$-dimensional Jimbo-Miwa-like (JM) equations ([25–34]) is given as

$$\Psi_{xxxy} + 3\Psi_{xx}\Psi_y + 3\Psi_x\Psi_{xy} + 2\Psi_{yt} \\ - 3(\Psi_{xz} + \Psi_{yz} + \Psi_{zz}) = 0, \tag{1}$$

and

$$\Psi_{xxxy} + 3\Psi_{xx}\Psi_y + 3\Psi_x\Psi_{xy} - 3\Psi_{xz} \\ + 2(\Psi_{xt} + \Psi_{yt} + \Psi_{zt}) = 0, \tag{2}$$

explains some $(3 + 1)$-dimensional waves in physics [25, 26], in which $\Psi(x, y, z, t)$ is a function including spaces x, y, z and time t. In addition, this equation cannot pass certain conventional integrability tests [27]. Qi et al. [32] studied the extended $(3 + 1)$-dimensional Jimbo-Miwa-like equation via the Hirota method and obtained the solitary-wave solutions and novel exact soliton solutions.

(ii) The extended $(2 + 1)$-dimensional Calogero-Bogoyavlenskii-Schiff (eCBS) equation [35] is given as

$$\Psi_{xt} + \Psi_{xxxy} + 6\Psi_{xx}\Psi_y + 6\Psi_x\Psi_{xy} + \delta_1\Psi_{xy} \\ + \delta_2\Psi_{xx} + \delta_3(\Psi_{xxxx} + 12\Psi_x\Psi_{xx}) = 0, \tag{3}$$

where δ_1, δ_2, and δ_3 are arbitrary values. The eCBS equation will be changed to a $(2 + 1)$-dimensional CBS equation when $\delta_1 = \delta_2 = \delta_3 = 0$ [36]. Ren et al. [35] investigated the extended $(2 + 1)$-dimensional Calogero-Bogoyavlenskii-Schiff-like equation by using the generalized bilinear operators based on a prime number $p = 3$.

(iii) The generalization of the $(2 + 1)$-dimensional Bogoyavlensky-Konopelchenko (BK) equation [37] is given as

$$\Psi_{xt} + \alpha(\Psi_{xxxx} + 6\Psi_x\Psi_{xx}) + \beta(\Psi_{xxxy} + 3\Psi_{xx}\Psi_y \\ + 3\Psi_x\Psi_{xy}) + \delta_1\Psi_{xx} + \delta_2\Psi_{xy} + \delta_3\Psi_{yy} = 0. \tag{4}$$

The $(3 + 1)$-dimensional VC B-type KP equation is extended from the KP equation and can explain some notable $(3 + 1)$-dimensional waves in fluid dynamics [37]. Also, Zhang and Pang [38] found the lump and lump-type solutions through the Hirota bilinear form for the variable-coefficient B-type Kadomtsev-Petviashvili equation.

(iv) The variable-coefficient extension of the DJKM (vDJKM) equation ([39–48]) is given as

$$\Psi_{xxxxy} + 4\Psi_x\Psi_{xxy} + 2\Psi_y\Psi_{xxx} + 6\Psi_{xx}\Psi_{xy} \\ - \alpha\Psi_{yyy} - 2\beta g(t)\Psi_{xxt} + h(t)\Psi_{xxy} = 0, \tag{5}$$

where the wave amplitude $u(x, y, t)$ is a function of variables and x, y, t, $g(t)$, and $h(t)$ are functions of t. The vD JKM equation (6) reduces to the DJKM equation when $g(t) = 1$ and $h(t) = 0$ as follows:

$$\Psi_{xxxxy} + 4\Psi_x\Psi_{xxy} + 2\Psi_y\Psi_{xxx} \\ + 6\Psi_{xx}\Psi_{xy} - \alpha\Psi_{yyy} - 2\beta\Psi_{xxt} = 0. \tag{6}$$

The soliton solutions to a few $(3 + 1)$-dimensional generalized nonlinear integrable equations have been constructed. The adopted approach is the multiple Exp-function method, which was presented earlier in the valuable work by Ma et al. [49]. Recently, special kinds of reductions of soliton solutions to rational functions that are being actively studied are lump solutions to nonlinear partial differential equations by Ma and Zhou [50] and their interactions with solitons to the Hirota-Satsuma-Ito equation in $(2 + 1)$-dimensions by Ma [51], even for linear PDEs by the same author [52]. Cordero et al. constructed the stability analysis of the fourth-order iterative method for finding multiple roots of nonlinear equations [53]. Gao et al. obtained the optical soliton solutions of the cubic-quartic nonlinear Schrödinger and resonant nonlinear Schrödinger equations with the parabolic law [54]. The truncated Painlevé expansion was employed to derive a Bäcklund transformation of a $(2 + 1)$-dimensional nonlinear system by Zhao and Han [55]. Also, Gao et al. studied on the conformable $(2 + 1)$-dimensional Ablowitz-KaupNewell-Segur equation in order to show the existence of complex combined dark-bright soliton solutions [56]. The same authors presented the nonlinear Zoomeron equation by using the newly extended direct algebraic technique [57]. The instability modulation for the $(2 + 1)$-dimension paraxial wave equation and its new optical soliton solutions in Kerr's media by utilizing the modified auxiliary expansion method have been studied in [58]. In [59], the authors found new complex solitons to the perturbed nonlinear Schrödinger model with the help of an analytical method. In this paper, we will study the multiple Exp-function method for determining the multiple soliton solutions. The multiple Exp-function method used by some of the powerful authors for various nonlinear equations include the nonlinear evolution equations [60], the $(2 + 1)$-dimensional Calogero-Bogoyavlenskii-Schiff equation [61], the generalized $(1 + 1)$-dimensional and $(2 + 1)$-dimensional Ito equations [62], a new generalization of the associated Camassa-Holm equation [63], the $(3 + 1)$-dimensional generalized KP and BKP equations [64], and the new $(2 + 1)$-dimensional Korteweg-de Vries equation [65]. In [65], Liu et al. utilized the multiple Exp-function method for the most well-known equation, namely, the Korteweg-de Vries (KdV) equation, and gained one-soliton-, two-soliton-, and three-soliton-type

solutions with interpretations for the obtained soliton solutions. Moreover, they analyzed and illustrated the propagation and interaction of some soliton solutions by selecting appropriate values. Also, the elasticity and being unchanged for some solutions when interacting among three solitons and after the collision were investigated.

The rest of this paper is structured as follows: the multiple Exp-function scheme is summarized in Section 2. In Sections 3–7, the extended JM equations, the extended CBS equation, the generalized BK equation, and the vDJKM equation, respectively, will be investigated to find one-soliton, two-soliton, and triple-soliton solutions. In the last section, the conclusions are given.

2. Multiple Exp-Function Method

This section elucidates a systematic explanation of the multiple Exp-function method [60–64] so that it can be further applied to the nonlinear PDEs in order to furnish its exact solutions:

Step 1. The following NLPDE are as follows:

$$\mathcal{N}\left(x, y, t, \Psi, \Psi_x, \Psi_y, \Psi_z, \Psi_t, \Psi_{xx}, \Psi_{tt}, \cdots\right) = 0. \quad (7)$$

We commence a sequence of novel variables $\xi_i = \xi_i(x, y, z, t)$, $1 \le i \le n$, by solvable PDEs, for example, the linear ones:

$$\begin{aligned}
\xi_{i,x} &= \alpha_i \xi_i, \\
\xi_{i,y} &= \beta_i \xi_i, \\
\xi_{i,z} &= \gamma_i \xi_i, \\
\xi_{i,t} &= \delta_i \xi_i, \\
& 1 \le i \le n,
\end{aligned} \quad (8)$$

where $\alpha_i, \beta_i, \gamma_i, 1 \le i \le n$ are the angular wave numbers and $\delta_i, 1 \le i \le n$ are the wave frequencies. It must be pointed out that this is frequently the initiating step for constructing the exact solutions to nonlinear partial differential equations; moreover, solving such linear equations redounds to the exponential function solutions,

$$\xi_i = \varpi_i e^{\theta_i}, \quad \theta_i = \alpha_i x + \beta_i y + \gamma_i z - \delta_i t, \ 1 \le i \le n, \quad (9)$$

in which $\varpi_i, 1 \le i \le n$, are undetermined values.

Step 2. Supposing the solution of equation (7) happens to be of the following form in terms of new variables $\xi_i, 1 \le i \le n$:

$$\Psi(x, y, z, t) = \frac{\Delta(\xi_1, \xi_2, \cdots, \xi_n)}{\Omega(\xi_1, \xi_2, \cdots, \xi_n)},$$

$$\Delta = \sum_{r,s=1}^{n} \sum_{i,j=0}^{M} \Delta_{rs,ij} \xi_r^i \xi_s^j, \ \Omega = \sum_{r,s=1}^{n} \sum_{i,j=0}^{N} \Omega_{rs,ij} \xi_r^i \xi_s^j, \quad (10)$$

in which $\Delta_{rs,ij}$ and $\Omega_{rs,ij}$ are values to be settled. Plugging equation (10) into equation (7) and ordering the numerator of the rational function to zero, we can gain a series of the nonlinear algebraic equations about the variables $\alpha_i, \beta_i, \gamma_i, \delta_i, \Delta_{rs,ij}$ and $\Omega_{rs,ij}$. Solving the solutions for these nonlinear algebraic equations and inserting these solutions into equation (10), the multiple soliton solutions for equation (7) can be achieved as follows:

$$\Psi(x, y, z, t) = \frac{\Delta\left(\varpi_1 e^{\alpha_1 x + \beta_1 y + \gamma_1 x - \delta_1 t}, \cdots, \varpi_n e^{\alpha_n x + \beta_n y + \gamma_n x - \delta_n t}\right)}{\Omega\left(\varpi_1 e^{\alpha_1 x + \beta_1 y + \gamma_1 x - \delta_1 t}, \cdots, \varpi_n e^{\alpha_n x + \beta_n y + \gamma_n x - \delta_n t}\right)}, \quad (11)$$

and also we have

$$\Delta_t = \sum_{i=1}^{n} \Delta_{\xi_i} \xi_{i,t},$$

$$\Omega_t = \sum_{i=1}^{n} \Omega_{\xi_i} \xi_{i,t},$$

$$\Delta_x = \sum_{i=1}^{n} \Delta_{\xi_i} \xi_{i,x}$$

$$\Omega_x = \sum_{i=1}^{n} \Omega_{\xi_i} \xi_{i,x}$$

$$\Delta_y = \sum_{i=1}^{n} \Delta_{\xi_i} \xi_{i,y},$$

$$\Omega_y = \sum_{i=1}^{n} \Omega_{\xi_i} \xi_{i,y},$$

$$\Delta = \Delta(\xi_1, \xi_2, \cdots, \xi_n),$$

$$\Omega = \Omega(\xi_1, \xi_2, \cdots, \xi_n),$$

$$\Delta_{\xi_i} = \frac{d\Delta}{d_{\xi_i}}, \quad (12)$$

$$\Omega_{\xi_i} = \frac{d\Omega}{d_{\xi_i}},$$

$$\xi_{i,t} = \frac{d\xi_i}{dt},$$

$$\xi_{i,x} = \frac{d\xi_i}{dx},$$

$$\xi_{i,y} = \frac{d\xi_i}{dy},$$

$$\Psi_t = \frac{\Omega \sum_{i=1}^{n} \Delta_{\xi_i} \xi_{i,t} - \Delta \sum_{i=1}^{n} \Omega_{\xi_i} \xi_{i,t}}{\Omega^2},$$

$$\Psi_x = \frac{\Omega \sum_{i=1}^{n} \Delta_{\xi_i} \xi_{i,x} - \Delta \sum_{i=1}^{n} \Omega_{\xi_i} \xi_{i,x}}{\Omega^2},$$

$$\Psi_y = \frac{\Omega \sum_{i=1}^{n} \Delta_{\xi_i} \xi_{i,y} - \Delta \sum_{i=1}^{n} \Omega_{\xi_i} \xi_{i,y}}{\Omega^2},$$

$$\Psi_z = \frac{\Omega \sum_{i=1}^{n} \Delta_{\xi_i} \xi_{i,z} - \Delta \sum_{i=1}^{n} \Omega_{\xi_i} \xi_{i,z}}{\Omega^2}.$$

3. Multiple Soliton Solutions for the Extended $(3+1)$ JM Equation (1)

3.1. Set I: One-Wave Solution. We commence with a one-wave function based on the statement in Step 2 in the previous section; we suppose that equation (1) has the rational function of the one-wave solution as shown in the following form:

$$\Psi(x, y, z, t) = \frac{\Delta_1}{\Omega_1}, \quad \Omega_1 = 1 + \rho_1 e^{\alpha_1 x + \beta_1 y + \gamma_1 z - \delta_1 t},$$
$$\Delta_1 = \sigma_1 + \rho_1 \sigma_2 e^{\alpha_1 x + \beta_1 y + \gamma_1 z - \delta_1 t}, \tag{13}$$

in which σ_1 and σ_2 are unfound constants. Plugging (13) into equation (1), we gain to the following case:

$$\rho_1 = \rho_1,$$
$$\sigma_1 = \sigma_1,$$
$$\sigma_2 = \frac{\rho_2 (2\alpha_1 \rho_1 + \sigma_1)}{\rho_1}, \tag{14}$$
$$\delta_1 = \frac{\alpha_1^3 \beta_1 - 3\alpha_1 \gamma_1 - 3\beta_1 \gamma_1 - 3\gamma_1^2}{2\beta_1}.$$

Therefore, the resulting one-wave solution reads as

$$\Psi(x, y, z, t) = \frac{\sigma_1 + \rho_2 (2\alpha_1 \rho_1 + \sigma_1) e^{\alpha_1 x + \beta_1 y + \gamma_1 z - \left(\left(\alpha_1^3 \beta_1 - 3\alpha_1 \gamma_1 - 3\beta_1 \gamma_1 - 3\gamma_1^2\right)/(2\beta_1)\right)t}}{1 + \rho_1 e^{\alpha_1 x + \beta_1 y + \gamma_1 z - \left(\left(\alpha_1^3 \beta_1 - 3\alpha_1 \gamma_1 - 3\beta_1 \gamma_1 - 3\gamma_1^2\right)/(2\beta_1)\right)t}}. \tag{15}$$

By selecting the suitable values of parameters, the graphic presentation of the periodic wave solution is presented in Figure 1 including the 3D plot, the contour plot, the density plot, and the 2D plot when three spaces arise at spaces $x = -1$, $x = 0$, and $x = 1$.

3.2. Set II: Two-Wave Solutions. We commence with the two-wave functions based on the statement in Step 2 in the previous section; we suppose that equation (1) has the rational function of two-wave solutions as shown in the following form:

$$\Psi(x, y, z, t) = \frac{\Delta_2}{\Omega_2}, \tag{16}$$

$$\Omega_2 = 1 + \sigma_1 e^{\alpha_1 x + \beta_1 y + \gamma_1 z - \delta_1 t} + \sigma_2 e^{\alpha_2 x + \beta_2 y + \gamma_2 z - \delta_2 t}$$
$$+ \sigma_1 \sigma_2 \sigma_{12} e^{(\alpha_1 + \alpha_2)x + (\beta_1 + \beta_2)y + (\gamma_1 + \gamma_2)z - (\delta_1 + \delta_2)t},$$

$$\Delta_2 = \rho_1 e^{\alpha_1 x + \beta_1 y + \gamma_1 z - \delta_1 t} + \rho_2 e^{\alpha_2 x + \beta_2 y + \gamma_2 z - \delta_2 t}$$
$$+ \rho_1 \rho_2 \rho_{12} e^{(\alpha_1 + \alpha_2)x + (\beta_1 + \beta_2)y + (\gamma_1 + \gamma_2)z - (\delta_1 + \delta_2)t}. \tag{17}$$

Plugging (16) along with (17) into equation (1), we get to the following case:

$$\sigma_1 = \sigma_1,$$
$$\sigma_2 = \sigma_2,$$
$$\delta_1 = \frac{1}{2}\alpha_1^3,$$
$$\delta_2 = \frac{1}{2}\alpha_2^3,$$
$$\rho_1 = 2\alpha_1 \sigma_1, \tag{18}$$
$$\rho_2 = 2\alpha_2 \sigma_2,$$
$$\rho_{12} = \frac{(\beta_1 - \beta_2)(\alpha_1 - \alpha_2)}{2\alpha_1 \alpha_2 (\beta_1 + \beta_2)},$$
$$\sigma_{12} = \frac{(\beta_1 - \beta_2)(\alpha_1 - \alpha_2)}{(\beta_1 + \beta_2)(\alpha_1 + \alpha_2)}.$$

Therefore, the resulting two-wave solution reads as

$$\Psi_2(x, y, z, t) = \frac{2\alpha_1 \sigma_1 e^{\Lambda_1} + 2\alpha_2 \sigma_2 e^{\Lambda_2} + 2\sigma_1 \sigma_2 (((\beta_1 - \beta_2)(\alpha_1 - \alpha_2))/(\beta_1 + \beta_2)) e^{\Lambda_1 + \Lambda_2}}{1 + \sigma_1 e^{\Lambda_1} + \sigma_2 e^{\Lambda_2} + \sigma_1 \sigma_2 (((\beta_1 - \beta_2)(\alpha_1 - \alpha_2))/((\beta_1 + \beta_2)(\alpha_1 + \alpha_2))) e^{\Lambda_1 + \Lambda_2}}, \tag{19}$$

in which $\Lambda_i = \alpha_i x + \beta_i y + \gamma_i z - (1/2)\alpha_i^3 t$, $i = 1, 2$. By selecting the suitable values of parameters, the graphic representation of the periodic wave solution is presented in Figure 2 includ-

ing the 3D plot, the contour plot, the density plot, and the 2D plot when three spaces arise at spaces $x = -10$, $x = 0$, and $x = 10$.
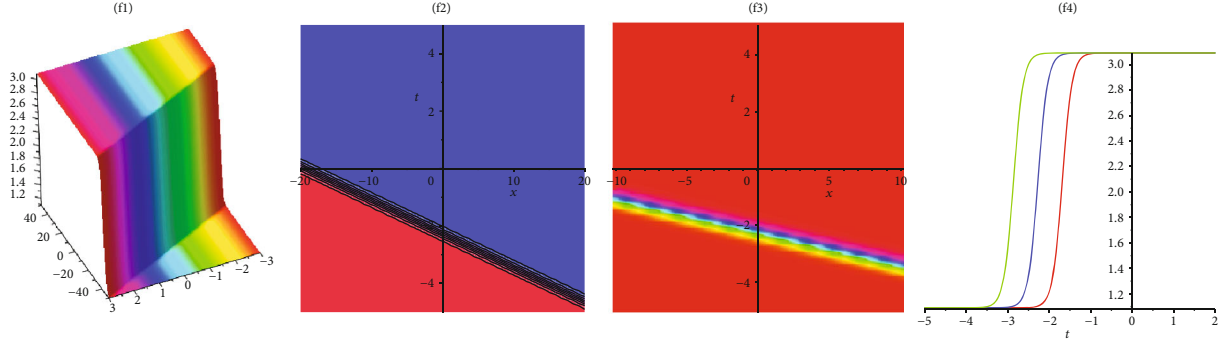
FIGURE 1: The one-wave solution (15) at $\alpha_1 = 1$, $\beta_1 = 1.5$, $\sigma_2 = 1.5$, $\rho_1 = 2.2$, $\gamma_1 = 2$, $\rho_2 = 3$, $y = 10$, and $z = 2$.
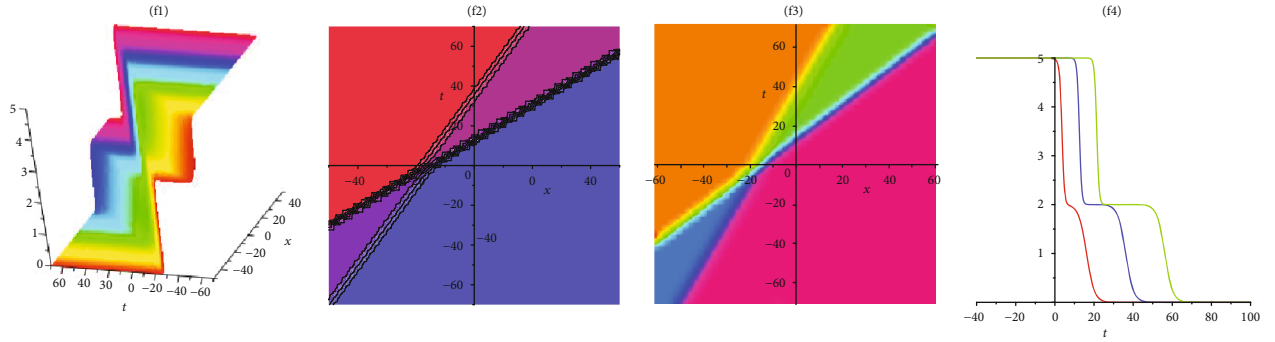
FIGURE 2: The two-wave solution (19) at $\alpha_1 = 1$, $\alpha_2 = 1.5$, $\beta_1 = 1.5$, $\beta_2 = 2$, $\sigma_1 = 1.1$, $\sigma_2 = 2$, $\gamma_1 = 1.5$, $\gamma_2 = 2$, $y = 10$, and $z = 2$.

3.3. Set III: Triple-Wave Solutions.

We commence with three-wave functions based on the statement in Step 2 in the previous section; we suppose that equation (1) has the rational function of triple-wave solutions as shown in the following form:

$$\Psi(x, y, z, t) = \frac{\Delta_3}{\Omega_3}, \tag{20}$$

$$\begin{aligned}
\Omega_3 &= 1 + \rho_1 e^{\Lambda_1} + \rho_2 e^{\Lambda_2} + \rho_3 e^{\Lambda_3} + \rho_1 \rho_2 \rho_{12} e^{\Lambda_1 + \Lambda_2} \\
&\quad + \rho_1 \rho_3 \rho_{13} e^{\Lambda_1 + \Lambda_3} + \rho_2 \rho_3 \rho_{23} e^{\Lambda_2 + \Lambda_3} \\
&\quad + \rho_1 \rho_2 \rho_3 \rho_{12} \rho_{13} \rho_{23} e^{\Lambda_1 + \Lambda_2 + \Lambda_3}, \\
\Delta_3 &= \sigma_1 e^{\Lambda_1} + \sigma_2 e^{\Lambda_2} + \sigma_3 e^{\Lambda_3} + \sigma_1 \sigma_2 \sigma_{12} e^{\Lambda_1 + \Lambda_2} \\
&\quad + \sigma_1 \sigma_3 \sigma_{13} e^{\Lambda_1 + \Lambda_3} + \sigma_2 \sigma_3 \sigma_{23} e^{\Lambda_2 + \Lambda_3} \\
&\quad + \sigma_1 \sigma_2 \sigma_3 \sigma_{12} \sigma_{13} \sigma_{23} e^{\Lambda_1 + \Lambda_2 + \Lambda_3},
\end{aligned} \tag{21}$$

in which $\Lambda_i = \alpha_i x + \beta_i y + \gamma_i x - \delta_i t$, $i = 1, 2, 3$. Plugging (20) along with (21) into equation (1), we obtain the following case:

$$\begin{aligned}
\sigma_i &= \sigma_i, \\
\delta_i &= \frac{1}{2} \alpha_i^3, \\
\rho_i &= 2\alpha_i \sigma_i, \\
i &= 1, 2, 3,
\end{aligned}$$

$$\rho_{12} = \frac{(\beta_1 - \beta_2)(\alpha_1 - \alpha_2)}{2\alpha_1 \alpha_2 (\beta_1 + \beta_2)},$$

$$\sigma_{12} = \frac{(\beta_1 - \beta_2)(\alpha_1 - \alpha_2)}{(\beta_1 + \beta_2)(\alpha_1 + \alpha_2)},$$

$$\rho_{13} = \frac{(\beta_1 - \beta_3)(\alpha_1 - \alpha_3)}{2\alpha_1 \alpha_3 (\beta_1 + \beta_3)},$$

$$\sigma_{13} = \frac{(\beta_1 - \beta_3)(\alpha_1 - \alpha_3)}{(\beta_1 + \beta_3)(\alpha_1 + \alpha_3)},$$

$$\rho_{23} = \frac{(\beta_2 - \beta_3)(\alpha_2 - \alpha_3)}{2\alpha_2 \alpha_3 (\beta_2 + \beta_3)},$$

$$\sigma_{23} = \frac{(\beta_2 - \beta_3)(\alpha_2 - \alpha_3)}{(\beta_2 + \beta_3)(\alpha_2 + \alpha_3)}. \tag{22}$$

Therefore, the resulting three-wave solution reads as

$$\Psi_3(x, y, z, t) = \frac{\Delta_3}{\Omega_3}, \tag{23}$$

in which Δ_3 and Ω_3 with their relations are given in (21) and (22). By selecting suitable values of parameters, the graphic representation of the periodic wave solution is presented in Figure 3 including the 3D plot, the contour plot,
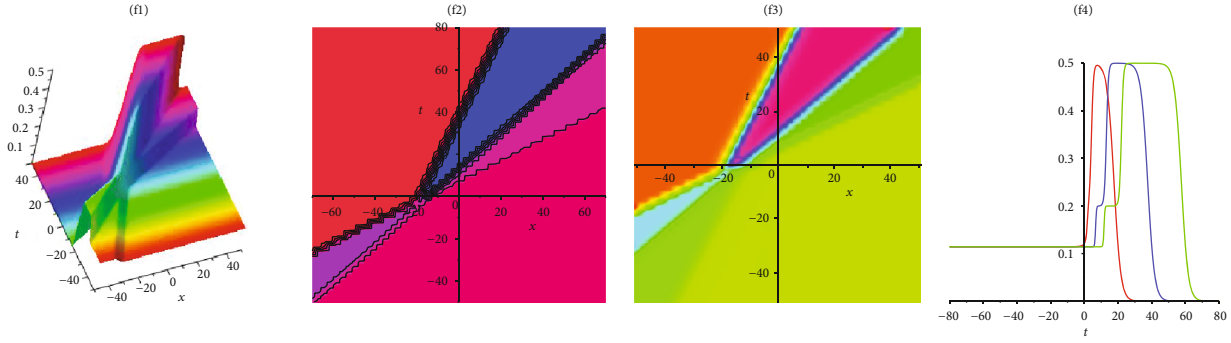
FIGURE 3: The three-wave solution (23) at $\alpha_1 = 1$, $\alpha_2 = 1.5$, $\alpha_3 = 2$, $\sigma_1 = 1.1$, $\sigma_2 = 2$, $\sigma_3 = 2.7$, $\beta_1 = 2.5$, $\beta_2 = 2$, $\beta_3 = 2.6$, $\gamma_1 = 1.5$, $\gamma_2 = 2$, $\gamma_3 = 3$, $y = 10$, and $z = 2$.

the density plot, and the 2D plot when three spaces arise at spaces $x = -10$, $x = 0$, and $x = 10$.

4. Multiple Soliton Solutions for the Extended (3 + 1) JM Equation (2)

4.1. Set I: One-Wave Solution. We commence with the one-wave function based on the statement in Step 2 in the previous section; we suppose that equation (2) has the rational function of the one-wave solution as shown in the following form:

$$\Psi(x, y, z, t) = \frac{\Delta_1}{\Omega_1}, \quad \Omega_1 = \rho_1 + \rho_2 e^{\alpha_1 x + \beta_1 y + \gamma_1 z - \delta_1 t},$$
$$\Delta_1 = \sigma_1 + \sigma_2 e^{\alpha_1 x + \beta_1 y + \gamma_1 z - \delta_1 t}, \tag{24}$$

in which ρ_1, ρ_2, σ_1, and σ_2 are unfound constants. Plugging (24) into equation (1), we gain the following case:

$$\rho_1 = \rho_1,$$
$$\rho_2 = \rho_2,$$
$$\sigma_1 = \sigma_1,$$
$$\sigma_2 = \frac{\rho_2(2\alpha_1\rho_1 + \sigma_1)}{\rho_1}, \tag{25}$$
$$\delta_1 = \frac{\alpha_1(\alpha_1{}^2\beta_1 - 3\gamma_1)}{2\alpha_1 + 2\beta_1 + 2\gamma_1}.$$

Therefore, the resulting one-wave solution reads as

$$\Psi(x, y, z, t) = \frac{\sigma_1 + \rho_2(2\alpha_1\rho_1 + \sigma_1)e^{\alpha_1 x + \beta_1 y + \gamma_1 z - \left(\left(\alpha_1\left(\alpha_1^2\beta_1 - 3\gamma_1\right)\right)/(2\alpha_1 + 2\beta_1 + 2\gamma_1)\right)t}}{\rho_1 + \rho_2\, e^{\alpha_1 x + \beta_1 y + \gamma_1 z - \left(\left(\alpha_1\left(\alpha_1^2\beta_1 - 3\gamma_1\right)\right)/(2\alpha_1 + 2\beta_1 + 2\gamma_1)\right)t}}. \tag{26}$$

By choosing the suitable values of parameters, the graphic representation of the periodic wave solution is presented in Figure 4 containing the 3D plot, the contour plot, the density plot, and the 2D plot when three spaces arise at spaces $x = -1$, $x = 0$, and $x = 1$.

4.2. Set II: Two-Wave Solutions. We commence with two-wave functions based on the statement in Step 2 in the previous section; we suppose that equation (2) has the rational function of two-wave solutions as shown in the following form:
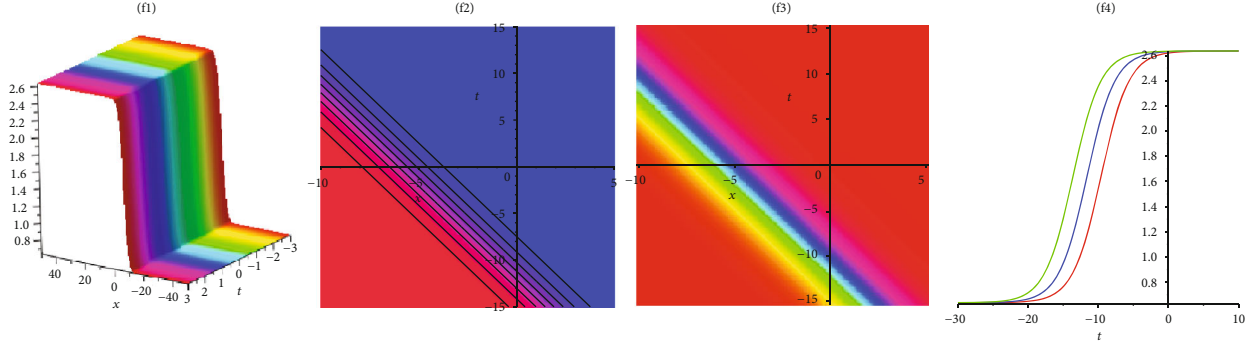
$$\Psi(x, y, z, t) = \frac{\Delta_2}{\Omega_2}, \tag{27}$$

$$\Omega_2 = 1 + \sigma_1 e^{\alpha_1 x + \beta_1 y + \gamma_1 z - \delta_1 t} + \sigma_2 e^{\alpha_2 x + \beta_2 y + \gamma_2 z - \delta_2 t}$$
$$+ \sigma_1\sigma_2\sigma_{12}e^{(\alpha_1 + \alpha_2)x + (\beta_1 + \beta_2)y + (\gamma_1 + \gamma_2)z - (\delta_1 + \delta_2)t},$$

$$\Delta_2 = \rho_1 e^{\alpha_1 x + \beta_1 y + \gamma_1 z - \delta_1 t} + \rho_2 e^{\alpha_2 x + \beta_2 y + \gamma_2 z - \delta_2 t}$$
$$+ \rho_1\rho_2\rho_{12}e^{(\alpha_1 + \alpha_2)x + (\beta_1 + \beta_2)y + (\gamma_1 + \gamma_2)z - (\delta_1 + \delta_2)t}. \tag{28}$$

Plugging (27) along with (28) into equation (2), we get to the following case:

$$\sigma_i = \sigma_i,$$
$$\delta_i = \frac{\alpha_i^3\beta_i}{2\alpha_i + 2\beta_i},$$
$$\rho_i = 2\alpha_i\sigma_i,$$
$$i = 1, 2,$$

FIGURE 4: The one-wave solution (26) at $\alpha_1 = 1$, $\beta_1 = 1.5$, $\sigma_1 = 1.4$, $\rho_1 = 2.2$, $\rho_2 = 3$, $\gamma_1 = 2$, $y = 1$, and $z = 2$.

$$\rho_{12} = \frac{\Sigma_{12}}{2\alpha_1\alpha_2\Gamma_{12}},$$

$$\sigma_{12} = \frac{Y_{12}}{\Phi_{12}}. \tag{29}$$

$$\begin{aligned}
\Sigma_{12} &= \alpha_1\alpha_2\big(\alpha_1^2\beta_2 + 2\alpha_1\alpha_2\beta_1 - 2\alpha_1\alpha_2\beta_2 - \alpha_2^2\beta_1 + 3\beta_1^2\beta_2 \\
&\quad - 3\beta_1\beta_2^2\big)\big(\alpha_1^2 - \alpha_2^2\big) + \alpha_2^2\beta_1^2(\alpha_1 + \alpha_2)\big(3\alpha_1^2 - 3\alpha_1\alpha_2 + \alpha_2^2\big) \\
&\quad + 2\,\alpha_1\alpha_2\beta_1\beta_2(\alpha_1 + \alpha_2)\big(\alpha_1^2 - 3\,\alpha_1\alpha_2 + \alpha_2^2\big) \\
&\quad + \alpha_1^2\beta_2^2(\alpha_1 + \alpha_2)\big(\alpha_1^2 - 3\,\alpha_1\alpha_2 + 3\,\alpha_2^2\big),
\end{aligned}$$

$$\begin{aligned}
\Gamma_{12} &= \alpha_1\alpha_2(\alpha_1 + \alpha_2)\big(\alpha_1^2\beta_2 + 2\,\alpha_1\alpha_2\beta_1 + 2\,\alpha_1\alpha_2\beta_2 + \alpha_2^2\beta_1 \\
&\quad + 3\,\beta_1^2\beta_2 + 3\,\beta_1\beta_2^2\big) + \alpha_2^2\beta_1^2\big(3\,\alpha_1^2 + 3\,\alpha_1\alpha_2 + \alpha_2^2\big) \\
&\quad + 2\,\alpha_1\alpha_2\beta_1\beta_2\big(\alpha_1^2 + 3\,\alpha_1\alpha_2 + \alpha_2^2\big) \\
&\quad + \alpha_1^2\beta_2^2\big(\alpha_1^2 + 3\,\alpha_1\alpha_2 + 3\,\alpha_2^2\big),
\end{aligned}$$

$$\begin{aligned}
Y_{12} &= \alpha_1\alpha_2(\alpha_1 - \alpha_2)\big(\alpha_1^2\beta_2 + 2\,\alpha_1\alpha_2\beta_1 - 2\,\alpha_1\alpha_2\beta_2 - \alpha_2^2\beta_1 \\
&\quad + 3\,\beta_1^2\beta_2 - 3\,\beta_1\beta_2^2\big) + \alpha_2^2\beta_1^2\big(3\,\alpha_1^2 - 3\,\alpha_1\alpha_2 + \alpha_2^2\big) \\
&\quad + 2\,\alpha_1\alpha_2\beta_1\beta_2\big(\alpha_1^2 - 3\,\alpha_1\alpha_2 + \alpha_2^2\big) \\
&\quad + \alpha_1^2\beta_2^2\big(\alpha_1^2 - 3\,\alpha_1\alpha_2 + 3\,\alpha_2^2\big),
\end{aligned}$$

$$\begin{aligned}
\Phi_{12} &= \alpha_1\alpha_2(\alpha_1 + \alpha_2)\big(\alpha_1^2\beta_2 + 2\,\alpha_1\alpha_2\beta_1 + 2\,\alpha_1\alpha_2\beta_2 + \alpha_2^2\beta_1 \\
&\quad + 3\,\beta_1^2\beta_2 + 3\,\beta_1\beta_2^2\big) + \alpha_2^2\beta_1^2\big(3\,\alpha_1^2 + 3\,\alpha_1\alpha_2 + \alpha_2^2\big) \\
&\quad + 2\,\alpha_1\alpha_2\beta_1\beta_2\big(\alpha_1^2 + 3\,\alpha_1\alpha_2 + \alpha_2^2\big) \\
&\quad + \alpha_1^2\beta_2^2\big(\alpha_1^2 + 3\,\alpha_1\alpha_2 + \alpha_2^2\big).
\end{aligned} \tag{30}$$

Therefore, the resulting two-wave solution reads as

$$\Psi_2(x, y, z, t) = \frac{2\alpha_1\sigma_1 e^{\Lambda_1} + 2\alpha_2\sigma_2 e^{\Lambda_2} + 2\sigma_1\sigma_2(\Sigma_{12}/\Gamma_{12})e^{\Lambda_1+\Lambda_2}}{1 + \sigma_1 e^{\Lambda_1} + \sigma_2 e^{\Lambda_2} + \sigma_1\sigma_2(Y_{12}/\Phi_{12})e^{\Lambda_1+\Lambda_2}}, \tag{31}$$

in which $\Lambda_i = \alpha_i x + \beta_i y + \gamma_i z - (\alpha_i^3\beta_i/(2\alpha_i + 2\beta_i))t$, $i = 1, 2$. By selecting the suitable values of parameters, the graphic representation of the periodic wave solution is presented in Figure 5 including the 3D plot, the contour plot, the density

plot, and the 2D plot when three spaces arise at spaces $x = -10$, $x = 0$, and $x = 10$.

4.3. Set III: Triple-Wave Solutions. We commence with three-wave functions based on the statement in Step 2 in the previous section; we suppose that equation (2) has the rational function of triple-wave solutions as shown in the following form:

$$\Psi(x, y, z, t) = \frac{\Delta_3}{\Omega_3}, \tag{32}$$

$$\begin{aligned}
\Omega_3 &= 1 + \rho_1 e^{\Lambda_1} + \rho_2 e^{\Lambda_2} + \rho_3 e^{\Lambda_3} + \rho_1\rho_2\rho_{12}e^{\Lambda_1+\Lambda_2} \\
&\quad + \rho_1\rho_3\rho_{13}e^{\Lambda_1+\Lambda_3} + \rho_2\rho_3\rho_{23}e^{\Lambda_2+\Lambda_3} \\
&\quad + \rho_1\rho_2\rho_3\rho_{12}\rho_{13}\rho_{23}e^{\Lambda_1+\Lambda_2+\Lambda_3},
\end{aligned}$$

$$\begin{aligned}
\Delta_3 &= \sigma_1 e^{\Lambda_1} + \sigma_2 e^{\Lambda_2} + \sigma_3 e^{\Lambda_3} + \sigma_1\sigma_2\sigma_{12}e^{\Lambda_1+\Lambda_2} \\
&\quad + \sigma_1\sigma_3\sigma_{13}e^{\Lambda_1+\Lambda_3} + \sigma_2\sigma_3\sigma_{23}e^{\Lambda_2+\Lambda_3} \\
&\quad + \sigma_1\sigma_2\sigma_3\sigma_{12}\sigma_{13}\sigma_{23}e^{\Lambda_1+\Lambda_2+\Lambda_3},
\end{aligned} \tag{33}$$

in which $\Lambda_i = \alpha_i x + \beta_i y + \gamma_i x - \delta_i t$, $i = 1, 2, 3$. Plugging (32) along with (33) into equation (2), we obtain the following case:

$$\sigma_i = \sigma_i,$$

$$\delta_i = \frac{\alpha_i^3\beta_i}{2\alpha_i + 2\beta_i},$$

$$\rho_i = 2\alpha_i\sigma_i,$$

$$i = 1, 2, 3, \tag{34}$$

$$\rho_{12} = \frac{\Sigma_{12}}{2\alpha_1\alpha_2\Gamma_{12}},$$

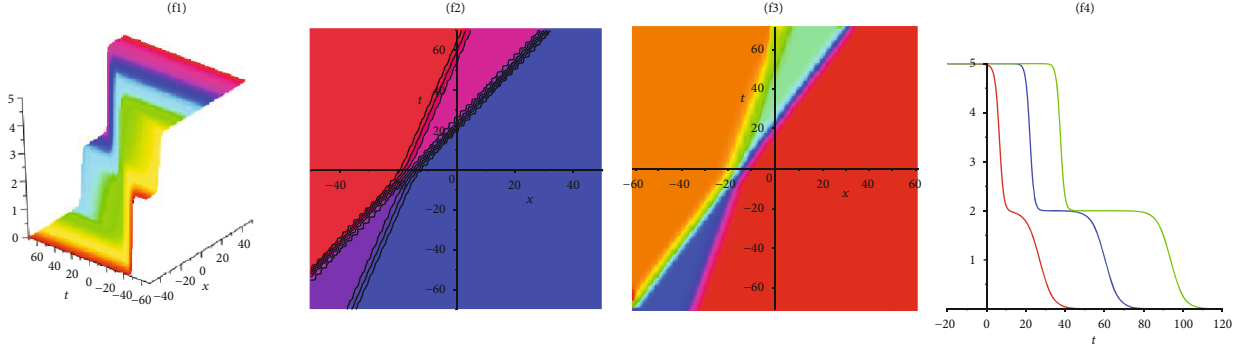$$\sigma_{12} = \frac{Y_{12}}{\Phi_{12}}.$$

FIGURE 5: The two-wave solution (31) at $\alpha_1 = 1$, $\alpha_2 = 1.5$, $\beta_1 = 1.5$, $\beta_2 = 2$, $\sigma_1 = 1.1$, $\sigma_2 = 2$, $\gamma_1 = 1.5$, $\gamma_2 = 2$, $y = 10$, and $z = 2$.

$$\rho_{13} = \frac{\Sigma_{13}}{2\alpha_1\alpha_3\Gamma_{13}},$$

$$\sigma_{13} = \frac{Y_{13}}{\Phi_{13}},$$

$$\rho_{23} = \frac{\Sigma_{23}}{2\alpha_2\alpha_3\Gamma_{23}},$$

$$\sigma_{23} = \frac{Y_{23}}{\Phi_{23}},$$

$$\Sigma_{13} = \text{substitute } \{\alpha_2 = \alpha_3, \beta_2 = \beta_3\}, \text{ at } \Sigma_{12}, \tag{35}$$

$$\Sigma_{23} = \text{substitute } \{\alpha_1 = \alpha_2, \beta_1 = \beta_2\}, \text{ at } \Sigma_{13},$$

$$\Gamma_{13} = \text{substitute } \{\alpha_2 = \alpha_3, \beta_2 = \beta_3\}, \text{ at } \Gamma_{12},$$

$$\Gamma_{23} = \text{substitute } \{\alpha_1 = \alpha_2, \beta_1 = \beta_2\}, \text{ at } \Gamma_{13},$$

$$Y_{13} = \text{substitute } \{\alpha_2 = \alpha_3, \beta_2 = \beta_3\}, \text{ at } Y_{12},$$

$$Y_{23} = \text{substitute } \{\alpha_1 = \alpha_2, \beta_1 = \beta_2\}, \text{ at } Y_{13},$$

$$\Phi_{13} = \text{substitute } \{\alpha_2 = \alpha_3, \beta_2 = \beta_3\}, \text{ at } \Phi_{12},$$

$$\Phi_{23} = \text{substitute } \{\alpha_1 = \alpha_2, \beta_1 = \beta_2\}, \text{ at } \Phi_{13}.$$

Therefore, the resulting three-wave solution reads as

$$\Psi_3(x, y, z, t) = \frac{\Delta_3}{\Omega_3}, \tag{36}$$

in which Δ_3 and Ω_3 with their relations are given in (33) and (34). By selecting the suitable values of parameters, the graphic representation of the periodic wave solution is presented in Figure 6 containing the 3D plot, the contour plot, the density plot, and the 2D plot when three spaces arise at spaces $x = -10$, $x = 0$, and $x = 10$.

5. Multiple Soliton Solutions for the (2 + 1) eCBS Equation

5.1. Set I: One-Wave Solution. We start with one-wave function based on the statement in Step 2 in the previous section;

we suppose that equation (3) has the rational function of the one-wave solution as shown in the following form:

$$\Psi(x, y, t) = \frac{\Delta_1}{\Omega_1}, \quad \Omega_1 = \rho_1 + \rho_2 e^{\alpha_1 x + \beta_1 y - \lambda_1 t}, \tag{37}$$

$$\Delta_1 = \sigma_1 + \sigma_2 e^{\alpha_1 x + \beta_1 y - \lambda_1 t},$$

in which σ_1 and σ_2 are unfound constants. Substituting (37) into equation (3), we get to the following case:

$$\rho_1 = \rho_1,$$

$$\rho_2 = \rho_2,$$

$$\sigma_1 = \sigma_1, \tag{38}$$

$$\sigma_2 = \frac{\rho_2(\alpha_1\rho_1 + \sigma_1)}{\rho_1},$$

$$\lambda_1 = \alpha_1^3\delta_3 + \alpha_1^2\beta_1 + \alpha_1\delta_2 + \beta_1\delta_1.$$

Therefore, the resulting one-wave solution reads as

$$\Psi(x, y, t) = \frac{\sigma_1 + (\rho_2(\alpha_1\rho_1 + \sigma_1)/\rho_1)e^{\alpha_1 x + \beta_1 y - (\alpha_1^3\delta_3 + \alpha_1^2\beta_1 + \alpha_1\delta_2 + \beta_1\delta_1)t}}{\rho_1 + \rho_2 e^{\alpha_1 x + \beta_1 y - (\alpha_1^3\delta_3 + \alpha_1^2\beta_1 + \alpha_1\delta_2 + \beta_1\delta_1)t}}. \tag{39}$$

By selecting the suitable values of parameters, the graphic representation of the periodic wave solution is presented in Figure 7 containing the 3D plot, the contour plot, the density plot, and the 2D plot when three spaces arise at spaces $x = -1$, $x = 0$, and $x = 1$.

5.2. Set II: Two-Wave Solutions. We begin with two-wave functions based on the statement in Step 2 in the previous section; we suppose that equation (3) has the rational function of two-wave solutions as shown in the following form:

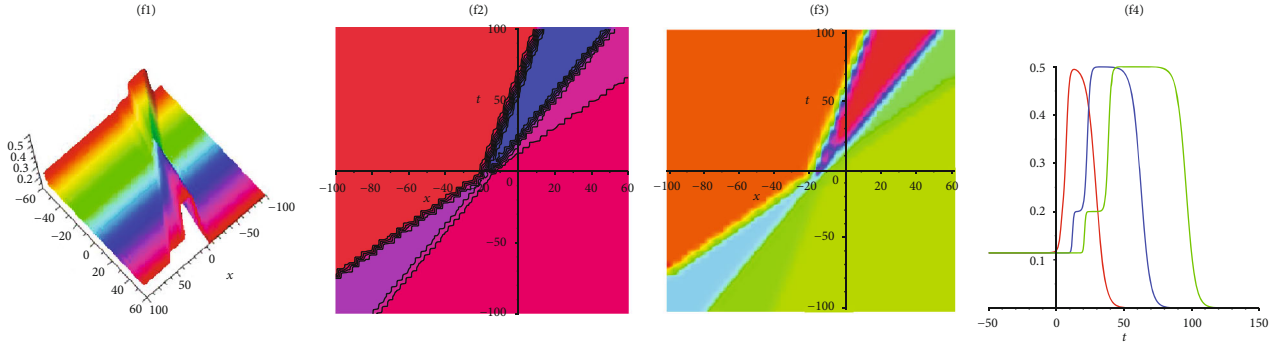$$\Psi(x, y, z, t) = \frac{\Delta_2}{\Omega_2}, \tag{40}$$

FIGURE 6: The three-wave solution (36) at $\alpha_1 = 1, \alpha_2 = 1.5, \alpha_3 = 2, \sigma_1 = 1.1, \sigma_2 = 2, \sigma_3 = 2.7, \beta_1 = 2.5, \beta_2 = 2, \beta_3 = 2.6, \gamma_1 = 1.5, \gamma_2 = 2, \gamma_3 = 3, y = 10,$ and $z = 2$.
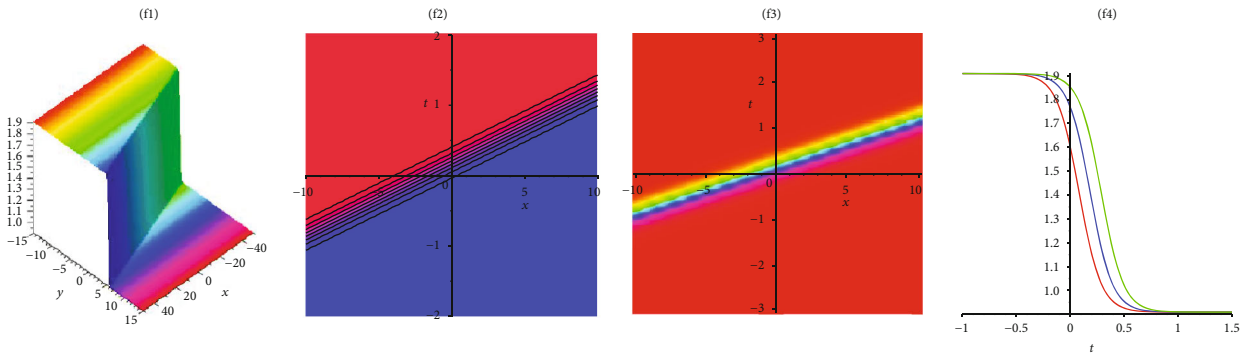

FIGURE 7: The one-wave solution (39) at $\alpha_1 = 1, \sigma_1 = 2, \rho_1 = 2.2, \rho_2 = 3, \rho_1 = 2, \beta_1 = 1.5, \delta_1 = 1.5, \delta_2 = 2.5, \delta_3 = 0.5,$ and $y = 1$.

$$\Omega_2 = 1 + \sigma_1 e^{\alpha_1 x + \beta_1 y - \lambda_1 t} + \sigma_2 e^{\alpha_2 x + \beta_2 y - \lambda_2 t}$$
$$+ \sigma_1 \sigma_2 \sigma_{12} e^{(\alpha_1 + \alpha_2)x + (\beta_1 + \beta_2)y + (\gamma_1 - (\lambda_1 + \lambda_2)t},$$

$$\Delta_2 = \rho_1 e^{\alpha_1 x + \beta_1 y - \lambda_1 t} + \rho_2 e^{\alpha_2 x + \beta_2 y - \lambda_2 t}$$
$$+ \rho_1 \rho_2 \rho_{12} e^{(\alpha_1 + \alpha_2)x + (\beta_1 + \beta_2)y - (\lambda_1 + \lambda_2)t}. \tag{41}$$

Inserting (40) along with (41) into equation (3), we obtain the following case:

$$\sigma_i = \sigma_i,$$
$$\lambda_i = \alpha_i^3 \delta_3 + \alpha_i^2 \beta_i + \alpha_i \delta_2 + \beta_i \delta_1,$$
$$\rho_i = 2\alpha_i \sigma_i,$$
$$i = 1, 2, \tag{42}$$

$$\rho_{12} = \frac{3\,\alpha_1 \alpha_2 \delta_3 (\alpha_1 - \alpha_2)^2 + (\alpha_1 - \alpha_2)(\alpha_1^2 \beta_2 + 2\alpha_1 \alpha_2 \beta_1 - 2\alpha_1 \alpha_2 \beta_2 - \alpha_2^2 \beta_1)}{\alpha_2 \alpha_1 (3\alpha_1 \alpha_2 \delta_3 (\alpha_1 + \alpha_2) + 2\alpha_1 \alpha_2 (\beta_1 + \beta_2) + \alpha_1^2 \beta_2 + \alpha_2^2 \beta_1)}.$$

$$\sigma_{12} = \frac{3\,\alpha_1 \alpha_2 \delta_3 (\alpha_1 - \alpha_2)^2 + (\alpha_1 - \alpha_2)(\alpha_1^2 \beta_2 + 2\,\alpha_1 \alpha_2 \beta_1 - 2\,\alpha_1 \alpha_2 \beta_2 - \alpha_2^2 \beta_1)}{3\,\alpha_1 \alpha_2 \delta_3 (\alpha_1 + \alpha_2)^2 + (\alpha_1 + \alpha_2)(\alpha_1^2 \beta_2 + 2\,\alpha_1 \alpha_2 \beta_1 + 2\,\alpha_1 \alpha_2 \beta_2 + \alpha_2^2 \beta_1)}. \tag{43}$$

Therefore, the resulting two-wave solution reads as

$$\Psi_2(x, y, t) = \frac{\sigma_1 e^{\Lambda_1} + \sigma_2 e^{\Lambda_2} + \sigma_1 \sigma_2 \sigma_{12} e^{\Lambda_1 + \Lambda_2}}{\rho_1 e^{\Lambda_1} + \rho_2 e^{\Lambda_2} + 4\alpha_1 \alpha_2 \sigma_1 \sigma_2 \rho_{12} e^{\Lambda_1 + \Lambda_2}}, \tag{44}$$

in which $\Lambda_i = \alpha_i x + \beta_i y - (\alpha_i^3 \delta_3 + \alpha_i^2 \beta_i + \alpha_i \delta_2 + \beta_i \delta_1)t, i = 1, 2$. By choosing the suitable values of parameters, the graphic representation of the periodic wave solution is presented in Figure 8 including the 3D plot, the contour plot,
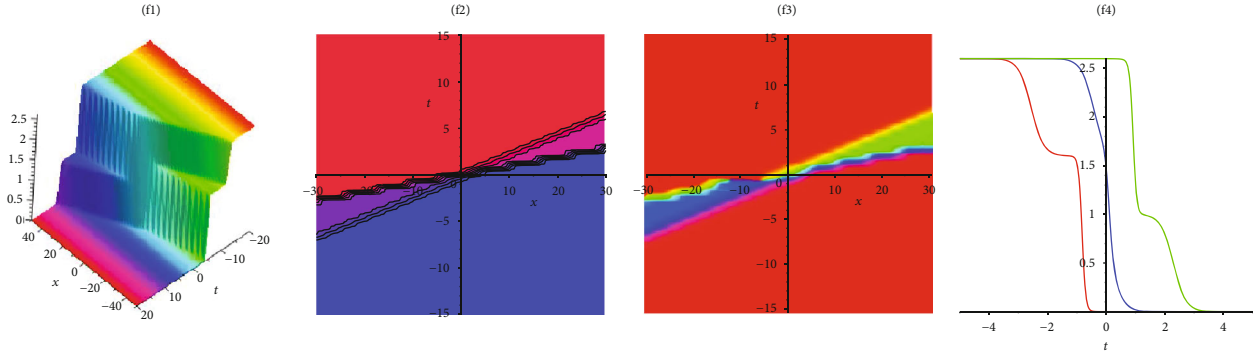
FIGURE 8: The two-wave solution (44) at $\alpha_1 = 1, \alpha_2 = 1.6, \sigma_2 = 1.5, \rho = 1.5, \rho_1 = 2, \rho_2 = 3, \beta_1 = 2.5, \beta_2 = 3.4, \gamma_1 = 0.75, \gamma_2 = 1.5, a(t) = \sin(t),$ $b(t) = \cos(t), y = 1,$ and $z = 2$.

the density plot, and the 2D plot when three spaces arise at spaces $x = -1$, $x = 0$, and $x = 1$.

5.3. Set III: Triple-Wave Solutions. We commence with three-wave functions based on the statement in Step 2 in the previous section; we suppose that equation (3) has the rational function of triple-wave solutions as shown in the following form:

$$\Psi(x, y, z, t) = \frac{\Delta_3}{\Omega_3}, \tag{45}$$

$$\Omega_3 = 1 + \rho_1 e^{\Lambda_1} + \rho_2 e^{\Lambda_2} + \rho_3 e^{\Lambda_3} + \rho_1 \rho_2 \rho_{12} e^{\Lambda_1 + \Lambda_2}$$
$$+ \rho_1 \rho_3 \rho_{13} e^{\Lambda_1 + \Lambda_3} + \rho_2 \rho_3 \rho_{23} e^{\Lambda_2 + \Lambda_3}$$
$$+ \rho_1 \rho_2 \rho_3 \rho_{12} \rho_{13} \rho_{23} e^{\Lambda_1 + \Lambda_2 + \Lambda_3},$$

$$\Delta_3 = \sigma_1 e^{\Lambda_1} + \sigma_2 e^{\Lambda_2} + \sigma_3 e^{\Lambda_3} + \sigma_1 \sigma_2 \sigma_{12} e^{\Lambda_1 + \Lambda_2}$$
$$+ \sigma_1 \sigma_3 \sigma_{13} e^{\Lambda_1 + \Lambda_3} + \sigma_2 \sigma_3 \sigma_{23} e^{\Lambda_2 + \Lambda_3}$$
$$+ \sigma_1 \sigma_2 \sigma_3 \sigma_{12} \sigma_{13} \sigma_{23} e^{\Lambda_1 + \Lambda_2 + \Lambda_3}, \tag{46}$$

in which $\Lambda_i = \alpha_i x + \beta_i y - \lambda_i t, i = 1, 2, 3$. Plugging (45) along with (46) into equation (3), we obtain the following case:

$$\sigma_i = \sigma_i,$$

$$\lambda_i = \alpha_i^3 \delta_3 + \alpha_i^2 \beta_i + \alpha_i \delta_2 + \beta_i \delta_1,$$

$$\rho_i = 2\alpha_i \sigma_i,$$

$$i = 1, 2, 3,$$

$$\rho_{12} = \frac{3\alpha_1 \alpha_2 \delta_3 (\alpha_1 - \alpha_2)^2 + (\alpha_1 - \alpha_2)(\alpha_1^2 \beta_2 + 2\alpha_1 \alpha_2 \beta_1 - 2\alpha_1 \alpha_2 \beta_2 - \alpha_2^2 \beta_1)}{\alpha_2 \alpha_1 (3\alpha_1 \alpha_2 \delta_3 (\alpha_1 + \alpha_2) + 2\alpha_1 \alpha_2 (\beta_1 + \beta_2) + \alpha_1^2 \beta_2 + \alpha_2^2 \beta_1)},$$

$$\sigma_{12} = \frac{3\alpha_1 \alpha_2 \delta_3 (\alpha_1 - \alpha_2)^2 + (\alpha_1 - \alpha_2)(\alpha_1^2 \beta_2 + 2\alpha_1 \alpha_2 \beta_1 - 2\alpha_1 \alpha_2 \beta_2 - \alpha_2^2 \beta_1)}{3\alpha_1 \alpha_2 \delta_3 (\alpha_1 + \alpha_2)^2 + (\alpha_1 + \alpha_2)(\alpha_1^2 \beta_2 + 2\alpha_1 \alpha_2 \beta_1 + 2\alpha_1 \alpha_2 \beta_2 + \alpha_2^2 \beta_1)},$$

$$\rho_{13} = \text{substitute } \{\alpha_2 = \alpha_3, \beta_2 = \beta_3\}, \text{ at } \rho_{12},$$

$$\sigma_{13} = \text{substitute } \{\alpha_2 = \alpha_3, \beta_2 = \beta_3\}, \text{ at } \sigma_{12},$$

$$\rho_{23} = \text{substitute } \{\alpha_1 = \alpha_2, \beta_1 = \beta_2\}, \text{ at } \rho_{13},$$

$$\sigma_{23} = \text{substitute } \{\alpha_1 = \alpha_2, \beta_1 = \beta_2\}, \text{ at } \sigma_{13}. \tag{47}$$

Therefore, the resulting three-wave solution reads as

$$\Psi_3(x, y, z, t) = \frac{\Delta_3}{\Omega_3}. \tag{48}$$

In (48), Δ_3 and Ω_3 with their relations are given in (46) and (47). By selecting the suitable values of parameters, the graphic representation of the periodic wave solution is presented in Figure 9 containing the 3D plot, the contour plot, the density plot, and the 2D plot when three spaces arise at spaces $x = -1$, $x = 0$, and $x = 1$.

6. Multiple Soliton Solutions for the Generalized $(2 + 1)$ BK Equation

6.1. Set I: One-Wave Solution. We start with one-wave function based on the statement in Step 2 in the previous section; we suppose that equation (4) has the rational function of the one-wave solution as shown in the following form:

$$\Psi(x, y, t) = \frac{\Delta_1}{\Omega_1}, \quad \Omega_1 = \rho_1 + \rho_2 e^{\alpha_1 x + \beta_1 y - \lambda_1 t},$$
$$\Delta_1 = \sigma_1 + \sigma_2 e^{\alpha_1 x + \beta_1 y - \lambda_1 t}, \tag{49}$$

By selecting the suitable values of parameters, the graphic representation of the periodic wave solution is presented in Figure 10 containing the 3D plot, the contour plot, the density plot, and the 2D plot when three spaces arise at spaces $x = -10$, $x = 0$, and $x = 10$.

6.2. Set II: Two-Wave Solutions. We begin with two-wave functions based on the statement in Step 2 in the previous section; we suppose that equation (4) has the rational function of two-wave solutions as shown in the following form:

in which σ_1 and σ_2 are unfound constants. Putting (49) into equation (4), we get to the following case:

$$\rho_1 = \rho_1,$$
$$\rho_2 = \rho_2,$$
$$\sigma_1 = \sigma_1,$$
$$\sigma_2 = \frac{\rho_2(2\,\alpha_1\rho_1 + \sigma_1)}{\rho_1}, \tag{50}$$
$$\lambda_1 = \frac{\alpha\alpha_1^4 + \beta\,\alpha_1^3\beta_1 + \alpha_1^2\delta_1 + \alpha_1\beta_1\delta_2 + \beta_1^2\delta_3}{\alpha_1}.$$

Therefore, the resulting one-wave solution reads as

$$\Psi(x, y, t) = \frac{\sigma_1 + (\rho_2(2\alpha_1\rho_1 + \sigma_1)/\rho_1)e^{\alpha_1 x + \beta_1 y - \left(\alpha\alpha_1^4 + \beta\alpha_1^3\beta_1 + \alpha_1^2\delta_1 + \alpha_1\beta_1\delta_2 + \beta_1^2\delta_3/\alpha_1\right)t}}{\rho_1 + \rho_2 e^{\alpha_1 x + \beta_1 y - \left(\alpha\alpha_1^4 + \beta\alpha_1^3\beta_1 + \alpha_1^2\delta_1 + \alpha_1\beta_1\delta_2 + \beta_1^2\delta_3/\alpha_1\right)t}}. \tag{51}$$

$$\Psi(x, y, z, t) = \frac{\Delta_2}{\Omega_2}, \tag{52}$$

$$\Omega_2 = 1 + \sigma_1 e^{\alpha_1 x + \beta_1 y - \lambda_1 t} + \sigma_2 e^{\alpha_2 x + \beta_2 y - \lambda_2 t}$$
$$+ \sigma_1\sigma_2\sigma_{12}e^{(\alpha_1 + \alpha_2)x + (\beta_1 + \beta_2)y + (\gamma_1 - (\lambda_1 + \lambda_2))t},$$
$$\Delta_2 = \rho_1 e^{\alpha_1 x + \beta_1 y - \lambda_1 t} + \rho_2 e^{\alpha_2 x + \beta_2 y - \lambda_2 t}$$
$$+ \rho_1\rho_2\rho_{12}e^{(\alpha_1 + \alpha_2)x + (\beta_1 + \beta_2)y - (\lambda_1 + \lambda_2)t}. \tag{53}$$

Inserting (52) along with (53) into equation (4), we obtain the following case:

$$\sigma_i = \sigma_i,$$
$$\lambda_i = \frac{\alpha\alpha_i^4 + \beta\,\alpha_i^3\beta_i + \alpha_i^2\delta_1 + \alpha_i\beta_i\delta_2 + \beta_i^2\delta_3}{\alpha_i},$$
$$\rho_i = 2\alpha_i\sigma_i,$$
$$i = 1, 2,$$
$$\rho_{12} = \frac{3\alpha\alpha_1^2\alpha_2^2(\alpha_2 + \alpha_1)(\alpha_1 - \alpha_2)^2 + (\alpha_2 + \alpha_1)\left(\beta\alpha_1\alpha_2(\alpha_1 - \alpha_2)(\alpha_1^2\beta_2 + 2\alpha_1\alpha_2\beta_1 - 2\alpha_1\alpha_2\beta_2 - \alpha_2^2\beta_1) - \delta_3(\alpha_1\beta_2 - \alpha_2\beta_1)^2\right)}{2\alpha_1\alpha_2(\alpha_1\alpha_2(\alpha_2 + \alpha_1)(3\alpha\alpha_1\alpha_2(\alpha_2 + \alpha_1) + \beta\left(\alpha_1^2\beta_2 + 2\alpha_1\alpha_2\beta_1 + 2\alpha_1\alpha_2\beta_2 + \alpha_2^2\beta_1\right)) - \delta_3(\alpha_1\beta_2 - \alpha_2\beta_1)^2},$$
$$\sigma_{12} = \frac{\alpha_1\alpha_2(-\alpha_2 + \alpha_1)\left(3\,\alpha\alpha_1\alpha_2(-\alpha_2 + \alpha_1) + \beta(\alpha_1^2\beta_2 + 2\alpha_1\alpha_2\beta_1 - 2\alpha_1\alpha_2\beta_2 - \alpha_2^2\beta_1)\right) - \delta_3(\alpha_1\beta_2 - \alpha_2\beta_1)^2}{\alpha_1\alpha_2(\alpha_2 + \alpha_1)\left(3\alpha\alpha_1\alpha_2(\alpha_2 + \alpha_1) + \beta(\alpha_1^2\beta_2 + 2\alpha_1\alpha_2\beta_1 + 2\alpha_1\alpha_2\beta_2 + \alpha_2^2\beta_1)\right) - \delta_3(\alpha_1\beta_2 - \alpha_2\beta_1)^2}. \tag{54}$$
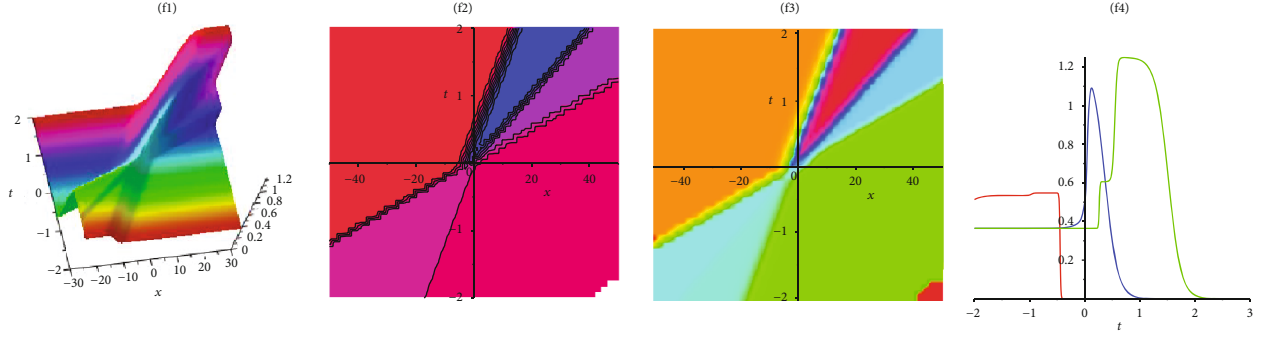
FIGURE 9: The three-wave solution (48) at $\alpha_1 = 1$, $\alpha_2 = 2.3$, $\alpha_3 = 3.6$, $\rho_1 = 2$, $\rho_2 = 2.3$, $\rho_3 = 4.2$, $\sigma_1 = 0.5$, $\sigma_2 = 1.6$, $\sigma_3 = 2.3$, $\delta_1 = 1$, $\delta_2 = 0.5$, $\delta_3 = 2.2$, $\beta_1 = 1.5$, $\beta_2 = 3.3$, and $\beta_3 = 5.4$.

Therefore, the resulting two-wave solution reads as

$$\Psi_2(x, y, t) = \frac{\sigma_1 e^{\Lambda_1} + \sigma_2 e^{\Lambda_2} + \sigma_1 \sigma_2 \sigma_{12} e^{\Lambda_1 + \Lambda_2}}{\rho_1 e^{\Lambda_1} + \rho_2 e^{\Lambda_2} + 4\alpha_1 \alpha_2 \sigma_1 \sigma_2 \rho_{12} e^{\Lambda_1 + \Lambda_2}}, \quad (55)$$

in which $\Lambda_i = \alpha_i x + \beta_i y - (\alpha\alpha_i^4 + \beta\alpha_i^3\beta_i + \alpha_i^2\delta_1 + \alpha_i\beta_i\delta_2 + \beta_i^2 \delta_3/\alpha_i)t$, $i = 1, 2$. By choosing suitable values of parameters, the analytical treatment of the periodic wave solution is presented in Figure 11 including the 3D plot, the contour plot, the density plot, and the 2D plot when three spaces arise at spaces $x = -10$, $x = 0$, and $x = 10$.

6.3. Set III: Triple-Wave Solutions.
We commence with three-wave functions based on the statement in Step 2 in the previous section; we suppose that equation (4) has the rational function of triple-wave solutions as shown in the following form:

$$\Psi(x, y, z, t) = \frac{\Delta_3}{\Omega_3}, \quad (56)$$

$$\Omega_3 = 1 + \rho_1 e^{\Lambda_1} + \rho_2 e^{\Lambda_2} + \rho_3 e^{\Lambda_3} + \rho_1 \rho_2 \rho_{12} e^{\Lambda_1 + \Lambda_2}$$
$$+ \rho_1 \rho_3 \rho_{13} e^{\Lambda_1 + \Lambda_3} + \rho_2 \rho_3 \rho_{23} e^{\Lambda_2 + \Lambda_3}$$
$$+ \rho_1 \rho_2 \rho_3 \rho_{12} \rho_{13} \rho_{23} e^{\Lambda_1 + \Lambda_2 + \Lambda_3},$$

$$\Delta_3 = \sigma_1 e^{\Lambda_1} + \sigma_2 e^{\Lambda_2} + \sigma_3 e^{\Lambda_3} + \sigma_1 \sigma_2 \sigma_{12} e^{\Lambda_1 + \Lambda_2}$$
$$+ \sigma_1 \sigma_3 \sigma_{13} e^{\Lambda_1 + \Lambda_3} + \sigma_2 \sigma_3 \sigma_{23} e^{\Lambda_2 + \Lambda_3}$$
$$+ \sigma_1 \sigma_2 \sigma_3 \sigma_{12} \sigma_{13} \sigma_{23} e^{\Lambda_1 + \Lambda_2 + \Lambda_3}, \quad (57)$$

in which $\Lambda_i = \alpha_i x + \beta_i y - \lambda_i t$, $i = 1, 2, 3$. Plugging (56) along with (57) into equation (4), we obtain the following case:

$$\sigma_i = \sigma_i,$$

$$\lambda_i = \frac{\alpha\alpha_i^4 + \beta\alpha_i^3\beta_i + \alpha_i^2\delta_1 + \alpha_i\beta_i\delta_2 + \beta_i^2\delta_3}{\alpha_i},$$

$$\rho_i = 2\alpha_i\sigma_i,$$

$$i = 1, 2, 3,$$

$$\rho_{12} = \frac{3\alpha\alpha_1^2\alpha_2^2(\alpha_2 + \alpha_1)(\alpha_1 - \alpha_2)^2 + (\alpha_2 + \alpha_1)(\beta\alpha_1\alpha_2(\alpha_1 - \alpha_2)(\alpha_1^2\beta_2 + 2\alpha_1\alpha_2\beta_1 - 2\alpha_1\alpha_2\beta_2 - \alpha_2^2\beta_1)) - \delta_3(\alpha_1\beta_2 - \alpha_2\beta_1)^2}{2\alpha_1\alpha_2(\alpha_1\alpha_2)(\alpha_2 + \alpha_1(3\alpha\alpha_1\alpha_2(\alpha_2 + \alpha_1) + \beta(\alpha_1^2\beta_2 + 2\alpha_1\alpha_2\beta_1 + 2\alpha_1\alpha_2\beta_2 + \alpha_2^2\beta_1)) - \delta_3(\alpha_1\beta_2 - \alpha_2\beta_1)^2)},$$

$$\sigma_{12} = \frac{\alpha_1\alpha_2(-\alpha_2 + \alpha_1)(3\,\alpha\alpha_1\alpha_2(-\alpha_2 + \alpha_1) + \beta(\alpha_1^2\beta_2 + 2\alpha_1\alpha_2\beta_1 - 2\alpha_1\alpha_2\beta_2 - \alpha_2^2\beta_1)) - \delta_3(\alpha_1\beta_2 - \alpha_2\beta_1)^2}{\alpha_1\alpha_2(\alpha_2 + \alpha_1)(3\alpha\alpha_1\alpha_2(\alpha_2 + \alpha_1) + \beta(\alpha_1^2\beta_2 + 2\alpha_1\alpha_2\beta_1 + 2\alpha_1\alpha_2\beta_2 + \alpha_2^2\beta_1)) - \delta_3(\alpha_1\beta_2 - \alpha_2\beta_1)^2},$$

$$\rho_{13} = \text{substitute } \{\alpha_2 = \alpha_3, \beta_2 = \beta_3\}, \text{ at } \rho_{12},$$

$$\sigma_{13} = \text{substitute } \{\alpha_2 = \alpha_3, \beta_2 = \beta_3\}, \text{ at } \sigma_{12},$$

$$\rho_{23} = \text{substitute } \{\alpha_1 = \alpha_2, \beta_1 = \beta_2\}, \text{ at } \rho_{13},$$

$$\sigma_{23} = \text{substitute } \{\alpha_1 = \alpha_2, \beta_1 = \beta_2\}, \text{ at } \sigma_{13}.$$
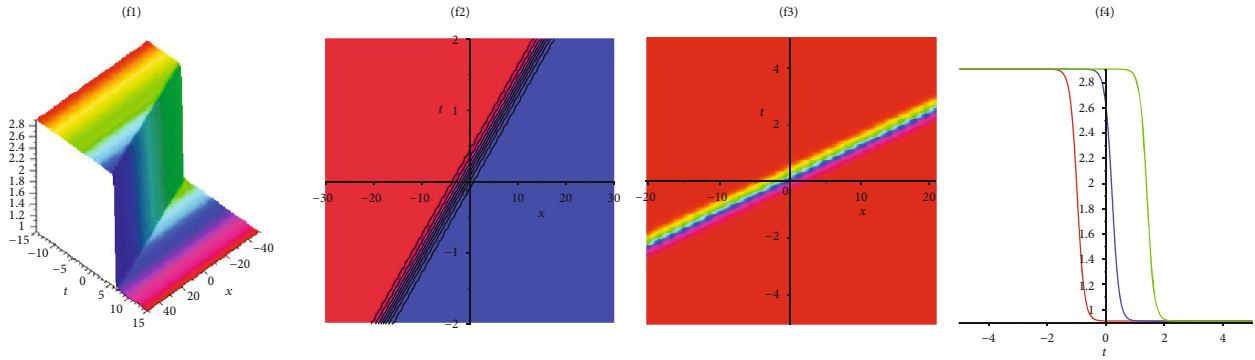
$$(58)$$

FIGURE 10: The one-wave solution (51) at $\alpha = 0.5, \beta = 0.1, \alpha_1 = 1, \beta_1 = 1.5, \sigma_1 = 2, \text{rho}_1 = 2.2, \text{rho}_2 = 3, \delta_1 = 1, \delta_2 = 1.5, \delta_3 = 2, \delta_1 = 1, \delta_2 = 1.5, \delta_3 = 2,$ and $y = 1.$
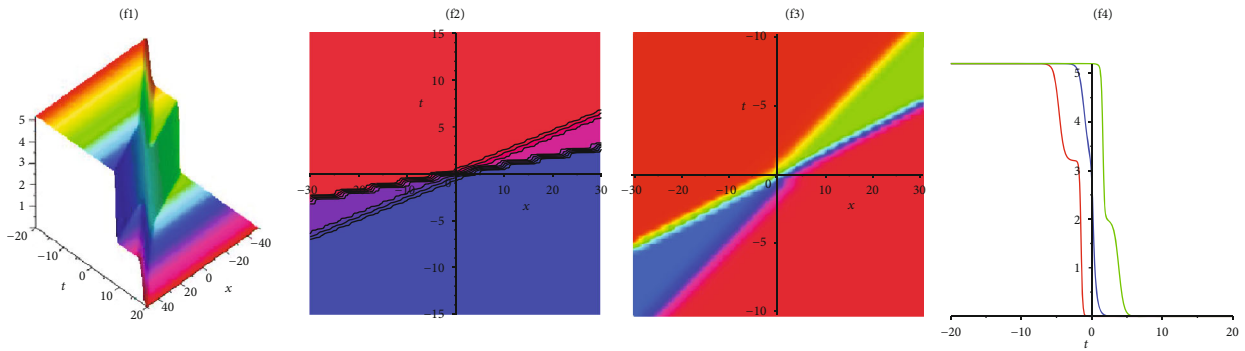

FIGURE 11: The two-wave solution (55) at $\alpha = 0.5, \beta = 0.1, \alpha_1 = 1, \alpha_2 = 1.6, \text{rho}_1 = 2, \text{rho}_2 = 2.3, \sigma_1 = 1.3, \sigma_2 = 2, \beta_1 = 0.5, \beta_2 = 1.5, \delta_1 = 0.1, \delta_2 = 0.5, \delta_3 = 0.2,$ and $y = 1.$
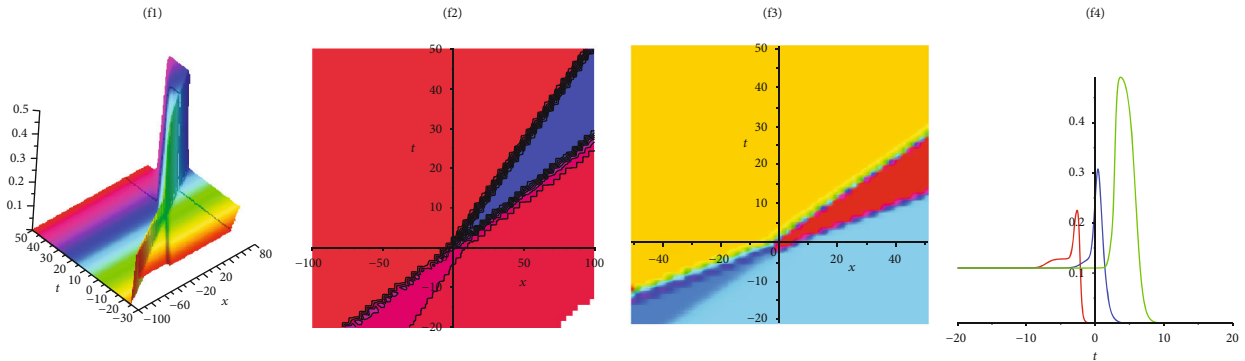

FIGURE 12: The three-wave solution (59) at alpha $= 0.5$, beta $= 0.1, \alpha_1 = 1, \alpha_2 = 2.3, \alpha_3 = 3.6, \rho_1 = 2, \rho_2 = 2.3, \rho_3 = 4.2, \sigma_1 = 0.5, \sigma_2 = 1.6, \sigma_3 = 2.3, \delta_1 = 1, \delta_2 = 0.5, \delta_3 = 2.2, \beta_1 = 1.5, \beta_2 = 3.3,$ and $\beta_3 = 5.4.$

Therefore, the resulting three-wave solution reads as

$$\Psi_3(x, y, z, t) = \frac{\Delta_3}{\Omega_3}, \tag{59}$$

in which Δ_3 and Ω_3 with their relations are given in (57) and (58). By selecting the suitable values of parameters, the graphic representation of the periodic wave solution is presented in Figure 12 containing the 3D plot, the contour plot, the density plot, and the 2D plot when three spaces arise at spaces $x = -10, x = 0,$ and $x = 10.$

7. Multiple Soliton Solutions for the $(2 + 1)$ VC DJKM Equation

7.1. Set I: One-Wave Solution. We start with one-wave function based on the statement in Step 2 in the previous section; we suppose that equation (6) has the rational function of one-wave solutions as shown in the following form:

$$\Psi(x, y, t) = \frac{\Delta_1}{\Omega_1}, \quad \Omega_1 = \rho_1 + \rho_2 e^{\alpha_1 x + \beta_1 y - \lambda_1 t},$$

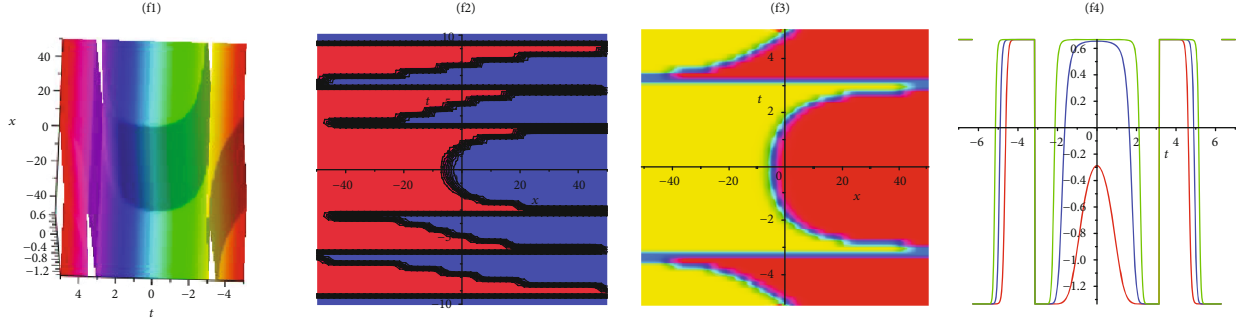$$\Delta_1 = \sigma_1 + \sigma_2 e^{\alpha_1 x + \beta_1 y - \lambda_1 t}, \tag{60}$$

FIGURE 13: The one-wave solution (62) at $\alpha = 0.5$, $\beta = 0.2$, $\alpha_1 = 1$, $\beta_1 = 1.5$, $\sigma_2 = 2$, $rho_1 = 2.2$, $\rho_2 = 3$, $g(t) = \sin(t)$, $h(t) = \cos(t)$, and $y = 1$.

in which ρ_1, ρ_2, σ_1 and σ_2 are unfound constants. Inserting (60) into equation (6), we gain the following case:

$$\rho_1 = \rho_1,$$

$$\rho_2 = \rho_2,$$

$$\sigma_1 = -\frac{\rho_1(2\alpha_1\rho_2 - \sigma_2)}{\rho_2},$$

$$\sigma_2 = \sigma_2,$$

$$\lambda_1 = \frac{\beta_1(-\alpha_1^4 + \alpha\beta_1^2 - h(t)\alpha_1^2)}{2\beta g(t)\alpha_1^2}. \tag{61}$$

Therefore, the resulting one-wave solution reads as

$$\Psi(x, y, t) = \frac{-(\rho_1(2\alpha_1\rho_2 - \sigma_2)/\rho_2) + \sigma_2 \, e^{\alpha_1 x + \beta_1 y - (\beta_1(-\alpha_1^4 + \alpha\beta_1^2 - h(t)\alpha_1^2)/2\beta \, g(t)\alpha_1^2)t}}{\rho_1 + \rho_2 \, e^{\alpha_1 x + \beta_1 y - (\beta_1(-\alpha_1^4 + \alpha\beta_1^2 - h(t)\alpha_1^2)/2\beta \, g(t)\alpha_1^2)t}}. \tag{62}$$

By choosing the suitable values of parameters, the graphic representation of the periodic wave solution is presented in Figure 13 containing the 3D plot, the contour plot, the density plot, and the 2D plot when three spaces arise at spaces $x = -5$, $x = 0$, and $x = 5$.

7.2. Set II: Two-Wave Solutions. We begin with two-wave functions based on the statement in Step 2 in the previous section; we suppose that equation (6) has the rational function of two-wave solutions as shown in the following form:

$$\Psi(x, y, z, t) = \frac{\Delta_2}{\Omega_2}, \tag{63}$$

$$\Omega_2 = 1 + \sigma_1 e^{\alpha_1 x + \beta_1 y - \lambda_1 t} + \sigma_2 e^{\alpha_2 x + \beta_2 y - \lambda_2 t}$$
$$+ \sigma_1 \sigma_2 \sigma_{12} e^{(\alpha_1 + \alpha_2)x + (\beta_1 + \beta_2)y + (\gamma_1 - (\lambda_1 + \lambda_2))t},$$
$$\Delta_2 = \rho_1 e^{\alpha_1 x + \beta_1 y - \lambda_1 t} + \rho_2 e^{\alpha_2 x + \beta_2 y - \lambda_2 t}$$
$$+ \rho_1 \rho_2 \rho_{12} e^{(\alpha_1 + \alpha_2)x + (\beta_1 + \beta_2)y - (\lambda_1 + \lambda_2)t}. \tag{64}$$

Inserting (63) along with (64) into equation (6), we obtain the following case:

$$\sigma_i = \sigma_i,$$
$$\lambda_i = \frac{\beta_i(-\alpha_i^4 + \alpha\beta_i^2 - h(t)\alpha_i^2)}{2\beta g(t)\alpha_i^2},$$
$$\rho_i = 2\alpha_i \sigma_i,$$
$$i = 1, 2. \tag{65}$$

$$\rho_{12} = \frac{1}{2}\frac{\alpha_1^2\alpha_2^2(\alpha_1 + \alpha_2)(-\alpha_2 + \alpha_1)^2 + \alpha(\alpha_1 + \alpha_2)(\alpha_1\beta_2 - \alpha_2\beta_1)^2}{\alpha_2\alpha_1(\alpha_1^2\alpha_2^2(\alpha_1 + \alpha_2)^2 + \alpha(\alpha_1\beta_2 - \alpha_2\beta_1)^2)},$$

$$\sigma_{12} = \frac{\alpha_1^2\alpha_2^2(-\alpha_2 + \alpha_1)^2 + \alpha(\alpha_1\beta_2 - \alpha_2\beta_1)^2}{\alpha_1^2\alpha_2^2(\alpha_1 + \alpha_2)^2 + \alpha(\alpha_1\beta_2 - \alpha_2\beta_1)^2}. \tag{66}$$

Therefore, the resulting two-wave solution reads as

$$\Psi_2(x, y, t) = \frac{\sigma_1 e^{\Lambda_1} + \sigma_2 e^{\Lambda_2} + \sigma_1\sigma_2\sigma_{12}e^{\Lambda_1 + \Lambda_2}}{\rho_1 e^{\Lambda_1} + \rho_2 e^{\Lambda_2} + 4\alpha_1\alpha_2\sigma_1\sigma_2\rho_{12}e^{\Lambda_1 + \Lambda_2}}, \tag{67}$$

in which $\Lambda_i = \alpha_i x + \beta_i y - (\beta_i(-\alpha_i^4 + \alpha\beta_i^2 - h(t)\alpha_i^2)/2\beta g(t)\alpha_i^2)t$, $i = 1, 2$. By choosing the suitable values of parameters, the graphic representation of the periodic wave solution is presented in Figure 14 containing the 3D plot, the contour plot,
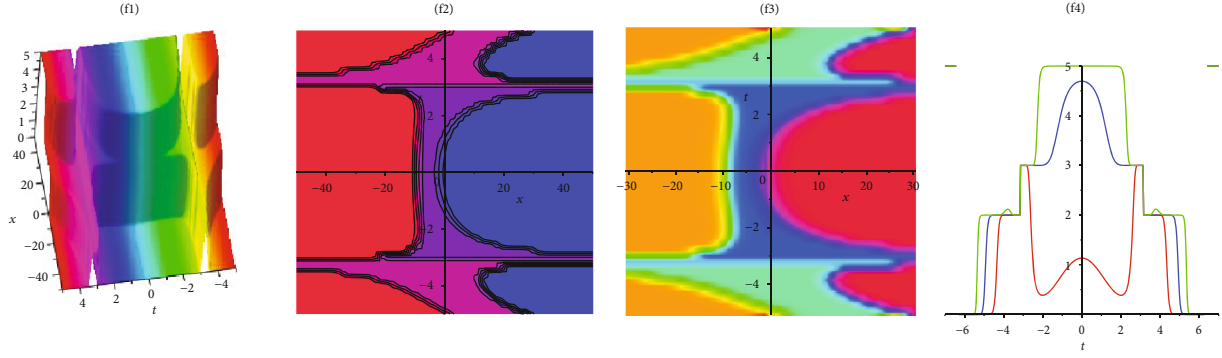
FIGURE 14: The two-wave solution (67) at $\alpha = 0.5, \beta = 0.2, \alpha_1 = 1, \alpha_2 = 1.5, \beta_1 = 1.5, \beta_2 = 2, \sigma_1 = 1.1, \sigma_2 = 2, \rho_1 = 2.2, \rho_2 = 3, g(t) = \sin(t), h(t) = \cos(t)$, and $y = 1$.
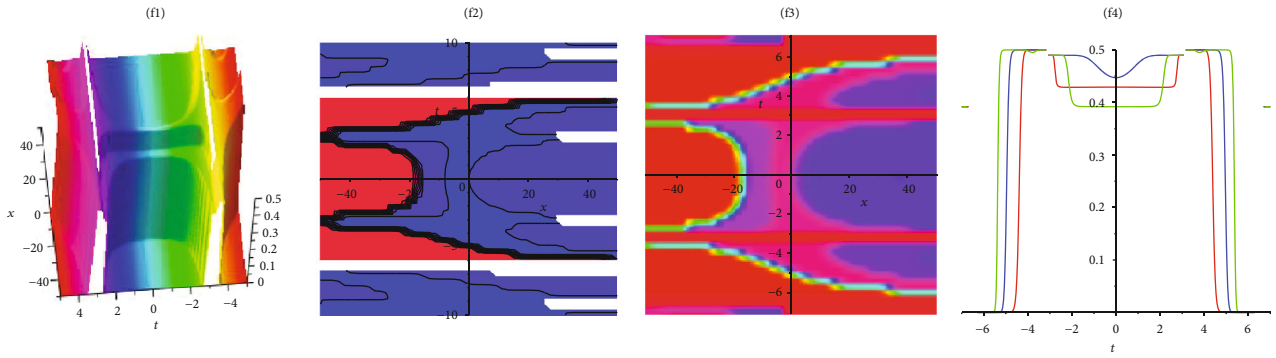

FIGURE 15: The three-wave solution (71) at $\alpha = 0.5, \beta = 0.2, \alpha_1 = 1, \alpha_2 = 1.5, \alpha_3 = 2, \beta_1 = 1.5, \beta_2 = 2, \beta_3 = 3.2, \sigma_1 = 1.1, \sigma_2 = 2, \sigma_3 = 1.5, \rho_1 = 2.2, \rho_2 = 3, g(t) = \sin(t), h(t) = \cos(t)$, and $y = 1$.

the density plot, and the 2D plot when three spaces arise at spaces $x = -10$, $x = 0$, and $x = 10$.

7.3. Set III: Triple-Wave Solutions.

We commence with three-wave functions based on the statement in Step 2 in the previous section; we suppose that equation (6) has the rational function of triple-wave solutions as shown in the following form:

$$\Psi(x, y, z, t) = \frac{\Delta_3}{\Omega_3}, \tag{68}$$

$$
\begin{aligned}
\Omega_3 = {} & 1 + \rho_1 e^{\Lambda_1} + \rho_2 e^{\Lambda_2} + \rho_3 e^{\Lambda_3} + \rho_1 \rho_2 \rho_{12} e^{\Lambda_1 + \Lambda_2} \\
& + \rho_1 \rho_3 \rho_{13} e^{\Lambda_1 + \Lambda_3} + \rho_2 \rho_3 \rho_{23} e^{\Lambda_2 + \Lambda_3} \\
& + \rho_1 \rho_2 \rho_3 \rho_{12} \rho_{13} \rho_{23} e^{\Lambda_1 + \Lambda_2 + \Lambda_3},
\end{aligned}
$$

$$
\begin{aligned}
\Delta_3 = {} & \sigma_1 e^{\Lambda_1} + \sigma_2 e^{\Lambda_2} + \sigma_3 e^{\Lambda_3} + \sigma_1 \sigma_2 \sigma_{12} e^{\Lambda_1 + \Lambda_2} \\
& + \sigma_1 \sigma_3 \sigma_{13} e^{\Lambda_1 + \Lambda_3} + \sigma_2 \sigma_3 \sigma_{23} e^{\Lambda_2 + \Lambda_3} \\
& + \sigma_1 \sigma_2 \sigma_3 \sigma_{12} \sigma_{13} \sigma_{23} e^{\Lambda_1 + \Lambda_2 + \Lambda_3},
\end{aligned} \tag{69}
$$

in which $\Lambda_i = \alpha_i x + \beta_i y - \lambda_i t, i = 1, 2, 3$. Plugging (68) along with (69) into equation (6), we obtain the following case:

$$\sigma_i = \sigma_i,$$

$$\lambda_i = \frac{\beta_i (-\alpha_i^4 + \alpha \beta_i^2 - h(t)\alpha_i^2)}{2\beta g(t)\alpha_i^2},$$

$$\rho_i = 2\alpha_i \sigma_i,$$

$$i = 1, 2, 3,$$

$$\rho_{12} = \frac{1}{2} \frac{\alpha_1^2 \alpha_2^2 (\alpha_1 + \alpha_2)(-\alpha_2 + \alpha_1)^2 + \alpha(\alpha_1 + \alpha_2)(\alpha_1 \beta_2 - \alpha_2 \beta_1)^2}{\alpha_2 \alpha_1 (\alpha_1^2 \alpha_2^2 (\alpha_1 + \alpha_2) + \alpha(\alpha_1 \beta_2 - \alpha_2 \beta_1)^2)},$$

$$\sigma_{12} = \frac{\alpha_1^2 \alpha_2^2 (-\alpha_2 + \alpha_1)^2 + \alpha(\alpha_1 \beta_2 - \alpha_2 \beta_1)^2}{\alpha_1^2 \alpha_2^2 (\alpha_1 + \alpha_2)^2 + \alpha(\alpha_1 \beta_2 - \alpha_2 \beta_1)^2},$$

$$\rho_{13} = \text{substitute } \{\alpha_2 = \alpha_3, \beta_2 = \beta_3\}, \text{ at } \rho_{12},$$

$$\sigma_{13} = \text{substitute } \{\alpha_2 = \alpha_3, \beta_2 = \beta_3\}, \text{ at } \sigma_{12},$$

$$\rho_{23} = \text{substitute } \{\alpha_1 = \alpha_2, \beta_1 = \beta_2\}, \text{ at } \rho_{13},$$

$$\sigma_{23} = \text{substitute } \{\alpha_1 = \alpha_2, \beta_1 = \beta_2\}, \text{ at } \sigma_{13}. \tag{70}$$

Therefore, the resulting three-wave solution reads as

$$\Psi_3(x, y, z, t) = \frac{\Delta_3}{\Omega_3}. \tag{71}$$

In (71), Δ_3 and Ω_3 with their relations are given in (69) and (70). By selecting the suitable values of parameters, the

graphic representation of the periodic wave solution is presented in Figure 15 containing the 3D plot, the contour plot, the density plot, and the 2D plot when three spaces arise at spaces $x = -10$, $x = 0$, and $x = 10$.

This paper finds many novel one-soliton-, two-soliton-, and triple-soliton-type solutions to governing models. With the help of some computations, surfaces of results reported have been observed in Figures 1–15. These figures are dependent on the family conditions which are of importance physically. It has been investigated that all figures plotted have been symbolized for the four types of the nonlinear PDEs containing the extended $(3 + 1)$-dimensional Jimbo-Miwa-like equation, the extended $(2 + 1)$-dimensional Calogero-Bogoyavlenskii-Schiff equation, the generalization of the $(2 + 1)$-dimensional Bogoyavlensky-Konopelchenko equation, and a variable-coefficient extension of the DJKM equation. These mathematical properties come from exponential function properties. In this sense, from the mathematical and physical points of view, these results play an important role in explaining wave propagation of nonlinear phenomena. Hence, we consider that surfaces plotted in this paper have proven such physical meaning of the obtained solutions.

8. Conclusion

In this article, we obtained the multiple soliton solutions of the novel extended $(3 + 1)$-dimensional Jimbo-Miwa-like equation, the extended $(2 + 1)$-dimensional Calogero-Bogoyavlenskii-Schiff equation, the generalization of the $(2 + 1)$-dimensional Bogoyavlensky-Konopelchenko equation, and a variable-coefficient extension of the DJKM equation via operating the multiple Exp-function method, including one-soliton-, two-soliton-, and triple-soliton-type solutions. It is quite visible that this novel scheme has plenty of family of solutions containing rational exponential functions by selecting particular parameters. Thus, this paper provides a lot of encouragement for future research in soliton topics. The behaviours of the solutions for the known nonlinear equations obtained by the multiple Exp-function method by choosing the suitable values are cited in Figures 1–15. Moreover, the numerical simulations include gaining the coefficients ρ_i, σ_i, and λ_i and have been carried out to show that the projected algorithm is applicable and efficient. The analytical study has been conducted for the solutions $\Psi(x, y, t)$ obtained by employing the aforementioned method, in which obtained solutions were as one-soliton, two-soliton, and triple-soliton. The results are beneficial to the study of wave propagation. All computations in this paper have been made quickly with the aid of Maple.

Data Availability

The datasets supporting the conclusions of this article are included within the article and its additional file.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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