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Research Article

The Existence of n Periodic Solutions on One Element n-Degree Polynomial Differential Equation

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This paper deals with a class of one element n-degree polynomial differential equations. By the fixed point theory, we obtain n periodic solutions of the equation. This paper generalizes some related conclusions of some papers.

1. Introduction

Consider the following one element n-degree polynomial differential equation:

$$\frac{dx}{dt} = \sum_{i=0}^{n} a_i(t)x^i, \quad (n \in N^+), \tag{1}$$

where $a_i(t)\ (i = 0, 1, 2, \cdots, n)$ is the ω-periodic continuous functions on R. When $n = 1$, equation (1) is a linear differential equation. With regard to the periodic solution of the equation, we propose the following:

Proposition 1 (see [1]). *Consider the following:*

$$\frac{dx}{dt} = a_1(t)x + a_0(t), \tag{2}$$

where $a_1(t)$ and $a_0(t)$ are ω-periodic continuous functions on R; if $\int_0^\omega a_1(t)dt \neq 0$, then equation (2) has a unique ω-periodic continuous solution $\eta(t)$, mod $(\eta) \subseteq$ mod $(a_1(t), a_0(t))$, and $\eta(t)$ can be written as follows:

$$\eta(t) = \begin{cases} \displaystyle\int_{-\infty}^{t} e^{\int_s^t a_1(\tau)d\tau} a_0(s)ds, & \displaystyle\int_0^\omega a_1(t)dt < 0, \\ \displaystyle -\int_{t}^{+\infty} e^{\int_s^t a_1(\tau)d\tau} a_0(s)ds, & \displaystyle\int_0^\omega a_1(t)dt > 0. \end{cases} \tag{3}$$

When $n = 2$, equation (1) is Riccati's equation. Riccati's equation plays an important role in fluid mechanics and in the theory of elastic vibration. There are many studies on this equation [2–9], and there is also a proposition about the periodic solutions of Riccati's equation, as follows:

Proposition 2 (see [2]). *Consider the following equation:*

$$\frac{dx}{dt} = a_2(t)x^2 + a_1(t)x + a_0(t), \tag{4}$$

where $a_2(t)$, $a_1(t)$, and $a_0(t)$ are all ω-periodic continuous functions on R. Suppose that the following conditions hold:

$$\begin{aligned} &(H_1)\ a_2(t) \neq 0, \\ &(H_2)\ a_1^2(t) - 4a_2(t)a_0(t) > 0, \\ &(H_3) \sup_{t\in[0,\omega]} \left(-\frac{a_1(t)}{2a_2(t)} - \sqrt{\left(\frac{a_1(t)}{2a_2(t)}\right)^2 - \frac{a_0(t)}{a_2(t)}} \right) \\ &< \inf_{t\in[0,\omega]} \left(-\frac{a_1(t)}{2a_2(t)} + \sqrt{\left(\frac{a_1(t)}{2a_2(t)}\right)^2 - \frac{a_0(t)}{a_2(t)}} \right), \end{aligned} \tag{5}$$

then equation (4) has exactly two ω-periodic continuous solutions.

When $a_1(t) \equiv 0$, in [10], the author obtained the existence and more accurate range of two periodic solutions of equation (4) by means of the fixed point theorem.

It is easy for us to guess under what conditions is equation (1) satisfied, and are there existing n periodic solutions of equation (1)?

In this paper, we consider the n-degree polynomial differential equation for the special case of equation (1) as follows:

$$\frac{dx}{dt} = a(t)(x - \gamma_1(t)) \cdots (x - \gamma_i(t)) \cdots (x - \gamma_n(t)), \quad (6)$$

and we give a new criterion to judge the existence of n periodic solutions on equation (6); these conclusions generalize the relevant conclusions of References [1, 2, 10].

The rest of the paper is arranged as follows: In Section 2, some lemmas and abbreviations are introduced to be used later. In Section 3, the existence of n periodic solutions on equation (6) is obtained. We end this paper with a short conclusion.

2. Some Lemmas and Abbreviations

Lemma 3 (see [11]). *Suppose that an ω-periodic sequence $\{f_n(t)\}$ is convergent uniformly on any compact set of R, $f(t)$ is an ω-periodic function, and $\operatorname{mod}(f_n) \subseteq \operatorname{mod}(f)$ $(n = 1, 2, \cdots)$, then $\{f_n(t)\}$ is convergent uniformly on R.*

Lemma 4 (see [12]). *Suppose V is a metric space, C is a convex closed set of V, and its boundary is ∂C; if $T : V \longrightarrow V$ is a continuous compact mapping, such that $T(\partial C) \subseteq C$, then T has a fixed point on C.*

For the sake of convenience, suppose that $f(t)$ is an ω-periodic continuous function on R; we denote

$$f_M = \sup_{t\in[0,\omega]} f(t), \qquad f_L = \inf_{t\in[0,\omega]} f(t). \quad (7)$$

3. Periodic Solutions of the Polynomial Differential Equation

In this section, we discuss the existence of n periodic solutions of equation (6).

Theorem 5. *Consider equation (6), $a(t), \gamma_i(t)$ $(i = 1, 2, \cdots, n)$ are all ω-periodic continuous functions on R; suppose that the following conditions hold:*

$$(H_1)\ a(t) \neq 0, \quad \forall t \in [0, \omega],$$
$$(H_2)\ (\gamma_i)_M < (\gamma_{i+1})_L, \quad i = 1, 2, \cdots, n-1, \quad (8)$$

then equation (6) has exactly n ω-periodic continuous solutions $\Phi_i(t)$ $(i = 1, 2, \cdots, n)$, and

$$(\gamma_i)_L \le \Phi_i(t) \le (\gamma_i)_M, \quad i = 1, 2, \cdots, n. \quad (9)$$

Proof. By (H_1), it follows $a(t) > 0$ or $a(t) < 0$. In order to avoid repetition, we only prove the case of $a(t) > 0$. As the proof of the existence of every periodic solution is the same, for the sake of simplicity, we only prove the existence of the n-th periodic solution $\Phi_n(t)$ of equation (6).

Here, we will divide the proof into two steps.

(1) We prove the existence of n periodic solutions of equation (6). Suppose

$$S = \{\varphi(t) \in C(R, R) \mid \varphi(t + \omega) = \varphi(t)\}, \quad (10)$$

given any $\varphi(t), \psi(t) \in S$, the distance is defined as follows:

$$\rho(\varphi, \psi) = \sup_{t\in[0,\omega]} |\varphi(t) - \psi(t)|. \quad (11)$$

Thus, (S, ρ) is a complete metric space. Take a convex closed set B_n of S as follows:

$$B_n = \{\varphi(t) \in S \mid (\gamma_n)_L \le \varphi(t) \le (\gamma_n)_M, \quad \operatorname{mod}(\varphi) \subseteq \operatorname{mod}(a, \gamma_1, \cdots, \gamma_n)\}. \quad (12)$$

Given any $\varphi(t) \in B_n$, consider the following:

$$\frac{dx}{dt} = a(t)(\varphi(t) - \gamma_1(t)) \cdots (\varphi(t) - \gamma_i(t)) \cdots (x - \gamma_n(t)) = f(t)(x - \gamma_n(t)) = f(t)x - f(t)\gamma_n(t). \quad (13)$$

Here

$$f(t) = a(t)(\varphi(t) - \gamma_1(t)) \cdots (\varphi(t) - \gamma_i(t)) \cdots (\varphi(t) - \gamma_{n-1}(t)). \quad (14)$$

By (H_1), (H_2), and equation (12), we get that

$$0 < a_L((\gamma_n)_L - (\gamma_1)_M) \cdots ((\gamma_n)_L - (\gamma_i)_M) \cdots ((\gamma_n)_L - (\gamma_{n-1})_M) \le f(t) \le a_M((\gamma_n)_M - (\gamma_1)_L) \cdots ((\gamma_n)_M - (\gamma_i)_L) \cdots ((\gamma_n)_M - (\gamma_{n-1})_L), \quad (15)$$

hence, we have

$$\int_0^\omega f(t)dt > 0, \quad (16)$$

and because $a(t), \gamma_i(t)$ $(i = 1, \cdots, n)$ are ω-periodic continuous functions on R, it follows that $f(t)$, $f(t)\gamma_n(t)$ are ω-periodic continuous functions on R, by equation (16). According to Proposition 1, equation

(13) has a unique ω-periodic continuous solution as follows:

$$\eta(t) = \int_t^{+\infty} e^{\int_s^t f(\tau)d\tau} f(s)\gamma_n(s)ds, \tag{17}$$

and

$$\text{mod}\,(\eta) \subseteq \text{mod}\,(f(t), f(t)\gamma_n(t)). \tag{18}$$

By equations (12) and (14), it follows that

$$\begin{aligned} \text{mod}\,(f(t)) &\subseteq \text{mod}\,(a, \gamma_1, \cdots, \gamma_n), \\ \text{mod}\,(f(t)\gamma_n(t)) &\subseteq \text{mod}\,(a, \gamma_1, \cdots, \gamma_n), \end{aligned} \tag{19}$$

hence we have

$$\text{mod}\,(\eta) \subseteq \text{mod}\,(a, \gamma_1, \cdots, \gamma_n). \tag{20}$$

By equations (12), (15), and (17), we get

$$\begin{aligned} \eta(t) &\geq (\gamma_n)_L \int_t^{+\infty} e^{\int_s^t f(\tau)d\tau} f(s)ds \\ &= -(\gamma_n)_L \int_t^{+\infty} e^{\int_s^t f(\tau)d\tau} d\left(\int_s^t f(\tau)d\tau\right) \\ &= -(\gamma_n)_L \left[e^{\int_s^t f(\tau)d\tau}\right]_t^{+\infty} \\ &= -(\gamma_n)_L \left[e^{\int_{+\infty}^t f(\tau)d\tau} - 1\right] = (\gamma_n)_L, \end{aligned} \tag{21}$$

and

$$\begin{aligned} \eta(t) &\leq (\gamma_n)_M \int_t^{+\infty} e^{\int_s^t f(\tau)d\tau} f(s)ds \\ &= -(\gamma_n)_M \int_t^{+\infty} e^{\int_s^t f(\tau)d\tau} d\left(\int_s^t f(\tau)d\tau\right) \\ &= -(\gamma_n)_M \left[e^{\int_s^t f(\tau)d\tau}\right]_t^{+\infty} \\ &= -(\gamma_n)_M \left[e^{\int_{+\infty}^t f(\tau)\,d\tau} - 1\right] = (\gamma_n)_M, \end{aligned} \tag{22}$$

hence, $\eta(t) \in B_n$.

Define a mapping as follows:

$$(T\varphi)(t) = \int_t^{+\infty} e^{\int_s^t f(\tau)d\tau} f(s)\gamma_n(s)ds. \tag{23}$$

Thus, if given any $\varphi(t) \in B_n$, then $(T\varphi)(t) \in B_n$, hence $T : B_n \longrightarrow B_n$.

Now, we prove that the mapping T is a compact mapping.

Consider any sequence $\{\varphi_k(t)\} \subseteq B_n (k = 1, 2, \cdots)$, then it follows that

$$\begin{gathered} (\gamma_n)_L \leq \varphi_k(t) \leq (\gamma_n)_M, \\ \text{mod}\,(\varphi_k) \subseteq \text{mod}\,(a, \gamma_1, \cdots, \gamma_n), \\ (k = 1, 2, \cdots), \end{gathered} \tag{24}$$

on the other hand, $(T\varphi_k)(t) = x_{\varphi_k}(t)$ satisfies

$$\begin{aligned} \frac{dx_{\varphi_k}(t)}{dt} = a(t)(\varphi_k(t) - \gamma_1(t)) \cdots (\varphi_k(t) \\ - \gamma_i(t)) \cdots \left(x_{\varphi_k}(t) - \gamma_n(t)\right). \end{aligned} \tag{25}$$

Thus, we have

$$\begin{aligned} \left|\frac{dx_{\varphi_k}(t)}{dt}\right| &\leq 2a_M\left((\gamma_n)_M - (\gamma_1)_L\right) \cdots \left((\gamma_n)_M \right.\\ &\quad \left. - (\gamma_i)_L\right) \cdots \left((\gamma_n)_M - (\gamma_{n-1})_L\right)|\gamma_n|_M, \\ \text{mod}\left(x_{\varphi_k}(t)\right) &\subseteq \text{mod}\,(a, \gamma_1, \cdots, \gamma_n), \end{aligned} \tag{26}$$

hence $\{(dx_{\varphi_k}(t))/dt\}$ is uniformly bounded; therefore, $\{x_{\varphi_k}(t)\}$ is uniformly bounded and equicontinuous on R, by the theorem of Ascoli-Arzela, for any sequence $\{x_{\varphi_k}(t)\} \subseteq B_n$, there exists a subsequence (also denoted by $\{x_{\varphi_k}(t)\}$) such that $\{x_{\varphi_k}(t)\}$ is convergent uniformly on any compact set of R, by equation (26), combined with Lemma 3, $\{x_{\varphi_k}(t)\}$ is convergent uniformly on R, that is to say, T is relatively compact on B_n.

Next, we prove that T is a continuous mapping.

Suppose $\{\varphi_k(t)\} \subseteq B_n$, $\varphi(t) \in B_n$, and

$$\varphi_k(t) \longrightarrow \varphi(t) \cdot (k \longrightarrow \infty). \tag{27}$$

Denote

$$\begin{aligned} f_k(t) = a(t)(\varphi_k(t) - \gamma_1(t)) \cdots (\varphi_k(t) \\ - \gamma_i(t)) \cdots (\varphi_k(t) - \gamma_{n-1}(t)), \end{aligned} \tag{28}$$

then we have

$$f_k(t) \longrightarrow f(t), \quad (k \longrightarrow \infty), \tag{29}$$

and

$$\begin{aligned} 0 &< a_L\left((\gamma_n)_L - (\gamma_1)_M\right) \cdots \left((\gamma_n)_L - (\gamma_i)_M\right) \cdots \left((\gamma_n)_L - (\gamma_{n-1})_M\right) \\ &\leq f_k(t) \\ &\leq a_M\left((\gamma_n)_M - (\gamma_1)_L\right) \cdots \left((\gamma_n)_M - (\gamma_i)_L\right) \cdots \left((\gamma_n)_M - (\gamma_{n-1})_L\right). \end{aligned} \tag{30}$$

By equation (23), we have

$$\begin{aligned}
&|(T\varphi_k)(t)-(T\varphi)(t)| \\
&= \left|\int_t^{+\infty} e^{\int_s^t f_k(\tau)d\tau} f_k(s)\gamma_n(s)ds - \int_t^{+\infty} e^{\int_s^t f(\tau)d\tau} f(s)\gamma_n(s)ds\right| \\
&= \left|\int_t^{+\infty} e^{\int_s^t f_k(\tau)d\tau}(f_k(s)-f(s))\gamma_n(s)ds \right.\\
&\quad \left.+ \int_t^{+\infty}\left(e^{\int_s^t f_k(\tau)d\tau} - e^{\int_s^t f(\tau)d\tau}\right) f(s)\gamma_n(s)ds\right| \\
&= \left|\int_t^{+\infty} e^{\int_s^t f_k(\tau)d\tau}(f_k(s)-f(s))\gamma_n(s)ds \right.\\
&\quad \left.+ \int_t^{+\infty}\left(e^{\xi}\int_s^t (f_k(\tau)-f(\tau))d\tau\right) f(s)\gamma_n(s)ds\right| \\
&\le \left(\int_t^{+\infty}\left|e^{\int_s^t f_k(\tau)d\tau}\gamma_n(s)\right|ds \right.\\
&\quad \left.+ \int_t^{+\infty}\left|\left(e^{\xi}\int_s^t d\tau\right) f(s)\gamma_n(s)\right|ds\right)\rho(f_k,f),
\end{aligned} \tag{31}$$

where ξ is between $\int_s^t f_k(\tau)d\tau$ and $\int_s^t f(\tau)d\tau$; thus, ξ is between

$$a_L\left((\gamma_n)_L-(\gamma_1)_M\right)\cdots\left((\gamma_n)_L-(\gamma_i)_M\right)\cdots\left((\gamma_n)_L - (\gamma_{n-1})_M\right)(t-s), \tag{32}$$

and

$$a_M\left((\gamma_n)_M-(\gamma_1)_L\right)\cdots\left((\gamma_n)_M-(\gamma_i)_L\right)\cdots\left((\gamma_n)_M - (\gamma_{n-1})_L\right)(t-s), \tag{33}$$

hence we have

$$\begin{aligned}
|(T\varphi_k)(t)-(T\varphi)(t)| &\le \left(\int_t^{+\infty} e^{a_L\left((\gamma_n)_L-(\gamma_1)_M\right)\cdots\left((\gamma_n)_L-(\gamma_i)_M\right)\cdots\left((\gamma_n)_L-(\gamma_{n-1})_M\right)(t-s)}|\gamma_n(s)|ds \right.\\
&\quad \left.+ \int_t^{+\infty} e^{a_L\left((\gamma_n)_L-(\gamma_1)_M\right)\cdots\left((\gamma_n)_L-(\gamma_i)_M\right)\cdots\left((\gamma_n)_L-(\gamma_{n-1})_M\right)(t-s)}(s-t)|f(s)\gamma_n(s)|ds\right)\rho(f_k,f) \\
&\le \left\{\frac{|\gamma_n|_M}{a_L\left((\gamma_n)_L-(\gamma_1)_M\right)\cdots\left((\gamma_n)_L-(\gamma_i)_M\right)\cdots\left((\gamma_n)_L-(\gamma_{n-1})_M\right)}\right. \\
&\quad \left.+ \frac{|\gamma_n|_M\left((\gamma_n)_M-(\gamma_1)_L\right)\cdots\left((\gamma_n)_M-(\gamma_i)_L\right)\cdots\left((\gamma_n)_M-(\gamma_{n-1})_L\right)}{\left[a_L\left((\gamma_n)_L-(\gamma_1)_M\right)\cdots\left((\gamma_n)_L-(\gamma_i)_M\right)\cdots\left((\gamma_n)_L-(\gamma_{n-1})_M\right)\right]^2}\right\}\rho(f_k,f).
\end{aligned} \tag{34}$$

By equation (29) and the above inequality, it follows that

$$(T\varphi_k)(t) \longrightarrow (T\varphi)(t)\ (k\longrightarrow\infty), \tag{35}$$

hence, T is continuous; therefore, $T: B_n \longrightarrow B_n$ is a continuous compact mapping, and by equation (23), it is easy to see, $T(\partial B_n)\subseteq B_n$; according to Lemma 4, T has a fixed point on B_n, and the fixed point is the ω-periodic continuous solution $\Phi_n(t)$ of equation (6), and

$$(\gamma_n)_L \le \Phi_n(t) \le (\gamma_n)_M. \tag{36}$$

Similarly, we can prove the existence of the periodic solutions $\Phi_i(t)\ (i=1,2,\cdots,n-1)$ of equation (6), and we have

$$(\gamma_i)_L \le \Phi_i(t) \le (\gamma_i)_M\ (i=1,2,\cdots\cdots,n-1). \tag{37}$$

(2) We prove that equation (6) has exactly n periodic solutions.

Let us discuss the possible range of $x(t)$ of equation (6); we divide the initial value $x(t_0)=x_0$ into the following parts:

$$\begin{gathered}
x_0 \in \left(-\infty,(\gamma_1)_L\right), \\
\left[(\gamma_1)_L,(\gamma_1)_M\right], \\
\left((\gamma_1)_M,(\gamma_2)_L\right), \\
\left[(\gamma_2)_L,(\gamma_2)_M\right], \\
\left((\gamma_2)_M,(\gamma_3)_L\right),\cdots,\left((\gamma_{n-1})_M,(\gamma_n)_L\right), \\
\left[(\gamma_n)_L,(\gamma_n)_M\right], \\
\left((\gamma_n)_M,+\infty\right).
\end{gathered} \tag{38}$$

We will only prove the following cases. For the sake of convenience, suppose n is an even number.

Remark 1. If n is an odd number, the proof is similar, and we omit it here.

Let

$$f(t,x)=a(t)(x-\gamma_1(t))\cdots(x-\gamma_i(t))\cdots(x-\gamma_n(t)). \quad (39)$$

(i) If $x_0 \in (-\infty,(\gamma_1)_L)$

Consider equation (6). We have $dx/dt|_{(t_0,x_0)}=f(t_0,x_0)>0$. Thus, $x(t)$ may stay at $(-\infty,(\gamma_1)_L)$ or enter into $[(\gamma_1)_L,(\gamma_1)_M]$ at some time t. If $x(t)$ stays at $(-\infty,(\gamma_1)_L)$, then $dx/dt=f(t,x)>0$. Thus, $x(t)$ cannot be a periodic solution of equation (6). If $x(t)$ enters into $[(\gamma_1)_L,(\gamma_1)_M]$ at some time t, then there is not a $t_1(t_1>t_0)$ such that $x(t_1)=x(t_0)=x_0$; thus $x(t)$ can also not be a periodic solution of equation (6).

(ii) If $x_0\in[(\gamma_1)_L,(\gamma_1)_M]$, then equation (6) has an ω-periodic continuous solution $x(t)=\Phi_1(t)$ with initial value $x(t_0)=\Phi_1(t_0)$

As $f(t,\gamma_1)=f(t,\gamma_2)=0$, by differential mean value theorem, it follows that

$$f_x{}'(t,\xi_1(t))=0,(\gamma_1(t)<\xi_1(t)<\gamma_2(t)). \quad (40)$$

By equation (39), we have

$$f_x{}'(t,\gamma_1(t))<0, \quad (41)$$

$$f_x{}'(t,\gamma_2(t))>0. \quad (42)$$

Note that

$$(\gamma_1)_L\le\Phi_1(t)\le(\gamma_1)_M. \quad (43)$$

By equations (41) and (43), it follows that

$$f_x{}'(t,\Phi_1(t))<0. \quad (44)$$

Now, suppose that there is another ω-periodic continuous solution $\Psi_1(t)$ of equation (6) which satisfies

$$(\gamma_1)_L\le\Psi_1(t)\le(\gamma_1)_M. \quad (45)$$

Because $f(t,x)$ is a polynomial function with continuous partial derivatives to x, equation (6) satisfies the existence and uniqueness of solutions to initial value problems of differential equations, thus

$$|\Phi_1(t)-\Psi_1(t)|>0(\forall t\in R). \quad (46)$$

By equations (41) and (45), it follows that

$$f_x{}'(t,\Psi_1(t))<0. \quad (47)$$

Consider the following equation:

$$\begin{aligned}\frac{d[\Phi_1(t)-\Psi_1(t)]}{dt}&=f(t,\Phi_1(t))-f(t,\Psi_1(t))\\&=f_x'[t,\Psi_1(t)+\theta(\Phi_1(t)-\Psi_1(t))](\Phi_1(t)\\&\quad-\Psi_1(t)),(0<\theta<1).\end{aligned} \quad (48)$$

Thus, we have

$$|\Phi_1(t)-\Psi_1(t)|=|\Phi_1(0)-\Psi_1(0)|e^{\int_0^t f_x{}'[s,\Psi_1(s)+\theta(\Phi_1(s)-\Psi_1(s))]ds}. \quad (49)$$

By equations (43) and (45), it follows that

$$(\gamma_1)_L\le\Psi_1(t)+\theta(\Phi_1(t)-\Psi_1(t))\le(\gamma_1)_M. \quad (50)$$

By equations (41) and (50), it follows that

$$f_x'[t,\Psi_1(t)+\theta(\Phi_1(t)-\Psi_1(t))]<0. \quad (51)$$

By equations (49) and (51), it follows that

$$|\Phi_1(t)-\Psi_1(t)|\longrightarrow 0(t\longrightarrow+\infty), \quad (52)$$

By equations (46) and (52), this is a contradiction, thus $\Psi_1(t)$ cannot be a periodic solution of equation (6), that is to say, equation (6) has exactly a unique ω-periodic continuous solution $\Phi_1(t)$ which satisfies $(\gamma_1)_L\le\Phi_1(t)\le(\gamma_1)_M$.

(iii) If $x_0\in((\gamma_1)_M,(\gamma_2)_L)$

Consider equation (6). We have $dx/dt|_{(t_0,x_0)}=f(t_0,x_0)<0$. Thus, $x(t)$ may stay at $((\gamma_1)_M,(\gamma_2)_L)$ or enter into $[(\gamma_1)_L,(\gamma_1)_M]$ at some time t. If $x(t)$ stays at $((\gamma_1)_M,(\gamma_2)_L)$, we have $dx/dt=f(t,x)<0$, then $x(t)$ cannot be a periodic solution of equation (6). If $x(t)$ enters into $[(\gamma_1)_L,(\gamma_1)_M]$ at some time t, then there is not a $t_1(t_1>t_0)$ such that $x(t_1)=x(t_0)=x_0$; thus, $x(t)$ can also not be a periodic solution of equation (6).

Similarly, if $x_0\in[(\gamma_i)_L,(\gamma_i)_M], i=2,3,\cdots,n,$ we can prove that equation (6) has exactly a unique ω-periodic continuous solution $\Phi_i(t)$ which satisfies $(\gamma_i)_L\le\Phi_i(t)\le(\gamma_i)_M$ $(i=2,3,\cdots,n)$.

Similarly, if $x_0\in((\gamma_i)_M,(\gamma_{i+1})_L), i=2,3,\cdots,n$, then the solution $x(t)$ of equation (6) with an initial value $x(t_0)=x_0$ cannot be a periodic solution of equation (6).

(iv) If $x_0 \in ((\gamma_n)_M, +\infty)$

Consider equation (6). We have $dx/dt|_{(t_0,x_0)} = f(t_0, x_0) > 0$. Thus, $x(t)$ may stay at $\in((\gamma_n), +\infty)$ or $x(t) \longrightarrow +\infty, (t \longrightarrow +\infty)$. If $x(t)$ stays at $((\gamma_n)_M, +\infty)$, we have $dx/dt = f(t, x) > 0$, then $x(t)$ cannot be a periodic solution of equation (6). If $x(t) \longrightarrow +\infty$, $(t \longrightarrow +\infty)$, then $x(t)$ can also not be a periodic solution of equation (6).

To sum up, equation (6) has exactly n ω-periodic continuous solutions $\Phi_i(t) (i = 1, 2, \cdots, n)$ which satisfy

$$(\gamma_i)_L \leq \Phi_i(t) \leq (\gamma_i)_M \quad (i = 1, 2, \cdots, n). \tag{53}$$

This is the end of the proof of Theorem 5.

Remark 2. If $n = 1$, Theorem 5 is exactly Proposition 1.

Remark 3. If $n = 2$, Theorem 5 is exactly Proposition 2.

4. Conclusion

Consider the following Riccati's differential equation:

$$\frac{dx}{dt} = a(t)x^2 + b(t)x + c(t), \tag{54}$$

about the periodic solutions on equation (54), there is a conclusion as follows:

Theorem 6 (see [13]). *Consider equation (54); $a(t)$, $b(t)$, and $c(t)$ are all ω-periodic continuous functions on R. Suppose that the following conditions hold:*

$$(H_1) a(t) > 0, \tag{55}$$

then equation (54) has at most two ω-periodic continuous solutions.

What conditions are the coefficient functions of the equation satisfied? The equation has exactly two periodic solutions. Proposition 2 gives the answer.

Consider the following Abel's differential equation:

$$\frac{dx}{dt} = a(t)x^3 + b(t)x^2 + c(t)x + d(t), \tag{56}$$

about the periodic solutions on equation (56), we have the following result.

Theorem 7 (see [13]). *Consider equation (56), $a(t)$, $b(t)$, $c(t)$, and $d(t)$ are all ω-periodic continuous functions on R. Suppose that the following conditions hold:*

$$(H_1) a(t) > 0, \tag{57}$$

then equation (56) has at most three ω-periodic continuous solutions.

We cannot help but ask: What conditions are the coefficient functions of the equation satisfied? The equation has exactly three periodic solutions. We show the following answer.

Corollary 8. *Consider the following Abel's differential equation:*

$$\frac{dx}{dt} = a(t)[x - \gamma_1(t)][x - \gamma_2(t)][x - \gamma_3(t)]. \tag{58}$$

Here, $a(t), \gamma_i(t)$ $(i = 1, 2, 3)$ are all ω-periodic continuous functions on R. Suppose that the following conditions hold:

$$\begin{aligned} &(H_1)\, a(t) \neq 0, \\ &(H_2)\, (\gamma_1)_M < (\gamma_2)_L \leq (\gamma_2)_M < (\gamma_3)_L, \end{aligned} \tag{59}$$

then equation (58) has exactly three ω-periodic continuous solutions $\Phi_1(t)$, $\Phi_2(t)$, and $\Phi_3(t)$, and

$$\begin{aligned} (\gamma_1)_L &\leq \Phi_1(t) \leq (\gamma_1)_M, \\ (\gamma_2)_L &\leq \Phi_2(t) \leq (\gamma_2)_M, \\ (\gamma_3)_L &\leq \Phi_3(t) \leq (\gamma_3)_M. \end{aligned} \tag{60}$$

For $n \geq 3, n \in N^+$, equation (1) has not always the most n periodic solutions (see [13]). But in this paper, we obtain a new criterion for the existence of n periodic solutions of periodic equation (6); the size range of n periodic solutions is also given. It can be said that this paper is a generalization of the conclusions of the related articles on periodic solutions.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no competing interest.

Authors' Contributions

The authors contributed to each part of this paper equally. The authors read and approved the final manuscript.

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