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Research Article

On Construction of a Super Hierarchy of the Wadati-Konno-Ichikawa Equation

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In this paper, a super Wadati-Konno-Ichikawa (WKI) hierarchy associated with a 3×3 matrix spectral problem is derived with the help of the zero-curvature equation. We obtain the super bi-Hamiltonian structures by using of the super trace identity. Infinitely, many conserved laws of the super WKI equation are constructed by using spectral parameter expansions.

1. Introduction

The super extensions of the standard integrable systems in two-dimensional spacetime have been investigated for the recent several decades. Many classical integrable equations have been extended to be the super completely integrable equations, such as the super Korteweg-de Vries (KdV) equation [1–3], super AKNS [4–7], super Kadomtsev-Petviashvili (KP) [8], super Kaup-Newell (KN) [9], super Camassa-Holm (CH) [10], super vector nonlinear Schrödinger equations [11], super Heisenberg [12], and so on [13–20].

The Wadati-Konno-Ichikawa (WKI) equation, proposed in [21], can be written in the form

$$\begin{aligned} iu_t &= \left(\frac{u}{\sqrt{1+uv}}\right)_{xx}, \\ iv_t &= -\left(\frac{v}{\sqrt{1+uv}}\right)_{xx}, \end{aligned} \tag{1}$$

which can be used to describe the nonlinear oscillation of elastic beam under tension. In this paper, we propose a super WKI hierarchy associated with a 3×3 matrix spectral problem, in which the first nontrivial member takes the following form:

$$u_t = i\left[\left(\frac{-u}{\sqrt{1+uv}}\right)_x + \frac{2u\alpha}{1+uv}\left(\frac{\beta}{\sqrt{1+uv}}\right)_x - \frac{u}{\sqrt{1+uv}}\left(\frac{\alpha\beta}{1+uv}\right)_x + \frac{2\alpha\alpha_x}{\left(\sqrt{1+uv}\right)^3}\right]_x,$$

$$v_t = i\left[\left(\frac{v}{\sqrt{1+uv}}\right)_x + \frac{2v\beta}{1+uv}\left(\frac{\alpha}{\sqrt{1+uv}}\right)_x + \frac{v}{\sqrt{1+uv}}\left(\frac{\alpha\beta}{1+uv}\right)_x - \frac{2\beta\beta_x}{\left(\sqrt{1+uv}\right)^3}\right]_x,$$

$$\alpha_t = i\left[\frac{-2}{1+uv}\left(\frac{\alpha}{\sqrt{1+uv}}\right)_x - \frac{2u}{1+uv}\left(\frac{\beta}{\sqrt{1+uv}}\right)_x + \frac{u\alpha}{1+uv}\left(\frac{v}{\sqrt{1+uv}}\right)_x - \frac{\beta}{1+uv}\left(\frac{u}{\sqrt{1+uv}}\right)_x - \frac{3u\alpha\beta\beta_x - (2-uv)\alpha\alpha_x\beta}{\left(\sqrt{1+uv}\right)^5}\right]_x,$$

$$\beta_t = i\left[\frac{2}{1+uv}\left(\frac{\beta}{\sqrt{1+uv}}\right)_x - \frac{2v}{1+uv}\left(\frac{\alpha}{\sqrt{1+uv}}\right)_x - \frac{v\beta}{1+uv}\left(\frac{u}{\sqrt{1+uv}}\right)_x - \frac{\alpha}{1+uv}\left(\frac{v}{\sqrt{1+uv}}\right)_x + \frac{3v\alpha\alpha_x\beta + (2-uv)\alpha\beta\beta_x}{\left(\sqrt{1+uv}\right)^5}\right]_x, \tag{2}$$

which is the well-known WKI equation (1) as $\alpha = 0$ and $\beta = 0$.

The outline of this paper is as follows. In Section 2, we introduce a 3×3 matrix spectral problem with two commuting potentials u and v, and two anticommuting potentials α and β. This spectral problem is an extension of the spectral problems associated with the WKI equation. From this spectral problem, a hierarchy of the super WKI equations are proposed with the aid of the zero-curvature equation. In Section 3, the super bi-Hamiltonian structures of the super WKI hierarchy are constructed by using the super trace identity [22–26]. In Section 4, we derive infinite conservation laws of the super WKI equation by resorting to the spectral parameter expansions. For the applied and analytic aspects on conservation laws, one can refer to [27–30]. We can refer to the two most recent results on the mixed method for the calculation of conservation laws studied in [29, 30].

2. Super WKI Equations

In this section, a hierarchy of super WKI equations will be obtained. We first introduce a 3×3 matrix spectral problem:

$$\phi_x = U\phi, \quad \phi = \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}, \quad U = \begin{pmatrix}\lambda & \lambda u & \lambda\alpha\\ \lambda v & -\lambda & \lambda\beta\\ -\lambda\beta & \lambda\alpha & 0\end{pmatrix}, \tag{3}$$

where u, v, λ, ϕ_1, and ϕ_2 are the commuting variables, which can be indicated by the degree (mod 2) p as $p(u) = p(v) = p(\lambda) = p(\phi_1) = p(\phi_2) = 0$; α, β, and ϕ_3 are the anticommuting variables which can be indicated by p as $p(\alpha) = p(\beta) = p(\phi_3) = 1$. In order to derive the hierarchy of super nonlinear evolution equations associated with the spectral problem (3), we need to solve the stationary zero-curvature equation:

$$V_x = [U, V], \quad V = \left(V_{ij}\right)_{3\times 3}, \tag{4}$$

where $p(V_{11}) = p(V_{12}) = p(V_{21}) = p(V_{22}) = p(V_{33}) = 0$; $p(V_{13}) = p(V_{23}) = p(V_{31}) = p(V_{32}) = 1$. We note that equation (4) is equivalent to

$$\begin{aligned}
V_{11x} &= -\lambda v V_{12} + \lambda u V_{21} + \lambda\alpha V_{31} - \lambda\beta V_{13},\\
V_{12x} &= 2\lambda V_{12} + \lambda u(V_{22} - V_{11}) + \lambda\alpha(V_{13} + V_{32}),\\
V_{13x} &= \lambda V_{13} + \lambda u V_{23} + \lambda\alpha(V_{33} - V_{11}) - \lambda\beta V_{12},\\
V_{21x} &= -2\lambda V_{21} + \lambda v(V_{11} - V_{22}) + \lambda\beta(V_{31} - V_{23}),\\
V_{22x} &= \lambda v V_{12} - \lambda u V_{21} + \lambda\alpha V_{23} + \lambda\beta V_{32},\\
V_{23x} &= -\lambda V_{23} + \lambda v V_{13} - \lambda\alpha V_{21} + \lambda\beta(V_{33} - V_{22}),\\
V_{31x} &= -\lambda V_{31} - \lambda v V_{32} + \lambda\alpha V_{21} + \lambda\beta(V_{33} - V_{11}),\\
V_{32x} &= \lambda V_{32} - \lambda u V_{31} + \lambda\alpha(V_{22} - V_{33}) - \lambda\beta V_{12},\\
V_{33x} &= \lambda\alpha(V_{23} + V_{31}) + \lambda\beta(V_{32} - V_{13}),
\end{aligned} \tag{5}$$

where each entry $V_{ij} = V_{ij}(A, B, C, \rho, \delta)$ is a function of A, B, C, ρ, and δ:

$$\begin{aligned}
V_{11} &= -A,\\
V_{12} &= B,\\
V_{13} &= \delta,\\
V_{21} &= C,\\
V_{22} &= A,\\
V_{23} &= \rho,\\
V_{31} &= -\rho,\\
V_{32} &= \delta,\\
V_{33} &= 0,
\end{aligned} \tag{6}$$

with $p(A) = p(B) = p(C) = 0, p(\rho) = p(\delta) = 1$. Substituting (6) into (5), we have

$$\begin{aligned}
A_x &= \lambda v B - \lambda u C + \lambda\alpha\rho + \lambda\beta\delta,\\
B_x &= 2\lambda B + 2\lambda u A + 2\lambda\alpha\delta,\\
C_x &= -2\lambda C - 2\lambda v A - 2\lambda\beta\rho,
\end{aligned}$$

$$\delta_x = \lambda\delta + \lambda u\rho + \lambda\alpha A - \lambda\beta B,$$
$$\rho_x = -\lambda\rho + \lambda v\delta - \lambda\alpha C - \lambda\beta A. \quad (7)$$

The functions A, B, C, ρ, and δ are expanded as the following Laurent series in λ:

$$A = \sum_{j=0}^{\infty} A_j\lambda^{-j},$$
$$B = \sum_{j=0}^{\infty} B_j\lambda^{-j},$$
$$C = \sum_{j=0}^{\infty} C_j\lambda^{-j},$$
$$\rho = \sum_{j=0}^{\infty} \rho_j\lambda^{-j},$$
$$\delta = \sum_{j=0}^{\infty} \delta_j\lambda^{-j}. \quad (8)$$

Substituting (8) into (7), we can get the Lenard recursion equation as follows:

$$KG_{j-1} = JG_j, JG_{-1} = 0, \quad j \geq 0, \quad (9)$$

where

$$G_{j-1} = \left(C_j, B_j, -2\rho_j, 2\delta_j, A_j\right)^{\mathrm{T}}, \quad (10)$$

and K and J are two operators defined by

$$K = \frac{1}{2}\left(K_{ij}\right)_{5\times5},$$
$$J = \frac{1}{2}\left(J_{ij}\right)_{5\times5}, \quad (11)$$

with

$$(K_{ij}) = \begin{pmatrix} 0 & \partial\left(\frac{1+uv-\alpha\beta}{1+uv}\right)\partial & \partial\left(\frac{u\alpha}{1+uv}\right)\partial & -\partial\left(\frac{\alpha}{1+uv}\right)\partial & 0 \\ -K_{12} & 0 & -\partial\left(\frac{\beta}{1+uv}\right)\partial & -\partial\left(\frac{v\beta}{1+uv}\right)\partial & 0 \\ -K_{13} & -K_{23} & -\partial\left(\frac{u(1+uv+2\alpha\beta)}{(1+uv)^2}\right)\partial & \partial\left(\frac{1+uv+(1-uv)\alpha\beta}{(1+uv)^2}\right)\partial & 0 \\ -K_{14} & -K_{24} & K_{34} & \partial\left(\frac{v(1+uv+2\alpha\beta)}{(1+uv)^2}\right)\partial & 0 \\ -u\partial & -v\partial & -\alpha\partial & -\beta\partial & 2\partial \end{pmatrix},$$
$$(J_{ij}) = \begin{pmatrix} \partial u\partial^{-1}u\partial & 2\partial + \partial u\partial^{-1}v\partial & \partial u\partial^{-1}\alpha\partial & \partial u\partial^{-1}\beta\partial & 0 \\ 2\partial + \partial v\partial^{-1}u\partial & \partial v\partial^{-1}v\partial & \partial v\partial^{-1}\alpha\partial & \partial v\partial^{-1}\beta\partial & 0 \\ \partial\alpha\partial^{-1}u\partial & \partial\alpha\partial^{-1}v\partial & \partial\alpha\partial^{-1}\alpha\partial & \partial + \partial\alpha\partial^{-1}\beta\partial & 0 \\ \partial\beta\partial^{-1}u\partial & \partial\beta\partial^{-1}v\partial & -\partial + \partial\beta\partial^{-1}\alpha\partial & \partial\beta\partial^{-1}\beta\partial & 0 \\ -u\partial & -v\partial & -\alpha\partial & -\beta\partial & 2\partial \end{pmatrix}, \quad (12)$$

and $\partial = \partial_x$.

To find a general representation of the solution for (9), we present a Lenard recursion equation as follows:

$$Kg_{j-1} = Jg_j, \quad j \geq 0, \quad (13)$$

with condition to identify constants of integration as zero when acting with operator J^{-1} upon Kg_j. This means that g_j is uniquely determined by the recursion equation (13). We choose $\ker J = c_0 g_{-1} = c_0(1+uv)^{-3/2}(1+uv+\alpha\beta)(-v, -u, 2\beta, -2\alpha, 1)^{\mathrm{T}}$. Through long calculations, we have

$$g_0 = \begin{pmatrix} \frac{v_x}{2\left(\sqrt{1+uv}\right)^3} + \frac{3v(\alpha\beta_x - \alpha_x\beta) - 3v^2\alpha\alpha_x - (2-uv)\beta\beta_x}{2\left(\sqrt{1+uv}\right)^5} \\ \frac{-u_x}{2\left(\sqrt{1+uv}\right)^3} + \frac{3u(\alpha\beta_x - \alpha_x\beta) + 3u^2\beta\beta_x + (2-uv)\alpha\alpha_x}{2\left(\sqrt{1+uv}\right)^5} \\ \frac{v_x\alpha + 2v\alpha_x - 2\beta_x}{\left(\sqrt{1+uv}\right)^3} - \frac{3(uv)_x(v\alpha-\beta)}{2\left(\sqrt{1+uv}\right)^5} \\ \frac{-u_x\beta - 2u\beta_x - 2\alpha_x}{\left(\sqrt{1+uv}\right)^3} + \frac{3(uv)_x(u\beta+\alpha)}{2\left(\sqrt{1+uv}\right)^5} \\ \frac{uv_x - u_x v}{4\left(\sqrt{1+uv}\right)^3} - \frac{(2-uv)(\alpha\beta_x - \alpha_x\beta) - 3(v\alpha\alpha_x - u\beta\beta_x)}{2\left(\sqrt{1+uv}\right)^5} \end{pmatrix}. \tag{14}$$

Operating with $(J^{-1}K)^j$ upon $G_{-1} = c_0 g_{-1}$, we get the general solution of (9).

$$G_j = c_0 g_j + c_1 g_{j-1} + \cdots + c_{j+1} g_{-1}, \quad j \geq -1, \tag{15}$$

where $c_0, c_1, \cdots, c_{j+1}$ are constants of integration and $g_j = (g_j^{(1)}, g_j^{(2)}, g_j^{(3)}, g_j^{(4)}, g_j^{(5)})^{\mathrm{T}}$.

Let ϕ satisfy the spectral problem (3) and the following auxiliary problem

$$\phi_{t_n} = V^{(n)}\phi, \tag{16}$$

where each entry $V_{ij}^{(n)} = V_{ij}(A^{(n)}, B^{(n)}, C^{(n)}, \rho^{(n)}, \delta^{(n)})$ in the matrix $V^{(n)}$ is a polynomial of eigenparameter λ with

$$\begin{aligned} A^{(n)} &= \sum_{j=0}^{n-1} A_j \lambda^{n+1-j}, \\ B^{(n)} &= \sum_{j=0}^{n-1} B_j \lambda^{n+1-j} + \lambda(B_n + uA_n), \\ C^{(n)} &= \sum_{j=0}^{n-1} C_j \lambda^{n+1-j} + \lambda(C_n + vA_n), \\ \rho^{(n)} &= \sum_{j=0}^{n-1} \rho_j \lambda^{n+1-j} + \lambda(\rho_n + \beta A_n), \\ \delta^{(n)} &= \sum_{j=0}^{n-1} \delta_j \lambda^{n+1-j} + \lambda(\delta_n + \alpha A_n), \end{aligned} \tag{17}$$

where A_j, B_j, C_j, ρ_j, and δ_j are determined by (15). Then, the compatibility condition of (3) and (16) yields the zero-curvature equation $U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0$, which is equivalent to the hierarchy of the super WKI equations.

$$\left(u_{t_n}, v_{t_n}, \alpha_{t_n}, \beta_{t_n}\right)^{\mathrm{T}} = PKG_{n-2} = PJG_{n-1}, \quad n \geq 1, \tag{18}$$

where P is a projective map $(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)^{\mathrm{T}} \longrightarrow (\eta_1, \eta_2, \eta_3, \eta_4)^{\mathrm{T}}$. This can be transformed into

$$\left(u_{t_n}, v_{t_n}, \alpha_{t_n}, \beta_{t_n}\right)^{\mathrm{T}} = c_0 X_{n-1} + c_1 X_{n-2} + \cdots + c_{n-1} X_0, \quad n \geq 1, \tag{19}$$

with $X_j = PKg_{j-1} = PJg_j$. The first nontrivial member in the hierarchy (19) is as follows:

$$\begin{aligned} u_{t_1} &= \frac{c_0}{2}\left[\left(\frac{-u}{\sqrt{1+uv}}\right)_x + \frac{2u\alpha}{1+uv}\left(\frac{\beta}{\sqrt{1+uv}}\right)_x - \frac{u}{\sqrt{1+uv}}\left(\frac{\alpha\beta}{1+uv}\right)_x + \frac{2\alpha\alpha_x}{\left(\sqrt{1+uv}\right)^3}\right]_x, \\ v_{t_1} &= \frac{c_0}{2}\left[\left(\frac{v}{\sqrt{1+uv}}\right)_x + \frac{2v\beta}{1+uv}\left(\frac{\alpha}{\sqrt{1+uv}}\right)_x + \frac{v}{\sqrt{1+uv}}\left(\frac{\alpha\beta}{1+uv}\right)_x - \frac{2\beta\beta_x}{\left(\sqrt{1+uv}\right)^3}\right]_x, \\ \alpha_{t_1} &= \frac{-c_0}{2}\left[\frac{2}{1+uv}\left(\frac{\alpha}{\sqrt{1+uv}}\right)_x + \frac{2u}{1+uv}\left(\frac{\beta}{\sqrt{1+uv}}\right)_x - \frac{u\alpha}{1+uv}\left(\frac{v}{\sqrt{1+uv}}\right)_x + \frac{\beta}{1+uv}\left(\frac{u}{\sqrt{1+uv}}\right)_x + \frac{3u\alpha\beta\beta_x - (2-uv)\alpha\alpha_x\beta}{\left(\sqrt{1+uv}\right)^5}\right]_x, \\ \beta_{t_1} &= \frac{-c_0}{2}\left[\frac{-2}{1+uv}\left(\frac{\beta}{\sqrt{1+uv}}\right)_x + \frac{2v}{1+uv}\left(\frac{\alpha}{\sqrt{1+uv}}\right)_x + \frac{v\beta}{1+uv}\left(\frac{u}{\sqrt{1+uv}}\right)_x + \frac{\alpha}{1+uv}\left(\frac{v}{\sqrt{1+uv}}\right)_x - \frac{3v\alpha\alpha_x\beta + (2-uv)\alpha\beta\beta_x}{\left(\sqrt{1+uv}\right)^5}\right]_x, \end{aligned} \tag{20}$$

which is reduced to the famous WKI equation (1) (see [21, 31]) as $t_1 = t$, $\alpha = 0$, $\beta = 0$, $c_0 = 2i$ or the super WKI equation (2) as $t_1 = t$, $c_0 = 2i$.

3. Super Bi-Hamiltonian Structures

In this section, the super bi-Hamiltonian structures of equation (19) will be established by using the super trace identity as follows [23–27]:

$$\frac{\delta}{\delta U_0}\int \mathrm{str}\left(V\frac{\partial U}{\partial \lambda}\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\left(\lambda^{\gamma}\mathrm{str}\left(\frac{\partial U}{\partial U_0}V\right)\right), \quad (21)$$

where γ is a constant to be determined and $U_0=(u,v,\alpha,\beta)^T$. It is easy to observe that

$$\begin{aligned}
\mathrm{str}\left(V\frac{\partial U}{\partial \lambda}\right) &= -2A + vB + uC - 2\alpha\rho + 2\beta\delta,\\
\mathrm{str}\left(\frac{\partial U}{\partial u}V\right) &= \lambda C,\\
\mathrm{str}\left(\frac{\partial U}{\partial v}V\right) &= \lambda B,\\
\mathrm{str}\left(\frac{\partial U}{\partial \alpha}V\right) &= -2\lambda\rho,\\
\mathrm{str}\left(\frac{\partial U}{\partial \beta}V\right) &= 2\lambda\delta.
\end{aligned} \quad (22)$$

Substituting (22) and (8) into (21), we arrive at

$$\begin{aligned}
&\frac{\delta}{\delta U_0}\int(-2A + vB + uC - 2\alpha\rho + 2\beta\delta)dx\\
&\quad = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\left(\lambda^{\gamma+1}(C,B,-2\rho,2\delta)^{\mathrm{T}}\right),\\
&\frac{\delta}{\delta U_0}\int\left(-2A_j + vB_j + uC_j - 2\alpha\rho_j + 2\beta\delta_j\right)dx\\
&\quad = (\gamma+1-j)\left(C_j, B_j, -2\rho_j, 2\delta_j\right)^{\mathrm{T}}, \quad j\ge 0.
\end{aligned} \quad (23)$$

Through direct calculations, we find that $\gamma = k$ as $c_k \neq 0$, $\{c_l\}_{l=0,l\neq k}^{j} = 0$, $0 \le k \le j$. Using (23) and noticing (15), we have

$$\frac{\delta}{\delta U_0}H_j = \left(C_j, B_j, -2\rho_j, 2\delta_j\right)^{\mathrm{T}}, \quad j \ge 0, \quad (24)$$

where

$$\begin{aligned}
H_0 &= -2c_0\int\frac{1+uv-\alpha\beta}{\sqrt{1+uv}}dx,\\
H_1 &= c_0\int\left(\frac{u_xv-uv_x}{2uv\sqrt{1+uv}} + \frac{\alpha_x\beta-\alpha\beta_x+v\alpha\alpha_x-u\beta\beta_x}{\left(\sqrt{1+uv}\right)^3}\right)dx\\
&\quad - 2c_1\int\frac{1+uv-\alpha\beta}{\sqrt{1+uv}}dx,\\
H_j &= \sum_{k=0}^{j-2}\frac{c_k}{k+1-j}\int\left(\left(u-\partial^{-1}u\partial\right)g_{j-k}^{(1)} + \left(v-\partial^{-1}v\partial\right)g_{j-k}^{(2)}\right.\\
&\quad \left. + \left(\alpha-\partial^{-1}\alpha\partial\right)g_{j-k}^{(3)} + \left(\beta-\partial^{-1}\beta\partial\right)g_{j-k}^{(4)}\right)dx\\
&\quad + c_{j-1}\int\left(\frac{u_xv-uv_x}{2uv\sqrt{1+uv}} + \frac{\alpha_x\beta-\alpha\beta_x+v\alpha\alpha_x-u\beta\beta_x}{\left(\sqrt{1+uv}\right)^3}\right)dx\\
&\quad - 2c_j\int\frac{1+uv-\alpha\beta}{\sqrt{1+uv}}dx, \quad j\ge 2.
\end{aligned} \quad (25)$$

From (24), we obtain the desired Hamiltonian form of (19) as follows:

$$\left(u_{t_n}, v_{t_n}, \alpha_{t_n}, \beta_{t_n}\right)^{\mathrm{T}} = K\frac{\delta}{\delta U_0}H_{n-1} = J\frac{\delta}{\delta U_0}H_n, \quad (26)$$

where K and J are two super-Hamiltonian operators defined by

$$\begin{aligned}
K &= \frac{1}{2}\left(K_{ij}\right)_{4\times4},\\
J &= \frac{1}{2}\left(J_{ij}\right)_{4\times4},
\end{aligned} \quad (27)$$

with K_{ij} and J_{ij} given by (11). Especially, the super WKI equation (20) can be written as follows:

$$\left(u_{t_1}, v_{t_1}, \alpha_{t_1}, \beta_{t_1}\right)^{T} = K\frac{\delta}{\delta U_0}H_0 = J\frac{\delta}{\delta U_0}H_1. \quad (28)$$

4. Conservation Laws

In this section, infinitely, many conservation laws of the super WKI equation (20) will be constructed. First, let us introduce the variables

$$\begin{aligned}
M &= \frac{\phi_2}{\phi_1},\\
\Gamma &= \frac{\phi_3}{\phi_1},
\end{aligned} \quad (29)$$

where ϕ_1, ϕ_2, and ϕ_3 satisfy (3) and (16) with $n=1$. Noticing that $\phi_3^2 = 0$, we get from (3) that

$$\begin{aligned}
M_x &= \lambda\left(v - 2M + \beta\Gamma - uM^2 - \alpha M\Gamma\right),\\
\Gamma_x &= \lambda(-\beta + \alpha M - \Gamma - uM\Gamma).
\end{aligned} \quad (30)$$

We expand M, Γ in powers of λ^{-1} as follows:

$$M = \sum_{j=0}^{\infty} M_j \lambda^{-j}, \quad \Gamma = \sum_{j=0}^{\infty} \Gamma_j \lambda^{-j}, \tag{31}$$

where $p(M_j) = 0, p(\Gamma_j) = 1$. Substituting (31) into (30) and comparing the coefficients of the same powers of λ, we obtain

$$M_0 = \frac{1}{u}\left(-1 \pm \frac{1}{\sqrt{1+uv}}\right), \quad \Gamma_0 = \frac{1}{u}\left(\alpha \mp \frac{u\beta + \alpha}{\sqrt{1+uv}}\right), \tag{32}$$

and a recursion formula for M_j and Γ_j

$$\begin{aligned} M_{j+1} &= \frac{\mp 1}{2\sqrt{1+uv}}\left(M_{jx} - \Gamma_0 \Gamma_{jx} + u\sum_{k=1}^{j} M_k M_{j-k+1} - (u\Gamma_0 - \alpha)\sum_{k=1}^{j} M_k \Gamma_{j-k+1}\right), \\ \Gamma_{j+1} &= \frac{\mp 1}{\sqrt{1+uv}}\left(\Gamma_{jx} - \alpha M_{j+1} + u\sum_{k=0}^{j} M_{j-k+1}\Gamma_k\right), \quad j \geq 0. \end{aligned} \tag{33}$$

Since

$$\frac{\partial}{\partial t}\frac{\phi_{1x}}{\phi_1} = \frac{\partial}{\partial x}\frac{\phi_{1t}}{\phi_1}, \tag{34}$$

we can derive the conservation law of (20) as follows:

$$\frac{\partial}{\partial t}(1 + uM + \alpha\Gamma)\lambda = \frac{\partial}{\partial x}\left(V_{11}^{(1)} + V_{12}^{(1)} M + V_{13}^{(1)}\Gamma\right), \tag{35}$$

where

$$\begin{aligned} V_{11}^{(1)} &= -c_0 g_{-1}^{(5)} \lambda^2, \\ V_{12}^{(1)} &= c_0\left(-u g_{-1}^{(5)}\lambda + g_0^{(2)} + u g_0^{(5)}\right)\lambda, \\ V_{13}^{(1)} &= c_0\left(-\alpha g_{-1}^{(5)}\lambda + \frac{g_0^{(4)}}{2} + \alpha g_0^{(5)}\right)\lambda. \end{aligned} \tag{36}$$

Assuming that $I = 1 + uM + \alpha\Gamma$, $F = (V_{11}^{(1)} + V_{12}^{(1)} M + V_{13}^{(1)}\Gamma)/\lambda$, (35) can be rewritten as $I_t = F_x$, which is the right form of conservation laws. We expand I and F as series in powers of λ with the coefficients which are called conserved densities and currents, respectively

$$I = \sum_{j=0}^{\infty} I_j \lambda^{-j}, \quad F = \mp c_0 \lambda + \sum_{j=0}^{\infty} F_j \lambda^{-j}, \tag{37}$$

where c_0 is a integration constant of (15). The first members of conserved densities and currents are as follows:

$$\begin{aligned} I_0 &= \pm \frac{1 + uv - \alpha\beta}{\sqrt{1+uv}}, \\ F_0 &= \pm \frac{c_0}{4}\left(\frac{uv_x - u_x v}{1+uv} - \frac{(uv_x - u_x v)\alpha\beta - 6v\alpha\alpha_x + 6u\beta\beta_x - 2(2-uv)(\alpha_x\beta - \alpha\beta_x)}{(1+uv)^2}\right). \end{aligned} \tag{38}$$

The recursion relations of I_j and $F_j (j \geq 1)$ are as follows:

$$\begin{aligned} F_j &= c_0\left(-u g_{-1}^{(5)} M_{j+1} + \left(g_0^{(2)} + u g_0^{(5)}\right) M_j - \alpha g_{-1}^{(5)}\Gamma_{j+1} + \left(\frac{g_0^{(4)}}{2} + \alpha g_0^{(5)}\right)\Gamma_j\right), \\ I_j &= uM_j + \alpha\Gamma_j, \quad j \geq 1, \end{aligned} \tag{39}$$

where M_j and Γ_j can be computed by (33).

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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