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Research Article

Some Properties of Generalized Einstein Tensor for a Pseudo-Ricci Symmetric Manifold

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The object of the paper is to study some properties of the generalized Einstein tensor $G(X, Y)$ which is recurrent and birecurrent on pseudo-Ricci symmetric manifolds $(\mathrm{PRS})_n$. Considering the generalized Einstein tensor $G(X, Y)$ as birecurrent but not recurrent, we state some theorems on the necessary and sufficient conditions for the birecurrency tensor of $G(X, Y)$ to be symmetric.

1. Introduction

In the late twenties, because of the important role of symmetric spaces in differential geometry, Cartan [1], who, in particular, obtained a classification of those spaces, established Riemannian symmetric spaces. The notion of the pseudosymmetric manifold was introduced by Chaki [2] and Deszcz [3]. Recently, some necessary and sufficient conditions for a Chaki pseudosymmetric (respectively, pseudo-Ricci symmetric [4]) manifold to be Deszcz pseudosymmetric (respectively, Ricci-pseudo symmetric [5]) have been examined in [6].

A nonflat n-dimensional Riemannian manifold (M, g), $(n > 3)$ is called a *pseudo-Ricci symmetric manifold* if the Ricci tensor S of type (0,2) is not identically zero and satisfies the condition [4]

$$(\nabla_Z S)(X, Y) = 2\pi(Z)S(X, Y) + \pi(X)S(Z, Y) + \pi(Y)S(X, Z), \tag{1}$$

where π is a nonzero 1-form, ρ is a vector field by

$$g(X, \rho) = \pi(X) \tag{2}$$

for all vector fields X, and ∇ denotes the operator of covariant differentiation with respect to the metric g. Such a manifold is denoted by $(\mathrm{PRS})_n$. The 1-form π is called the associated 1-form of the manifold. If $\pi = 0$, then the manifold reduces to a Ricci symmetric manifold or covariantly constant

$$(\nabla_Z S)(X, Y) = 0. \tag{3}$$

The notion of pseudo-Ricci symmetry is different from that of Deszcz [3].

The pseudo-Ricci symmetric manifolds have some importance in the general theory of relativity. So, pseudo-Ricci symmetric manifolds on some structures have been studied by many authors (see, e.g., [7, 8]).

A nonflat Riemannian manifold (M, g), $(n > 2)$, is called generalized recurrent if the Ricci tensor S is nonzero and satisfies the condition

$$(\nabla_Z S)(X, Y) = A(Z)S(X, Y) + B(Z)g(X, Y), \tag{4}$$

where A and B are nonzero 1-forms [9]. If the associated 1-form B becomes zero, then the manifold reduces to Ricci recurrent, i.e.,

$$(\nabla_Z S)(X, Y) = A(Z)S(X, Y). \tag{5}$$

A Riemannian manifold (M, g), $(n \geq 2)$, is said to be an Einstein manifold if the following condition:

$$S = \frac{r}{n} g, \tag{6}$$

holds on M, where S and r denote the Ricci tensor and scalar curvature of (M, g), respectively. According to [10], equation (6) is called the Einstein metric condition. Also, Einstein manifolds form a natural subclass of various classes of Riemannian manifolds by a curvature condition imposed on their Ricci tensor [10]. For instance, every Einstein manifold belongs to the class of Riemannian manifolds (M, g) realizing the following relation:

$$S(X, Y) = ag(X, Y) + bA(X)A(Y), \tag{7}$$

where a, b are real numbers and A is a nonzero 1-form such that

$$g(X, U) = A(X) \tag{8}$$

for all vector fields X.

A nonflat Riemannian manifold (M, g), $(n > 2)$, is defined to be a quasi-Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition (7).

2. Recurrent Generalized Einstein Tensor $G(X, Y)$ in $(\mathrm{PRS})_n$

It is well known that the Einstein tensor $E(X, Y)$ for a Riemannian manifold is defined by

$$E(X, Y) = S(X, Y) - rg(X, Y), \tag{9}$$

where $S(X, Y)$ and r are, respectively, the Ricci tensor and the scalar curvature of the manifold, playing an important part in Einstein's theory of gravitation as well as in proving some theorems in Riemannian geometry [10]. Moreover, the Einstein tensor can be obtained from Yano's tensor of concircular curvature. In [11], by using this approach, some generalizations of the Einstein tensor were achieved.

In this section, we consider the generalized Einstein tensor

$$G(X, Y) = S(X, Y) - \kappa r g(X, Y), \tag{10}$$

where κ is constant [12].

Now, we assume that our manifold $(\mathrm{PRS})_n$ has nonzero $G(X, Y)$-Einstein tensor. By taking the covariant derivative of (10), in the local coordinates, we get

$$\nabla_k G_{ij} = \nabla_k S_{ij} - \kappa g_{ij} \nabla_k r. \tag{11}$$

If we contract (1) over X and Y, then we obtain

$$\nabla_k r = 2\pi_k r + 2\pi^h S_{hk}. \tag{12}$$

Substituting (1) and (12) into (11), we achieve

$$\nabla_k G_{ij} = 2\pi_k S_{ij} + \pi_i S_{kj} + \pi_j S_{ik} - \left(2\pi_k r + 2\pi^h S_{hk}\right)\kappa g_{ij}. \tag{13}$$

Now, contracting (13) with respect to i and k, we obtain

$$\operatorname{div} G_{ij} = \nabla_k G_j^k = (3 - 2\kappa)\pi^h S_{hj} + (1 - 2\kappa) r \pi_j. \tag{14}$$

If we assume that $G(X, Y)$ is conservative [13], i.e., div $G = 0$, then from (14), we have

$$(3 - 2\kappa)P_j + (1 - 2\kappa) r \pi_j = 0, \tag{15}$$

where $P_j = \pi^h S_{hj}$.

If $(1 - 2\kappa)r$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector $\pi(X)$, then $(3 - 2\kappa)$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector P_j. Conversely, if equation (15) holds, then the form (14) the generalized Einstein tensor $G(X, Y)$ is conservative. We have thus proved the following.

Theorem 1. *For a $(PRS)_n$ manifold, the necessary and sufficient condition of the generalized Einstein tensor $G(X, Y)$ be conservative is that $(1 - 2\kappa)r$ and $(3 - 2\kappa)$ be eigenvalues of the Ricci tensor S corresponding to the eigenvectors π_j and $P_j = \pi^h S_{hj}$, respectively.*

Let $G(X, Y)$ be recurrent, i.e., from (5),

$$\nabla_k G_{ij} = A_k G_{ij}. \tag{16}$$

Substituting equations (10) and (13) into equation (16) yields

$$2\pi_k S_{ij} + \pi_i S_{jk} + \pi_j S_{ik} - \left(2\pi_k r + 2\pi^h S_{hk}\right)\kappa g_{ij} = A_k \left(S_{ij} - r\kappa g_{ij}\right). \tag{17}$$

If we contract (17) over i and k, then we have

$$(3 - 2\kappa)P_j + (1 - 2\kappa) r \pi_j = A_k S_j^k - r\kappa A_j. \tag{18}$$

This leads to the following result:

Theorem 2. *In a $(PRS)_n$ manifold, let us assume that the generalized Einstein tensor $G(X, Y)$ is recurrent with the recurrence vector generated by the 1-form A. Then, the recurrency vector A satisfies equation ((18)).*

Now, we assume that the generalized Einstein tensor $G(X, Y)$ is conservative. From (15) and (18), we get

$$Q_j - r\kappa A_j = 0, \tag{19}$$

where $Q_j = A_k S_j^k$.

Then, the following theorem holds true:

Theorem 3. *In a $(PRS)_n$ manifold, let the generalized Einstein tenso $G(X, Y)$ be recurrent with the recurrence vector generated by the 1-form A. If the generalized Einstein tensor $G(X, Y)$ is also conservative, then the vectors Q_j and A_j are linearly dependent.*

Let $G(X, Y)$ be a generalized recurrent. Then from (4),

$$\nabla_k G_{ij} = A_k G_{ij} + B_k g_{ij}. \tag{20}$$

Using (1) and (10), we get

$$\begin{aligned} 2\pi_k S_{ij} + \pi_i S_{kj} + \pi_j S_{ik} + \kappa g_{ij}\left(2\pi_k r + 2\pi^h S_{kh}\right) \\ = A_k\left(S_{ij} + \kappa r g_{ij}\right) + B_k g_{ij}. \end{aligned} \tag{21}$$

If we contract (21) over i and j, then we have

$$(1 + \kappa n)(2\pi_k r + 2P_k - A_k r) = nB_k. \tag{22}$$

If $1 + \kappa n = 0$, then $B_k = 0$.

This leads to the following result:

Theorem 4. *If $\kappa = -1/n$, a $(PRS)_n$ manifold admitting the generalized Einstein tensor $G(X, Y)$ which is the generalized recurrent cannot exist.*

3. Birecurrent Generalized Einstein Tensor $G(X, Y)$ in $(PRS)_n$

In this section, we examine some properties of the generalized Einstein tensor $G(X, Y)$ in $(\mathrm{PRS})_n$ which is birecurrent. If the generalized Einstein tensor $G(X, Y)$ satisfies the condition

$$\nabla_l \nabla_k G_{ij} = \mu_{lk} G_{ij} \tag{23}$$

for some nonzero covariant tensor field μ_{lk}, then we call G_{ij} as birecurrent generalized Einstein tensor.

It is easy to see that a recurrent generalized Einstein tensor $G(X, Y)$ is birecurrent. In fact, by taking the covariant derivative of (16) with respect to U^l, we get

$$\nabla_l \nabla_k G_{ij} = (\nabla_l A_k + A_k A_l) G_{ij} \tag{24}$$

with $\mu_{lk} = \nabla_l A_k + A_k A_l$.

Now, we assume that $(\mathrm{PRS})_n$ admitting the generalized Einstein tensor $G(X, Y)$ satisfies (24), but not (16). Changing the order of indices l and k in (23) and subtracting the expression so obtained from (23), we have

$$\nabla_{[l}\nabla_{k]} G_{ij} = \mu_{[lk]} G_{ij}, \tag{25}$$

where the bracket indicates antisymmetrization. If μ_{lk} is a symmetric tensor, then $\nabla_{[l}\nabla_{k]} G_{ij} = 0$, and vice versa.

Thus, we have the following result:

Lemma 5. *The birecurrency tensor of the generalized Einstein tensor $G(X, Y)$ is symmetric if and only if the equation*

$$\nabla_{[l}\nabla_{k]} G_{ij} = 0 \tag{26}$$

holds.

Now, by taking the covariant derivative of (13), we obtain

$$\begin{aligned} \nabla_l \nabla_k G_{ij} = (4\pi_k \pi_l + 2\nabla_l \pi_k)\left(S_{ij} - r\kappa g_{ij}\right) + 3\pi_k \pi_i S_{jl} \\ + 3\pi_k \pi_j S_{il} + 2\pi_i \pi_j S_{lk} + (\nabla_l \pi_i + 2\pi_i \pi_l) S_{jk} \\ + \left(\nabla_l \pi_j + 2\pi_l \pi_j\right) S_{ik} - 4P_l \pi_k \kappa g_{ij} - 2\nabla_l P_k \kappa g_{ij}, \end{aligned} \tag{27}$$

where $P_k = \pi_i S_k^i$.

The covariant derivative of P_k is

$$\nabla_l P_k = \nabla_l\left(\pi_h S_k^h\right) = (\nabla_l \pi_h) S_k^h + \pi_h\left(\nabla_l S_k^h\right). \tag{28}$$

Writing (1) as

$$\nabla_l S_k^i = 2\pi_l S_k^i + \pi_k S_k^i + \pi^i S_{kl,} \tag{29}$$

using (28) and (29), we achieve

$$\nabla_l P_k = (\nabla_l \pi_h) S_k^h + 2\pi_l P_k + \pi_k P_l + \pi S_{kl}, \tag{30}$$

$$(\nabla_l \pi_h) S_k^h = \nabla_l P_k - 2\pi_l P_k - \pi_k P_l - \pi S_{kl}. \tag{31}$$

Now, we apply Lemma 5, and by using equation (26), we obtain

$$\begin{aligned} 2(\nabla_l \pi_k - \nabla_k \pi_l)\left(S_{ij} - r\kappa g_{ij}\right) + (\pi_i \pi_k - \nabla_k \pi_i) S_{jl} \\ - (\pi_l \pi_i - \nabla_l \pi_i) S_{jk} + \left(3\pi_k \pi_j - \nabla_k \pi_j\right) S_{il} \\ - \left(3\pi_l \pi_j - \nabla_l \pi_j\right) S_{ik} + 4(\pi_l P_k - \pi_k P_l)\kappa g_{ij} \\ - 2(\nabla_l P_k - \nabla_k P_l)\kappa g_{ij} = 0. \end{aligned} \tag{32}$$

Contracting (32) with respect to i and j, we get

$$\begin{aligned} r(\nabla_l \pi_k - \nabla_k \pi_l)(1 - \kappa n) + 2(1 + \kappa n)(\pi_l P_k - \pi_k P_l) \\ + (\nabla_l \pi_h) S_k^h - (\nabla_k \pi_h) S_l^h - (\nabla_l P_k - \nabla_k P_l)\kappa n = 0. \end{aligned} \tag{33}$$

Substituting (31) into (33) yields

$$(1-\kappa n)[r(\nabla_l\pi_k-\nabla_k\pi_l)+(\nabla_l P_k-\nabla_k P_l)] + (1+2\kappa n)(\pi_l P_k-\pi_k P_l)=0. \quad (34)$$

If $\kappa = 1/n$, the generalized Einstein tensor $G(X, Y)$ reduces to the Einstein tensor $E(X, Y)$. So, we can state the following:

Theorem 6. *In $(PRS)_n$, the birecurrency tensor of Einstein tensor $E(X, Y)$ is symmetric if and only if the vector fields π_k and P_k are linearly dependent.*

Let us now recall that a $\varphi(\mathrm{Ric})$ vector field was introduced by Hinterleitner and Kiosak as a vector field satisfying the condition $\nabla\varphi = \mu\mathrm{Ric}$ [14], where μ is some constant, Ric is the Ricci tensor, and ∇ is the Levi-Civita connection.

If $\kappa = -1/2n$, then it follows from (34) that

$$r(\nabla_l\pi_k-\nabla_k\pi_l)+(\nabla_l P_k-\nabla_k P_l)=0. \quad (35)$$

It is evident that π_k and P_k are closed or $\pi(\mathrm{Ric})$ and $P(\mathrm{Ric})$ vector fields.

Therefore, we have

Theorem 7. *In $(PRS)_n$, the birecurrency tensor of generalized Einstein tensor $G(X, Y)$ with $\kappa = -1/2n$ is symmetric if and only if the vector fields π_k and P_k are closed or $\pi(Ric)$ and $P(Ric)$.*

Theorem 8. *In $(PRS)_n$, the birecurrency tensor of generalized Einstein tensor $G(X, Y)$ with $\kappa \neq -1/2n$ is symmetric if and only if the vector fields π_k and P_k are linearly dependent, and the vector field π_k is closed or $\pi(Ric)$.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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