Hindawi
Advances in Mathematical Physics
Volume 2020, Article ID 6617517, 10 pages
https://doi.org/10.1155/2020/6617517

Research Article

Fixed-Point Theorem for Nonlinear F-Contraction via w-Distance

Abdelkarim Kari,[1] **Mohamed Rossafi**,[2] **El Miloudi Marhrani**,[1] **and Mohamed Aamri**[1]

[1]*Laboratory of Algebra, Analysis and Applications Faculty of Sciences Ben M'Sik, Hassan II University, Casablanca, Morocco*
[2]*Department of Mathematics, University of Ibn Tofail, B.P. 133 Kenitra, Morocco*

Correspondence should be addressed to Mohamed Rossafi; rossafimohamed@gmail.com

Received 2 October 2020; Revised 3 November 2020; Accepted 5 November 2020; Published 19 November 2020

Academic Editor: Leopoldo Greco

The aim of this paper is to introduce a notion of (ϕ, F)-contraction defined on a metric space with w-distance. Moreover, fixed-point theorems are given in this framework. As an application, we prove the existence and uniqueness of a solution for the nonlinear Fredholm integral equations. Some illustrative examples are provided to advocate the usability of our results.

1. Introduction

By a contraction on a metric space (X, d), we understand a mapping $T : X \longrightarrow X$ satisfying for all $x, y \in X : d(Tx, Ty) \leq kd(x, y)$, where k is a real in $[0, 1)$.

In 1922, Banach proved the following theorem.

Theorem 1 (see [1]). *Let (X, d) be a complete metric space. Let $T : X \longrightarrow X$ be a contraction. Then,*

(i) *T has a unique fixed point $x \in X$*

(ii) *For every $x_0 \in X$, the sequence (x_n), where $x_{n+1} = Tx_n$, converges to x*

(iii) *We have the following estimate: for every $x \in X$, $d(x_n, x) \leq (k^n/1 - k)d(x_0, x_1)$, $n \in \mathbb{N}$*

As a result of its intelligibility and profitableness, the previous theorem has become a very celebrated and popular tool in solving the existence problems in many branches of mathematical analysis.

Many mathematicians extended the Banach contraction principle in two major directions, one by stating the conditions on the mapping T and second by taking the set X as more general structure.

Recently, Kari et al. [2] give some fixed-point results for generalized $\theta - \phi$-contraction in the framework of (α, η)-compete rectangular b-metric spaces.

In 2012, Wardowski [3] introduced the concept of F-contraction, using this concept, he proved the existence and uniqueness of a fixed point in complete metric spaces. This direction has been studied and generalized in different spaces, and various fixed-point theorems are developed [4, 5]. Cosentino and Vetro [6] presented some fixed-point results of Hardy-Rogers type for self-mappings on complete metric spaces or complete ordered metric spaces. In 2016, Piri and Kumam [7] introduced the modified generalized F-contractions, by combining the ideas of Dung and Hang [8], Piri and Kumam [9], Wardowski [3], and Wardowski and Van Dung [10], and gave some fixed-point result for these type mappings on complete metric space.

In 1996, Kada et al. initiated the notion of w-distance on a metric space; then, many authors used this concept to prove some results of fixed-point theory [11, 12].

Recently, Wongyat and Sintunavarat [13] introduced a special w-distance called ceiling distance and proved some fixed point for generalized contraction mappings with respect to this distance.

Later, Wardowski [14] studied a new type of contractions called nonlinear F-contraction.

In this paper, we shall obtain a fixed-point theorem for (ϕ, F)-contraction with respect to w-distance on complete metric spaces. Various examples are constructed to illustrate our results. As an application, we prove the existence and uniqueness of a solution for the nonlinear Fredholm integral equations.

The paper is structured as follows:

In Section 2, we briefly recall some definitions and basic properties used to prove our main results.

In Section 3, we present our results.

Section 4 is devoted to the application of the result in nonlinear integral equations.

2. Preliminaries

Kada et al. [15] introduced the concept of w-distance on a metric space as follows:

Definition 2 (see [15]). Let (X, d) be a metric space. A function $q : X \times X \longrightarrow \mathbb{R}^+$ is called a w -distance on X, if it satisfies the following three conditions for all $x, y, z \in X$:

(W_1) $q(x, y) \leq q(x, z) + q(z, y)$

(W_2) $q(x, .)$: $X \longrightarrow \mathbb{R}^+$ is lower semicontinuous on for all $x \in X$

(W_3) For each $\varepsilon > 0$, there exists $\delta > 0$ such that $q(x, y) \leq \delta$ and $q(x, z) \leq \delta$ imply $d(y, z) \leq \varepsilon$

Remark 3. Each metric on a nonempty set X is a w-distance on X.

Example 1 (see [13]). Let (X, d) be a metric space. The function $q : X \times X \longrightarrow \mathbb{R}^+$ defined by $q(x, y) = c$ for every $x, y \in X$ is a w-distance on X, where c is a positive real number. But q is not a metric since $q(x, x) = c \neq 0$ for any $x \in X$.

The following lemma is a useful tool for proving our results.

Lemma 4 (see [15]). *Let (X, d) be a metric space, q be a w -distance on X, $\{x_n\}$ and $\{y_n\}$ be two sequences in X, and $x, y, z \in X$.*

(i) If $\lim_{n \longrightarrow +\infty} q(x_n, x) = \lim_{n \longrightarrow +\infty} q(x_n, y) = 0$ then $x = y$. In particular, if $q(z, x) = q(z, y) = 0$, then $x = y$

(ii) If $d(x_n, y_n) \leq \alpha_n$ and $d(x_n, y) \leq \beta_n$ for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, +\infty[$ converging to 0, then $\{y_n\}$ converges to y

(iii) If for each $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $m > n > N_\varepsilon$ implies $q(x_n, x_m) < \varepsilon$, then $\{x_n\}$ is a Cauchy sequence

Definition 5 (see [13]). A w-distance q on a metric space (X, d) is said to be a ceiling distance of d if and only if

$$q(x, y) \geq d(x, y), \tag{1}$$

for all $x, y \in X$.

Example 2 (see [13]). Let $X = \mathbb{R}$ with the metric $d : X \times X \longrightarrow \mathbb{R}^+$ defined by $d(x, y) = |x - y|$ for all $x, y \in X$, and let a, $b \geq 1$. Define the function $q : X \times X \longrightarrow \mathbb{R}^+$ by

$$q(x, y) = \max \{a(y - x), b(x - y)\}, \tag{2}$$

for all $x, y \in X$. Then, q is a ceiling distance of d.

The following definition was introduced by Wardowski.

Definition 6 (see [3]). Let F be the family of all functions $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that

(i) F is strictly increasing

(ii) For each sequence $(x_n)_{n\in\mathbb{N}}$ of positive numbers,

$$\lim_{n \longrightarrow \infty} x_n = 0, \text{ if and only if } \lim_{n \longrightarrow \infty} F(x_n) = -\infty \tag{3}$$

(iii) There exists $k \in]0, 1[$ such that $\lim_{x \longrightarrow 0} x^k F(x) = 0$

Recently, Piri and Kuman [9] extended the result of Wardowski [3] by changing the condition (iii) in Definition 6 as follows.

Definition 7 (see [9]). Let Γ be the family of all functions $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that

(i) F is strictly increasing

(ii) For each sequence $(x_n)_{n\in\mathbb{N}}$ of positive numbers,

$$\lim_{n \longrightarrow \infty} x_n = 0, \text{ if and only if } \lim_{n \longrightarrow \infty} F(x_n) = -\infty \tag{4}$$

(iii) F is continuous

The following definition introduced by Wardowski [14] will be used to prove our result.

Definition 8 (see [14]). Let $\mathbb{F}$ be the family of all functions $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ and Φ be the family of all functions $\phi :]0, +\infty[\longrightarrow]0, +\infty[$ satisfy the following conditions:

(i) F is strictly increasing

(ii) For each sequence $(x_n)_{n\in\mathbb{N}}$ of positive numbers,

$$\lim_{n \longrightarrow \infty} x_n = 0, \text{ if and only if } \lim_{n \longrightarrow \infty} F(x_n) = -\infty \tag{5}$$

(iii) $\liminf_{\alpha \longrightarrow s^+} \phi(s) > 0$ for all $s > 0$

(iv) There exists $k \in]0, 1[$ such that

$$\lim_{x\longrightarrow 0^+} x^k F(x) = 0 \tag{6}$$

By replacing the condition (iii) in Definition 8, we introduce a new class of F-contraction.

Definition 9. Let $\mathfrak{F}$ be the family of all functions $F:\mathbb{R}^+ \longrightarrow \mathbb{R}$ and Φ be the family of all functions $\phi:]0,+\infty[\longrightarrow]0,+\infty[$ satisfy the following conditions:

(i) F is strictly increasing

(ii) $\liminf_{\alpha\longrightarrow s^+}\phi(s) > 0$ for all $s > 0$

(iii) For each sequence $(x_n)_{n\in\mathbb{N}}$ of positive numbers,

$$\lim_{n\longrightarrow\infty} x_n = 0, \text{ if and only if } \lim_{n\longrightarrow\infty} F(x_n) = -\infty \tag{7}$$

(iv) F is continuous

Example 3.

(i) Let $F_1(t) = (-1/\sqrt{t}) + t$, $F_2(t) = (-1/(e^t - 1)) + e^{\sqrt{t}}$. Then, $F_1, F_2 \in \mathfrak{F}$

(ii) Let $\phi_1(t) = 1/(t+1)$, $\phi_2(t) = 1/(\sqrt{t}+1)$, and $\phi_3(t) = 1/(t^2+1)$. Then, ϕ_1, ϕ_2 and $\phi_3 \in \Phi$

Definition 10 (see [14]). Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called a (ϕ, F) -contraction on (X, d), if there exist $F \in \mathbb{F}$ and $\phi \in \Phi$ such that

$$F(d(Tx, Ty)) + \phi(d(x, y)) \le F(d(x, y)), \tag{8}$$

for all $x, y \in X$ for which $Tx \neq Ty$.

3. Main Result

In this paper, using the idea introduced by Wongyat and Sintunavarat [13], we presented the concept of (ϕ, F)-contraction on a complete metric space with w-distance.

Definition 11. Let q be a w-distance on a metric space (X, d). A mapping $T : X \longrightarrow X$ is said to be a w-generalized (ϕ, F) -contraction of type $(\mathbb{F})$ on (X, d) if there exist $F \in \mathbf{F}$ and $\phi \in \Phi$ such that

$$F(q(Tx, Ty)) + \phi(q(x, y)) \le F(q(x, y)), \tag{9}$$

for all $x, y \in X$ for which $Tx \neq Ty$.

Theorem 12. *Let (X, d) be a complete metric space and $q : X \times X \longrightarrow [0,+\infty[$ be a w -distance on X and a ceiling distance of d, supposing that $T : X \longrightarrow X$ is a w -generalized (ϕ, F) -contraction of type $(\mathbb{F})$. Then, T has a unique fixed point on X.*

Proof. Let $x_0 \in X$ be an arbitrary point in X; define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by

$$x_{n+1} = Tx_n = T^{n+1}x_0, \tag{10}$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then the proof is finished.

We can suppose that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$.

Since q is a ceiling distance of d, we obtain $q(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$.

Substituting $x = x_{n-1}$ and $y = x_n$, from (9), for all $n \in \mathbb{N}$, we have

$$F[q(x_n, x_{n+1})] + \phi(q(x_{n-1}, x_n)) \le [F(q(x_{n-1}, x_n))], \quad \forall n \in \mathbb{N}. \tag{11}$$

Imply that

$$F[q(x_n, x_{n+1})] \le [F(q(x_{n-1}, x_n))] - \phi(q(x_{n-1}, x_n)) < [F(q(x_{n-1}, x_n))]. \tag{12}$$

Since F is increasing, then $q(x_n, x_{n+1}) < q(x_{n-1}, x_n)$. Therefore, $q(x_n, x_{n+1})_{n\in\mathbb{N}}$ is monotone strictly decreasing sequence of nonnegative real numbers. Consequently, there exists $\alpha \ge 0$ such that

$$\lim_{n\longrightarrow\infty} q(x_n, x_{n+1}) = \alpha. \tag{13}$$

The inequality (11) implies

$$\begin{aligned} F(q(x_n, x_{\mathrm{n}+1})) &\le (F(q(x_{n-1}, x_n))) - \phi(q(x_{n-1}, x_n)) \\ &\le (F(q(x_{n-2}, x_{n-1})) - \phi(q(x_{n-1}, x_n)) \\ &\quad - \phi(q(x_{n-2}, x_{n-1})) \le \cdots \\ &\le F(q(x_0, x_1)) - \sum_{i=0}^{n} \phi(q(x_i, x_{i+1})). \end{aligned} \tag{14}$$

Since $\liminf_{\alpha\longrightarrow s^+}\phi(\alpha) > 0$, we have $\liminf_{n\longrightarrow\infty}\phi(q(x_{n-1}, x_n)) > 0$; then from the definition of the limit, there exists $n_0 \in \mathbb{N}$ and $A > 0$ such that for all $n \ge n_0$, $\phi(q(x_{n-1}, x_n)) > A$, hence

$$\begin{aligned} F(q(x_n, x_{n+1})) &\le F(q(x_0, x_1)) - \sum_{i=0}^{n_0-1} \phi(q(x_i, x_{i+1})) \\ &\quad - \sum_{i=n_0-1}^{n} \phi(q(x_i, x_{i+1}) \le F(q(x_0, x_1)) \\ &\quad - \sum_{i=n_0-1}^{n} A = F(q(x_0, x_1)) - (n - n_0)A, \end{aligned} \tag{15}$$

for all $n \ge n_0$. Taking the limit as $n \longrightarrow \infty$ in the above inequality, we get

$$\lim_{n\longrightarrow\infty} F(q(x_n, x_{n+1})) \le \lim_{n\longrightarrow\infty} [F(q(x_0, x_1)) - (n-n_0)A], \tag{16}$$

that is, $\lim_{n\longrightarrow\infty} F(q(x_n, x_{n+1})) = -\infty$; then, from the condition (ii) of Definition 8, we conclude that

$$\lim_{n\longrightarrow\infty} q(x_n, x_{n+1}) = 0. \tag{17}$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e., $\lim_{n,m\longrightarrow\infty} d(x_n, x_m) = 0$, for all $n, m \in \mathbb{N}$.

The condition (iv) of Definition 8 implies that there exists $k \in]0, 1[$ such that

$$\lim_{n\longrightarrow\infty} [q(x_n, x_{n+1})]^k q(x_n, x_{n+1}) = 0. \tag{18}$$

From the inequality (12), we get

$$[q(x_n, x_{n+1})]^k F(q(x_n, x_{n+1})) \le [q(x_n, x_{n+1})]^k [F(q(x_0, x_1)) - (n-n_0)A]. \tag{19}$$

Hence,

$$\begin{aligned} &[q(x_n, x_{n+1})]^k F(q(x_n, x_{n+1})) - [q(x_n, x_{n+1})]^k F(q(x_0, x_1)) \\ &\quad \le -[q(x_n, x_{n+1})]^k (n-n_0)A \le 0. \end{aligned} \tag{20}$$

Taking limit $n \longrightarrow \infty$ in the above inequality, we conclude that

$$\lim_{n\longrightarrow\infty} q(x_n, x_{n+1})^k (n-n_0)A = 0. \tag{21}$$

Then, there exists $n_1 \in \mathbb{N}$, such that for all $n \ge n_1$,

$$q(x_n, x_{n+1}) \le \frac{1}{[(n-n_0)A]^k}. \tag{22}$$

Therefore, for $m > n \ge \max\{n_0, n_1\}$, we have

$$\begin{aligned} q(x_n, x_m) &\le q(x_n, x_{n+1}) + q(x_{m+1}, x_{m+2}) + \cdots + q(x_{m-1}, x_m) \\ &= \sum_{i=1}^{m-1} q(x_i, x_{i+1}) \le \sum_{i=1}^{\infty} q(x_i, x_{i+1}) \le \sum_{i=1}^{\infty} \frac{1}{[(i-n_0)A]^k}. \end{aligned} \tag{23}$$

Since $0 < k < 1$, then

$$\lim_{n,m\longrightarrow\infty} q(x_n, x_m) = 0. \tag{24}$$

By Lemma 4, we can conclude that $\{x_n\}$ is a Cauchy sequence in X. By the completeness of (X, d), there exists $z \in X$ such that

$$\lim_{n\longrightarrow\infty} d(x_n, z) = 0. \tag{25}$$

Now, we show that $d(Tz, z) = 0$; arguing by contradiction, we assume that

$$d(Tz, z) > 0 \Rightarrow q(Tz, z) > 0. \tag{26}$$

From (24), for each $r > 0$, there is $n_r \in \mathbb{N}$ such that

$$q(x_{n_r}, x_n) < \frac{1}{r}, \tag{27}$$

for all $n_r > r$. Since $q(x_{n_r}, .)$ is lower semicontinuous and $x_n \longrightarrow x$ as $n \longrightarrow \infty$, we get

$$q(x_{n_r}, z) \le \liminf_{x\longrightarrow\infty} q(x_{n_r}, x_n) \le \frac{1}{r}, \tag{28}$$

implying that

$$\liminf_{n\longrightarrow\infty} q(x_{n_r}, z) = 0. \tag{29}$$

Now, by triangular inequality, we get,

$$q(Tx_{n_r}, Tz) \le q(Tx_{n_r}, z) + q(z, Tz), \tag{30}$$

$$q(z, Tz) \le q(z, x_{n_r}) + q(x_{n_r}, Tz). \tag{31}$$

By letting $n \longrightarrow \infty$ in inequality (30) and (31), we obtain

$$q(z, Tz) \le \lim_{n\longrightarrow\infty} q(Tx_n, Tz) \le q(z, Tz). \tag{32}$$

Therefore,

$$\lim_{n\longrightarrow\infty} q(Tx_{n_r}, Tz) = q(z, Tz). \tag{33}$$

Let $A = d(z, Tz) > 0$, from the definition of the limit, there exists $n_2 \in \mathbb{N}$ such that

$$\left| q(Tx_{n_r}, Tz) - q(z, Tz) \right| \le A, \quad \forall n \ge n_2, \tag{34}$$

which implies that

$$q(Tx_{n_r}, Tz) > 0, \quad \forall n \ge n_2. \tag{35}$$

Applying (9) with $x = z$ and $y = x_{n_r}$, we have

$$F(q(Tz, Tx_{n_r})) + \phi(q(z, x_{n_r})) \le F(q(z, x_{n_r})), \tag{36}$$

which implies that

$$F(q(Tz, Tx_{n_k})) \le F(q(z, x_{n_r})). \tag{37}$$

Since F is increasing, we get

$$q(Tz, Tx_{n_r}) \le q(z, x_{n_r}). \tag{38}$$

By letting $n \longrightarrow \infty$ in the above inequality, we obtain

$$\lim_{n\longrightarrow\infty} q(Tx_{n_r}, Tz) = q(z, Tz) = 0. \quad (39)$$

Which is a contradiction, then $d(z, Tz) = q(z, Tz) = 0$, so $Tz = z$.

For uniqueness, now, suppose that $z, u \in X$ are two fixed points of T such that $u \neq z$. Therefore, we have

$$q(Tz, Tu) = q(z, u) > 0. \quad (40)$$

Applying (9) with $x = z$ and $y = u$, we have

$$F(q(Tu, Tz)) = F(q(z, u)) + \phi(q(z, u)) \le [F(q(z, u))], \quad (41)$$

implying

$$q(u, z) < q(u, z), \quad (42)$$

which is a contradiction. Therefore, $u = z$.

Example 4. Let $X = [0,+\infty[$ with the metric $d : X \times X \longrightarrow [0,+\infty[$ defined by

$$d(x, y) = |x - y|, \quad (43)$$

for all $x, y \in X$. Define a mapping $T : X \longrightarrow X$ by

$$Tx = \frac{x}{3}. \quad (44)$$

Suppose that $F(t) = \ln(t)$ and $\phi(t) = 1/(t + 1)$, clearly $F \in \mathbb{F}$ and $\phi \in \Phi$. Also, we define a w-distance $q : X \times X \longrightarrow [0,+\infty[$ by

$$q(x, y) = \max\{x, y\}, \quad (45)$$

for all $x, y \in X$. It is easy to see that q is a ceiling distance of d. Now, we will show that T satisfies the condition (9).

Case 1. If $x \ge y$, then $q(x, y) = x$, $q(Tx, Ty) = x/3$, and $\phi(q(x, y)) = 1/(x + 1)$. Thus,

$$\begin{aligned} F(q(x, y)) &= \ln(x) \\ F(q(Tx, Ty)) &= \ln\left(\frac{x}{3}\right). \end{aligned} \quad (46)$$

We prove that T is a (ϕ, F) contraction mapping of type $(\mathbb{F})$. Indeed,

$$\begin{aligned} &F(q(Tx, Ty) + \phi(q(x, y)) - F(q(x, y)) \\ &\quad = \ln\left(\frac{x}{3}\right) + \frac{1}{x+1} - \ln(x) = \frac{1}{x+1} - \ln(3) \le 0. \end{aligned} \quad (47)$$

Therefore,

$$F(q(Tx, Ty)) + \phi(q(x, y)) \le F(q(x, y)). \quad (48)$$

Case 2. If $x < y$, then $q(x, y) = y$, $q(Tx, Ty) = y/3$, and $\phi(q(x, y)) = 1/(y + 1)$. Thus,

$$\begin{aligned} F(q(x, y)) &= \ln(y) \\ F(q(Tx, Ty)) &= \ln\left(\frac{y}{3}\right). \end{aligned} \quad (49)$$

Therefore,

$$F(q(Tx, Ty)) + \phi(q(x, y)) \le F(q(x, y)). \quad (50)$$

Hence, 0 is the unique fixed point of T.

Example 5. Consider the sequence $(S_n)_{n\in\mathbb{N}^*}$ defined as follows:

$$\begin{aligned} S_1 &= 1 \times 2, \\ S_2 &= 1 \times 2 + 2 \times 3, \cdots, \\ S_n &= 1 \times 2 + 2 \times 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}. \end{aligned} \quad (51)$$

Let the metric $d : X \times X \longrightarrow [0,+\infty[$ defined by

$$d(x, y) = |x - y|, \quad (52)$$

for all $x, y \in X$.

Define a mapping $T : X \longrightarrow X$ by

$$T(S_n) = \begin{cases} S_1 & \text{if } n = 1, \\ S_{n-1} & \text{if } n \ge 2. \end{cases} \quad (53)$$

Clearly, the Banach contraction is not satisfied. In fact, we can check easily that

$$\lim_{n\longrightarrow\infty} \frac{d(T(S_n), T(S_1))}{d(S_n, S_1)} = \lim_{n\longrightarrow\infty} \frac{n(n-1)(n+1) - 6}{n(n+1)(n+2) - 6} = 1. \quad (54)$$

Suppose that $F(t) = \ln(t)$ and $\phi(t) = 1/(t + 1)$, clearly $F \in \mathbb{F}$ and $\phi \in \Phi$. Also, we define a w-distance $q : X \times X \longrightarrow [0,+\infty[$ by

$$q(x, y) = \max\{x, y\}, \quad (55)$$

for all $x, y \in X$. It is easy to see that q is a ceiling distance of d.

Now, we will show that T satisfies the condition (9).

Case 1. $n = 1$ and $m \ge 2$. In this case, we have

$$\begin{aligned} q(S_n, S_1) &= \frac{n(n+1)(n+2)}{3}, \\ q(T(S_n), T(S_1)) &= \frac{n(n-1)(n+2)}{3}, \\ \phi(q(S_n, S_1)) &= \frac{3}{n(n+1)(n+2) + 3}. \end{aligned} \quad (56)$$

Thus,

$$F(q(S_n, S_1)) = \ln\left[\frac{n(n+1)(n+2)}{3}\right],$$
$$F(q(T(S_n), T(S_1))) = \ln\left[\frac{n(n-1)(n+1)}{3}\right]. \quad (57)$$

On the other hand,

$$\begin{aligned}
&F(q(T(S_n), T(S_1))) + \phi(d(S_n, S_1))) - F(q(S_n, S_1)) \\
&= \ln\left[\frac{n(n-1)(n+1)}{3}\right] - \ln\left[\frac{n(n+1)(n+2)}{3}\right] \\
&+ \frac{3}{n.(n+1)(n+2)+3} = \ln\left[\frac{n-1}{n+2}\right] + \frac{3}{n(n+1)(n+2)+3} \\
&\leq 0, \text{for all } n \geq 2.
\end{aligned} \quad (58)$$

Therefore,

$$F(q(T(S_n), T(S_1))) + \phi(q(S_n, S_1))) \leq F(q(S_n, S_1)). \quad (59)$$

Case 2. $m > n > 1$. In this case, we have

$$q(S_n, S_m) = \frac{m(m+1)}{3},$$
$$q(T(S_n), T(S_m)) = \frac{m(m-1)}{3}, \quad (60)$$
$$\phi(q(S_n, S_m)) = \frac{3}{m^2 + m + 3}.$$

Thus,

$$F(q(S_m, S_1)) = \ln\left[\frac{m(m+1)(m+2)}{3}\right],$$
$$F(q(T(S_n), T(S_1))) = \ln\left[\frac{m(m-1)(m+1)}{3}\right]. \quad (61)$$

On the other hand,

$$\begin{aligned}
&F(q(T(S_n), T(S_m))) + \phi(q(S_n, S_n)) - F(q(S_n, S_n)) \\
&= \ln\left[\frac{m(m-1)(m+1)}{3}\right] - \ln\left[\frac{m(m+1)(m+2)}{3}\right] \\
&+ \frac{3}{m(m+1)(m+2)+3} = \ln\left[\frac{m-1}{m+2}\right] + \frac{3}{n(m+1)(m+2)+3} \\
&\leq 0, \text{for all } n \geq 2.
\end{aligned} \quad (62)$$

Therefore,

$$F(q(T(S_n), T(S_m))) + \phi(q(S_n, S_m))) \leq F(q(S_n, S_m)). \quad (63)$$

Thus, the inequality (9) that is satisfied implies that T has a unique fixed point. In this example, S_1 is the unique fixed point of T.

Definition 13. Let q be a w -distance on a metric space (X, d). A mapping $T : X \longrightarrow X$ is said to be a w -generalized (ϕ, F) -contraction of type $(\mathfrak{F})$ on (X, d) if there exist $F \in \mathfrak{F}$ and $\phi \in \Phi$ such that

$$F(q(Tx, Ty)) + \phi(q(x, y)) \leq F(q(x, y)), \quad (64)$$

for all $x, y \in X$ for which $Tx \neq Ty$.

Theorem 14. *Let (X, d) be a complete metric space and $q : X \times X \longrightarrow]0,+\infty[$ be a w -distance on X and a ceiling distance of d. Suppose that $T : X \longrightarrow X$ is a (ϕ, F) -contraction of type $(\mathfrak{F})$. Then, T has a unique fixed point on X.*

Proof. As in the proof of Theorem 12, we can conclude that

$$\lim_{n \longrightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (65)$$

Next, we show that (x_n) is a Cauchy sequence, i.e.,

$$\lim_{n,m \longrightarrow \infty} d(x_m, x_n) = 0. \quad (66)$$

Now, we claim that $\lim_{n,m \longrightarrow \infty} q(x_m, x_n) = 0$. Arguing by contradiction, we assume that there exists $\varepsilon > 0$ we can find and sequences $(m(k))_k$ and $(n(k))_k$ of positive integers such that for all positive integers, $n(k) > m(k) > k$,

$$q\left(x_{m(k)}, x_{n(k)}\right) \geq \varepsilon, \quad (67)$$

$$q\left(x_{m(k)}, x_{n(k)-1}\right) < \varepsilon. \quad (68)$$

Again by triangular inequality and using (65), (67), and (68), we get

$$\begin{aligned}
\varepsilon \leq q\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &\leq q\left(x_{m_{(k)}}, x_{n_{(k)-1}}\right) + q\left(x_{n_{(k)-1}}, x_{m_{(k)}}\right) \\
&< \varepsilon + q\left(x_{n(k)-1}, x_{m_{(k)}}\right).
\end{aligned} \quad (69)$$

So,

$$\lim_{k \longrightarrow \infty} q\left(x_{m(k)}, x_{n(k)}\right) = \varepsilon. \quad (70)$$

Again by the triangular inequality, for all $n \in \mathbb{N}$, we have the following two inequalities

$$\begin{aligned}
&q\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq q\left(x_{m_{(k)+1}}, x_{m(k)}\right) + q\left(x_{m(k)}, x_{n_{(k)}}\right) + q\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right), \\
&q\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq q\left(x_{m_{(k)}}, x_{m(k)+1}\right) + q\left(x_{m(k)+1}, x_{n_{(k)+1}}\right) + q\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right).
\end{aligned} \quad (71)$$

Letting $k \longrightarrow \infty$ in the above inequalities, using (65) and (70), we obtain

$$\lim_{k \longrightarrow \infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) = \varepsilon. \quad (72)$$

Hence, from the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \geq \varepsilon, \text{ for all } n \geq n_0. \tag{73}$$

Applying (64) with $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$, we obtain

$$F\left[q\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right] + \phi\left[q\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right] \leq F\left(q\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right). \tag{74}$$

Letting $k \longrightarrow \infty$ the above inequality, we obtain

$$\begin{aligned}\lim_{k\longrightarrow\infty} F\left[q\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right] &\leq \lim_{k\longrightarrow\infty} F\left(q\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right) - \lim_{k\longrightarrow\infty} \phi \cdot \left[q\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right] \\ &= \lim_{k\longrightarrow\infty} F\left(q\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right) - \liminf_{k\longrightarrow\infty} \phi \cdot \left[q\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right].\end{aligned} \tag{75}$$

Since F is a continuous and $\lim\inf_{k\longrightarrow\infty}\phi[q(x_{m_{(k)}}, x_{n_{(k)}})] > 0$, we conclude that

$$\varepsilon < \varepsilon, \tag{76}$$

which is a contradiction. Then,

$$\lim_{n,m\longrightarrow\infty} q(x_m, x_n) = 0. \tag{77}$$

By the condition (iii) of Lemma 4, we can conclude that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $u \in X$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$.

As in the proof of Theorem 12, we conclude that for each $l > 0$, there exists n_2 such that $\lim_{n\longrightarrow\infty} q(Tx_{n_l}, Tu) = q(u, Tu)$, $\lim_{n\longrightarrow\infty} q(x_{n_l}, u) = 0$, and $q(Tx_{n_l}, Tu) > 0$ for all $n_l \geq n_2$.

Now applying (64) with $x = x_{n_l}$ and $y = u$, we get

$$F\left[q\left(Tx_{n_l}, Tu\right)\right] + \phi\left[q\left(x_{n_l}, u\right)\right] \leq F\left[q\left(x_{n_l}, u\right)\right], \tag{78}$$

which implies that

$$\lim_{n\longrightarrow\infty} F\left[q\left(Tx_{n_l}, Tu\right)\right] \leq \lim_{n\longrightarrow\infty} F\left[q\left(x_{n_l}, u\right)\right]. \tag{79}$$

Therefore, $\lim_{n\longrightarrow+\infty} F[q(Tx_{n_l}, Tu)] = 0$. Hence, $q(u, Tu) = 0$, so $Tu = u$.

Following the proof of Theorem 12, we know that u is a unique fixed point of T. This complete the proof.

Example 6. Let $X = [0,+\infty[$ with the metric $d : X \times X \longrightarrow [0,+\infty[$ defined by

$$d(x, y) = |x - y|, \tag{80}$$

for all $x, y \in X$. Define a mapping $T : X \longrightarrow X$ by

$$Tx = \frac{x}{2}. \tag{81}$$

Suppose that $F(t) = t - (1/t)$ and $\phi(t) = 1/(t + 1)$, clearly $F \in \mathfrak{I}$ and $\phi \in \Phi$. Also, we define a w-distance $q : X \times X \longrightarrow [0,+\infty[$ by

$$q(x, y) = \max\{x, y\}, \tag{82}$$

for all $x, y \in X$. It is easy to see that q is a ceiling distance of d. Now, we will show that T satisfies the condition (64).

We prove that T is a (ϕ, F)-contraction mapping of type $(\mathfrak{I})$.

Case 1. If $x \geq y$, then $q(x, y) = x$, $q(Tx, Ty) = (x/2) - (2/x)$, and $\phi(q(x, y)) = 1/(x + 1)$. Thus,

$$\begin{aligned}F(q(x, y)) &= x - \frac{1}{x}, \\ F(q(Tx, Ty)) &= \frac{x}{2} - \frac{2}{x}.\end{aligned} \tag{83}$$

On the other hand,

$$F(q(Tx, Ty)) + \phi(q(x, y)) - F(q(x, y)) = \frac{x}{2} - \frac{2}{x} + \frac{1}{x + 1} - x + \frac{1}{x} \leq 0. \tag{84}$$

Therefore,

$$F(q(Tx, Ty)) + \phi(q(x, y)) \leq F(q(x, y)). \tag{85}$$

Case 2. If $x < y$, then $q(x, y) = y$, $q(Tx, Ty) = (y/2) - (2/y)$, and $\phi(q(x, y)) = 1/(y + 1)$. Thus,

$$\begin{aligned}F(q(x, y)) &= y - \frac{1}{y}, \\ F(q(Tx, Ty)) &= \frac{y}{2} - \frac{2}{y}.\end{aligned} \tag{86}$$

On the other hand,

$$\begin{aligned}&F(q(Tx, Ty)) + \phi(q(x, y)) - F(q(x, y)) \\ &\quad = \frac{y}{2} - \frac{2}{y} + \frac{1}{y + 1} - y + \frac{1}{y} = -\frac{y^2 + 2}{2y} + \frac{1}{y + 1} \leq 0 \leq 0.\end{aligned} \tag{87}$$

Therefore,

$$F(q(Tx, Ty)) + \phi(q(x, y)) \leq F(q(x, y)). \tag{88}$$

Hence, 0 is the unique fixed point of T.

Example 7. Let X be the set defined by

$$X = \{\lambda_n : n \in \mathbb{N}^*\}, \tag{89}$$

where

$$\lambda_n = \frac{n(n+1)}{2}. \tag{90}$$

Let the metric $d : X \times X \longrightarrow [0,+\infty[$ defined by

$$d(x, y) = |x - y|, \tag{91}$$

for all $x, y \in X$. Define a mapping $T : X \longrightarrow X$ by

$$T(\lambda_n) = \begin{cases} 1 & \text{if } n = 1, \\ \dfrac{n(n-1)}{2} & \text{if } n \geq 2. \end{cases} \tag{92}$$

Clearly, the Banach contraction is not satisfied. In fact, we can check easily that

$$\lim_{n \longrightarrow \infty} \frac{d(T(\lambda_n), T(\lambda_1))}{d(\lambda_n, \lambda_1)} = \lim_{n \longrightarrow \infty} \frac{n^2 - n - 2}{n^2 + n - 2} = 1. \tag{93}$$

Suppose that $F(t) = (-1/t) + t$ and $\phi(t) = 1/(t+1)$, clearly $F \in \mathfrak{F}$ and $\phi \in \Phi$. Also, we define a w-distance $q : X \times X \longrightarrow [0,+\infty[$ by

$$q(x, y) = \max \{x, y\}, \tag{94}$$

for all $x, y \in X$. It is easy to see that q is a ceiling distance of d. Now, we will show that T satisfies the condition (64).

Case 1. $n = 1$ and $m \geq 2$. In this case, we have

$$\begin{aligned} q(\lambda_n, \lambda_1) &= \frac{n(n+1)}{2}, \\ q(T(\lambda_n), T(\lambda_1)) &= \frac{n(n-1)}{2}, \\ \phi(q(\lambda_n, \lambda_1)) &= \frac{1}{(n(n+1))/2 + 1}. \end{aligned} \tag{95}$$

Thus,

$$\begin{aligned} F(q(\lambda_n, \lambda_1)) &= \frac{2}{-n^2 - n} + \frac{n^2 + n}{2}, \\ F(q(T(\lambda_n), T(\lambda_1))) &= \frac{2}{-n^2 + n} + \frac{n^2 - n}{2}, \end{aligned} \tag{96}$$

On the other hand,

$$\begin{aligned} &F(q(T(\lambda_n), T(\lambda_1))) + \phi(q(\lambda_n, \lambda_1))) - F(q(\lambda_n, \lambda_1)) \\ &= \frac{-4n}{(n-1)(n^2 + n)} - n + \frac{2}{n^2 + n + 2} \leq 0. \end{aligned} \tag{97}$$

Therefore,

$$F(q(T\lambda_n, T\lambda_1)) + \phi(\lambda_n, \lambda_1)) \leq F(q(\lambda_n, \lambda_1)). \tag{98}$$

Case 2. $m > n > 1$. In this case, we have

$$\begin{aligned} q(\lambda_n, \lambda_m) &= \frac{m(m+1)}{2}, \\ q(T(\lambda_n), T(\lambda_m)) &= \frac{m(m-1)}{2}, \\ \phi(q(\lambda_n, \lambda_m)) &= \frac{2}{m^2 + m + 2}. \end{aligned} \tag{99}$$

Thus,

$$\begin{aligned} F(q(\lambda_n, \lambda_m)) &= \frac{2}{-m^2 - m} + \frac{m^2 + m}{2}, \\ F(q(T\lambda_n, T\lambda_m)) &= \frac{2}{-m^2 + m} + \frac{m^2 - m}{2}, \end{aligned} \tag{100}$$

On the other hand,

$$\begin{aligned} &F(q(T(\lambda_n), T(\lambda_m))) + \phi(\lambda_n, \lambda_m)) - F(q(\lambda_n, \lambda_m)) \\ &= \frac{-4m}{(m^2 + m)(m^2 - m)} - m + \frac{2}{m^2 + m + 2} \leq 0. \end{aligned} \tag{101}$$

Therefore,

$$F(q((T\lambda_n), T(\lambda_m))) + \phi(\lambda_n, \lambda_m)) \leq F(q(\lambda_n, \lambda_m)). \tag{102}$$

Thus, the inequality (64) that is satisfied implies that T has a unique fixed point. In this example, λ_1 is the unique fixed point of T.

Taking $q = d$ in Theorems 12 and 14, we obtain the following result.

Corollary 15. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a (ϕ, F)-contraction of type $\mathbb{F}$. Then, T has a unique fixed point.*

Corollary 16. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a (ϕ, F)-contraction of type $\mathfrak{F}$. Then, T has a unique fixed point.*

4. Application to Nonlinear Integral Equations

In this section, we endeavor to apply Theorems 12 and 14 to prove the existence and uniqueness of the integral equation of Fredholm type:

$$x(t) = \lambda \int_a^b K(t, s, x(s))ds, \tag{103}$$

where $a, b \in \mathbb{R}$, $x \in C([a, b], \mathbb{R})$, and $K : [a, b]^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function.

Theorem 17. *Consider the Fredholm integral equation (103) and assume that the kernel function K satisfies the condition $|K(t,s,x(s))| + |K(t,s,y(s))| \le e^{-1/(|x(t)|+|y(t)|+1)}(|x(t)| + |y(t)|$) for all $t, s \in [a,b]$ and $x, y \in \mathbb{R}$. Then, the equation (4.1) has a unique solution $x \in C([a,b]$ for some constant λ depending on the constants a, b.*

Proof. Let $X = C([a,b]$ and $T : X \longrightarrow X$ defined by

$$T(x)(t) = \lambda \int_a^b K(t,s,x(s))ds, \tag{104}$$

for all $x \in X$. Clearly, X with the metric $d : X \times X \longrightarrow [0,+\infty[$ given by

$$d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|, \tag{105}$$

for all $x, y \in X$, is a complete metric space. Next, define the function $q : X \times X \longrightarrow [0,+\infty[$ by

$$q(x,y) = \sup_{t\in[a,b]} |x(t)|+|y(t)|, \tag{106}$$

for all $x, y \in X$.

Clearly, q is a w-distance on X and a ceiling distance of d. We will find the condition on λ under which the operator has a unique fixed point which will be the solution of the integral equation (103). Assume that $x, y \in X$ and $t, s \in [a,b]$. Then, we get

$$\begin{aligned} |Tx(t)| + |Ty(t)| &= |\lambda| \left(\left| \int_a^b K(t,s,x(s))ds \right| + \left| \int_a^b K(t,s,y(s))ds \right| \right) \\ &\le |\lambda| \int_a^b |K(t,s,x(s))ds| + |\lambda| \int_a^b |K(t,s,y(s))ds| \\ &\le |\lambda| \int_a^b \left(e^{-1/(|x(s)|+|y(s)|+1)}(|x(s)| + |y(s)|) \right) ds, \end{aligned} \tag{107}$$

which implies that

$$\begin{aligned} \sup_{t\in[a,b]} (|Tx(t)| + |Ty(t)|) &= \sup_{t\in[a,b]} \left(|\lambda| \left(\left(\int_a^b K(t,x(s))ds \right| + \left| \int_a^b K(t,y(s))ds \right| \right) \right) \\ &\le \sup_{t\in[a,b]} \le |\lambda| \int_a^b |K(t,s,x(s))ds| \\ &\quad + |\lambda| \int_a^b |K(t,s,y(s))ds| \le |\lambda| \int_a^b \\ &\quad \cdot \left(e^{-\sup_{s\in[a,b]} (1/(|x(s)|+|y(s)|+1))} \left(\sup_{s\in[a,b]} |x(s)| + |y(s)| \right) \right) ds. \end{aligned} \tag{108}$$

Since by the definition of the w-distance on X and a ceiling distance of d, we have $q(Tx,Ty) > 0$ and $q(x,y) > 0$ for any $x \ne y$, then we can take natural logarithm sides and get

$$\begin{aligned} \ln [q(Tx,Ty)] &\le \ln \left[\lambda(b-a)e^{-1/q(x,y)} q(x,y) \right] \\ &= -\frac{1}{q(x,y)+1} + \ln (|\lambda|(b-a)q(x,y)), \end{aligned} \tag{109}$$

provided that $|\lambda|(b-a) \le 1$, which implies that

$$\ln [q(Tx,Ty)] \le -\frac{1}{q(x,y)+1} + \ln (q(x,y)). \tag{110}$$

Hence,

$$F(q(Tx,Ty)) + \phi(q(x,y)) \le F(q(x,y)), \tag{111}$$

for all $x, y \in X$. It follows that T satisfies the condition (9). Therefore, there exists a unique solution of the nonlinear Fredholm inequality (103).

Example 8. Let $[a,b] = [0,e^2]$. Consider the equation

$$x(t) = \int_0^{e^2} \frac{x(s)}{1+x(s)^2} ds. \tag{112}$$

Here, $K(s,t,x(s)) = x(s)/(11 + x(s)^2)$ and we have

$$\begin{aligned} |K(s,t,x(s))| + |K(s,t,y(s))| &= \left| \frac{x(s)}{1+x(s)^2} \right| + \left| \frac{y(s)}{1+y(s)^2} \right| \\ &\le (|x(s)| + |x(s)|) \\ &\le e^{-1/(|x(s)|+|y(s)|+1)}(|x(s)| + |x(s)|). \end{aligned} \tag{113}$$

Then, the condition (9) holds. From Theorem 17, the nonlinear integral equation (103) has a unique solution for $|\lambda| \le 1/e^2$.

By using direct computation, let

$$\beta = \int_0^{e^2} \frac{x(s)}{1+x(s)^2} ds. \tag{114}$$

Then, we have $x(t) = \lambda$ and hence,

$$\beta = \int_0^{e^2} \frac{\beta\lambda}{1+(\beta\lambda)^2} ds = e^2 . \frac{\beta\lambda}{1+(\beta\lambda)^2}, \tag{115}$$

which implies that

$$\lambda^2\beta^3 - (e^2\lambda - 1)\beta = 0 \Leftrightarrow \beta[1 + \lambda^2\beta^2 - e^2\lambda] = 0. \tag{116}$$

Therefore,

$$\begin{cases} \beta = 0, \\ \lambda^2\beta^2 + 1 - e^2\lambda = 0, \end{cases} \tag{117}$$

we obtain that this equation has a unique solution when $\lambda < 1/(b-a) = 1/e^2$.

Data Availability

No data were used to support this study.

Conflicts of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[2] A. Kari, M. Rossafi, E. Marhrani, and M. Aamri, "θ-ϕ-Contraction on α, η-Complete Rectangularb-Metric spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 2020, Article ID 5689458, 9 pages, 2020.

[3] D. Wardowski, "Fixed points of a new type of contractive mappings in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2012, no. 1, Article ID 94, 2012.

[4] N. A. Secelean, "Iterated function systems consisting of F-contractions," *Fixed Point Theory and Applications*, vol. 2013, no. 1, Article ID 277, 2013.

[5] M. Sgroi and C. Vetro, "Multi-valued F-contractions and the solutions of certain functional and integral equations," *Filomat*, vol. 27, no. 7, pp. 1259–1268, 2013.

[6] M. Cosentino and P. Vetroa, "Fixed point results for F-contractive mappings of Hardy-Rogers-type," *Filomat*, vol. 28, no. 4, pp. 715–722, 2014.

[7] H. Piri and P. Kumam, "Wardowski type fixed point theorems in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2016, no. 1, Article ID 45, 2016.

[8] N. V. Dung and V. T. L. Hang, "A fixed point theorem for generalized *F*-contractions on complete metric spaces," *Vietnam Journal of Mathematics*, vol. 43, no. 4, pp. 743–753, 2015.

[9] H. Piri and P. Kumam, "Some fixed point theorems concerning *F*-contraction in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2014, no. 1, Article ID 210, 2014.

[10] D. Wardowski and N. Van Dung, "Fixed points of F-weak contractions on complete metric spaces," *Demonstratio Mathematica*, vol. 47, no. 1, pp. 146–155, 2014.

[11] C. Alegre, J. Marín, and S. Romaguera, "A fixed point theorem for generalized contractions involving *w*-distances on complete quasi-metric spaces," *Fixed Point Theory and Applications*, vol. 2014, no. 1, 2014.

[12] J. Marín, S. Romaguera, and P. Tirado, "Generalized contractive set-valued maps on complete preordered quasi-metric spaces," *Journal of Function Spaces and Applications*, vol. 2013, pp. 1–6, 2013.

[13] T. Wongyat and W. Sintunavarat, "The existence and uniqueness of the solution for nonlinear Fredholm and Volterra integral equations together with nonlinear fractional differential equations via *w*-distances," *Advances in Difference Equations*, vol. 2017, no. 1, Article ID 211, 2017.

[14] D. Wardowski, "Solving existence problems via *F*-contractions," *Proceedings of American Mathematical Society*, vol. 146, no. 4, pp. 1585–1598, 2018.

[15] O. Kada, T. Suzuki, and W. Takahashi, "Nonconvex minimization theorems and fixed point theorems in complete metric spaces," *Mathematica Japonica*, vol. 44, pp. 381–391, 1996.