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Research Article

Pressureless Magnetohydrodynamics System: Riemann Problem and Vanishing Magnetic Field Limit

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This paper proposes the pressureless magnetohydrodynamics (MHD) system by neglecting the effect of pressure difference in the MHD system. Firstly, the Riemann problem for the pressureless MHD system is solved with five kinds of structures of solutions consisting of combinations of shock, rarefaction wave, contact discontinuity, and vacuum state. Secondly, the limit behavior of the obtained Riemann solutions as the magnetic field drops to zero is studied. It is shown that, as the magnetic field vanishes, the Riemann solutions of the pressureless MHD system just tend to the corresponding Riemann solutions of the Euler equations for pressureless fluids. The formation processes of delta shocks and vacuum states are clarified. For the delta shock, both the intermediate density and internal energy simultaneously develop delta measures.

1. Introduction

Magnetohydrodynamics has been the subject of great interest from both mathematical and physical points of view due to its applications in the variety of fields such as astrophysics, nuclear science, engineering physics, and plasma physics. Ideal magnetohydrodynamics neglects the viscous and thermal dissipation effects and assumes a perfectly conducting fluid. The ideal unsteady compressible MHD system reads [1]

$$\begin{cases} \rho_t + \operatorname{div}\ (\rho \mathbf{U}) = 0, \\ (\rho \mathbf{U})_t + \operatorname{div}\ (\rho \mathbf{U} \otimes \mathbf{U} + pI) - (\, rot\, \mathbf{B}) \times \mathbf{B} = 0, \\ \left(\rho E + \dfrac{1}{2}\mathbf{B}^2\right)_t + \operatorname{div}\ (\rho \mathbf{U} E + \mathbf{U} p) - \operatorname{div}\ ((\mathbf{U} \times \mathbf{B}) \times \mathbf{B}) = 0, \\ \mathbf{B}_t - \operatorname{rot}(\mathbf{U} \times \mathbf{B}) = 0, \\ \operatorname{div}\ \mathbf{B} = 0, \end{cases} \tag{1}$$

where ρ is the fluid density, $\mathbf{U}$ the velocity vector of the fluid, p the pressure, $E = |U|^2/2 + e$ the total energy per unit mass with e being the internal energy per unit mass, $\mathbf{B}$ the magnetic field vector, and I the unit matrix.

The system (1) is highly nonlinear and complicated; therefore, it is difficult to do a direct investigation on it. To make a simplification, the condition $\mathbf{U} \perp \mathbf{B}$ has been applied extensively [2, 3]. As indicated by the momentum equations, the particle motion is dictated by momentum transport (inertia) and pressure gradients. When the effect of pressure difference is very small, for example, at low temperature and low pressure in the adhesion particle dynamics, the effect of pressure difference may be neglected. The well-known pressureless Euler equations have been obtained just by neglecting the effect of pressure difference in the Euler equations. Let us consider the one-dimensional motion with plane symmetry permeated by a magnetic field orthogonal to the trajectories of the fluid, that is, $\mathbf{U} = (u(x,t), 0, 0)$, $\rho = \rho(x,t)$, $p = p(x,t)$, and $B = (0, b(x,t), 0)$, and neglect the effect of pressure difference, then we reach the following pressureless MHD system:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \left(\rho u^2 + \dfrac{b^2}{2}\right)_x = 0, \\ \left(\rho E + \dfrac{b^2}{2}\right)_t + \left(\rho u E + u b^2\right)_x = 0, \\ b_t + (bu)_x = 0. \end{cases} \tag{2}$$

In consideration of the frozen-in law in physics, we are concerned with the pressureless MHD system (2) under the assumption $b = k\rho$ ($k > 0$ is a constant). In addition, for convenience, we take the total internal energy $H = \rho e$ as an independent variable. Then, the pressureless MHD system (2) can be rewritten as

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \left(\rho u^2 + \dfrac{b^2}{2}\right)_x = 0, \\ \left(\dfrac{\rho u^2}{2} + H + \dfrac{b^2}{2}\right)_t + \left(\left(\dfrac{\rho u^2}{2} + H\right) u + u b^2\right)_x = 0. \end{cases} \tag{3}$$

The first aim of this paper is to solve the Riemann problem for the system (3) with initial data

$$(\rho, u, H)(x, t = 0) = \begin{cases} (\rho_-, u_-, H_-), & x < 0, \\ (\rho_+, u_+, H_+), & x > 0. \end{cases} \tag{4}$$

For the three characteristics of (3), two are genuinely nonlinear and one is linearly degenerate; thus, the classical basic waves contain shocks, rarefaction waves, and contact discontinuities. For the Riemann problem, by the analysis method in phase space, with the help of the pseudointersection points of wave curves, we establish the existence and uniqueness of solutions with five different structures consisting of combinations of shock, rarefaction wave, contact discontinuity, and vacuum state.

It is well known that the MHD system formally tends to the Euler equations in fluid dynamics as the magnetic field vanishes. In particular, letting the magnetic field vanish, the pressureless MHD system (3) becomes the Euler equations for pressureless fluids:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \left(\rho u^2\right)_x = 0, \\ \left(\dfrac{\rho u^2}{2} + H\right)_t + \left(\left(\dfrac{\rho u^2}{2} + H\right) u\right)_x = 0, \end{cases} \tag{5}$$

which consist of the mass, momentum, and energy conservation laws. It is generally known that for the media which can be regarded as having no pressure, one must take into account energy transport [4]. In [5, 6], to study delta shock solutions of (5), special integral identities were introduced and the Rankine-Hugoniot conditions were obtained. In [7], the Riemann problem (5) and (4) was solved constructively and the solution exactly includes two kinds: delta shock solution and vacuum solution. For the delta shock, both the density and internal energy simultaneously develop delta measures. As to the Euler equations for pressureless fluids only consisting of the mass and momentum conservation laws, please refer to [8–11].

Let us recall some knowledge with respect to delta shocks and vacuum states. Delta shock is a kind of nonclassical wave on which at least one state variable may develop a Dirac delta measure. Mathematically, they are characterized by the delta functions appearing in the state variables. Physically, they can be used to express the concentration phenomenon. As for delta shocks, besides the papers cited above, also see [12–19]. The other situation is the vacuum state which is a state with $\rho = 0$. It describes the cavitation phenomenon. The phenomena of concentration and cavitation and the formation of the delta shock and the vacuum state have attracted wide attention from researchers. For instance, Li and Chen and Liu discussed this topic by considering the vanishing pressure limits of solutions to the isentropic [20, 21] and nonisentropic Euler equations [22]; Mitrović and Nedeljkov [23] discussed this topic by perturbing the generalized pressureless gas dynamics model; Shen and Sun [24] discussed this topic by studying the vanishing pressure limit of Riemann solutions to the perturbed Aw-Rascle model; Cheng and Yang [25] discussed this topic by investigating the partly vanishing pressure limits of solutions to a nonsymmetric Keyfitz-Kranzer system of conservation laws with generalized and modified Chaplygin gas; Yin and Sheng [26, 27] discussed this topic by considering the vanishing pressure limits of solutions to the relativistic Euler equations; Yang and Liu [28, 29] discussed this topic by introducing some flux approximations in the isentropic and nonisentropic classical Euler equations; and Sahoo and Sen [30] discussed this topic by considering the limiting behavior of two strictly hyperbolic systems of conservation laws. Compared with the delta shocks, we also refer the readers to [31–33] for δ^n-shocks and [34] for noncompressible δ-waves.

The second aim of this paper is to discuss the limit behavior of the solutions to the Riemann problem in the pressureless MHD system (3) as the magnetic field vanishes, that is, $k \to 0$. It is shown that when $u_- > u_+$, any Riemann solution containing two shocks and possibly a contact discontinuity to (3) tends to the delta shock Riemann solution to (5), where the intermediate density and internal energy between the two shocks tend to the weighted δ-measure which forms the delta shock. Here, we firstly take a sloping test function to obtain the limits (56) and (58), then approximate an arbitrary test function by this sloping test function to get the desired limits (64) and (65). At this point, it is a little different from Chen and Liu [21, 22], etc. By contrast, when $u_- < u_+$, we show that any Riemann solution containing two rarefaction waves and possibly a contact discontinuity to (3) tends to the vacuum Riemann solution to (5) even when the initial data stay away from the vacuum.

This paper is organized as follows. In Section 2, we solve the Riemann problem for (3). In Section 3, we review the

Riemann problem for (5). In Sections 4 and 5, we discuss the limit behavior of solutions to the Riemann problem for the pressureless MHD system (3) as the magnetic field vanishes. Finally, we give the conclusions and further discussions in Section 6.

2. Riemann Problem of the Pressureless MHD System

In this section, we consider the Riemann problem for (3) with initial data (4). The eigenvalues of (3) are

$$\lambda_0 = u, \lambda_\pm = u \pm k\sqrt{\rho}, \tag{6}$$

and the corresponding right eigenvectors are

$$\begin{aligned} \vec{r}_0 &= (0,0,1)^T, \\ \vec{r}_\pm &= \left(1, \pm\frac{k}{\sqrt{\rho}}, \frac{H}{\rho}\right)^T, \end{aligned} \tag{7}$$

satisfying

$$\begin{aligned} \nabla\lambda_0 \cdot \vec{r}_0 &\equiv 0, \\ \nabla\lambda_\pm \cdot \vec{r}_\pm &= \pm\frac{3}{2}\cdot\frac{k}{\sqrt{\rho}} \neq 0. \end{aligned} \tag{8}$$

Therefore, (3) is strictly hyperbolic and the λ_0-field is linearly degenerate, while the $\lambda_\pm$-fields are genuinely nonlinear. By seeking the self-similar solution $(\rho, u, H)(x,t) = (\rho, u, H)(\xi)$, $\xi = x/t$, the Riemann problem becomes the boundary value problem:

$$\begin{cases} -\xi\rho_\xi + (\rho u)_\xi = 0, \\ -\xi(\rho u)_\xi + \left(\rho u^2 + \dfrac{b^2}{2}\right)_\xi = 0, \\ -\xi\left(\dfrac{\rho u^2}{2} + H + \dfrac{b^2}{2}\right)_\xi + \left(\left(\dfrac{\rho u^2}{2} + H\right)u + ub^2\right)_\xi = 0, \end{cases} \tag{9}$$

$$(\rho, u, H)(\pm\infty) = (\rho_\pm, u_\pm, H_\pm). \tag{10}$$

Let us first solve the elementary waves.

Any smooth solution of (9) satisfies

$$\begin{pmatrix} u-\xi & \rho & 0 \\ k^2\rho & \rho(u-\xi) & 0 \\ 0 & H & u-\xi \end{pmatrix} \begin{pmatrix} \rho \\ u \\ H \end{pmatrix}_\xi = 0, \tag{11}$$

which provides the general solution (the constant state):

$$(\rho, u, H)(\xi) = \text{constant}, \tag{12}$$

the vacuum state:

$$\begin{aligned} \xi &= u, \\ \rho &= 0, \\ H &= 0, \end{aligned} \tag{13}$$

the singular solutions:

$$\begin{cases} \xi = \lambda_- = u - k\sqrt{\rho}, \\ \dfrac{du}{d\rho} = -\dfrac{k}{\sqrt{\rho}}, \\ \dfrac{du}{dH} = -\dfrac{k\sqrt{\rho}}{H}, \\ \dfrac{dH}{d\rho} = \dfrac{H}{\rho}, \end{cases} \tag{14}$$

$$\begin{cases} \xi = \lambda_+ = u + k\sqrt{\rho}, \\ \dfrac{du}{d\rho} = \dfrac{k}{\sqrt{\rho}}, \\ \dfrac{du}{dH} = \dfrac{k\sqrt{\rho}}{H}, \\ \dfrac{dH}{d\rho} = \dfrac{H}{\rho}. \end{cases} \tag{15}$$

For the singular solutions, it holds that

$$\frac{d\lambda_\pm}{d\rho} = \frac{\partial\lambda_\pm}{\partial u}\frac{du}{d\rho} + \frac{\partial\lambda_\pm}{\partial\rho} + \frac{\partial\lambda_\pm}{\partial H}\frac{dH}{d\rho} = \pm\frac{3k}{2\sqrt{\rho}}. \tag{16}$$

Let (ρ_l, u_l, H_l) and (ρ_r, u_r, H_r) denote the left and right states of the singular solutions, then from (16), one has that $\lambda_\pm(\rho_r, u_r, H_r) > \lambda_\pm(\rho_l, u_l, H_l) \Leftrightarrow \rho_r \gtrless \rho_l$.

The singular solution (14) with $\rho_l > \rho_r$ is called as the backward rarefaction wave, symbolized by $\overleftarrow{R}$, and the singular solution (15) with $\rho_l < \rho_r$ is called as the forward rarefaction wave, symbolized by $\overrightarrow{R}$. They can be rewritten as

$$\overleftarrow{R}: \begin{cases} \xi = \lambda_- = u - k\sqrt{\rho}, \\ u_r = u_l - 2k(\sqrt{\rho_r} - \sqrt{\rho_l}), \\ \dfrac{H_r}{H_l} = \dfrac{\rho_r}{\rho_l}, \\ \rho_r < \rho_l, \end{cases} \tag{17}$$

$$\overrightarrow{R}: \begin{cases} \xi = \lambda_- = u + k\sqrt{\rho}, \\ u_r = u_l + 2k(\sqrt{\rho_r} - \sqrt{\rho_l}), \\ \dfrac{H_r}{H_l} = \dfrac{\rho_r}{\rho_l}, \\ \rho_r > \rho_l. \end{cases} \tag{18}$$

For a bounded discontinuity at $\xi = \sigma$ with (ρ_l, u_l, H_l) and (ρ_r, u_r, H_r) on the left and right sides, the Rankine-Hugoniot relation reads

$$\begin{cases} -\sigma[\rho] + [\rho u] = 0, \\ -\sigma[\rho u] + \left[\rho u^2 + \dfrac{b^2}{2}\right] = 0, \\ -\sigma\left[\dfrac{\rho u^2}{2} + H + \dfrac{b^2}{2}\right] + \left[\left(\dfrac{\rho u^2}{2} + H + b^2\right)u\right] = 0, \end{cases} \tag{19}$$

where $[g] = g_l - g_r$ is the jump of g across the discontinuity. By solving (19), we obtain three kinds of discontinuities. The first is

$$\sigma_0 = u_r = u_l, \rho_r = \rho_l, H_l \neq H_r, \tag{20}$$

which is a contact discontinuity associating with λ_0, symbolized by J. The remaining two are

$$\begin{cases} \sigma_- = u_r - \dfrac{k}{\sqrt{2}}\rho_l\sqrt{\dfrac{1}{\rho_r} + \dfrac{1}{\rho_l}} = u_l - \dfrac{k}{\sqrt{2}}\rho_r\sqrt{\dfrac{1}{\rho_r} + \dfrac{1}{\rho_l}}, \\ u_r - u_l = -\dfrac{k}{\sqrt{2}}\sqrt{\dfrac{1}{\rho_r} + \dfrac{1}{\rho_l}}(\rho_r - \rho_l), \\ \dfrac{H_r}{H_l} = \dfrac{\rho_r}{\rho_l} + \dfrac{k^2}{4\rho_l H_l}(\rho_r - \rho_l)^3, \end{cases} \tag{21}$$

$$\begin{cases} \sigma_+ = u_r + \dfrac{k}{\sqrt{2}}\rho_l\sqrt{\dfrac{1}{\rho_r} + \dfrac{1}{\rho_l}} = u_l + \dfrac{k}{\sqrt{2}}\rho_r\sqrt{\dfrac{1}{\rho_r} + \dfrac{1}{\rho_l}}, \\ u_r - u_l = \dfrac{k}{\sqrt{2}}\sqrt{\dfrac{1}{\rho_r} + \dfrac{1}{\rho_l}}(\rho_r - \rho_l), \\ \dfrac{H_r}{H_l} = \dfrac{\rho_r}{\rho_l} + \dfrac{k^2}{4\rho_l H_l}(\rho_r - \rho_l)^3. \end{cases} \tag{22}$$

By the Lax entropy inequalities, the discontinuity (21) with λ_- should satisfy

$$\begin{aligned} \sigma_- < \lambda_-(\rho_l, u_l, H_l) < \lambda_+(\rho_l, u_l, H_l), \\ \lambda_-(\rho_r, u_r, H_r) < \sigma_- < \lambda_+(\rho_r, u_r, H_r), \end{aligned} \tag{23}$$

while the discontinuity (22) associating with λ_+ should satisfy

$$\begin{aligned} \lambda_-(\rho_l, u_l, H_l) < \sigma_+ < \lambda_+(\rho_l, u_l, H_l), \\ \lambda_-(\rho_r, u_r, H_r) < \lambda_+(\rho_r, u_r, H_r) < \sigma_+. \end{aligned} \tag{24}$$

It is easy to check that the inequalities (23) and (24) are equivalent to $\rho_r > \rho_l$ and $\rho_r < \rho_l$, respectively.

The discontinuity (21) with $\rho_l < \rho_r$ is called as the backward shock, symbolized by $\overleftarrow{S}$, and the discontinuity (22) with $\rho_l > \rho_r$ is called as the forward shock, symbolized by $\overrightarrow{S}$.

For a given left state $V_l = (\rho_l, u_l, H_l)$, all possible states which can connect to V_l on the right by a backward rarefaction wave must be located on the curve

$$\overleftarrow{R}(V_l): \begin{cases} u = u_l - 2k(\sqrt{\rho} - \sqrt{\rho_l}), \\ \dfrac{H}{H_l} = \dfrac{\rho}{\rho_l}, \\ \rho < \rho_l, \end{cases} \tag{25}$$

and all possible states which can connect to V_l on the right by a backward shock must be located on the curve

$$\overleftarrow{S}(V_l): \begin{cases} u - u_l = -\dfrac{k}{\sqrt{2}}\sqrt{\dfrac{1}{\rho} + \dfrac{1}{\rho_l}}(\rho - \rho_l), \\ \dfrac{H}{H_l} = \dfrac{\rho}{\rho_l} + \dfrac{k^2}{4\rho_l H_l}(\rho - \rho_l)^3, \\ \rho > \rho_l. \end{cases} \tag{26}$$

For a given right state $V_r = (\rho_r, u_r, H_r)$, all possible states which can connect to V_r on the left by a forward rarefaction wave must be located on the curve

$$\overrightarrow{R}(V_r): \begin{cases} u = u_r + 2k(\sqrt{\rho} - \sqrt{\rho_r}), \\ \dfrac{H}{H_r} = \dfrac{\rho}{\rho_r}, \\ \rho < \rho_r, \end{cases} \tag{27}$$

and all possible states which can connect to V_r on the left by a forward shock must be located on the curve

$$\overrightarrow{S}(V_r): \begin{cases} u - u_r = \dfrac{k}{\sqrt{2}}\sqrt{\dfrac{1}{\rho_r} + \dfrac{1}{\rho}}(\rho - \rho_r), \\ \dfrac{H}{H_r} = \dfrac{\rho}{\rho_r} + \dfrac{k^2}{4\rho_r H_r}(\rho - \rho_r)^3, \\ \rho > \rho_r. \end{cases} \tag{28}$$

Denote $\overleftarrow{W}(V_l) = \overleftarrow{R}(V_l) \cup \overleftarrow{S}(V_l)$ and $\overrightarrow{W}(V_r) = \overrightarrow{R}(V_r) \cup \overrightarrow{S}(V_r)$. For $\overleftarrow{W}(V_l)$, it is easy to check that when ρ increases, u decreases and H increases; for $\overrightarrow{W}(V_r)$, it holds that when ρ increases, u increases and H increases. Besides, it can also be calculated that $\lim_{\rho\to+\infty} u = -\infty$, $\lim_{\rho\to+\infty} H = +\infty$ for $\overleftarrow{W}(V_l)$ while $\lim_{\rho\to+\infty} u = +\infty$, $\lim_{\rho\to+\infty} H = +\infty$ for $\overrightarrow{W}(V_r)$. In addition, the curve $\overleftarrow{W}(V_l)$ interacts with the u-axis at $u = u_l + 2k\sqrt{\rho_l}$ and $\overrightarrow{W}(V_r)$ interacts with the u-axis at $u = u_r - 2k\sqrt{\rho_r}$.

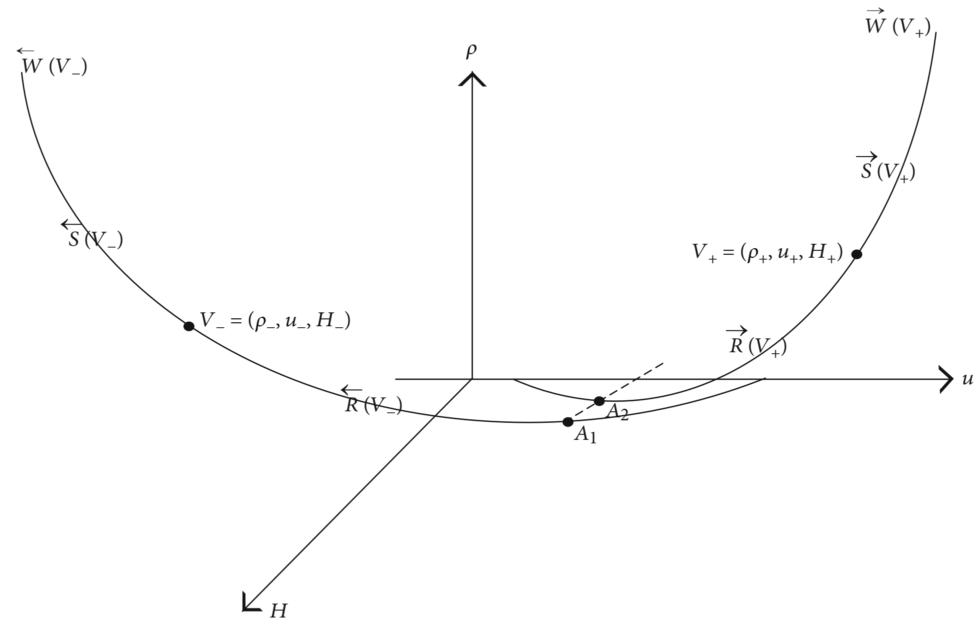

Figure 1: Pseudointersection points of curves of elementary waves.

We next construct the solutions of the Riemann problem (3)–(4) by using the above elementary waves.

Draw the backward wave curve $\overleftarrow{W}(V_-)$ passing the left state $V_- = (\rho_-, u_-, H_-)$ and the forward wave curve $\overrightarrow{W}(V_+)$ passing the right state $V_+ = (\rho_+, u_+, H_+)$. We call the points $A_1 \in \overleftarrow{W}(V_-)$ and $A_2 \in \overrightarrow{W}(V_+)$ the pseudointersection points if the ρ and u coordinates are the same and the H coordinate may be different (see Figure 1). The projections on the (ρ, u)-plane of the pseudointersection points are just the interaction point of the projections on the (ρ, u)-plane of $\overleftarrow{W}(V_-)$ and $\overrightarrow{W}(V_+)$. The states at the pseudointersection points can be connected by a contact discontinuity because the density and velocity across the contact discontinuity do not change (see (20)), and then the solution is allowed to transition from points A_1 to A_2 in the phase space.

When $u_- + 2k\sqrt{\rho_-} \le u_+ - 2k\sqrt{\rho_+}$, it is easy to see that $\overleftarrow{W}(V_-)$ and $\overrightarrow{W}(V_+)$ do not have pseudointersection points. Notice that $\overleftarrow{W}(V_-)$ and $\overrightarrow{W}(V_+)$ interact with the u-axis. Then, the Riemann solution consists of a backward rarefaction wave, a vacuum intermediate state, and a forward rarefaction wave.

When $u_- + 2k\sqrt{\rho_-} > u_+ - 2k\sqrt{\rho_+}$, it is known that $\overleftarrow{W}(V_-)$ and $\overrightarrow{W}(V_+)$ must have pseudointersection points. Then, the Riemann solutions can be constructed according to the different locations on $\overleftarrow{W}(V_-)$ and $\overrightarrow{W}(V_+)$ of the pseudointersection points. To be precise, the Riemann solution contains a backward rarefaction wave, a contact discontinuity, and a forward rarefaction wave when the pseudointersection points lie on $\overleftarrow{R}(V_-)$ and $\overrightarrow{R}(V_+)$; it contains a backward rarefaction wave, a contact discontinuity, and a forward shock wave when the pseudointersection points lie on $\overleftarrow{R}(V_-)$ and $\overrightarrow{S}(V_+)$; it includes a backward shock wave, a contact discontinuity, and a forward rarefaction wave when the pseudointersection points lie on $\overleftarrow{S}(V_-)$ and $\overrightarrow{R}(V_+)$; and it consists of a backward shock wave, a contact discontinuity, and a forward shock wave when the pseudointersection points lie on $\overleftarrow{S}(V_-)$ and $\overrightarrow{S}(V_+)$.

The conclusion can be stated in the following theorem.

Theorem 1. *There exists a unique piecewise smooth solution, which includes shock, rarefaction wave, contact discontinuity, and vacuum state, of the Riemann problem for (3) with initial data (4).*

3. Riemann Problem of the Euler Equations for Pressureless Fluids

In order to well understand the limit behavior of solutions to (3) and (4) as the magnetic field vanishes, we give a sketch of the results for the Riemann problem (5) and (4). For more details, see [7].

The system (5) has a triple eigenvalue $\lambda = u$ with two right eigenvectors $\overrightarrow{r}_1 = (1, 0, 0)^T$, $\overrightarrow{r}_2 = (0, 0, 1)^T$ satisfying $\nabla\lambda_j \cdot \overrightarrow{r}_j \equiv 0\ (j = 1, 2)$. Hence, the system (5) is extremely nonstrictly hyperbolic and λ is linearly degenerate.

As usual, we look for the self-similar solution $(\rho, u, H)(x, t) = (\rho, u, H)(\xi), \xi = x/t$, then the Riemann problem is reduced to the boundary value problem

$$\begin{cases} -\xi\rho_\xi + (\rho u)_\xi = 0, \\ -\xi(\rho u)_\xi + \left(\rho u^2\right)_\xi = 0, \\ -\xi\left(\dfrac{\rho u^2}{2} + H\right)_\xi + \left(\left(\dfrac{\rho u^2}{2} + H\right)u\right)_\xi = 0, \end{cases} \tag{29}$$

$$(\rho, u, H)(\pm\infty) = (\rho_\pm, u_\pm, H_\pm). \tag{30}$$

It can be checked that besides the constant state $(\rho, u, H)(\xi) = \text{constant}$, the system (29) admits the vacuum state

$$\xi = u, \rho = 0, H = 0, \tag{31}$$

and the contact discontinuity

$$\xi = u_l = u_r, \tag{32}$$

where u_r and u_l denote the right and left states, respectively. With the constant, vacuum state, and contact discontinuity, it can be obtained that for the case $u_- < u_+$, the Riemann solution consists of two contact discontinuities and a vacuum state besides two constant states, which can be expressed as

$$(\rho, u, H)(\xi) = \begin{cases} (\rho_-, u_-, H_-), & -\infty < \xi < u_-, \\ (0, \xi, 0), & u_- \leq \xi \leq u_+, \\ (\rho_+, u_+, H_+), & u_+ < \xi < +\infty. \end{cases} \tag{33}$$

However, for the case $u_- > u_+$, the characteristics lines from the x-axis will overlap in the domain $\Omega = \{(x, t) \mid u_+ \leq x/t \leq u_-\}$ in the (x, t)-plane. So the singularity of solution must develop in Ω. One can furthermore prove that ρ, H, and $\partial u/\partial x$ blow up simultaneously in a finite time even starting from smooth initial data. Therefore, no solution exists in the bounded variation space. Indeed, a solution containing weighted δ-measures (i.e., delta shock) supported on a line should be introduced in order to establish the existence in a space of measure from the mathematical point of view.

In order to define the measure solutions, the weighted δ-measure $w(s)\delta_L$ supported on a smooth curve L parameterized as $x = x(s), t = t(s) (c \leq s \leq d)$ is defined by

$$\langle w(s)\delta_L, \psi(x, t)\rangle = \int_c^d w(s)\psi(x(s), t(s))ds, \tag{34}$$

for all test functions $\psi(x, t) \in C_0^\infty((-\infty, +\infty) \times (0, \infty))$.

Then, for the case $u_- > u_+$, the Riemann solution is the following delta shock solution:

$$(\rho, u, H)(x, t) = \begin{cases} (\rho_-, u_-, H_-), & x < \sigma t, \\ (w(t)\delta(x - \sigma t), \sigma, h(t)\delta(x - \sigma t)), & x = \sigma t, \\ (\rho_+, u_+, H_+), & x > \sigma t, \end{cases} \tag{35}$$

where the weights $w(t)$ and $h(t)$ and velocity σ satisfy the generalized Rankine-Hugoniot relation

$$\begin{cases} \dfrac{dw(t)}{dt} = -\sigma[\rho] + [\rho u], \\ \dfrac{dw(t)\sigma}{dt} = -\sigma[\rho u] + [\rho u^2], \\ \dfrac{d(w(t)\sigma^2/2 + h(t))}{dt} = -\sigma[\rho u^2/2 + H] + [(\rho u^2/2 + H)u], \end{cases} \tag{36}$$

and the entropy condition

$$u_+ < \sigma < u_-, \tag{37}$$

with $[a] = a_- - a_+$ being the jump of a across the discontinuity. In [7], it was shown that the solution (35) satisfies the system (5) in the sense of measures. Under (37), solving (36) with initial data $w(0) = 0$ and $h(0) = 0$ yields

$$\begin{cases} \sigma = \dfrac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \\ w(t) = \sqrt{\rho_-\rho_+}(u_- - u_+)t, \\ h(t) = \dfrac{\rho_-\rho_+(u_- - u_+)^2 + 2(\sqrt{\rho_-} + \sqrt{\rho_+})(H_-\sqrt{\rho_+} + H_+\sqrt{\rho_-})}{2(\sqrt{\rho_-} + \sqrt{\rho_+})^2}(u_- - u_+)t. \end{cases} \tag{38}$$

We state the results in the following theorem.

Theorem 2. *There exists a unique piecewise smooth solution, which includes two contact discontinuities and a vacuum state when* $u_- < u_+$ *and a delta shock when* $u_- > u_+$, *of the Riemann problem for (5) with initial data (4).*

4. Limit of Solution to (3)–(4) for $u_- > u_+$

In this section, we study the limit behavior of the Riemann solution to the pressureless MHD system for the case $u_- > u_+$ and $\rho_\pm, H_\pm > 0$ as the magnetic field vanishes.

Let $k_0 > 0$ be the constant satisfying

$$u_+ - u_- = \pm\frac{k_0}{\sqrt{2}}\sqrt{\frac{1}{\rho_+} + \frac{1}{\rho_-}}(\rho_+ - \rho_-), \tag{39}$$

then, for any fixed $k \in (0, k_0)$, the solution of (3) and (4) includes two shocks and one contact discontinuity, which can be expressed as

$$\left(\rho^k, u^k, H^k\right)(\xi) = \begin{cases} (\rho_-, u_-, H_-), & -\infty < \xi < \sigma_-^k, \\ \left(\rho_*^k, u_*^k, H_{*1}^k\right), & \sigma_-^k < \xi < \sigma_0^k, \\ \left(\rho_*^k, u_*^k, H_{*2}^k\right), & \sigma_0^k < \xi < \sigma_+^k, \\ (\rho_+, u_+, H_+), & \sigma_+^k < \xi < +\infty, \end{cases} \tag{40}$$

where (ρ_-, u_-, H_-) and $(\rho_*^k, u_*^k, H_{*1}^k)$ are connected by a backward shock $\overleftarrow{S}$ with speed σ_-^k

$$\overleftarrow{S}: \begin{cases} \sigma_-^k = u_*^k - \dfrac{k}{\sqrt{2}}\rho_-\sqrt{\dfrac{1}{\rho_*^k} + \dfrac{1}{\rho_-}} = u_- - \dfrac{k}{\sqrt{2}}\rho_*^k\sqrt{\dfrac{1}{\rho_*^k} + \dfrac{1}{\rho_-}}, \\ u_*^k - u_- = -\dfrac{k}{\sqrt{2}}\sqrt{\dfrac{1}{\rho_*^k} + \dfrac{1}{\rho_-}}\left(\rho_*^k - \rho_-\right), \\ \dfrac{H_{*1}^k}{H_-} = \dfrac{\rho_*^k}{\rho_-} + \dfrac{k^2}{4\rho_- H_-}\left(\rho_*^k - \rho_-\right)^3, \\ \rho_*^k > \rho_-. \end{cases} \tag{41}$$

$(\rho_*^k, u_*^k, H_{*1}^k)$ and $(\rho_*^k, u_*^k, H_{*2}^k)$ are connected by a contact discontinuity J with speed $\sigma_0^k = u_*^k$; $(\rho_*^k, u_*^k, H_{*2}^k)$ and (ρ_+, u_+, H_+) are connected by a forward shock $\overrightarrow{S}$ with speed σ_+^k

$$\overrightarrow{S}: \begin{cases} \sigma_+^k = u_+ + \dfrac{k}{\sqrt{2}}\rho_*^k\sqrt{\dfrac{1}{\rho_*^k} + \dfrac{1}{\rho_+}} = u_*^k + \dfrac{k}{\sqrt{2}}\rho_+\sqrt{\dfrac{1}{\rho_*^k} + \dfrac{1}{\rho_+}}, \\ u_+ - u_*^k = \dfrac{k}{\sqrt{2}}\sqrt{\dfrac{1}{\rho_*^k} + \dfrac{1}{\rho_+}}\left(\rho_+ - \rho_*^k\right), \\ \dfrac{H_{*2}^k}{H_+} = \dfrac{\rho_*^k}{\rho_+} + \dfrac{k^2}{4\rho_+ H_+}\left(\rho_*^k - \rho_+\right)^3, \\ \rho_*^k > \rho_+. \end{cases} \tag{42}$$

It follows from (41) and (42) that

$$\begin{aligned} u_- - u_+ &= \frac{k}{\sqrt{2}}\left(\sqrt{\frac{1}{\rho_-} + \frac{1}{\rho_*^k}}\left(\rho_*^k - \rho_-\right) + \sqrt{\frac{1}{\rho_+} + \frac{1}{\rho_*^k}}\left(\rho_*^k - \rho_+\right)\right) \\ &= \frac{k}{\sqrt{2}}\left(\sqrt{\left(\frac{1}{\rho_-} - \frac{1}{\rho_*^k}\right)\left(\left(\rho_*^k\right)^2 - \rho_-^2\right)} + \sqrt{\left(\frac{1}{\rho_+} - \frac{1}{\rho_*^k}\right)\left(\left(\rho_*^k\right)^2 - \rho_+^2\right)}\right) := J\left(k, \rho_*^k\right). \end{aligned} \tag{43}$$

With (43), we have the following conclusions for the intermediate density ρ_*^k.

Lemma 3. *ρ_*^k is monotonic decreasing with respect to k, and $\lim_{k\to 0}\rho_*^k = +\infty$.*

Proof. Let $k_1 > k_2$. Assume $\rho_*^{k_1} \geq \rho_*^{k_2}$, then from (43), we obtain $J(k_1, \rho_*^{k_1}) > J(k_2, \rho_*^{k_2})$, which contradicts with $J(k_1, \rho_*^{k_1}) = J(k_2, \rho_*^{k_2}) = u_- - u_+$. Thus we have $\rho_*^{k_1} < \rho_*^{k_2}$.

If ρ_*^k is bounded, then from (43), we can get $u_- = u_+$, which contradicts with $u_- > u_+$. Thus, we get the unboundedness of ρ_*^k, which gives $\lim_{k\to 0}\rho_*^k = +\infty$.

Lemma 4. *$k\rho_*^k$ is monotonic increasing with respect to k, and*

$$\lim_{k\to 0} k\rho_*^k = \frac{\sqrt{2\rho_-\rho_+}}{\sqrt{\rho_-} + \sqrt{\rho_+}}(u_- - u_+) := L. \tag{44}$$

Proof. Let $k_1 > k_2$. If $k_1\rho_*^{k_1} \leq k_2\rho_*^{k_2}$, one can deduce $J(k_1, \rho_*^{k_1}) < J(k_2, \rho_*^{k_2})$, which contradicts with $J(k_1, \rho_*^{k_1}) = J(k_2, \rho_*^{k_2}) = u_- - u_+$. So $k_1\rho_*^{k_1} > k_2\rho_*^{k_2}$ must hold. In addition, the limit $\lim_{k\to 0} k\rho_*^k$ can be directly obtained from (43).

With Lemmas 3 and 4 and

$$\begin{aligned} H_{*1}^k &= \frac{H_-}{\rho_-}\rho_*^k + \frac{\left(k\rho_*^k - k\rho_-\right)^3}{4k\rho_-}, \\ H_{*2}^k &= \frac{H_+}{\rho_+}\rho_*^k + \frac{\left(k\rho_*^k - k\rho_+\right)^3}{4k\rho_+}, \end{aligned} \tag{45}$$

one can easily get the limits of the intermediate internal energies H_{*1}^k and H_{*2}^k.

Lemma 5. $\lim_{k\to 0} H_{*1}^k = \lim_{k\to 0} H_{*2}^k = +\infty$.

Lemma 6. $\lim_{k\to 0} kH_{*1}^k = L\cdot(H_-/\rho_- + L^2/4\rho_-)$ *and* $\lim_{k\to 0} kH_{*2}^k = L\cdot(H_+/\rho_+ + L^2/4\rho_+)$.

From the second equation in (41) (or (42)) and Lemmas 3 and 4, we obtain the limit of the intermediate velocity u_*^k.

Lemma 7.

$$\lim_{k\to 0} u_*^k = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}} := \sigma. \tag{46}$$

With Lemma 3 and (42) and (43), it follows $\lim_{k\to 0}\sigma_-^k = \lim_{k\to 0}\sigma_+^k = \lim_{k\to 0}\sigma_0^k = \lim_{k\to 0}u_*^k$, which shows the limits of the speeds of shocks and contact discontinuity in the following lemma.

Lemma 8.

$$\lim_{k\to 0}\sigma_-^k = \lim_{k\to 0}\sigma_+^k = \lim_{k\to 0}\sigma_0^k = \sigma. \tag{47}$$

Furthermore, we have

Lemma 9.

$$\lim_{k\to 0}\int_{\sigma_-^k}^{\sigma_+^k}\rho_*^k d\xi = \lim_{k\to 0}\rho_*^k\left(\sigma_+^k-\sigma_-^k\right) = M,$$
$$\lim_{k\to 0}\int_{\sigma_-^k}^{\sigma_+^k}H_*^k d\xi = \lim_{k\to 0}\left(H_{*1}^k\left(\sigma_0^k-\sigma_-^k\right)+H_{*2}^k\left(\sigma_+^k-\sigma_0^k\right)\right) = N, \tag{48}$$

where

$$M = \sqrt{\rho_-\rho_+}(u_- - u_+),$$
$$N = \frac{\rho_-\rho_+(u_- - u_+)^2 + 2\left(\sqrt{\rho_-}+\sqrt{\rho_+}\right)\left(H_-\sqrt{\rho_+}+H_+\sqrt{\rho_-}\right)}{2\left(\sqrt{\rho_-}+\sqrt{\rho_+}\right)^2} \cdot (u_- - u_+). \tag{49}$$

Proof. With (41) and (42), we have

$$\rho_*^k\left(\sigma_+^k-\sigma_-^k\right) = \frac{1}{\sqrt{2}}k\rho_*^k\left(\rho_-\sqrt{\frac{1}{\rho_*^k}+\frac{1}{\rho_-}}+\rho_+\sqrt{\frac{1}{\rho_*^k}+\frac{1}{\rho_+}}\right),$$
$$H_{*1}^k\left(\sigma_0^k-\sigma_-^k\right) = \frac{1}{\sqrt{2}}kH_{*1}^k\rho_-\sqrt{\frac{1}{\rho_*^k}+\frac{1}{\rho_-}}, \tag{50}$$

$$H_{*2}^k\left(\sigma_+^k-\sigma_0^k\right) = \frac{1}{\sqrt{2}}kH_{*2}^k\rho_+\sqrt{\frac{1}{\rho_*^k}+\frac{1}{\rho_+}}. \tag{51}$$

Taking the limit $k\to 0$ in the above expressions gives the results.

Lemmas 3–9 show that, as k drops to zero, the intermediate velocity and all of the speeds of $\overleftarrow{S}$, J, and $\overrightarrow{S}$ tend to the constant σ, which means that $\overleftarrow{S}$, J, and $\overrightarrow{S}$ coincide. Correspondingly, the intermediate density ρ_*^k and internal energies H_{*1}^k and H_{*2}^k simultaneously develop delta measures.

Now, we are ready to characterize the limit of solutions of (3) (4) as $k\to 0$ for the case $u_- > u_+$.

Let us take a sloping test function $\phi(\xi)\in C_0^\infty(-\infty,+\infty)$ such that $\phi(\xi)\equiv\phi(\sigma)$ for ξ in a neighborhood Ω of $\xi=\sigma$. Then, there exists $k_1\in(0,k_0)$ such that when $0<k<k_1$, it holds $\sigma_-^k\in\Omega$, $\sigma_0^k\in\Omega$, and $\sigma_+^k\in\Omega$. For $k\in(0,k_1)$, we have

$$\int_{-\infty}^{+\infty}\rho^k\phi d\xi = \left(\int_{-\infty}^{\sigma_-^k}+\int_{\sigma_+^k}^{+\infty}\right)\rho^k\phi d\xi + \int_{\sigma_-^k}^{\sigma_+^k}\rho^k\phi d\xi, \tag{52}$$

in which

$$\lim_{k\to 0}\left(\int_{-\infty}^{\sigma_-^k}+\int_{\sigma_+^k}^{+\infty}\right)\rho^k\phi d\xi = \lim_{k\to 0}\int_{-\infty}^{\sigma_-^k}\rho_-\phi d\xi + \lim_{k\to 0}\int_{\sigma_+^k}^{+\infty}\rho_+\phi d\xi = \int_{-\infty}^{+\infty}\rho_0(\xi-\sigma)\phi d\xi, \tag{53}$$

where

$$\rho_0(x) = \begin{cases}\rho_-, & x<0,\\ \rho_+, & x>0,\end{cases} \tag{54}$$

$$\lim_{k\to 0}\int_{\sigma_-^k}^{\sigma_+^k}\rho^k\phi d\xi = \lim_{k\to 0}\rho_*^k\left(\sigma_+^k-\sigma_-^k\right)\phi(\sigma) = M\phi(\sigma), \tag{55}$$

by virtue of the Lemma 9. Thus, we have obtained

$$\lim_{k\to 0}\int_{-\infty}^{+\infty}\left(\rho^k-\rho_0(\xi-\sigma)\right)\phi d\xi = M\phi(\sigma). \tag{56}$$

For $k\in(0,k_1)$, we also have

$$\int_{-\infty}^{+\infty}H^k\phi d\xi = \left(\int_{-\infty}^{\sigma_-^k}+\int_{\sigma_+^k}^{+\infty}\right)H^k\phi d\xi + \int_{\sigma_-^k}^{\sigma_0^k}H_{*1}^k\phi d\xi + \int_{\sigma_0^k}^{\sigma_+^k}H_{*2}^k\phi d\xi, \tag{57}$$

which gives

$$\lim_{k\to 0}\int_{-\infty}^{+\infty}\left(H^k-H_0(\xi-\sigma)\right)\phi d\xi = N\phi(\sigma), \tag{58}$$

where

$$H_0(x) = \begin{cases}H_-, & x<0,\\ H_+, & x>0.\end{cases} \tag{59}$$

For an arbitrary test function $\varphi(\xi)\in C_0^\infty(-\infty,+\infty)$, we take a sloping test function ϕ such that $\phi(\sigma)=\varphi(\sigma)$ and

$$\max_{\xi\in(-\infty,+\infty)}|\phi-\varphi|<\mu. \tag{60}$$

We have

$$\begin{aligned}\lim_{k\to 0}&\int_{-\infty}^{+\infty}\left(\rho^k-\rho_0(\xi-\sigma)\right)\varphi d\xi\\&=\lim_{k\to 0}\left\{\int_{-\infty}^{+\infty}\left(\rho^k-\rho_0(\xi-\sigma)\right)\phi d\xi\right.\\&\quad\left.+\int_{-\infty}^{+\infty}\left(\rho^k-\rho_0(\xi-\sigma)\right)(\varphi-\phi)d\xi\right\}.\end{aligned}\tag{61}$$

The first term on the right side

$$\lim_{k\to 0}\int_{-\infty}^{+\infty}\left(\rho^k-\rho_0(\xi-\sigma)\right)\phi d\xi=M\phi(\sigma)=M\varphi(\sigma).\tag{62}$$

The second term on the right side

$$\begin{aligned}\int_{-\infty}^{+\infty}&\left(\rho^k-\rho_0(\xi-\sigma)\right)(\varphi-\phi)d\xi\\&=\int_{\sigma_-^k}^{\sigma_+^k}\left(\rho^k-\rho_0(\xi-\sigma)\right)(\varphi-\phi)d\xi\\&=\int_{\sigma_-^k}^{\sigma_+^k}\rho^k(\varphi-\phi)d\xi-\int_{\sigma_-^k}^{\sigma_+^k}\rho_0(\xi-\sigma)(\varphi-\phi)d\xi,\end{aligned}\tag{63}$$

which converges to 0 as $k\to 0$ by sending $\mu\to 0$ and recalling Lemma 9. Thus, we have that

$$\lim_{k\to 0}\int_{-\infty}^{+\infty}\left(\rho^k-\rho_0(\xi-\sigma)\right)\varphi d\xi=M\varphi(\sigma),\tag{64}$$

for all test functions $\varphi\in C_0^\infty(-\infty,+\infty)$. Similarly, we have

$$\lim_{k\to 0}\int_{-\infty}^{+\infty}\left(H^k-H_0(\xi-\sigma)\right)\varphi d\xi=N\varphi(\sigma),\tag{65}$$

for all test functions $\varphi\in C_0^\infty(-\infty,+\infty)$.

Let $\psi(x,t)\in C_0^\infty((-\infty,+\infty)\times[0,+\infty))$ be an arbitrary test function, and let $\tilde{\psi}(\xi,t):=\psi(\xi t,t)$. Then, it follows that

$$\begin{aligned}\lim_{k\to 0}&\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho^k\left(\frac{x}{t}\right)\psi(x,t)dxdt\\&=\lim_{k\to 0}\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho^k(\xi)\psi(\xi t,t)d(\xi t)dt\\&=\lim_{k\to 0}\int_0^{+\infty}t\left(\int_{-\infty}^{+\infty}\rho^k(\xi)\tilde{\psi}(\xi,t)d\xi\right)dt,\end{aligned}\tag{66}$$

and with (64), we have

$$\begin{aligned}\lim_{k\to 0}&\int_{-\infty}^{+\infty}\rho^k(\xi)\tilde{\psi}(\xi,t)d\xi\\&=\int_{-\infty}^{+\infty}\rho_0(\xi-\sigma)\tilde{\psi}(\xi,t)d\xi+M\tilde{\psi}(\sigma,t)\\&=t^{-1}\int_{-\infty}^{+\infty}\rho_0(x-\sigma t)\psi(x,t)dx+M\psi(\sigma t,t).\end{aligned}\tag{67}$$

Combining the two relations above yields

$$\begin{aligned}\lim_{k\to 0}&\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho^k\left(\frac{x}{t}\right)\psi(x,t)dxdt\\&=\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho_0(x-\sigma t)\psi(x,t)dxdt\\&\quad+\int_0^{+\infty}Mt\psi(\sigma t,t)dt.\end{aligned}\tag{68}$$

By the definition, the last term

$$\int_0^{+\infty}Mt\psi(\sigma t,t)dt=\langle Mt\delta_{x=\sigma t},\psi(x,t)\rangle.\tag{69}$$

Similarly, we can show from (65) that

$$\begin{aligned}\lim_{k\to 0}&\int_0^{+\infty}\int_{-\infty}^{+\infty}H^k\left(\frac{x}{t}\right)\psi(x,t)dxdt\\&=\int_0^{+\infty}\int_{-\infty}^{+\infty}H_0(x-\sigma t)\psi(x,t)dxdt\\&\quad+\int_0^{+\infty}Nt\psi(\sigma t,t)dt,\end{aligned}\tag{70}$$

with

$$\int_0^{+\infty}Nt\psi(\sigma t,t)dt=\langle Nt\delta_{x=\sigma t},\psi(x,t)\rangle.\tag{71}$$

Thus, we obtain the following conclusion.

Theorem 10. *Let*$u_->u_+$*and*$\rho_\pm, H_\pm>0$. *For fixed*$k>0$, *assume that*$(\rho^k,u^k,H^k)(x,t)$*is the*$\overleftarrow{S}J\overrightarrow{S}$*solution to (3) and (4). Then,*

$$\lim_{k\to 0}u^k(x,t)=\begin{cases}u_-, & x<\sigma t\\ \sigma, & x=\sigma t,\\ u_+, & x>\sigma t.\end{cases}\tag{72}$$

ρ^k*and*H^k*converge in the sense of distributions, and the limit functions are the sum of a step function and a Dirac delta function supported on*$x=\sigma t$*with weights*

$$\sqrt{\rho_-\rho_+}(u_--u_+)t,\tag{73}$$

$$\frac{\rho_-\rho_+(u_- - u_+)^2 + 2(\sqrt{\rho_-}+\sqrt{\rho_+})(H_-\sqrt{\rho_+}+H_+\sqrt{\rho_-})}{2(\sqrt{\rho_-}+\sqrt{\rho_+})^2} \cdot (u_- - u_+)t, \tag{74}$$

respectively, where $\sigma = (\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+)/(\sqrt{\rho_-}+\sqrt{\rho_+})$.

It can be seen that the limit of $(\rho^k, u^k, H^k)(x,t)$ is just the delta shock solution of the Riemann problem of the Euler equations for pressureless fluids (5).

5. Limit of Solution to (3)–(4) for $u_- < u_+$

This section discusses the limit behavior of the Riemann solution to the pressureless MHD system for $u_- < u_+$ and $\rho_\pm, H_\pm > 0$ as the magnetic field vanishes.

Let k_2 be the constant satisfying

$$u_+ - u_- = \pm 2k_2(\sqrt{\rho_+} - \sqrt{\rho_-}), \tag{75}$$

then, for any $k \in (0, k_2)$, the solution to (3) and (4) contains two rarefaction waves. Furthermore, let k_3 be the constant satisfying

$$u_- + 2k_3\sqrt{\rho_-} = u_+ - 2k_3\sqrt{\rho_+}, \tag{76}$$

then, for any $k \in (k_3, k_2)$, the Riemann solution contains non-vacuum intermediate states between two rarefaction waves

$$\left(\rho^k, u^k, H^k\right)(\xi) = \begin{cases} (\rho_-, u_-, H_-), & -\infty < \xi < u_- - k\sqrt{\rho_-}, \\ \overleftarrow{R}, & u_- - k\sqrt{\rho_-} \le \xi \le u_- + 2k\sqrt{\rho_-} - 3k\sqrt{\rho_*^k}, \\ \left(\rho_*^k, u_*^k, H_{*1}^k\right), & u_- + 2k\sqrt{\rho_-} - 3k\sqrt{\rho_*^k} \le \xi \le u_- + 2k\sqrt{\rho_-} - 2k\sqrt{\rho_*^k}, \\ \left(\rho_*^k, u_*^k, H_{*2}^k\right), & u_+ - 2k\sqrt{\rho_+} + 2k\sqrt{\rho_*^k} \le \xi \le u_+ - 2k\sqrt{\rho_+} + 3k\sqrt{\rho_*^k}, \\ \overrightarrow{R}, & u_+ - 2k\sqrt{\rho_+} + 3k\sqrt{\rho_*^k} \le \xi \le u_+ + k\sqrt{\rho_+}, \\ (\rho_+, u_+, H_+), & u_+ + k\sqrt{\rho_+} < \xi < +\infty, \end{cases} \tag{77}$$

where the backward rarefaction wave $\overleftarrow{R}$ connecting (ρ_-, u_-, H_-) and $(\rho_*^k, u_*^k, H_{*1}^k)$ can be expressed as

$$\overleftarrow{R}: \begin{cases} \xi = \lambda_- = u - k\sqrt{\rho}, \\ u - u_- = -2k(\sqrt{\rho} - \sqrt{\rho_-}), \\ \dfrac{H}{H_-} = \dfrac{\rho}{\rho_-}, \\ \rho < \rho_-. \end{cases} \tag{78}$$

The forward rarefaction wave $\overrightarrow{R}$ connecting $(\rho_*^k, u_*^k, H_{*2}^k)$ and (ρ_+, u_+, H_+) can be expressed as

$$\overrightarrow{R}: \begin{cases} \xi = \lambda_+ = u + k\sqrt{\rho}, \\ u_+ - u = 2k(\sqrt{\rho_+} - \sqrt{\rho}), \\ \dfrac{H}{H_+} = \dfrac{\rho}{\rho_+}, \\ \rho_+ > \rho. \end{cases} \tag{79}$$

$(\rho_*^k, u_*^k, H_{*1}^k)$ and $(\rho_*^k, u_*^k, H_{*2}^k)$ are connected by a contact discontinuity J with speed

$$\begin{aligned} \sigma_0^k = u_*^k &= u_- + 2k\sqrt{\rho_-} - 2k\sqrt{\rho_*^k} \\ &= u_+ - 2k\sqrt{\rho_+} + 2k\sqrt{\rho_*^k}. \end{aligned} \tag{80}$$

From (80), we have

$$\begin{aligned} u_+ - u_- &= 2k\left(\sqrt{\rho_-} - \sqrt{\rho_*^k}\right) + 2k\left(\sqrt{\rho_+} - \sqrt{\rho_*^k}\right) \\ &:= G\left(k, \rho_*^k\right), \end{aligned} \tag{81}$$

or

$$(u_- + 2k\sqrt{\rho_-}) - (u_+ - 2k\sqrt{\rho_+}) = 4k\sqrt{\rho_*^k}, \tag{82}$$

with which it is obvious that ρ_*^k is monotonic increasing with respect to k. Taking the limit $k \to k_3$ on both sides of (82), we have

$$\lim_{k \to k_3} \rho_*^k = 0. \tag{83}$$

Besides, from $H_{*1}^k/H_- = \rho_*^k/\rho_-$ and $H_{*2}^k/H_+ = \rho_*^k/\rho_+$, we have

$$\lim_{k \to k_3} H_{*1}^k = \lim_{k \to k_3} H_{*2}^k = 0. \tag{84}$$

Furthermore, from (80), one has

$$\lim_{k \to k_3} \sigma_0^k = \lim_{k \to k_3} u_*^k = u_- + 2k_3\sqrt{\rho_-} = u_+ - 2k_3\sqrt{\rho_+} := u_1. \tag{85}$$

Moreover, as $k \to k_3$, both the wave front $\xi_{\text{front}} = u_- + 2k\sqrt{\rho_-} - 3k\sqrt{\rho_*^k}$ of $\overleftarrow{R}$ and the wave back $\xi_{\text{back}} = u_+ - 2k\sqrt{\rho_+} + 3k\sqrt{\rho_*^k}$ of $\overrightarrow{R}$ tend to $\xi = u_1$. Therefore, as $k \to k_3$, the vacuum begins to appear and the wave front of $\overleftarrow{R}$, the wave back of $\overrightarrow{R}$, and the contact discontinuity J coincide. When k decreases so that $0 < k < k_3$, the solution to (3) and (4) contains a vacuum intermediate state between two rarefaction waves

$$\left(\rho^k, u^k, H^k\right)(\xi) = \begin{cases} (\rho_-, u_-, H_-), & -\infty < \xi < u_- - k\sqrt{\rho_-}, \\ \overleftarrow{R}, & u_- - k\sqrt{\rho_-} \le \xi \le u_- + 2k\sqrt{\rho_-}, \\ (0, \xi, 0), & u_- + 2k\sqrt{\rho_-} \le \xi \le u_+ - 2k\sqrt{\rho_+}, \\ \overrightarrow{R}, & u_+ - 2k\sqrt{\rho_+} \le \xi \le u_+ + k\sqrt{\rho_+}, \\ (\rho_+, u_+, H_+), & u_+ + k\sqrt{\rho_+} < \xi < +\infty, \end{cases} \tag{86}$$

with (78) and (79). Then, when k continues to decrease, the rarefaction waves become narrower and narrower and the vacuum region in between becomes wider and wider. Finally, when k drops to zero, the rarefaction waves become two lines with $\xi = u_-$ and $\xi = u_+$ and between which is a vacuum state. In summary, we have

$$\lim_{k \to 0} \left(\rho^k, u^k, H^k\right)(\xi) = \begin{cases} (\rho_-, u_-, H_-), & -\infty < \xi \le u_-, \\ (0, \xi, 0), & u_- \le \xi \le u_+, \\ (\rho_+, u_+, H_+), & u_+ \le \xi < +\infty, \end{cases} \tag{87}$$

which is just the vacuum Riemann solution to the Euler equations for pressureless fluids.

6. Conclusions and Further Discussions

In this paper, we propose the pressureless MHD system. As the magnetic field vanishes, its limit system is just the Euler equations for pressureless fluids consisting of the mass, momentum, and energy conservation laws, which is one of the popular models admitting delta shocks, an interesting topic. Using classical methods of hyperbolic conservation laws, we solve the Riemann problem for the pressureless MHD system. In the main part, we investigate the limits of the Riemann solutions to the pressureless MHD system as the magnetic field vanishes. It is shown that the vanishing magnetic field limits of the Riemann solutions to the pressureless MHD system are just the Riemann solutions to its limit system. From another point of view, we show how the delta shock solution of the Euler equations for pressureless fluids appears as the vanishing magnetic field limit of solution to the pressureless MHD system containing two shocks and possibly a contact discontinuity and how the vacuum solution to the Euler equations for pressureless fluids appears as the vanishing magnetic field limit of solution to the pressureless MHD system containing two rarefaction waves and possibly a contact discontinuity.

Following the investigations in this paper, two interesting topics are put forward. In (3), instead of the linear relation $b = k\rho$, if $b = kg(\rho)$, where $g(\rho)$ is a smooth function satisfying some growth conditions, then the discussion in this paper can be carried out. Besides, for general types of initial data, one can study the solution for the Euler equations of pressureless fluid by considering the vanishing magnetic field limit of solution to the pressureless MHD system. We will study them in the future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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