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Research Article

Existence and Uniqueness Theorems for a Fractional Differential Equation with Impulsive Effect under Band-Like Integral Boundary Conditions

Zihan Gao,[1] Tianlin Hu,[2] and Huihui Pang [2]

[1]College of Economics and Management, China Agricultural University, Beijing, China
[2]College of Science, China Agricultural University, Beijing, China

Correspondence should be addressed to Huihui Pang; phh2000@163.com

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In this paper, we consider a class of nonlinear Caputo fractional differential equations with impulsive effect under multiple band-like integral boundary conditions. By constructing an available completely continuous operator, we establish some criteria for judging the existence and uniqueness of solutions. Finally, an example is presented to demonstrate our main results.

1. Introduction

Researches on fractional differential equations have witnessed an unprecedented boom in recent years on account of the far-reaching application in various subjects, such as physics, biology, nuclear dynamics, chemistry, etc., for more details, see [1–3] and the references therein. Considering the impulse effect in the continuous differential equation can quantify the impact of the instantaneous mutation of the model and provide a theoretical basis for the practical application. Therefore, impulsive differential equation problems also attract great attention from scholars. For the theories of impulsive differential equations, the readers can refer to [4–7]. In addition, there have been some excellent results concerning the existence, uniqueness, and multiplicity of solutions or positive solutions to some nonlinear fractional differential equations with various nonlocal boundary conditions. As for some recent bibliographies, we refer readers to see [8–11] and the reference therein.

Yang and Zhang in [12] studied the following impulsive fractional differential equation

$$\begin{aligned} &{}^cD_{0+}^{\alpha}x(t) = f(t, x(t)), \quad t \in J = (0,1),\ t \neq t_k,\\ &\Delta x|_{t=\xi_k} = I_k(x(\xi_k)), \quad \Delta x'|_{t=\xi_k} = \bar{I}_k(x(\xi_k)),\ k = 1, 2, \ldots, m,\\ &x(0) = h(x),\ x(1) = g(x), \end{aligned} \tag{1}$$

where ${}^cD_{0+}^{\alpha}$ is the Caputo fractional derivative, $\alpha \in \mathbb{R}$, $1 < \alpha \leq 2$. $f : [0,1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $I_k, \bar{I}_k$ are continuous functions, $g(x) = \max_j((|x(\xi_j)|)/(\lambda + |x(\xi_j)|))$, $h(x) = \min_j((|x(\xi_j)|)/(\kappa + |x(\xi_j)|))$. By transforming the boundary value problem into an equivalent integral equation and employing some fixed point theorems, existence result is obtained.

The research results of fractional differential equations with integral boundary conditions are also quite rich, and the research on those questions remains as a hotpot among many scholars in recent years. We refer readers to see [13–16] and the reference therein.

In [13], Song and Bai considered the following boundary value problem of fractional differential equation with Riemann–Stieltjes integral boundary condition

$$\begin{aligned} &D_{0+}^{\alpha}u(t) + \lambda f(t, u(t), u(t)) = 0,\ 0 < t < 1,\ n-1 < \alpha \leq n,\\ &u^k(0) = 0, \quad 0 \leq k \leq n-2, \quad u(1) = \int_0^1 u(s)dA(s), \end{aligned} \tag{2}$$

where $n-1 < \alpha \leq n$, $\lambda > 0$, D_{0+}^{α} is the Riemann–Liouville fractional derivative, A is a function of bounded variation, $\int_0^1 u(s)dA(s)$ denotes the Riemann–Stieltjes integral of u with respect to A. By the use of fixed point theorem and the properties of mixed monotone operator theory, the existence and uniqueness of positive solutions for the problem are acquired.

Moreover, Zhao and Liang in [14] added impulsive effect to fractional equations with integral boundary conditions and discussed the existence of solutions

$$\begin{gathered} {}_{t_k}D_t^{\alpha}u(t) = f\left(t, u, u', D^{\alpha-1}u\right), \qquad t \neq t_k, \\ \Delta D^{\alpha-1}u(t_k)\Big|_{t=\xi_k} = I_k\left(u(t_k)\right), \quad k = 1, 2, \ldots, m, \\ u(0) = u'(0) = 0, \quad u'(1) = \int_0^{\eta} g(s, u(s))ds, \end{gathered} \tag{3}$$

where ${}_{t_k}D_t^{\alpha}$ is the Riemann-iouville fractional derivative of order $2 < \alpha \le 3$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$. $f \in C\left(J \times \mathbb{R}^3, \mathbb{R}\right)$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $0 < \eta < 1$, $g \in C(J \times \mathbb{R}, \mathbb{R})$. By applying the contraction mapping principle and the fixed point theorem, some sufficient criteria for the existence of solutions are obtained.

Inspired by the works above, we will study the impulsive fractional differential equation with band-like integral boundary conditions

$$ {}^cD_{0+}^{\alpha}x(t) = f\left(t, x(t), {}^cD_{0+}^{\beta}x(t)\right), \quad t \in (0, 1), t \neq \xi_k, \tag{4}$$

$$\begin{gathered} -\Delta x|_{t=\xi_k} = I_k\left(x(\xi_k)\right), \ k = 1, 2, \ldots, n, \\ x(0) = x(1) = \sum_{i=0}^{n} \alpha_i \int_{\xi_i}^{\xi_{i+1}} x(t)g(t)dt, \end{gathered} \tag{5}$$

where ${}^cD_{0+}^{\alpha}$, ${}^cD_{0+}^{\beta}$ are the Caputo fractional derivatives of order $1 < \alpha \le 2$, $0 < \beta < 1$, $J = [0, 1]$, $J_k = (\xi_k, \xi_{k+1}]$, $f \in C\left(J_k \times \mathbb{R}^2, \mathbb{R}\right)$, for $k = 0, 1, 2, \ldots, n$, and α_i is a nonnegative constant, $g \in C([0, 1], \mathbb{R}^+)$ satisfying $0 < \sum_{i=0}^{n}\alpha_i \int_{\xi_i}^{\xi_{i+1}} g(t)dt < 1$, $0 = \xi_0 < \xi_1 < \cdots < \xi_n < \xi_{n+1} = 1$, $x(\xi_k^+) = \lim_{h\to 0^+} x(\xi_k + h)$, and $x(\xi_k^-) = \lim_{h\to 0^-} x(\xi_k + h)$ represent the right and the left limits of $x(\xi_k)$ at $t = \xi_k$, $\Delta x|_{t=\xi_k} = x(\xi_k^+) - x(\xi_k^-)$. $I_k(x(\xi_k)) \in C(\mathbb{R}, \mathbb{R})$. By using the Leray–Schauder alternative theorem and the Banach contraction mapping principle, the existence and uniqueness theorems of solutions to problem (4) can be established.

We emphasize that the discontinuous points caused by impulse are just the upper and lower limits of the band-like integral values in the boundary conditions of (4). In other words, the value of the unknown function at the endpoint of the interval [0,1] is related to the linear combination of the integral values of the unknown function between the discontinuous points.

Another thing worth mentioning is that despite the complicated boundary conditions and the interference of the impulse, we use a piecewise function to represent the operator F in a concise form based on the form of the Green's function and accurately estimate the upper bound of its absolute value, which is fully prepared for the establishment of the main theorem.

Accordingly, the conclusions we reached are extensive results compared with the reference [4–7, 15–20] and a meaningful supplement to the theory of impulsive fractional differential equations.

2. Preliminaries

In this section, we present some definitions, lemmas, and some prerequisite results that will be used to prove our results.

Definition 1 [19]. The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is defined as

$$ I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \tag{6}$$

if the right-hand side is pointwise defined on $(0, \infty)$, where $\Gamma(\alpha)$ is the Euler gamma function satisfying $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt$, for $\alpha > 0$.

Definition 2 [16]. The Caputo fractional derivative of order $\alpha > 0$ for a function $f : (0, \infty) \to \mathbb{R}$ is defined as

$$ {}^cD_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t f^n(s)(t-s)^{n-\alpha-1}ds, \tag{7}$$

where $n = [\alpha] + 1$ and $[\alpha]$ stands for the largest integer that not greater than α.

Lemma 1. *For $h \in L^1(0, 1)$, the solution of the fractional differential equation ${}^cD_{0+}^{\alpha}u(t) + h(t) = 0$, $0 < t < 1$ can be expressed as*

$$\begin{aligned} u(t) = {} & -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}h(s)ds \\ & + c_1 + c_2 t + \cdots + c_n t^{n-1}, \quad 0 < t < 1, \end{aligned} \tag{8}$$

where $c_i \in \mathbb{R}$, for $i = 1, 2, \ldots, n$.

Lemma 2. *For any $v \in L^1(0, 1)$, the following boundary value problem*

$$ {}^cD_{0+}^{\alpha}x(t) = -v(t), \quad t \in J_k, \tag{9}$$

$$\begin{gathered} -\Delta x|_{t=\xi_k} = I_k\left(x(\xi_k)\right), \quad k = 1, 2, \ldots, n, \\ x(0) = x(1) = \sum_{i=0}^{n} \alpha_i \int_{\xi_i}^{\xi_{i+1}} x(t)g(t)dt, \end{gathered} \tag{10}$$

has a unique solution

$$\begin{aligned} x(t) = {} & A_v^1(t) + A_v^2(t) + B_1(v) + B_2(v) \\ & + \sum_{k=1}^{n} G(t, \xi_k) I_k\left(x(\xi_k)\right) \\ & + \frac{1}{\Delta} \sum_{i=0}^{n} \alpha_i \int_{\xi_i}^{\xi_i+1} \sum_{k=1}^{n} G(t, \xi_k) I_k\left(x(\xi_k)\right)dt, \end{aligned} \tag{11}$$

where

$$\begin{aligned} A_v^1(t) = {} & \frac{-1}{\Gamma(\alpha-1)} \int_{\xi_k}^{t} \int_0^s (s-\tau)^{\alpha-2} v(\tau)d\tau ds, \\ & \text{for } t \in J_k, k = 0, 1, 2, \ldots, n, \\ A_v^2(t) = {} & \frac{1}{\Gamma(\alpha-1)} \sum_{k=1}^{n+1} G\left(t, \xi_k\right) \int_{\xi_{k-1}}^{\xi_k} \int_0^s (s-\tau)^{\alpha-2} v(\tau)d\tau ds, \\ & \text{for } t \in J_k, k = 0, 1, 2, \ldots, n, \\ B_1(v) = {} & -\frac{1}{\Delta\Gamma(\alpha-1)} \sum_{i=0}^{n} \alpha_i \int_{\xi_i}^{\xi_{i+1}} \int_{\xi_i}^{t} \int_0^s (s-\tau)^{\alpha-2} v(\tau)d\tau ds dt, \\ B_2(v) = {} & \frac{1}{\Delta\Gamma(\alpha-1)} \sum_{i=0}^{n} \alpha_i \int_{\xi_i}^{\xi_{i+1}} \sum_{k=1}^{n+1} G\left(t, \xi_k\right) \\ & \cdot \int_{\xi_{k-1}}^{\xi_k} \int_0^s (s-\tau)^{\alpha-2} v(\tau)d\tau ds dt, \end{aligned} \tag{12}$$

where

$$G(t,\xi_k)=\begin{cases} t-1, & 0<\xi_k<t, \quad k=1,2,\dots,n,\\ t, & t\le \xi_k<1, \quad k=1,2,\dots,n,\end{cases}$$
$$\Delta = 1-\sum_{n}^{i=0}\alpha_i\int_{\xi_i}^{\xi_{i+1}} g(t)dt. \tag{13}$$

Proof. From equation (9), through calculation we have

$$x'(t)=\frac{-1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}v(s)ds+a, \tag{14}$$

where a is an arbitrary real constant.

For $t\in J_0$, according to (15), we can obtain

$$x(t)=\frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds+b+at, \tag{15}$$

where b is an arbitrary real constant.

For $t\in J_1$, based on (15) and (16), we have

$$\begin{aligned} x(t) &= \tfrac{-1}{\Gamma(\alpha-1)}\int_{\xi_1}^t\int_0^s (s-\tau)^{\alpha-2}v(\tau)d\tau ds\\ &-\tfrac{1}{\Gamma(\alpha)}\int_0^{t_1}(\xi_1-s)^{\alpha-1}v(s)ds+at+b-I_1(x(\xi_1)). \end{aligned} \tag{16}$$

Analogously, for $t\in J_k, k=2,3\dots,n$, it holds that

$$\begin{aligned} x(t) &= \frac{-1}{\Gamma(\alpha-1)}\int_{\xi_k}^t\int_0^s (s-\tau)^{\alpha-2}v(\tau)d\tau ds\\ &-\frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^n\int_{\xi_{k-1}}^{\xi_k}\int_0^s (s-\tau)^{\alpha-2}v(\tau)d\tau ds\\ &+at+b-\sum_{\xi_k<t} I_k(x(\xi_k)). \end{aligned} \tag{17}$$

Since $x(0)=x(1)=b$, together with (16) and (18), we receive that

$$a=\frac{-1}{\Gamma(\alpha-1)}\sum_{k=1}^{n+1}\int_{\xi_{k-1}}^{\xi_k}\int_0^s (s-\tau)^{\alpha-2}v(\tau)d\tau ds+\sum_{k=1}^n I_k(x(\xi_k)). \tag{18}$$

Substituting a into (18), and based on the form of Green's function, we get

$$\begin{aligned} x(t) &= b-\frac{1}{\Gamma(\alpha-1)}\int_{\xi_k}^t\int_0^s (s-\tau)^{\alpha-2}v(\tau)d\tau ds\\ &+\frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^{n+1}G(t,\xi_k)\int_{\xi_k}^t\int_0^s (s-\tau)^{\alpha-2}v(\tau)d\tau ds\\ &+\sum_{k=1}^n G(t,\xi_k)I_k(x(\xi_k)), \quad \text{for } t\in J_k. \end{aligned} \tag{19}$$

Subject to (20), using boundary conditions of (10), we have

$$\begin{aligned} b &= b\sum_{i=0}^n\alpha_i\int_{\xi_i}^{\xi_{i+1}} g(t)dt-\sum_{i=0}^n\alpha_i\int_{\xi_i}^{\xi_{i+1}}\frac{1}{\Gamma(\alpha-1)}\\ &\cdot\int_{\xi_i}^t\int_0^s (s-\tau)^{\alpha-2}v(\tau)g(t)d\tau dsdt\\ &+\sum_{i=0}^n\alpha_i\int_{\xi_i}^{\xi_{i+1}}\frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^{n+1}G(t,\xi_k)\\ &\cdot\int_{\xi_k}^t\int_0^s (s-\tau)^{\alpha-2}v(\tau)g(t)d\tau dsdt\\ &+\sum_{i=0}^n\alpha_i\int_{\xi_i}^{\xi_{i+1}}\sum_{k=1}^n G(t,\xi_k)I_k(x(\xi_k))g(t)dt. \end{aligned} \tag{20}$$

Consequently,

$$\begin{aligned} b &= \frac{-1}{\Delta\Gamma(\alpha-1)}\sum_{i=0}^n\alpha_i\int_{\xi_i}^{\xi_{i+1}}\int_{\xi_i}^t\int_0^s g(t)(s-\tau)^{\alpha-2}v(\tau)d\tau dsdt\\ &+\frac{1}{\Delta\Gamma(\alpha-1)}\sum_{i=0}^n\alpha_i\int_{\xi_i}^{\xi_{i+1}}\sum_{k=1}^{n+1}G(t,\xi_k)\\ &\cdot\int_{\xi_k}^t\int_0^s (s-\tau)^{\alpha-2}g(t)v(\tau)d\tau dsdt\\ &+\frac{1}{\Delta}\sum_{i=0}^n\alpha_i\int_{\xi_i}^{\xi_{i+1}} g(t)\sum_{k=1}^n G(t,\xi_k)I_k(x(\xi_k))dt, \end{aligned} \tag{21}$$

where $\Delta=1-\sum_{i=0}^n\alpha_i\int_{\xi_i}^{\xi_{i+1}}g(t)dt$. In what follows, we always assume that $0<\sum_{i=0}^n\alpha_i\int_{\xi_i}^{\xi_{i+1}}g(t)dt<1$.

From (18)–(20), and (22), it can be received that

$$\begin{aligned} x(t) &= A_v^1(t)+A_v^2(t)+B_1(v)+B_2(v)\\ &+\sum_{k=1}^n G(t,\xi_k)I_k(x(\xi_k))\\ &+\frac{1}{\Delta}\sum_{i=0}^n\alpha_i\int_{\xi_i}^{\xi_{i+1}}\sum_{k=1}^n G(t,\xi_k)I_k(x(\xi_k))dt, \end{aligned} \tag{22}$$

where $A_v^1(t), A_v^2(t), B_1(v), B_2(v)$ are denoted by (12). □

Define $X=\{x(t): x(t)\in C(J_k), {}^cD_{0+}^{\beta}x(t)\in C(J), x(\xi_k^+), x(\xi_k^-)$ exit and $x(\xi_k^-)=x(\xi_k), k=0,1,\dots,n\}$. Obviously, X is a Banach space endowed with the norm $\|x\|_X=\|x\|+\left\|{}^cD_{0+}^{\beta}x\right\|=\sup_{t\in[0,1]}|x(t)|+\sup_{t\in[0,1]}|{}^cD_{0+}^{\beta}x(t)|$.

Lemma 3. *For any $v\in L^1[0,1]$, the following results are true*

(1) $\left|A_v^1(t)\right|\le(\|v\|/\Gamma(\alpha+1))(1-\xi_k^{\alpha})$, *for* $t\in J_k, k=0,1,2,\dots,n$

(2) $\left|A_v^2(t)\right|\le\|v\|/\Gamma(\alpha+1)$, for $t\in J_k, k=0,1,2,\dots,n$

(3) $\left|B_1(v)\right|\le(\|v\|/\Delta\Gamma(\alpha+1))\sum_{i=0}^n\alpha_i[(1/(\alpha+1))(\xi_{i+1}^{\alpha+1}-\xi_i^{\alpha+1})+\xi_{i+1}(\xi_{i+1}^{\alpha}-\xi_i^{\alpha})]$

(4) $\left|B_2(v)\right|\le(\|v\|/\Delta\Gamma(\alpha+1))\sum_{i=0}^n\alpha_i(\xi_{i+1}-\xi_i)$.

Proof. For $t\in J_k, k=0,1,2,\dots,n$, we have

$$\begin{aligned} \left|A_v^1(t)\right| &\le \frac{1}{\Gamma(\alpha-1)}\int_{\xi_k}^1\int_0^s (s-\tau)^{\alpha-2}|v(\tau)|d\tau ds\\ &\le \frac{\|v\|}{\Gamma(\alpha+1)}(1-\xi_k^{\alpha}), \end{aligned} \tag{23}$$

$$\begin{aligned} \left|A_v^2(t)\right| &\le \frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^{n+1}\left|G(t,\xi_k)\right|\int_{t_{k-1}}^{\xi_k}\int_0^s (s-\tau)^{\alpha-2}|v(\tau)|d\tau ds\\ &\le \frac{\|v\|}{\Gamma(\alpha+1)}\sum_{k=1}^{n+1}(\xi_k^{\alpha}-\xi_{k-1}^{\alpha})\\ &= \frac{\|v\|}{\Gamma(\alpha+1)}. \end{aligned} \tag{24}$$

According to (24) and (25), we get

$$\begin{aligned}|B_1(v)| &\le \frac{1}{\Delta\Gamma(\alpha-1)}\sum_{i=0}^{n}\alpha_i\int_{\xi_i}^{\xi_{i+1}}\int_{\xi_i}^{t}\int_0^s (s-\tau)^{\alpha-2}|v(\tau)|d\tau ds dt\\ &\le \frac{\|v\|}{\Delta\Gamma(\alpha-1)}\sum_{i=0}^{n}\alpha_i\int_{\xi_i}^{\xi_{i+1}}(t^\alpha-\xi_i^\alpha)dt\\ &= \frac{\|v\|}{\Delta\Gamma(\alpha+1)}\sum_{i=0}^{n}\alpha_i\left[\frac{1}{\alpha+1}\left(\xi_{i+1}^{\alpha+1}-\xi_i^{\alpha+1}\right)-\xi_i^\alpha\xi_{i+1}+\xi_i^{\alpha+1}\right]\\ &= \frac{\|v\|}{\Delta\Gamma(\alpha+1)}\sum_{i=0}^{n}\alpha_i\left[\frac{1}{\alpha+1}\left(\xi_{i+1}^{\alpha+1}-\xi_i^{\alpha+1}\right)+\xi_{i+1}(\xi_{i+1}^\alpha-\xi_i^\alpha)\right]\end{aligned} \tag{25}$$

and

$$\begin{aligned}|B_2(v)| &\le \frac{\|v\|}{\Delta\Gamma(\alpha-1)}\sum_{i=0}^{n}\alpha_i\int_{\xi_i}^{\xi_{i+1}}\sum_{k=1}^{n+1}|G(t,\xi_k)|\int_{\xi_{k-1}}^{\xi_k}\int_0^s(s-\tau)^{\alpha-2}d\tau ds dt\\ &\le \frac{\|v\|}{\Delta\Gamma(\alpha+1)}\sum_{i=0}^{n}\alpha_i\int_{\xi_i}^{\xi_{i+1}}\sum_{k=1}^{n+1}(\xi_k^\alpha-\xi_{k-1}^\alpha)dt\\ &= \frac{\|v\|}{\Delta\Gamma(\alpha+1)}\sum_{i=0}^{n}\alpha_i(\xi_{i+1}-\xi_i).\end{aligned} \tag{26}$$

□

Apparently, $|G(t,\xi_k)|\le 1$, for $t\in[0,1]$ and $k=1,\dots,n$, so in view of Lemma 3, and combing with (11), we can write

$$\begin{aligned}|x(t)| \le{}& \frac{2\|v\|}{\Gamma(\alpha+1)}+\frac{\|v\|}{\Delta\Gamma(\alpha+1)}\sum_{i=0}^{n}\alpha_i(\xi_{i+1}-\xi_i)\\ &+\frac{\|v\|}{\Delta\Gamma(\alpha+1)}\sum_{i=0}^{n}\alpha_i\left[\frac{1}{\alpha+1}\left(\xi_{i+1}^{\alpha+1}-\xi_i^{\alpha+1}\right)+\xi_{i+1}(\xi_{i+1}^\alpha-\xi_i^\alpha)\right]\\ &+\sum_{k=1}^{n}\left|I_k(x(\xi_k))\right|+\frac{1}{\Delta}\sum_{i=0}^{n}\alpha_i\int_{\xi_i}^{\xi_{i+1}}\sum_{k=1}^{n}\left|I_k(x(\xi_k))\right|dt.\end{aligned} \tag{27}$$

Lemma 4. (the Leray–Schauder alternative theorem). *Let $F: X\to X$ be a completely continuous operator (i.e., a map that restricted to any bounded set in X is compact). Let*

$$\varepsilon(F)=\{x\in X: x=\lambda F(x), 0<\lambda<1\}. \tag{28}$$

Then either the set $\varepsilon(F)$ is unbounded, or F has at least one fixed point.

The operator $F: X\to X$ is defined by

$$\begin{aligned}F(x)(t) ={}& A^1_{f_x}(t)+A^2_{f_x}(t)+B_1(f_x)+B_2(f_x)\\ &+\sum_{k=1}^{n}G(t,\xi_k)I_k(x(\xi_k))\\ &+\frac{1}{\Delta}\sum_{i=0}^{n}\alpha_i\int_{\xi_i}^{\xi_{i+1}}\sum_{k=1}^{n}G(t,\xi_k)I_k(x(\xi_k))dt,\end{aligned} \tag{29}$$

where $f_x=f(t,x(t),{}^cD^\alpha_{0+}x(t))$. Accordingly, we know that

$$\begin{aligned}F(x)'(t) ={}& \frac{-1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}f\left(t,x(t),{}^cD^\beta_{0+}x(t)\right)ds\\ &+\frac{-1}{\Gamma(\alpha-1)}\sum_{k=1}^{n+1}\int_{\xi_{k-1}}^{\xi_k}\int_0^s(s-\tau)^{\alpha-2}\\ &\cdot f\left(t,x(t),{}^cD^\beta_{0+}x(t)\right)d\tau ds+\sum_{k=1}^{n}I_k(x(\xi_k))\end{aligned} \tag{30}$$

and

$$\begin{aligned}{}^cD^\beta_{0+}F(x)(t) &= \int_0^t\frac{1}{\Gamma(n-\beta)}F^{(n)}(s)(t-s)^{n-\beta-1}ds\\ &= \int_0^t\frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}F'(x)(s)ds.\end{aligned} \tag{31}$$

Lemma 5. *The operator $F: X\to X$ is completely continuous.*

Proof. The operator F is continuous in view of the continuity of $G(t,\xi_k)$, $f(t,x(t),{}^cD^\beta_{0+}x(t))$, and $I_k(x)$. Let $\Omega\subset X$ be bounded. Then there are positive constants T_1 and T_2 such that

$$\left|f\left(t,x(t),{}^cD^\beta_{0+}x(t)\right)\right|\le T_1,\quad \left|I_k(x(\xi_k))\right|\le T_2,\quad \text{for } x\in\Omega. \tag{32}$$

For convenience, we set

$$\begin{aligned}&T=\max\{T_1,T_2\},\quad R=\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}+n,\\ &N=\frac{2}{\Gamma(\alpha+1)}+\frac{1}{\Delta\Gamma(\alpha+1)}\sum_{i=0}^{n}\alpha_i(\xi_{i+1}-\xi_i)\\ &\quad+\frac{1}{\Delta\Gamma(\alpha+1)}\sum_{i=0}^{n}\alpha_i\left[\frac{1}{\alpha+1}\left(\xi_{i+1}^{\alpha+1}-\xi_i^{\alpha+1}\right)+\xi_{i+1}(\xi_{i+1}^\alpha-\xi_i^\alpha)\right]\\ &\quad+n\left[1+\frac{1}{\Delta}\sum_{i=0}^{n}\alpha_i(\xi_{i+1}-\xi_i)\right].\end{aligned} \tag{33}$$

For any $x\in\Omega$, we have

$$\begin{aligned}|F(x)(t)| \le{}& \frac{1}{\Gamma(\alpha-1)}\int_{\xi_k}^{t}\int_0^s(s-\tau)^{\alpha-2}\left|f_x(\tau)\right|d\tau ds\\ &+\frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^{n+1}\left|G(t,\xi_k)\right|\int_{\xi_{k-1}}^{\xi_k}\int_0^s(s-\tau)^{\alpha-2}\left|f_x(\tau)\right|d\tau ds\\ &+\frac{1}{\Delta\Gamma(\alpha-1)}\sum_{i=0}^{n}\alpha_i\int_{\xi_i}^{\xi_{i+1}}\int_{\xi_i}^{t}\int_0^s(s-\tau)^{\alpha-2}\left|f_x(\tau)\right|d\tau ds dt\\ &+\frac{1}{\Delta\Gamma(\alpha-1)}\sum_{i=0}^{n}\alpha_i\int_{\xi_i}^{\xi_{i+1}}\sum_{k=1}^{n+1}\left|G(t,\xi_k)\right|\\ &\cdot\int_{\xi_{k-1}}^{\xi_k}\int_0^s(s-\tau)^{\alpha-2}\left|f_x(\tau)\right|d\tau ds dt\\ &+\frac{1}{\Delta}\sum_{i=0}^{n}\alpha_i\int_{\xi_i}^{\xi_{i+1}}\sum_{k=1}^{n}\left|G(t,\xi_k)\right|I_k(x(\xi_k))dt\\ &+\sum_{k=1}^{n}\left|G(t,\xi_k)I_k(x(\xi_k))\right|\\ \le{}& T_1\left\{\frac{2-\xi_k^\alpha}{\Gamma(\alpha+1)}+\frac{1}{\Delta\Gamma(\alpha+1)}\sum_{i=0}^{n}\alpha_i(\xi_{i+1}-\xi_i)+\frac{1}{\Delta\Gamma(\alpha+1)}\right.\\ &\left.\cdot\sum_{i=0}^{n}\alpha_i\left[\frac{1}{\alpha+1}\left(\xi_{i+1}^{\alpha+1}-\xi_i^{\alpha+1}\right)+\xi_{i+1}(\xi_{i+1}^\alpha-\xi_i^\alpha)\right]\right\}\\ &+nT_2\left(\frac{1}{\Delta}\sum_{i=0}^{n}\alpha_i(\xi_{i+1}-\xi_i)+1\right)\le TN.\end{aligned} \tag{34}$$

Meanwhile, for $x\in\Omega$, we can get

$$\left|F(x)'(t)\right| \le T_1\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} + n\right) \le TR. \tag{35}$$

Furthermore, for $x \in \Omega$, we have

$$\begin{aligned}\left|{}^cD_{0+}^{\beta}F(x)(t)\right| &= \frac{1}{\Gamma(1-\beta)}\left|\int_0^t F(x)'(s)(t-s)^{-\beta}ds\right| \\ &\le \frac{TR}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}ds \\ &\le \frac{TR}{\Gamma(1-\beta)}\frac{1}{1-\beta} \\ &= \frac{TR}{\Gamma(2-\beta)}.\end{aligned} \tag{36}$$

Hence, the following result can be derived

$$\begin{aligned}\|F(x)\|_X &= \|F(x)\| + \left\|{}^cD_{0+}^{\beta}F(x)\right\| \\ &\le T\left[N + \frac{R}{\Gamma(2-\beta)}\right].\end{aligned} \tag{37}$$

Thus, we have shown the operator F is uniformly bounded.

Next, we will show that F is equicontinuous. Let $t_1, t_2 \in J_k$ with $t_1 \le t_2$, then we have

$$\begin{aligned}\left|(Fx)(t_2) - (Fx)(t_1)\right| &\le T_1\frac{1}{\Gamma(\alpha+1)}\left(t_2^{\alpha} - t_1^{\alpha}\right) \\ &\quad + T_1\frac{1}{\Gamma(\alpha+1)}\left|t_2 - t_1\right| \\ &\quad + T_1\frac{1}{\Delta\Gamma(\alpha+1)}\left(t_2^{\alpha} - t_1^{\alpha}\right)\sum_{i=0}^{n}\alpha_i(\xi_{i+1} - \xi_i) \\ &\quad + T_1\frac{1}{\Delta\Gamma(\alpha+1)}\left|t_2 - t_1\right|\sum_{i=0}^{n}\alpha_i(\xi_{i+1} - \xi_i) \\ &\quad + nT_2\left|t_2 - t_1\right|\frac{1}{\Delta}\sum_{i=0}^{n}\alpha_i(\xi_{i+1} - \xi_i) + T_2\left|t_2 - t_1\right|n \\ &\le \left|t_2 - t_1\right|\left[\frac{2T_1}{\Gamma(\alpha+1)} + \frac{2T_1}{\Delta\Gamma(\alpha+1)}\sum_{i=0}^{n}\alpha_i(\xi_{i+1} - \xi_i)\right. \\ &\quad \left. + nT_2\frac{1}{\Delta}\sum_{i=0}^{n}\alpha_i(\xi_{i+1} - \xi_i) + T_2 n\right]\end{aligned} \tag{38}$$

and

$$\begin{aligned}\left|F(x)'(t_2) - F(x)'(t_1)\right| &= \frac{1}{\Gamma(\alpha-1)}\int_0^{t_2}(s-\tau)^{\alpha-2}f_x(\tau)d\tau \\ &\quad - \frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}(s-\tau)^{\alpha-2}f_x(\tau)d\tau \\ &\le T_1\frac{1}{\Gamma(\alpha)}\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) \\ &\le T_1\frac{1}{\Gamma(\alpha)}\left|t_2 - t_1\right|.\end{aligned} \tag{39}$$

So we have

$$\begin{aligned}\left|{}^cD_{0+}^{\beta}F(x)(t_2) - {}^cD_{0+}^{\beta}F(x)(t_1)\right| &= \int_0^{t_2}\frac{(t_2-s)^{-\beta}}{\Gamma(1-\beta)}F(x)'(s)ds \\ &\quad - \int_0^{t_1}\frac{(t_2-s)^{-\beta}}{\Gamma(1-\beta)}F(x)'(s)ds \\ &= \frac{F(x)'(s)}{\Gamma(2-\beta)}\left(t_2^{1-\beta} - t_1^{1-\beta}\right) \\ &\le \frac{T_1\left|t_2 - t_1\right|^2}{\Gamma(\alpha)\Gamma(2-\beta)t_1^{\beta}}.\end{aligned} \tag{40}$$

Hence, we can get

$$\begin{aligned}&\left\|F(x)(t_2) - F(x)(t_1)\right\|_X \\ &\quad \le \left|t_2 - t_1\right|\left[\frac{2T_1}{\Gamma(\alpha+1)} + \frac{2T_1}{\Delta\Gamma(\alpha+1)}\sum_{i=0}^{n}\alpha_i(\xi_{i+1} - \xi_i)\right. \\ &\quad \left. + nT_2\frac{1}{\Delta}\sum_{i=0}^{n}\alpha_i(\xi_{i+1} - \xi_i) + T_2 n + \frac{T_1\left|t_2 - t_1\right|}{\Gamma(\alpha)\Gamma(2-\beta)t_1^{\beta}}\right],\end{aligned} \tag{41}$$

which implies that $\left\|F(x)(t_2) - F(x)(t_2)\right\|_X \to 0$ as $t_2 \to t_1$. Therefore, the operator F is equicontinuous, and the operator F is completely continuous. □

3. Main Results

In the following discussion, we assume that the following hypotheses are valid, where ρ_i, L_j and M_j are positive constants, for $i = 1, \ldots, 4$, $j = 1, 2$.

(H_1) $|f(t,x,y)| \le \rho_1 + \rho_2|x| + \rho_3|y|$
(H_2) $|I_k x(\xi_k)| \le \rho_4|x(\xi_k)|$, for $k = 1, \ldots, n$
(H_3) $|f(t,x,y) - f(t,x_1,y_1)| \le L_1[|x - x_1| + |y - y_1|]$
(H_4) $|I_k(u) - I_k(v)| \le L_2\|u - v\|$, for $k = 1, \ldots, n$
(H_5) $M_1 = \sup_{t\in[0,1]}|f(t,0,0)|$, $M_2 = I_k(0)$, for $k = 1, \ldots, n$.

The first result is based on the Letay–Schauder alternative theorem.

Theorem 1. *Assume that (H_1) and (H_2) hold. In addition it is assumed that*

$$\rho_0\left[N + \frac{R}{\Gamma(2-\beta)}\right] < 1, \tag{42}$$

where $\rho_0 = \max\{\rho_2, \rho_3, \rho_4\}$. Then boundary value problems (4) and (5) have at least one solution.

Proof. It will be verified that the set $\varepsilon = \{x \in Xx = \lambda F(x), 0 \le \lambda \le 1\}$ is bounded. Let $x \in \varepsilon$, then $x = \lambda F(x)$. For all $t \in [0,1]$, we have

$$x(t) = \lambda(Fx)(t). \tag{43}$$

According to (H_1), (H_2) and Lemma 3, for $t \in J_k$, $k = 0, 1, \ldots, n$, we have

$$|x(t)| = |\lambda(Fx)(t)| \le |(Fx)(t)|. \tag{44}$$

and

$$\begin{aligned}
|(Fx)(t)| &\le \left|A^1_{f_x}(t)\right| + \left|A^2_{f_x}(t)\right| + |B_1(f_x)| + |B_2(f_x)| \\
&\quad + \sum_{k=1}^{n} |G(t,\xi_k)||I_k(x(\xi_k))| \\
&\quad + \frac{1}{\Delta}\sum_{i=0}^{n}\alpha_i\int_{\xi_i}^{\xi_{i+1}}\sum_{k=1}^{n}|G(t,\xi_k)||I_k(x(\xi_k))|dt \\
&\le \left(\rho_1 + \rho_2|x(t)| + \rho_3\left|{}^cD^{\beta}_{0+}x(t)\right|\right) \\
&\quad \left\{\frac{2-\xi_k^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{\Delta\Gamma(\alpha+1)}\sum_{i=0}^{n}\alpha_i(\xi_{i+1}-\xi_i)\right. \\
&\quad + \frac{1}{\Delta\Gamma(\alpha+1)}\sum_{i=0}^{n}\alpha_i\left[\frac{1}{\alpha+1}\left(\xi_{i+1}^{\alpha+1}-\xi_i^{\alpha+1}\right)\right. \\
&\quad \left.\left. + \xi_{i+1}\left(\xi_{i+1}^{\alpha}-\xi_i^{\alpha}\right)\right]\right\} \\
&\quad + n\rho_4|x(\xi_k)|\left(\frac{1}{\Delta}\sum_{i=0}^{n}\alpha_i(\xi_{i+1}-\xi_i) + 1\right) \\
&\le (\rho_1 + \rho_0\|x\|_X)N.
\end{aligned} \tag{45}$$

Analogously, we have

$$\begin{aligned}
\left|F(x)'(t)\right| &\le \left(\rho_1 + \rho_2|x(t)| + \rho_3\left|{}^cD^{\beta}_{0+}x(t)\right|\right) \\
&\quad \cdot\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)}\right) + \rho_4|x(\xi_k)|n \\
&\le (\rho_1 + \rho_0\|x\|_X)R,
\end{aligned} \tag{46}$$

accordingly, we can get

$$\begin{aligned}
\left|{}^cD^{\beta}_{0+}Fx(t)\right| &= \left|\int_0^t \frac{1}{\Gamma(-\beta)}F'(s)(t-s)^{-\beta}ds\right| \\
&\le (\rho_1 + \rho_0\|x\|_X)R\int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}ds \\
&= (\rho_1 + \rho_0\|x\|_X)\frac{R}{\Gamma(2-\beta)}.
\end{aligned} \tag{47}$$

Hence, we have

$$\begin{aligned}
\|x\|_X &= \|F(x)\| + \left\|{}^cD^{\beta}_{0+}F(x)\right\| \\
&\le (\rho_1 + \rho_0\|x\|_X)\left[N + \frac{R}{\Gamma(2-\beta)}\right],
\end{aligned} \tag{48}$$

which means that $\rho_0[N + R/\Gamma(2-\beta)] < 1$ and ε is bounded. Therefore, by Lemma 4, the operator F has at least one fixed point. So boundary value problems (4) and (5) have at least one solution. □

Next, we will prove the uniqueness of solutions to boundary value problems (4) and (5) via the Banach contraction mapping principle.

Theorem 2. *Suppose that (H_3)–(H_5) are true, in addition that*

$$L < \left[N + \frac{R}{\Gamma(2-\beta)}\right]^{-1}, \tag{49}$$

then there is a unique solution for boundary value problem (4) and (5).

Proof. For convenience, we denote

$$L = \max(L_1, L_2), \quad M = \max(M_1, M_2). \tag{50}$$

We set $B_\theta = \{x \in X : \|x\|_X \le \theta\}$, for $x \in B_\theta$, on the basis of (H_1) and (H_3), we have

$$\begin{aligned}
\left|f\left(t, x(t), {}^cD^{\beta}_{0+}x(t)\right)\right| &\le \left|f\left(t, x(t), {}^cD^{\beta}_{0+}x(t)\right) - f(t,0,0)\right| + |f(t,0,0)| \\
&\le L_1\left[\|x\| + \left\|{}^cD^{\beta}_{0+}x\right\|\right] + M_1 \\
&= L_1\|x\|_X + M_1.
\end{aligned} \tag{51}$$

According to (H_4) and (H_5), we have

$$\begin{aligned}
\left|I_k(x_k)\right| &\le \left|I_k(x_k) - I_k(0)\right| + \left|I_k(0)\right| \le L_2\|x\| + M_2 \\
&\le L_2\|x\|_X + M_2.
\end{aligned} \tag{52}$$

So for $x \in B_\theta$, we have

$$\begin{aligned}
|F(x)(t)| &\le (L_1\|x\|_X + M_1)\left\{\frac{2-\xi_k^{\alpha}}{\Gamma(\alpha+1)}\right. \\
&\quad + \frac{1}{\Delta\Gamma(\alpha+1)}\sum_{i=0}^{n}\alpha_i(\xi_{i+1}-\xi_i) + \frac{1}{\Delta\Gamma(\alpha+1)} \\
&\quad \left.\cdot\sum_{i=0}^{n}\alpha_i\left[\frac{1}{\alpha+1}\left(\xi_{i+1}^{\alpha+1}-\xi_i^{\alpha+1}\right) + \xi_{i+1}\left(\xi_{i+1}^{\alpha}-\xi_i^{\alpha}\right)\right]\right\} \\
&\quad + (L_2\|x\|_X + M_2)n\left(1 + \frac{1}{\Delta}\sum_{i=0}^{n}\alpha_i(\xi_{i+1}-\xi_i)\right) \\
&\le (L\|x\|_X + M)N.
\end{aligned} \tag{53}$$

On the other hand, we get

$$\begin{aligned}
\left|F(x)'(t)\right| &\le (L_1\|x\|_X + M_1)\left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)}\right] \\
&\quad + \sum_{k=1}^{n}(L_2\|x\|_X + M_2) \\
&\le (L\|x\|_X + M)\left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} + n\right] \\
&= (L\theta + M)R,
\end{aligned} \tag{54}$$

further,

$$\begin{aligned}
\left|{}^cD^{\beta}_{0+}F(x)(t)\right| &= \left|\int_0^t \frac{1}{\Gamma(n-\beta)}F'(s)(t-s)^{-\beta}ds\right| \\
&\le (L\theta + M)R\int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}ds \\
&= \frac{1}{\Gamma(2-\beta)}(L\theta + M)R.
\end{aligned} \tag{55}$$

In consequence,

$$\begin{aligned}\|Fx\|_X &= \|F(x)\| + \left\|{}^cD_{0+}^{\beta}F(x)\right\| \leq (L\theta + M)N \\ &+ \frac{1}{\Gamma(2-\beta)}(L\theta + M)R \\ &= (L\theta + M)\left[N + \frac{R}{\Gamma(2-\beta)}\right]. \end{aligned} \quad (56)$$

Therefore, provided

$$\theta \geq \frac{M(N\Gamma(2-\beta) + R)}{\Gamma(2-\beta) - L[N\Gamma(2-\beta) + R]}, \quad (57)$$

we have $FB_\theta \subset B_\theta$.
For $x, y \in X$, $t \in J_k$, $k = 0, 1, \dots, n$, we obtain

$$\begin{aligned} &|F(x)(t) - F(y)(t)| \\ &\leq \frac{1}{\Gamma(\alpha-1)}\int_0^1\int_0^s (s-\tau)^{\alpha-2}\left|f_x(\tau) - f_y(\tau)\right| d\tau ds \\ &+ \frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^{n+1}|G(t,\xi_k)|\int_{\xi_{k-1}}^{\xi_k}\int_0^s (s-\tau)^{\alpha-2}\left|f_x(\tau) - f_y(\tau)\right| d\tau ds \\ &+ \frac{1}{\Delta\Gamma(\alpha-1)}\sum_{i=0}^{n}\alpha_i\int_{\xi_i}^{\xi_{i+1}}\int_{\xi_k}^{t}\int_0^s (s-\tau)^{\alpha-2}\left|f_x(\tau) - f_y(\tau)\right| d\tau ds dt \\ &+ \frac{1}{\Delta\Gamma(\alpha-1)}\sum_{i=0}^{n}\alpha_i\int_{\xi_i}^{\xi_{i+1}}\sum_{k=1}^{n+1}|G(t,\xi_k)| \\ &\cdot \int_{\xi_{k-1}}^{\xi_k}\int_0^s (s-\tau)^{\alpha-2}\left|f_x(\tau) - f_y(\tau)\right| d\tau ds dt \\ &+ \sum_{k=1}^{n}|G(t,\xi_k)||I_k(x_k) - I_k(y_k)| \\ &+ \frac{1}{\Delta}\sum_{i=0}^{n}\alpha_i\int_{\xi_i}^{\xi_{i+1}}\sum_{k=1}^{n}|G(t,\xi_k)||I_k(x_k) - I_k(y_k)| dt. \end{aligned} \quad (58)$$

According to (H_3)–(H_5), we get

$$\begin{aligned} &|F(x)(t) - F(y)(t)| \\ &\leq L_1\|x-y\|_X\left\{\frac{2-\xi_k^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Delta\Gamma(\alpha+1)}\sum_{i=0}^{n}\alpha_i(\xi_{i+1}-\xi_i)\right. \\ &\left.+ \frac{1}{\Delta\Gamma(\alpha+1)}\sum_{i=0}^{n}\alpha_i\left[\frac{1}{\alpha+1}\left(\xi_{i+1}^{\alpha+1} - \xi_i^{\alpha+1}\right) + \xi_{i+1}(\xi_{i+1}^{\alpha} - \xi_i^{\alpha})\right]\right\} \\ &+ L_2\|x-y\|_X n\left(1 + \frac{1}{\Delta}\sum_{i=0}^{n}\alpha_i(\xi_{i+1}-\xi_i)\right) \\ &\leq L\|x-y\|_X N. \end{aligned} \quad (59)$$

Similarly, it holds that

$$\begin{aligned} &|F'(x)(t) - F'(y)(t)| \leq L\|x-y\|_X R, \\ &\left|{}^cD_{0+}^{\beta}F(x)(t) - {}^cD_{0+}^{\beta}F(y)(t)\right| \leq L\|x-y\|_X \frac{R}{\Gamma(2-\beta)}. \end{aligned} \quad (60)$$

Based on the above derivation, we conclude that

$$\begin{aligned} \|F(x) - F(y)\|_X &\leq L\|x-y\|_X N + L\|x-y\|_X \frac{R}{\Gamma(2-\beta)} \\ &\leq L\|x-y\|_X\left[N + \frac{R}{\Gamma(2-\beta)}\right], \end{aligned} \quad (61)$$

So if

$$L < \left[N + \frac{R}{\Gamma(2-\beta)}\right]^{-1}, \quad (62)$$

then boundary value problem (4) and (5) has one and only one solution.

Example 1. Consider the following fractional order boundary value problem

$${}^cD_{0+}^{1.5}x(t) = f\left(t, x(t), {}^cD_{0+}^{0.5}x(t)\right), \quad t \in (0,1), t \neq \xi_1, \quad (63)$$

with multistripe and band-like boundary conditions

$$\begin{aligned} -\Delta x|_{t=\xi_1} &= I_1(x(\xi_1)), \\ x(0) = x(1) &= \sum_{i=0}^{1}\alpha_i\int_{\xi_i}^{\xi_{i+1}} x(t)g(t)dt, \end{aligned} \quad (64)$$

where

$$\begin{aligned} f\left(t, x(t), {}^cD_{0+}^{0.5}x(t)\right) &= \frac{1}{12(t^2+1)}x(t) + \frac{1}{12(t^2+1)}{}^cD_{0+}^{0.5}x(t) + \frac{1}{12}, \\ g(t) = t, \quad &I_k(x(\xi_1)) = \frac{1}{16} + \frac{1}{16}x(\xi_1), \\ \xi_0 = 0, \quad \xi_1 = \frac{1}{2}, \quad &\xi_2 = 1, \quad \alpha_0 = \frac{1}{4}, \quad \alpha_1 = \frac{1}{6}. \end{aligned} \quad (65)$$

Clearly,

$$\begin{aligned} &\left|f_x(t)\right| \leq \frac{1}{12}|x(t)| + \frac{1}{12}\left|{}^cD_{0+}^{0.5}x(t)\right| + \frac{1}{12}, \\ &\left|I(x(\xi_1))\right| \leq \frac{1}{16} + \frac{1}{16}\left|x(\xi_1)\right|, \\ &\left|f(t,x,y) - f(t,x_1,y_1)\right| \leq \frac{1}{12}\left[|x-x_1| + |y-y_1|\right], \\ &\left|I_1(u) - I_1(v)\right| \leq \frac{1}{16}\|u-v\|, \\ &M_1 = \sup_{t\in[0,1]}\left|f(t,0,0)\right| = \frac{1}{12}, \quad M_2 = I_k(0) = \frac{1}{16}. \end{aligned} \quad (66)$$

It is easy to verify that (H_1)–(H_5) hold. And by calculation, the following results can be obtained,

$$\begin{aligned} &\rho_1 = \frac{1}{12}, \quad \rho_2 = \frac{1}{12}, \quad \rho_3 = \frac{1}{12}, \quad \rho_4 = \frac{1}{16}, \quad \rho_0 = \frac{1}{12}, \\ &L_1 = M_1 = \frac{1}{12}, \quad L_2 = M_2 = \frac{1}{16}, \quad L = M = \frac{1}{12}. \end{aligned} \quad (67)$$

Furthermore, we have

$$\begin{aligned} \Delta = 0.9063, \quad R &= 2.8811, \quad N = 3.0941, \\ \rho_0\left[N + \frac{R}{\Gamma(2-\beta)}\right] &= 0.5288 < 1. \end{aligned} \quad (68)$$

By Theorem 1, boundary value problems (63) and (64) have at least one solution. We also have

$$\left[N + \frac{R}{\Gamma(2-\beta)}\right]^{-1} = 0.1576 > L. \quad (69)$$

By Theorem 2, boundary value problems (63) and (64) have a unique solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the manuscript, read, and approved the final draft.

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