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Research Article

Edge $\delta-$ Graceful Labeling for Some Cyclic-Related Graphs

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In this paper, we introduce a new type of labeling of a graph G with p vertices and q edges called edge $\delta-$ graceful labeling, for any positive integer δ, as a bijective mapping f of the edge set $E(G)$ into the set $\{\delta, 2\delta, 3\delta, \cdots, q\delta\}$ such that the induced mapping $f^* : V(G) \rightarrow \{0, \delta, 2\delta, 3\delta, \cdots, q\delta - \delta\}$, given by $f^*(u) = (\sum_{uv \in E(G)} f(uv)) \bmod (\delta k)$, where $k = \max(p, q)$, is an injective function. We prove the existence of an edge $\delta-$ graceful labeling, for any positive integer δ, for some cycle-related graphs like the wheel graph, alternate triangular cycle, double wheel graph $W_{n,n}$, the prism graph Π_n, the prism of the wheel $P(W_n)$, the gear graph G_n, the closed helm CH_n, the butterfly graph B_n, and the friendship Fr_n.

1. Introduction

The graphs considered here will be finite, undirected, and simple where $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G, respectively, $p = |V(G)|$ and $q = |E(G)|$.

A labeling of a graph is a mapping that carries graph elements (edges or vertices, or both) to positive integers, subject to certain constraints. A labeling of a graph is called edge labeling if the domain of the mapping is the edge set. Graph labeling methods are used for application problems in a communication network addressing system, for fasting communication in sensor networks, and for designing fault-tolerant systems with facility graphs, in coding theory for the design of good radar type codes, and can also be used for issues in mobile ad hoc networks [1–3].

In the early 1960s, the idea of graph labelings was introduced by Rosa in [4]. Following this paper, different techniques have been studied in graph labelings. One such graph labeling technique is the edge graceful labeling introduced by Lo [5]. In 1985 as a bijective, f from the set of edges to the set $\{1, 2, \cdots, q\}$ such that the induced map f^* from $V(G)$ to $\{0, 1, 2, \cdots, p-1\}$ given by $f^*(u) = (\sum_{uv \in E(G)} f(uv)) \bmod (p)$ is a bijective. The graph that admits a graceful labeling is called a graceful graph.

Solairaju and Chithra [6] in 2009 introduced a labeling of G called edge odd graceful labeling, which is a bijection f from the set of edges $E(G)$ to the set $\{1, 3, \cdots, 2q-1\}$ such that the induced map f^* from $V(G)$ to $\{0, 1, 2, \cdots, 2q-1\}$ given by $f^*(u) = (\sum_{uv \in E(G)} f(uv)) \bmod (2q)$ is an injective. For many results on this type of labeling, see [7–10].

Recently, a new type of labeling is introduced by Elsonbaty and Daoud [11] called edge even graceful labeling, which is a bijective f from the set of edges $E(G)$ to the set $\{2, 4, \cdots, 2q\}$ such that the induced map f^* from $V(G)$ to $\{0, 2, 4, \cdots, 2k-2\}$ given by $f^*(u) = (\sum_{uv \in E(G)} f(uv)) \bmod (2k)$ where $k = \max(p, q)$ is an injective. Several results have been published on edge even graceful labeling, see [12–15]. For a detailed survey on graph labeling, refer to a dynamic survey of graph labeling [16].

Now, we introduce a generalization of the edge graceful labeling to edge $\delta-$ graceful labeling for any positive integer δ.

Definition 1. An edge $\delta-$ graceful labeling of a graph $G = (V(G), E(G))$, with $p = |V(G)|$ vertices and $q = |E(G)|$ edges, for any positive integer δ, is a bijective mapping f of the edge set $E(G)$ into the set $\{\delta, 2\delta, 3\delta, \cdots, q\delta\}$ such that the induced

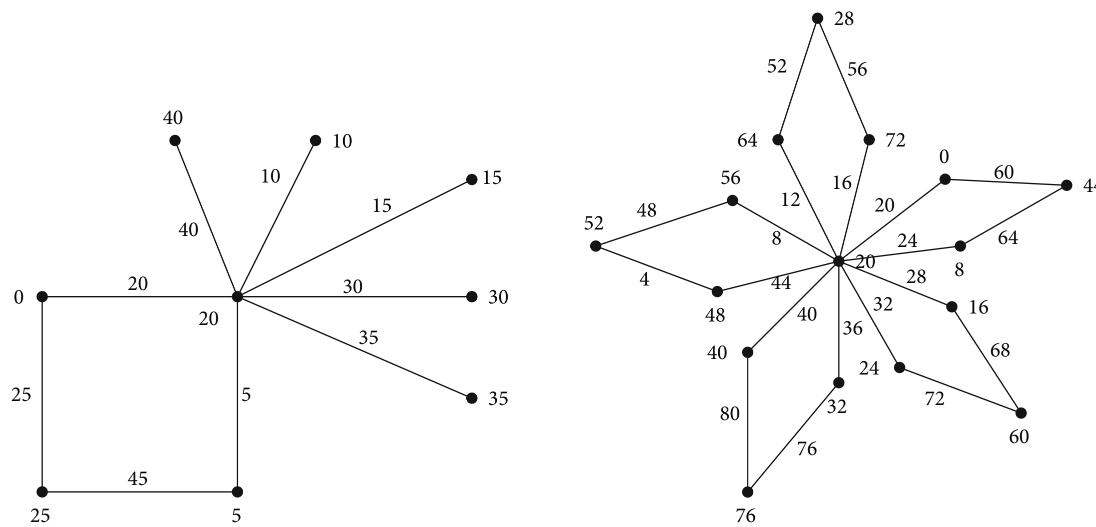

Figure 1: A graph with edge 5– graceful labeling and an edge 4– graceful labeling of another graph.

mapping $f^*: V(G) \rightarrow \{0, \delta, 2\delta, 3\delta, \cdots, q\delta - \delta\}$, given by $f^*(u) = (\sum_{uv \in E(G)} f(uv)) \bmod (\delta k)$, where $k = \max(p, q)$, is an injective function. The graph that admits an edge $\delta-$ graceful labeling is called an edge $\delta-$ graceful graph.

Note that, if $\delta = 2$, we have the edge even labeling; also, if $\delta = 3$, we have the edge triple labeling and so on. The odd cycle C_n whose vertices $v_1, v_2, \cdots, v_n$ and edges $(v_i v_{i+1})$ where $i = 1, 2, \cdots, n$ is an edge $\delta-$ graceful graph if and only if n is odd. Labeling the edges, respectively, by $\delta, 2\delta, 3\delta, \cdots, n\delta$ yields the labels $\delta, 3\delta, 5\delta, \cdots, 0\,; 2\delta, 4\delta, \cdots, (n-1)\delta$ on vertices, respectively. In Figure 1, we present an edge 5– graceful labeling of a graph and an edge 4– graceful labeling of another graph.

2. Edge $\delta-$ Graceful Labeling of the Wheel Graph W_n

Theorem 2. *For any positive integer δ, the wheel graph W_n, $n > 3$ is an edge $\delta-$ graceful graph.*

Proof. Let $\{v_0, v_1, v_2, \cdots, v_n\}$ be the vertices of W_n with hub vertex v_0, and the edges of W_n will be $\{v_0 v_i, v_i v_{i+1}, i = 1, 2, \cdots, n\}$. So, $p|V(W_n)| = n + 1$ and $q = |E(W_n)| = 2n$. There are three cases:

Case 1. $n \equiv 0 (\bmod\ 4)$, or $n \equiv 1 (\bmod\ 4)$.

We define the labeling function $f: E(W_n) \rightarrow \{\delta, 2\delta, 3\delta, \cdots, 2n\delta\}$ as follows:

$$\begin{aligned} &f(v_0 v_i) = i\delta, \quad \text{for } i \in \{1, 2, 3, \cdots, n-1\}, \\ &f(v_1 v_n) = (n+1)\delta, \quad f(v_i v_{i+1}) \\ &\qquad = \delta(2n - i + 1), \text{for } i \in \{1, 2, 3, \cdots, n-1\}. \end{aligned} \tag{1}$$

Then, the induced vertex labels are

$$\begin{aligned} &f^*(v_1) = \delta(n+2), \quad f^*(v_i) \\ &\qquad = \delta[2n - i + 3] \bmod (2n\delta), i = 2, \cdots, n. \end{aligned} \tag{2}$$

Hence, the labels of the vertices $v_2, v_3, v_4, \cdots, v_{n-1}, v_n$ will be $\delta, 0, \delta(2n-1), \cdots, \delta(n+4), \delta(n+3)$, respectively, which are all a multiple of δ and distinct numbers.

$$\begin{aligned} f^*(v_0) &= \left[\sum_{i=1}^{n} f(v_0 v_i) \bmod (2n\delta) = \sum_{i=1}^{n} (\delta i)\right] \bmod (2n\delta) \\ &= \left[\frac{\delta}{2}(n^2 + n)\right] \bmod (2n\delta). \end{aligned} \tag{3}$$

(i) If $n \equiv 0 (\bmod\ 4) \Rightarrow n = 4r \Rightarrow \delta q = 2n\delta = 8r\delta$, then

$$f^*(v_0) = \left[8r^2\delta + 2r\delta\right] \bmod (8r\delta) \equiv (2r\delta) \bmod (8r\delta) = \left(\frac{n\delta}{2}\right) \tag{4}$$

(ii) If $n \equiv 1 (\bmod\ 4)$, then $f^*(v_0) = n\,\delta$

Case 2. $n \equiv 2 (\bmod\ 4)$. Define the labeling function f as follows:

$$\begin{aligned} &f(v_1 v_n) = n\delta, \quad f(v_i v_{i+1}) = i\delta, \text{for } i \in \{1, 2, 3, \cdots, n-1\}, \\ &f(v_0 v_1) = (n+1)\delta, \quad f(v_0 v_i) \\ &\qquad = \delta(2n - i + 2), \text{for } i \in \{1, 2, 3, \cdots, n-1\}. \end{aligned} \tag{5}$$

Then, the induced vertex labels are $f^*(v_1) = 2\delta$ and $f^*(v_i) = [\delta(i+1)] \bmod (2n\delta), i \in \{2, 3, \cdots, n\}$. Hence, the labels of the vertices $v_2, v_3, \cdots, v_{n-1}, v_n$ will be $3\delta, 4\delta, \cdots, \delta n, \delta(n+1)$, respectively.

$$f^*(v_0) = \left[\sum_{i=1}^{n} f(v_0 v_i)\right] \bmod (2n\delta) = \left[\sum_{i=1}^{n} (n\delta + i\delta)\right] \bmod (2n\delta) = \left[\frac{\delta}{2}\left(3n^2 + n\right)\right] \bmod (2n\delta). \quad (6)$$

(i) If $n \equiv 2 (\bmod\ 4) \Rightarrow n = 4r + 2 \Rightarrow \delta q = 2n\delta = 8r\delta + 4\delta$, then

$$f^*(v_0) = \left[24r^2\delta + 26r\delta + 7\delta\right] \bmod (8r + 4)\delta \equiv (6r\delta + 3\delta) \bmod (8r\delta + 4\delta) = \left(\frac{3n\delta}{2}\right) \quad (7)$$

Case 3. $n \equiv 3 (\bmod\ 4)$. Define the labeling function f as follows:

$$f(v_0 v_1) = 2\delta, \quad f(v_0 v_i) = \delta(n - i + 3), \text{for } i \in \{2, 3, \cdots, n\},$$
$$f(v_1 v_n) = \delta, \quad f(v_i v_{i=1}) = \delta(n + i), \text{for } i \in \{1, 2, 3, \cdots, n - 1\}. \quad (8)$$

Then, the induced vertex labels are $f^*(v_1) = 3\delta$, $f^*(v_2) = 4\delta$, and $f^*(v_i) = [\delta(n + i + 2)] \bmod (2n\delta)$, $i = 3, 4, \cdots, n$. Hence, the labels of the vertices $v_3, v_4, \cdots, v_{n-1}, v_n$ will be $\delta(n + 5), \delta(n + 6), \cdots, \delta, 2\delta$, respectively.

$$f^*(v_0) = \left[\sum_{i=2}^{n+1} f(v_0 v_i)\right] \bmod (2n\delta) = \left[n\delta + \sum_{i=1}^{n} (i\delta)\right] \bmod (2n\delta) = \left[\frac{\delta}{2}\left(n^2 + 3n\right)\right] \bmod (2n\delta). \quad (9)$$

(i) If $n \equiv 3 (\bmod\ 4) \Rightarrow n = 4r + 3 \Rightarrow \delta q = 2n\delta = 8r\delta + 6\delta$, then

$$f^*(v_0) = \left[18r^2\delta + 18r\delta + 9\delta\right] \bmod (8r\delta + 6\delta) \equiv (4r\delta + 3\delta) \bmod (8r\delta + 6\delta) = n\delta \quad (10)$$

It is clear that, in all cases, for all $i \in \{1, 2, 3, \cdots, n\}$, the labels of the vertices v_i are all distinct, multiple of δ, and different from $f^*(v_0)$ which complete the proof.

It should be noted that W_3 is not an edge $\delta-$ graceful graph because for any bijective function $f : E(W_3) \to \{\delta, 2\delta, \cdots, 6\delta\}$, there is no injective induced function f^* that satisfied the requirements

Illustration 1. In Figure 2, we present W_{13} with edge 5− graceful labeling, W_{16} with edge 3− graceful labeling, W_{14} with edge 4− graceful labeling, and W_{15} with edge 6− graceful labeling, respectively.

3. Edge $\delta-$ Graceful Labeling of the Alternate Triangular Cycle $A(C_{2n})$

Definition 3 (see [16]). An alternate triangular cycle $A(C_{2n})$ is obtained from an even cycle $C_{2n} = \{v_1, u_1, v_2, u_2, \cdots, v_n, u_n\}$ by joining v_i and u_i to a new vertex w_i. That is, every alternate edge of a cycle is replaced by C_3.

Theorem 4. *For any positive integer δ, the alternate triangular cycle $A(C_{2n})$ is an edge $\delta-$ graceful graph.*

Proof. Let the alternate triangular cycle $A(C_{2n})$ be given as in Figure 3 with the edge set will be $\{v_i u_i, u_i, v_{i+1}, v_i w_i, u_i w_i, i = 1, 2, \cdots, n\}$. The graph $A(C_{2n})$ has $p = 3n$ and $q = 4n$. We define the labeling function $f : E(A(C_{2n})) \to \{\delta, 2\delta, 3\delta, \cdots, 4n\delta\}$ by

$$\begin{aligned}
&f(v_i u_i) = \delta i, \quad \text{for } i \in \{1, 2, 3, \cdots, n\},\\
&f(u_n v_1) = (3n + 1)\delta, \quad f(u_i v_{i+1}) = (4n - i + 1)\delta, \text{for } i \in \{1, 2, 3, \cdots, n - 1\},\\
&f(v_1 w_1) = (2n + 2)\delta, \quad f(v_i w_i) = \delta(n + 1 + i), \text{for } i \in \{2, 3, \cdots, n\},\\
&f(u_1 w_1) = (n + 2)\delta, \quad f(u_n w_n) = (n + 1)\delta, f(u_i w_i) = \delta(2n + 1 + i), \text{for } i \in \{2, 3, \cdots, n - 1\}.
\end{aligned} \quad (11)$$

By using the above labeling pattern, the induced vertex labels will be

$$\begin{aligned}
&f^*(v_1) = \delta(n + 4), \quad f^*(v_n) = \delta(2n + 3) \text{ and for } i = 2, 3, \cdots, n - 1,\\
&f^*(v_i) = [f(v_i u_i) + f(v_i w_i) + f(u_{i-1} v_i)] \bmod (4n\delta) = [\delta(n + 3 + i)] \bmod (4n\delta),\\
&f^*(u_1) = \delta(n + 3), \quad f^*(u_n) = \delta(n + 2) \text{ and for } i = 2, 3, \cdots, n - 1,\\
&f^*(u_i) = [f(v_i u_i) + f(u_i w_i) + f(u_i v_{i+1})] \bmod (4n\delta) = [\delta(2n + 2 + i)] \bmod (4n\delta),\\
&f^*(w_1) = \delta(3n + 4), \quad f^*(w_n) = \delta(3n + 2),\\
&f^*(w_i) = [\delta(3n + 2 + 2i)] \bmod (4n\delta), \quad i = 2, 3, \cdots, n - 1.
\end{aligned} \quad (12)$$

Hence, the labels of the vertices $v_2, v_3, \cdots, v_{n-1}$ will be $\delta(n + 5), \delta(n + 6), \cdots, \delta(2n + 2)$, respectively, the labels of the vertices $u_2, u_3, \cdots, u_{n-1}, u_n$ will be $\delta(2n + 4), \delta(2n + 5), \cdots, \delta(3n + 1)$, respectively, and the labels of the vertices $w_2, w_3, \cdots, w_{n-1}$ will be $\delta(3n + 6), \delta(3n + 8), \cdots, \delta n$, respectively.

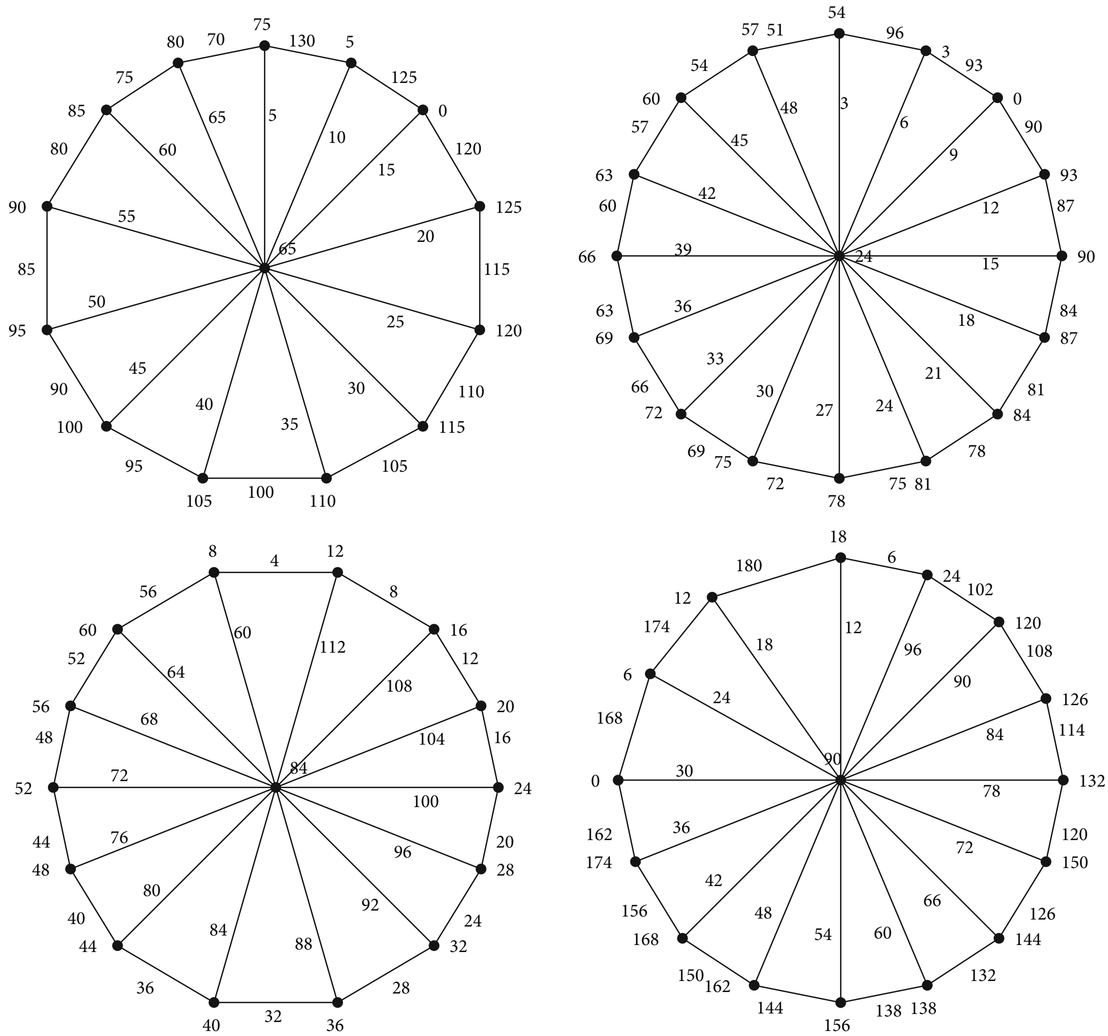

FIGURE 2: W_{13} with edge 5– graceful labeling, W_{16} with edge 3– graceful labeling, W_{14} with edge 4– graceful labeling, and W_{15} with edge 6– graceful labeling.

Hence, there are no repeated vertex labels which complete the proof.

Illustration 2. In Figure 4, we present an alternate triangular cycle $A(C_{16})$ with an edge 4– graceful labeling.

4. Edge δ– Graceful Labeling of the Double Wheel Graph $W_{n,n}$

Theorem 5. *For any positive integer δ, the double wheel graph $W_{n,n}$ is an edge δ– graceful graph.*

Proof. The double wheel graph $W_{n,n}$ consists of two cycles of n vertices connected to common hub. Let $\{v_1, v_2, \cdots, v_n\}$ be the vertices of one wheel and $\{u_1, u_2, \cdots, u_n\}$ be the vertices of other wheel with hub vertex v_0; the edges of $W_{n,n}$ will be $\{v_0v_i, v_iv_{i+1}, v_0u_i, u_iu_{i+1}, i = 1, 2, \cdots, n\}$. So, $p = |V(W_{n,n})| = 2n + 1$ and $q = |E(W_{n,n})| = 4n$, see Figure 5. There are two cases:

Case 1. When n is even. We define the labeling function $f : E(W_{n,n}) \rightarrow \{\delta, 2\delta, 3\delta, \cdots, 4n\delta\}$ as follows:

$$\begin{aligned} f(v_0v_i) &= (2n+i)\delta, \quad f(v_0u_i) \\ &= (3n+i)\delta, \text{for } i \in \{1,2,3,\cdots,n\}, \\ f(v_1v_n) &= n\delta, \quad f(v_iv_{i+1}) \\ &= i\delta, \text{for } i \in \{1,2,3,\cdots,n-1\}, \\ f(u_1u_n) &= 2n\delta, \quad f(u_iu_{i+1}) \\ &= (n+i)\delta, \text{for } i \in \{1,2,3,\cdots,n-1\}. \end{aligned} \tag{13}$$

Then, the induced vertex labels are

$$\begin{aligned} f^*(v_1) &= \delta(3n+2), \quad f^*(v_i) \\ &= [\delta(2n+3i-1)] \bmod (2n\delta), i = 2, \cdots, n, \\ f^*(u_1) &= \delta(2n+2), \quad f^*(u_i) \\ &= \delta[n+3i-1] \bmod (4n\delta), i = 2, \cdots, n. \end{aligned} \tag{14}$$

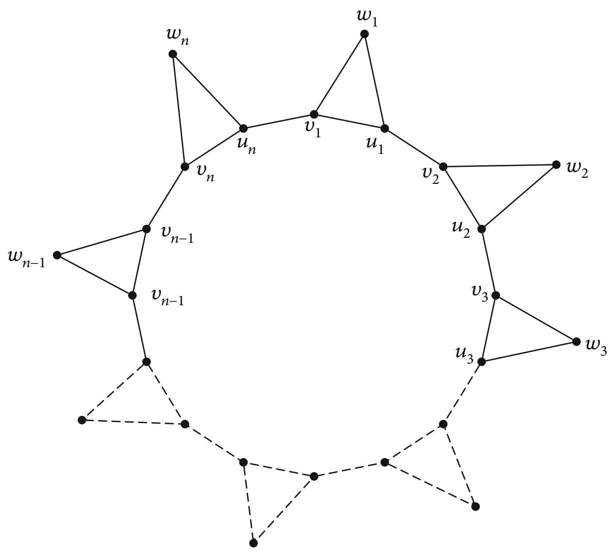

FIGURE 3: An alternate triangular cycle $A(C_{2n})$ with edge ordinary labeling.

Hence, the labels of the vertices $v_2, v_3, \cdots, v_n$ are $\delta(2n+5), \delta(2n+8), \cdots, \delta(n-4), \delta(n-1)$, respectively, and the labels of the vertices $u_2, u_3, \cdots, u_{n-1}, u_n$ are $\delta(n+5), \delta(n+8), \cdots, \delta(4n-4), \delta(4n-1)$, respectively.

$$\begin{aligned} f^*(v_0) &= \left[\sum_{i=1}^{n}(f(v_0v_i)+f(v_0v_i))\right] \bmod (4n\delta) \\ &= \left[\sum_{i=1}^{n}(2i\delta+5n\delta)\right] \bmod (4n\delta) \\ &= \left[5n^2\delta+n(n+1)\delta\right] \bmod (4n\delta) \\ &= \left[\delta\left(6n^2+n\right)\right] \bmod (4n\delta). \end{aligned} \tag{15}$$

Since $n\equiv 0(\bmod\ 2) \Rightarrow n=2r \Rightarrow \delta q=4n\delta=8r\delta$, then

$$\begin{aligned} f^*(v_0) &= \left[24r^2\delta+2r\delta\right] \bmod (8r\delta) \\ &\equiv 2r\delta \bmod (8r\delta)=n\delta. \end{aligned} \tag{16}$$

Case 2. If n is odd number, define the labeling function f as follows:

$$\begin{aligned} &f(v_0v_i)=i\delta,\quad f(v_0u_i)=(n+i)\delta, \text{for } i\in\{1,2,3,\cdots,n\}, \\ &f(v_1v_n)=(2n+1)\delta,\quad f(v_iv_{i+1}) \\ &\qquad=(3n-i+1)\delta, \text{for } i\in\{1,2,3,\cdots,n-1\}, \\ &f(u_1u_n)=(3n+1)\delta,\quad f(u_iu_{i+1}) \\ &\qquad=(4n-i+1)\delta, \text{for } i\in\{1,2,3,\cdots,n-1\}. \end{aligned} \tag{17}$$

Then, the induced vertex labels are

$$\begin{aligned} &f^*(v_1)=\delta(n+2),\quad f^*(v_i) \\ &\qquad=[\delta(2n-i+3)] \bmod (4n\delta), i=2,\cdots,n, \\ &f^*(u_1)=2\delta,\quad f^*(u_i) \\ &\qquad=[\delta(n-i+3)] \bmod (4n\delta), i=2,\cdots,n. \end{aligned} \tag{18}$$

Hence, the labels of the vertices $v_2, v_3, \cdots, v_{n-1}, v_n$ will be $\delta(2n+1), \delta(2n), \cdots, \delta(n+4), \delta(n+3)$, respectively, and the labels of the vertices $u_2, u_3, \cdots, u_{n-1}, u_n$ are $(n+1)\delta, n\delta, \cdots, 4\delta, 3\delta$, respectively.

$$\begin{aligned} f^*(v_0) &= \left[\sum_{i=1}^{n}(f(v_0v_i)+f(v_0v_i))\right] \bmod (4n\delta) \\ &= \left[\sum_{i=1}^{n}(i\delta+(n+i)\delta)\right] \bmod (4n\delta) \\ &= \left[n^2\delta+n(n+1)\delta\right] \bmod (4n\delta) \\ &= \left[\delta\left(2n^2+n\right)\right] \bmod (4n\delta). \end{aligned} \tag{19}$$

Clearly, the vertex labels are all distinct, multiple of δ, and different from $f^*(v_0)$. Thus, the double wheel graph is an edge $\delta-$ graceful graph for any positive integer δ.

Illustration 3. In Figure 6, we present $W_{9,9}$ with edge 4– graceful labeling and $W_{10,10}$ with edge 5– graceful labeling.

5. Edge $\delta-$ Graceful Labeling of the Prism Graph Π_n

Theorem 6. *For any positive integer δ, the prism graph Π_n is an edge $\delta-$ graceful graph.*

Proof. Let $\{v_0, v_1, v_2, \cdots, v_n\}$ be the vertices of C_n and $C_n' = \{u_0, u_1, u_2, \cdots, u_n\}$ be a copy of C_n. The prism Π_n is defined by joining each v_i of C_n to the corresponding vertex u_i of C_n' for all $i\in\{1,2,3,\cdots,n\}$; edges of W_n will be $\{v_i v_{i+1}, v_i u_i, u_i u_{i+1}, i=1,2,\cdots,n\}$. Thus, an n-prism graph has $p=2n$ vertices and $q=3n$ edges.

Let the vertex and edge symbols be given as in Figure 7.

Define the labeling mapping $f : E(\Pi_n) \to \{\delta, 2\delta, 3\delta, \cdots, 3n\delta\}$ by

$$\begin{aligned} &f(v_1v_n)=n\delta,\quad f(v_iv_{i+1}) \\ &\qquad=i\delta, \text{for } i\in\{1,2,3,\cdots,n-1\}, \\ &f(u_1u_n)=3n\delta,\quad f(u_iu_{i+1}) \\ &\qquad=\delta(2n+i), \text{for } i\in\{1,2,3,\cdots,n-1\}, \\ &f(v_1u_1)=\delta(n+1),\quad f(v_iu_i) \\ &\qquad=\delta(2n+2-i), \text{for } i\in\{2,3,\cdots,n\}. \end{aligned} \tag{20}$$

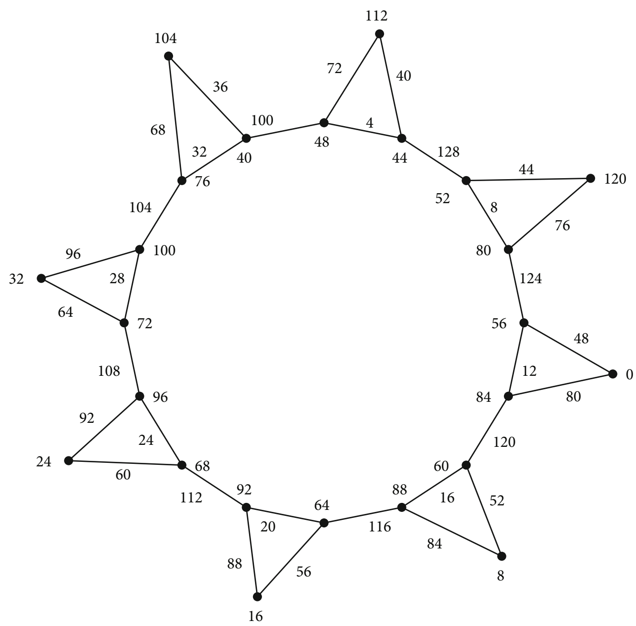

FIGURE 4: An alternate triangular cycle $A(C_{16})$ with an edge 4− graceful labeling.

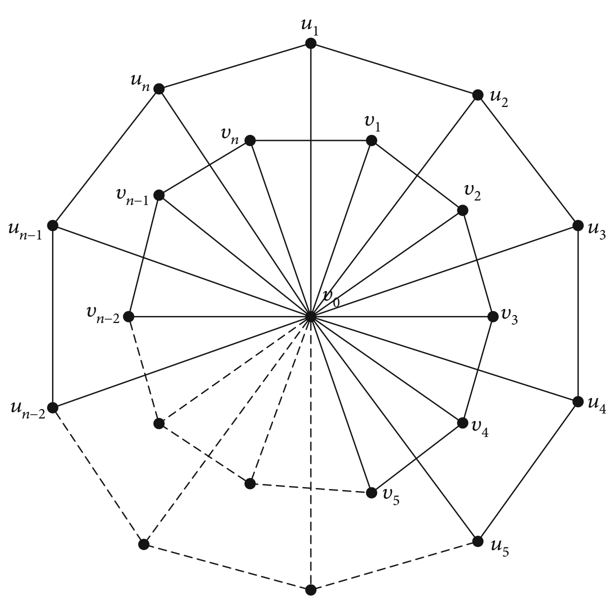

FIGURE 5: The double wheel graph $W_{n,n}$ with ordinary labeling.

So the induced vertex labels will be

$$\begin{aligned} f^*(v_1) &= \delta(2n+2), \quad f^*(v_i) \\ &= \delta(2n+1+i) \bmod (3n\delta), \text{ for } i \in \{2, 3, \cdots, n\}, \\ f^*(u_1) &= 2\delta, \quad f^*(u_i) = \delta(i+1), \text{ for } i \in \{2, 3, \cdots, n\}. \end{aligned} \tag{21}$$

Hence, the labels of the vertices $v_2, v_3, \cdots, v_{n-1}, v_n$ will be $\delta\,(2n+3), \delta\,(2n+4), \cdots, 0, \delta$, respectively, and the labels of the vertices $u_2, u_3, \cdots, u_{n-1}, u_n$ will be $3\,\delta, 4\,\delta, \cdots, \delta\, n, \delta\,(n+1)$, respectively.

Overall, the labels of the vertices are all different and multiplies of δ. Thus, the prism graph Π_n is an edge δ− graceful graph for any positive integer δ.

Illustration 4. In Figure 8, we present Π_{10} with edge 7− graceful labeling.

6. Edge δ− Graceful Labeling of the Prism of the Wheel $P(W_n)$

Definition 7 (see [3]). For $n \geq 3$, let $\{v_0, v_1, v_2, \cdots, v_n\}$ be the vertices of W_n with hub vertex v_0 and $W_n' = \{u_0, u_1, u_2, \cdots, u_n\}$ be a copy of W_n. Define Prism(W_n), called the *prism* of W_n, by $P(W_n) = K_2 \times W_n$, i.e., joining v_0 of W_n to the corresponding vertex u_0 of W_n' and each v_i of W_n to the corresponding vertex u_i of W_n' for all $i \in \{1,2,3, \cdots, n\}$. Thus, $E(P(W_n)) = E(W_n) \cup E(W_n') \cup \{v_i u_i | i \in \{1, 2, 3, \cdots, n\}\} \cup \{v_0 u_0\}$.

Theorem 8. *For any positive integer δ, the prism of the wheel $P(W_n)$ is an edge δ− graceful graph, when n is even.*

Proof. The prism of the wheel $P(W_n)$ has $p = 2n+2$ vertices and $q = 5n+1$ edges, the edges will be $\{v_0 v_i, v_i v_{i+1}, u_0 u_i, u_i u_{i+1}, v_i u_i, v_0 u_i, i = 1, 2, \cdots, n\}$. Let the vertex and edge symbols be given as in Figure 9.

We define the labeling function $f : E(P(W_n)) \to \{\delta, 2\delta, 3\delta, \cdots, (5n+1)\delta\}$ by

$$\begin{aligned} f(v_0 u_0) &= \delta(5n+1), \quad f(v_i u_i) \\ &= \delta i \text{ for } i \in \{1, 2, 3, \cdots, n\}, \\ f(v_1 v_n) &= \delta(n+1), \quad f(v_i v_{i+1}) \\ &= \delta(2n-i+1) \text{ for } i \in \{1, 2, 3, \cdots, n-1\}, \\ f(v_0 v_1) &= \delta(2n+1), \quad f(v_0 v_i) \\ &= \delta(3n-i+2) \text{ for } i \in \{2, 3, \cdots, n\}, \\ f(u_1 u_n) &= \delta(3n+1), \quad f(u_i u_{i+1}) \\ &= \delta(4n-i+1) \text{ for } i \in \{1, 2, 3, \cdots, n-1\}, \\ f(u_0 u_1) &= \delta(4n+1), \quad f(u_0 u_i) \\ &= \delta(5n-i+2) \text{ for } i \in \{2, 3, \cdots, n\}. \end{aligned} \tag{22}$$

In view of the above labeling pattern, then the induced vertex labels are

$$\begin{aligned} f^*(v_1) &= 2\delta, \quad f^*(v_i) \\ &= [\delta(2n-2i+4)] \bmod [(5n+1)\delta], i = 2, 3, \cdots, n, \\ f^*(u_1) &= (n+1)\delta, \quad f^*(u_i) \\ &= [\delta(3n-2i+3)] \bmod [(5n+1)\delta], i = 2, 3, \cdots, n. \end{aligned} \tag{23}$$

Hence, the labels of the vertices $v_2, v_3, \cdots, v_{n-1}, v_n$ are $2n\,\delta, (2n-2)\delta, \cdots, 6\,\delta, 4\delta$, respectively, and the labels of the vertices $u_2, u_3, \cdots, u_{n-1}, u_n$ are $(3n-1)\,\delta, (3n-3)\,\delta, \cdots, (n+5)\delta, (n+3)\delta$, respectively.

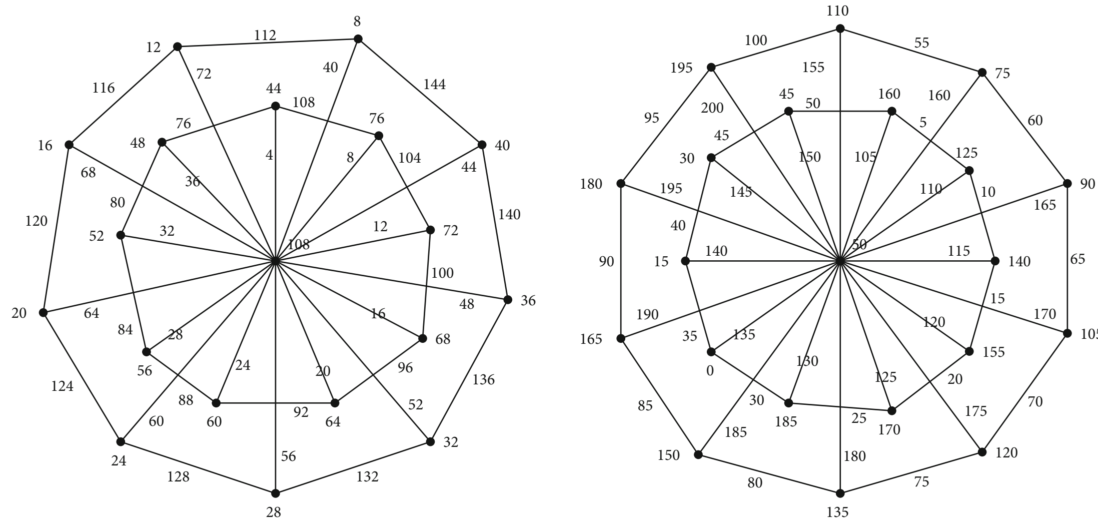

Figure 6: The graphs $W_{9,9}$ with edge 4− graceful labeling and $W_{10,10}$ with edge 5− graceful labeling.

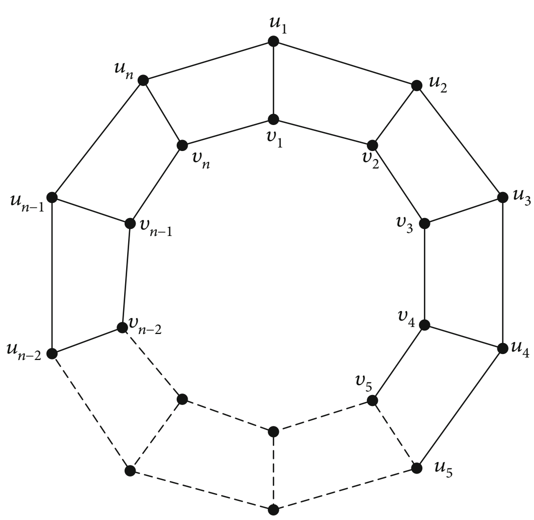

Figure 7: The prism graph Π_n with ordinary labeling.

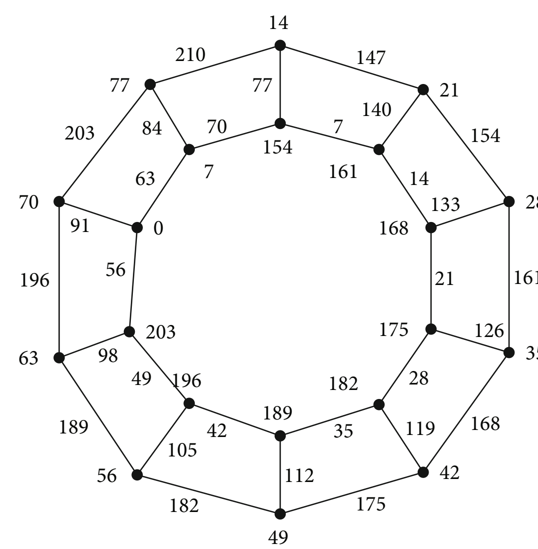

Figure 8: The Π_{10} with edge 7− graceful labeling.

Hence, there are no repeated vertex labels. Also in this labeling, we can check that

(i) $[f(v_0 v_1) + f(v_0 v_2)]$ mod $[(5n+1)\delta] = 0$

(ii) $[f(v_0 v_1) + f(v_0 v_{n-i+3})]$ mod $[(5n+1)\delta] = 0$, $i = 3, 4, \cdots, (n/2)+1$

Then, $f^*(v_0) = [\sum_{i=1}^{n} f(v_0 v_i) + f(v_0 u_0)]$ mod $[(5n+1)\delta] = 0$, finally

$$\begin{aligned} f^*(u_0) &= \left[\sum_{i=1}^{n} f(u_0 u_i) + f(v_0 u_0)\right] \bmod [(5n+1)\delta] \\ &= \left[\frac{\delta}{2}\left(9n^2+n\right)\right] \bmod [(5n+1)\delta]. \end{aligned} \tag{24}$$

There are three cases:

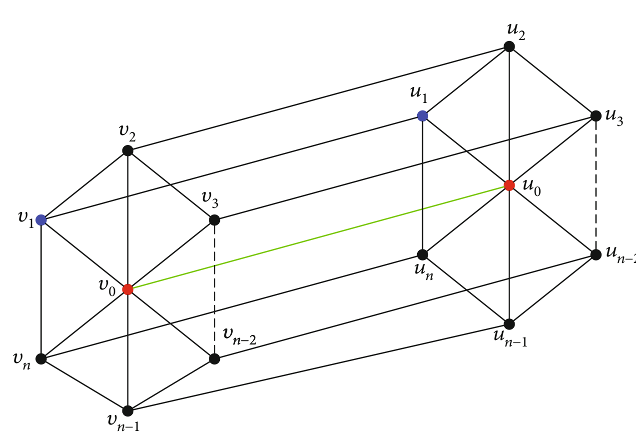

Figure 9: The prism of the wheel $P(W_n)$ with ordinary labeling.

Case 1. When $n \equiv 0 \pmod{10}$ or $n \equiv 2 \pmod{10}$ or $n \equiv 8 \pmod{10}$.

(i) If $n \equiv 0 \pmod{10} \Rightarrow n = 10r \Rightarrow \delta q = (5n+1)\delta = (50r+1)\delta$, then $f^*(u_0) = [450r^2\delta + 5r\delta] \bmod (50r+1)\delta \equiv (46r\delta + \delta) \bmod (50r+1)\delta = [23n + 5/5]\delta$

(ii) If $n \equiv 2 \pmod{10}$, then $f^*(u_0) = [(18n+4)/5]\delta$

(iii) If $n \equiv 8 \pmod{10}$, then $f^*(u_0) = [(3n+1)/5]\delta$

Case 2. When $n \equiv 4 \pmod{10}, \Rightarrow n = 10r+4 \Rightarrow \delta q = (5n+1)\delta = (50r+21)\delta$, then

$$f^*(u_0) = \left[450r^2\delta + 365r\delta + 74\delta\right] \bmod (50r+21)\delta \equiv (26r\delta + 11\delta) \bmod (50r+21)\delta = \left[\frac{13n+3}{5}\right]\delta. \quad (25)$$

In this case, we have $f^*(u_0)$ will equal to $f^*(u_i)$ when $i = (n+6)/5$, so we change the labeling of two edges (u_0v_0) and $(u_0u_{(n+6)/5})$ as follows: $f(u_0v_0) = ((24n+4)/5)\delta$ and $f(u_0v_{(n+6)/5}) = \delta(5n+1)$.

Then, $f^*(v_0) = ((24n+4)/5)\delta$, and $f^*(u_{(n+6)/5}) = ((14n+4)/5)\delta$.

Case 3. When $n \equiv 6 \pmod{10}, \Rightarrow n = 10r+6 \Rightarrow \delta q = (5n+1)\delta = (50r+31)\delta$, then

$$f^*(u_0) = \left[450r^2\delta + 545r\delta + 165\delta\right] \bmod (50r+31)\delta \equiv (16r\delta + 10\delta) \bmod (50r+31)\delta = \left[\frac{8n+2}{5}\right]\delta. \quad (26)$$

In this case, we have $f^*(u_0)$ will equal to $f^*(v_i)$ when $i = (n+9)/5$, so we change the labeling of two edges (u_0v_0) and $(v_0v_{(n+9)/5})$ as follows: $f(u_0v_0) = ((14n+1)/5)\delta$ and $f(v_0v_{(n+9)/5}) = \delta(5n+1)$.

Then, $f^*(u_0) = [(22n+3)/5]\delta$, and $f^*(v_{(n+9)/5}) = [(19n+6)/5]\delta$.

In all cases, $f^*(v_0)$ and $f^*(u_0)$ are not congruent to $f^*(v_i)$ nor $f^*(ui) \bmod [(5n+1)\delta]$, $i = 1, 2, 3, \cdots, n$ which completes the proof.

Illustration 5. In Figure 10, we present $P(W_6)$ with an edge 3− graceful labeling and $P(W_8)$ with an edge 4− graceful labeling.

Theorem 9. *For any positive integer δ, the prism of the wheel $P(W_n)$ is an edge δ− graceful graph, when n is odd.*

Proof. Let the vertex and edge symbols be given as in Figure 9. Define the labeling function f by

$$\begin{aligned}
&f(v_0u_0) = \delta, \quad f(v_iu_i) = \delta(n+i) \text{ for } i \in \{1, 2, 3, \cdots, n\},\\
&f(v_1v_n) = 3n\delta, \quad f(v_iv_{i+1}) = \delta(2n+i) \text{ for } i \in \{1, 2, 3, \cdots, n-1\},\\
&f(u_nu_1) = 4n\delta, \quad f(u_iu_{i+1}) = \delta(3n+i) \text{ for } i \in \{1, 2, 3, \cdots, n-1\},\\
&f(v_0v_1) = 5n\delta, \quad f(v_0v_i) = \begin{cases} \delta i & \text{if } i = 2, 3, \cdots, \dfrac{n+1}{2}, \\ \delta(4n+i-1) & \text{if } i = \dfrac{n+3}{2}, \cdots, n, \end{cases}\\
&f(u_0u_n) = (5n+1)\delta, \quad f(u_0u_i) = \begin{cases} \delta\left(\dfrac{n+1}{2}+i\right) & \text{if } i = 1, 2, 3, \cdots, \dfrac{\mathrm{n}-1}{2}, \\ \delta\left(\dfrac{7n+1}{2}+i\right) & \text{if } i = \dfrac{n+1}{2}, \cdots, \mathrm{n}-1. \end{cases}
\end{aligned} \quad (27)$$

In view of the above labeling pattern, we can check that

(i) $[f(v_0v_i) + f(v_0v_{n-i+2})] \bmod [(5n+1)\delta] = 0$, $i = 2, 3, \cdots, (n+1)/2$

(ii) $[f(u_0u_i) + f(u_0u_{n-i})] \bmod [(5n+1)\delta] = 0$, $i = 1, 2, \cdots, (n-1)/2$

(iii) $[f(v_0u_0) + f(v_0v_1)] \bmod [(5n+1)\delta] = 0$ then, the induced vertex labels are

$$\begin{aligned}
&f^*(v_0) = 0, \quad f^*(u_0) = \delta, f^*(v_1) = n\delta, f^*(u_n) = (5n-2)\delta, f^*\left(v_{(n+1)/2}\right) = 2n\delta,\\
&f^*(v_i) = \begin{cases} \delta(4i-2) & \text{if } i = 2, 3, \cdots, \dfrac{n+1}{2}, \\ \delta(4i-n-4) & \text{if } i = \dfrac{n+3}{2}, \cdots, n, \end{cases}\\
&f^*(u_i) = \begin{cases} \delta\left(\dfrac{5n-3}{2}+4i\right) & \text{if } i = 2, 3, \cdots, \dfrac{n-1}{2}, \\ \delta\left(\dfrac{n-5}{2}+4i\right) & \text{if } i = \dfrac{n+1}{2}, \cdots, n-1. \end{cases}
\end{aligned} \quad (28)$$

Hence, the labels of the vertices $v_2, v_3, \cdots, v_{(n+1)/2}$ are 6δ, $10\delta, \cdots, 2n\delta$ and the labels of the vertices $v_{(n+3)/2}, v_{(n+5)/2}, \cdots, v_n$ are $(n+2)\delta, (n+6)\delta, \cdots, (3n+4)\delta$, respectively. Also, the labels of the vertices $u_2, u_3, \cdots, u_{(n-1)/2}$ are $((5n+13)/2)\delta, ((5n+21)/2)\delta, \cdots, ((9n-7)/2)\delta$, and the labels of the vertices $u_{(n+1)/2}, u_{(n+3)/2}, \cdots, u_{n-1}$ are $((5n-1)/2)\delta, ((5n+7)/2)$

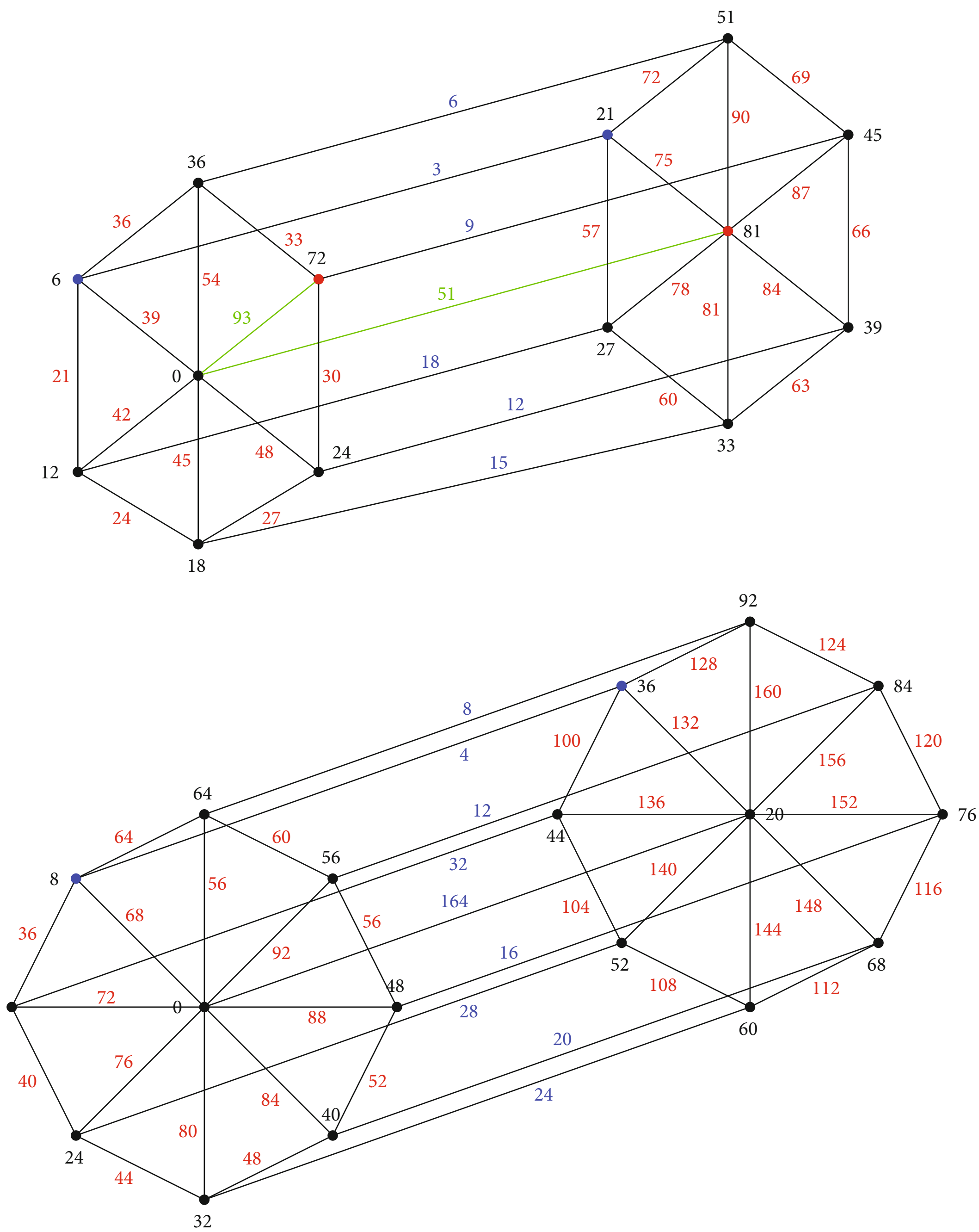

FIGURE 10: $P(W_6)$ with an edge 3– graceful labeling and $P(W_8)$ with an edge 4– graceful labeling.

$\delta, \cdots, ((3n-13)/2)\delta$, respectively. There are no repeated vertex labels when $n \equiv 3 \pmod 8$, *i.e.*, $n = 8r + 3$.

Also in this labeling, there are two cases:

Case 1. When $n \equiv 1 \pmod 4$, *i.e.*, $n = 4r + 1$.

In this case, we have $f^*(u_1)$ will equal to $f^*(u_i)$ when $i = (3n+5)/4$, so we change the labeling of two edges $(u_0 u_1)$ and $(u_0\, u_2)$ as follows: $f(u_0 u_1) = ((n+5)/2)\delta$ and $f(u_0 u_2) = ((n+3)/2)\delta$.

Then, $f^*(u_1) = ((7n+7)/2)\delta$, and $f^*(u_2) = ((5n+11)/2)\delta$.

Case 2. When $n \equiv 7 \pmod 8$, *i.e.*, $n = 8r + 7$.

In this case, we have $f^*(u_{(n+1)/2})$ will equal to $f^*(v_i)$ when $i = (7n+7)/8$, so we change the labeling of two edges $(u_0 u_{(n+1)/2})$ and $(u_0 u_{(n+3)/2})$ as follows: $f(u_0 u_{(n+1)/2}) = (4n+2)\delta$ and $f(u_0 u_{(n+3)/2}) = (4n+1)\delta$.

Then, $f^*(u_{(n+1)/2}) = ((5n+1)/2)\delta$, and $f^*(u_{(n+3)/2}) = ((5n+5)/2)\delta$.

Note that $P(W_3)$ does not follow this rule; however, it is an edge δ– graceful graph, as in Figure 11. In all cases, $f^*(v_0)$ and $f^*(u_0)$ are not congruent to $f^*(v_i)$ nor $f^*(ui) \bmod [(5n+1)\,\delta]$, $i = 1, 2, 3, \cdots, n$ which completes the proof.

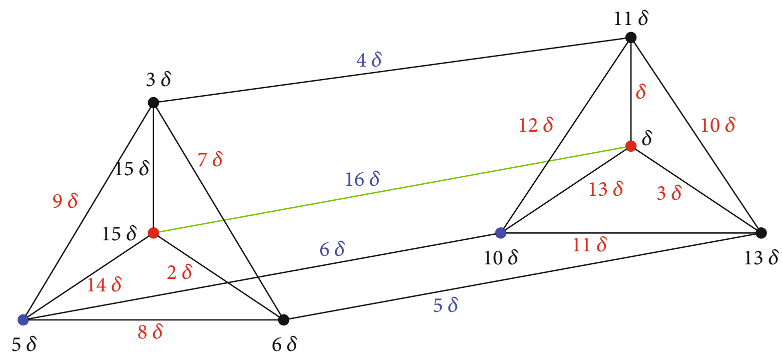

FIGURE 11: An edge $\delta-$ graceful labeling of the graph $P(W_3)$.

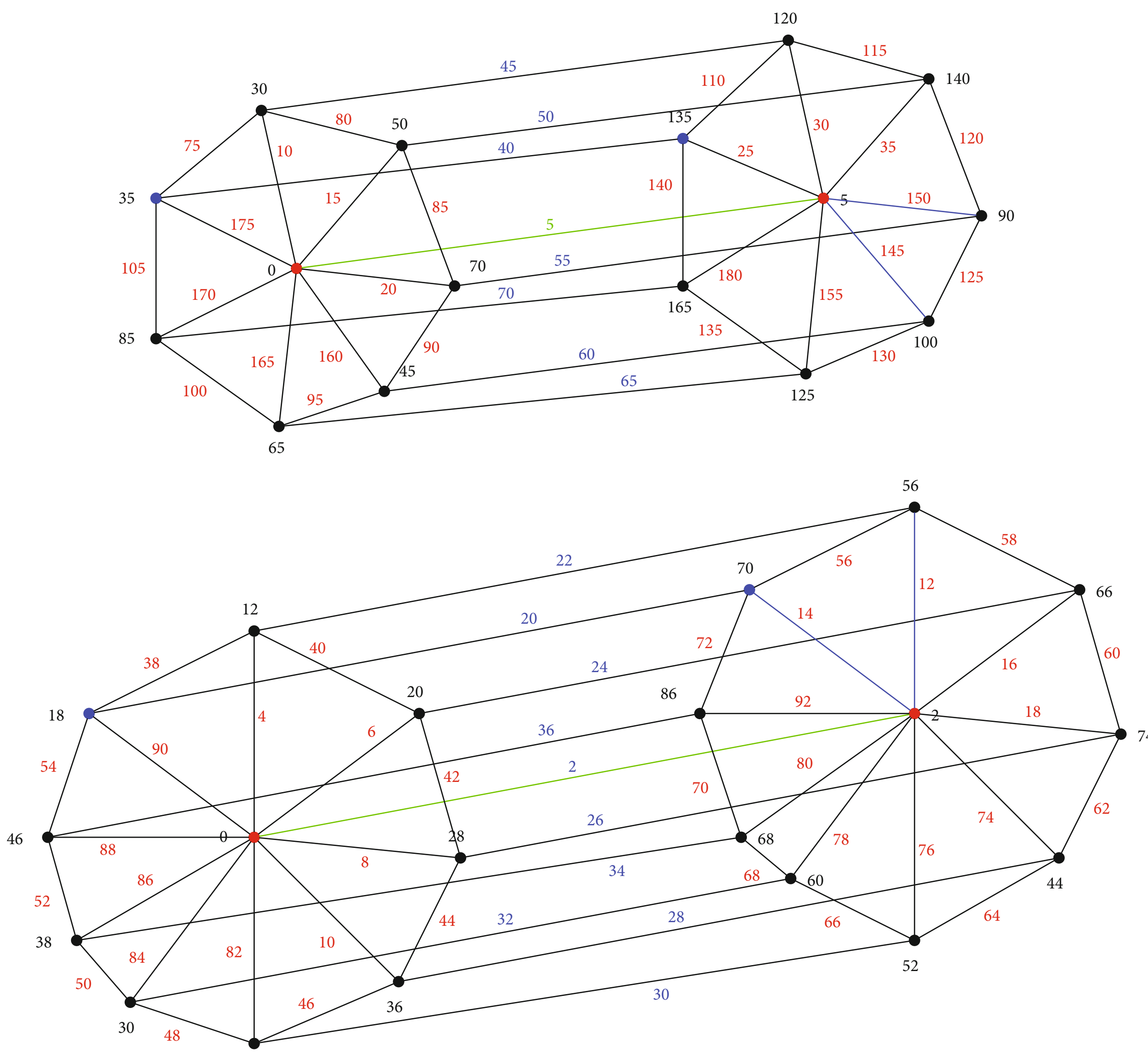

FIGURE 12: $P(W_7)$ with an edge 5– graceful labeling and $P(W_9)$ with an edge 2– graceful labeling.

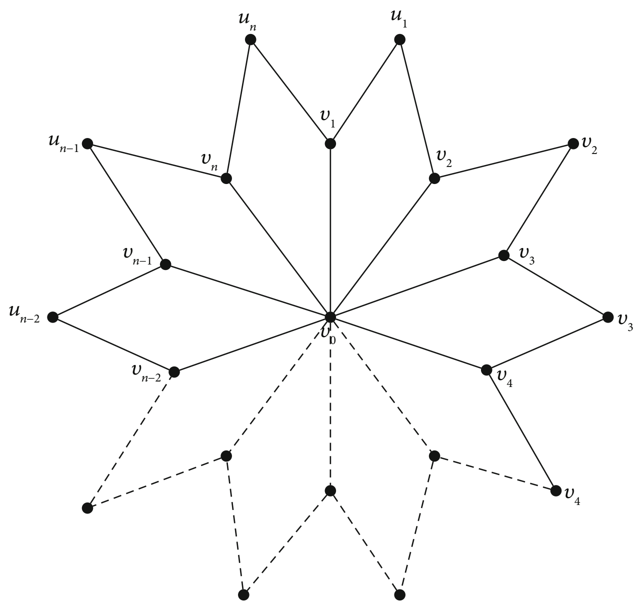

Figure 13: The gear graph G_n with ordinary labeling.

Illustration 6. In Figure 12, we present $P(W_7)$ with an edge 5 – graceful labeling and $P(W_9)$ with an edge 2– graceful labeling.

7. Edge δ– Graceful Labeling of the Gear Graph G_n

Theorem 10. *For any positive integer δ, the gear graph G_n is an edge δ– graceful graph.*

Proof. The gear graph G_n is the graph obtained from the wheel W_n by inserting a vertex between any two adjacent vertices in its cycle C_n; the gear graph G_n has $p = 2n + 1$ vertices and $q = 3n$ edges. Let the vertices of the wheel be $\{v_1, v_2, \cdots, v_n\}$, the hub vertex be v_0, and the new vertices be $\{u_1, u_2, \cdots, u_n\}$, so the edges will be $\{v_0v_i, v_iu_i, u_iv_{i+1}, i = 1, 2, \cdots, n\}$, see Figure 13.

We define the labeling function $f : E(G_n) \rightarrow \{\delta, 2\delta, 3\delta, \cdots, 3n\delta\}$ by

$$\begin{aligned} & f(v_iu_i) = \delta i \quad \text{for } i = 1, 2, \cdots, n, \\ & f(v_0v_1) = \delta(3n-1), \quad f(v_0v_i) = \delta(n + 2i - 3), i = 2, 3, \cdots, n, \\ & f(u_1v_2) = 3n\delta, \quad f(u_iv_{i+1}) = \delta(3n - 2i), i = 2, 3, \cdots, n. \end{aligned} \tag{29}$$

So the induced vertex labels will be

$$\begin{aligned} & f^*(v_1) = 0, \quad f^*(v_i) \\ & \quad = [\delta(n + i - 1)] \bmod (3n\delta), i = 2, 3, \cdots, n, \\ & f^*(u_n) = n\delta, \quad f^*(u_i) = \delta(3n - i), i = 2, 3, \cdots, n. \end{aligned} \tag{30}$$

Hence, the labels of the vertices $v_2, v_3, \cdots, v_{n-1}, v_n$ are $\delta(n+1), \delta(n+2), \cdots, \delta(2n-2), \delta(2n-1)$, respectively, and the labels of the vertices $u_1, u_2, \cdots, u_{n-2}, u_{n-1}$ are $(3n - 1)\delta, (3n-2)\delta, \cdots, \delta(2n+2), \delta(2n+1)$, respectively. Hence, there are no repeated vertex labels.

$$\begin{aligned} f^*(v_0) &= \sum_{i=1}^{n} (f(v_0v_i)) \bmod (3n\delta) \\ &= [(\delta n + \delta) + (\delta n + 3\delta) + (\delta n + 5\delta) + \cdots \\ &\quad + (\delta n + (2n-3)\delta) + (3\delta n - \delta)] \bmod (3n\delta) \\ &= [\delta n(n-1) + (\delta + 3\delta + 5\delta + \cdots + (2n-3)\delta) \\ &\quad + (3\delta n - \delta)] \bmod (3n\delta) = 2n^2\delta \bmod (3n\delta). \end{aligned} \tag{31}$$

Case 1. When $n \equiv 1 \pmod 6$ or $n \equiv 4 \pmod 6$.

(i) If $n \equiv 1 \pmod 6 \Rightarrow n = 6r + 1 \Rightarrow \delta q = 3n\delta = 18r\delta + 3\delta$, then $f^*(v_0) = [72r^2\delta + 24r\delta + 2\delta] \bmod (18r\delta + 3\delta) \equiv (12r\delta + 2\delta) \bmod (18r\delta + 3\delta) = 2n\delta$

(ii) If $n \equiv 4 \pmod 6$, then $f^*(v_0) = 2n\delta$

Case 2. When $n \equiv 0 \pmod 6$ or $n \equiv 3 \pmod 6$.

(i) If $n \equiv 0 \pmod 6 \Rightarrow n = 6r \Rightarrow \delta q = 3n\delta = 18r\delta$, then $f^*(v0) = [72\, r2\, \delta] \bmod (18r\delta) \equiv 0$

(ii) If $n \equiv 3 \pmod 6$, then $f^*(v_0) = 0$

In this case, we have $f^*(v_0)$ will equal to $f^*(v_1)$, so we change the labeling of two edges (v_0v_1) and $(u_n v_1)$ as follows: $f(v_0v_1) = 3n\delta$ and $f(u_n v_1) = \delta(3n-1)$. Then, $f^*(v_0) = \delta$ and $f^*(u_n) = (n-1)\delta$.

Case 3. When $n \equiv 2 \pmod 6$ or $n \equiv 5 \pmod 6$.

(i) If $n \equiv 2 \pmod 6 \Rightarrow n = 6r + 2 \Rightarrow \delta q = 3n\delta = 18r\delta + 6\delta$, then $f^*(v_0) = [72r^2\delta + 48r\delta + 8\delta] \bmod (18r\delta + 6\delta) \equiv (6r\delta + 2\delta) \bmod (18r\delta + 6\delta) = n\delta$

(ii) If $n \equiv 5 \pmod 6$, then $f^*(v_0) = n\delta$

In this case, we have $f^*(v_0)$ will equal to $f^*(u_n)$, so we change the labeling of two edges (v_0v_2) and $(v_n u_n)$ as follows: $f(v_0v_2) = n\delta$ and $f(v_n u_n) = \delta(n+1)$. Then, $f^*(v_0) = \delta(n-1)$, $f^*(u_n) = (n+1)\delta$, $f^*(v_n) = 2n\delta$, and $f^*(v_2) = n\delta$.

In all cases, $f^*(v_0)$ is not congruent to $f^*(v_i)$ nor $f^*(ui)$ mod $(3n\delta)$, $i = 1, 2, 3, \cdots, n$, and there are no repetition in the vertex labels which completes the proof.

Illustration 7. In Figure 14, we present G_{10} with edge 7– graceful labeling, G_9 with edge 4– graceful labeling, and G_{11} with edge 5– graceful labeling.

8. Edge δ– Graceful Labeling of the Closed Helm CH_n

The closed helm CH_n is the graph obtained from a helm H_n by joining each pendent vertex to form a cycle; a closed helm has $p = 2n + 1$ vertices and $q = 4n$ edges.

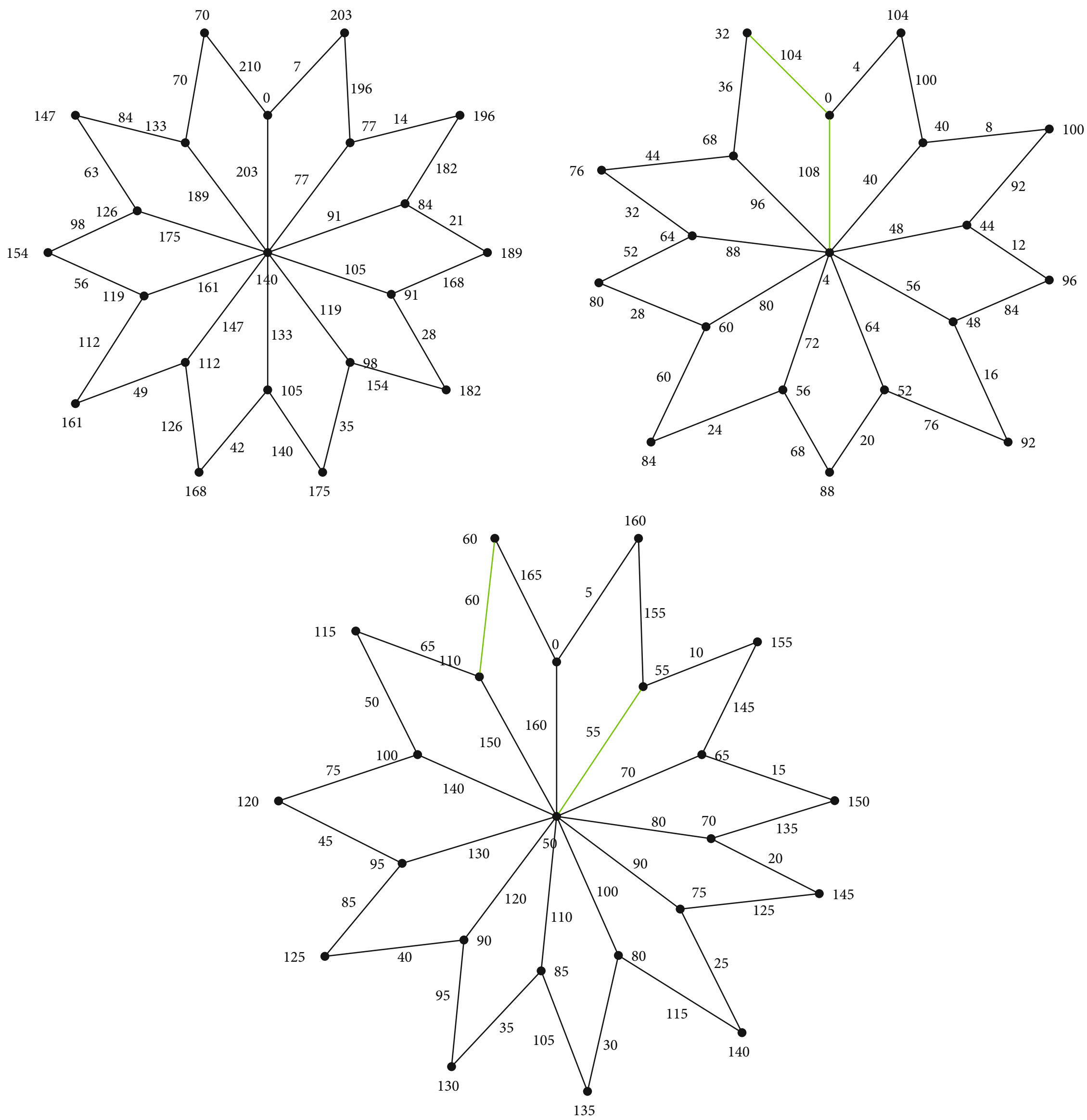

Figure 14: The gear graph G_{10} with edge 7− graceful labeling, G_9 with edge 4− graceful labeling, and G_{11} with edge 5− graceful labeling.

Theorem 11. *For any positive integer δ, the closed helm CH_n is an edge δ− graceful graph.*

Proof. In the closed helm CH_n, we have two copies of the cycle C_n; let the vertices in the inside copy be $\{v_1, v_2, \cdots, v_n\}$ and the vertices on the outside copy be $\{u_1, u_2, \cdots, u_n\}$ and a hub vertex v_0; the edges will be $\{v_0v_i, v_iu_i, v_iv_{i+1}, u_iu_{i+1}, i = 1, 2, \cdots, n\}$, see Figure 15.

We define the labeling function $f : E(\mathrm{CH}_n) \rightarrow \{\delta, 2\delta, 3\delta, \cdots, 4n\delta\}$ by

$$\begin{aligned} &f(v_0v_i) = \delta i \quad \text{for } i \in \{1, 2, 3, \cdots, n\}, \\ &f(v_1v_n) = 3n\delta, \quad f(v_iv_{i+1}) = \delta(2n + i) \text{ for } i \in \{1, 2, 3, \cdots, n-1\}, \\ &f(u_1u_n) = 4n\delta, \quad f(u_iu_{i+1}) = \delta(3n + i) \text{ for } i \in \{1, 2, 3, \cdots, n-1\}, \\ &f(v_1u_1) = (n+1)\delta, \quad f(u_iu_i) = \delta(2n - i + 2) \text{ for } i \in \{2, 3, \cdots, n\}. \end{aligned} \tag{32}$$

In view of the above labeling pattern, then the induced vertex labels are

$$\begin{aligned} &f^*(v_1) = \delta(2n+3), \quad f^*(v_i) \\ &\quad = [\delta(2n + 2i + 1)] \bmod (4n\delta), i = 2, 3, \cdots, n, \\ &f^*(u_1) = 2\delta, \quad f^*(u_i) = \delta(i+1), i = 2, 3, \cdots, n. \end{aligned} \tag{33}$$

Hence, the labels of the vertices $v_2, v_3, \cdots, v_{n-1}, v_n$ will be $\delta(2n+5), \delta(2n+7), \cdots, \delta(4n-1), \delta$, respectively, and the labels of the vertices $u_2, u_3, \cdots, u_{n-1}, u_n$ will be $3\delta, 4\delta, \cdots, \delta n, \delta(n+1)$, respectively.

$$\begin{aligned} f^*(v_0) &= \sum_{i=1}^{n} (f(v_0v_i)) \bmod (4n\delta) = \sum_{i=1}^{n} (\delta i) \bmod (4n\delta) \\ &= \left[\frac{\delta}{2}(n^2 + n)\right] \bmod (4n\delta). \end{aligned} \tag{34}$$

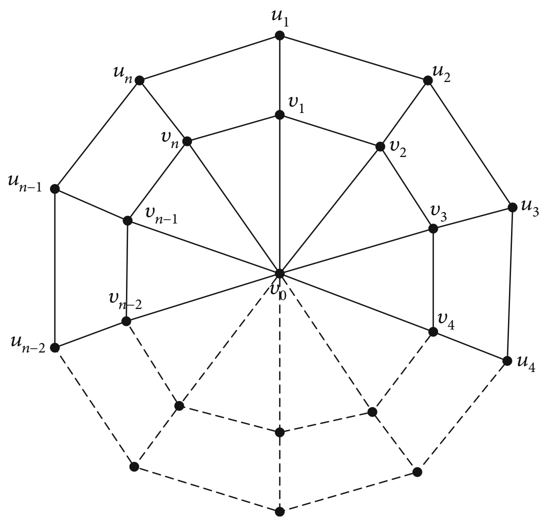

FIGURE 15: The closed helm CH_n with ordinary labeling.

Case 1. When $n \equiv 2 (\text{mod } 8)$ or $n \equiv 4 (\text{mod } 8)$ or $n \equiv 3 (\text{mod } 8)$ or $n \equiv 7 (\text{mod } 8)$.

(i) If $n \equiv 2 (\text{mod } 8) \Rightarrow n = 8r + 2 \Rightarrow \delta q = 4n\delta = 32r\delta + 8\delta$, then

$$f^*(v_0) = \left[32r^2\delta + 20r\delta + 3\delta\right] \text{mod } (32r\delta + 8\delta) \equiv (12r\delta + 3\delta) \text{ mod } (32r\delta + 8\delta) = \frac{3n\delta}{2} \tag{35}$$

(ii) If $n \equiv 4 (\text{mod } 8)$, then $f^*(v_0) = 5n\delta/2$

(iii) If $n \equiv 3 (\text{mod } 8)$, then $f^*(v_0) = 2n\delta$

(iv) If $n \equiv 7 (\text{mod } 8)$, then $f^*(v_0) = 0$

Case 2. When $n \equiv 0 (\text{mod } 8)$. If $n \equiv 0 (\text{mod } 8) \Rightarrow n = 8r \Rightarrow \delta q = 4n\delta = 32r\delta$, then

$$f^*(v_0) = \left[32r^2\delta + 4r\delta\right] \text{mod } (32r\delta) \equiv (4r\delta) \text{ mod } (32r\delta) = \frac{n\delta}{2}. \tag{36}$$

In this case, we have $f^*(v_0)$ will equal to $f^*(u_{(n-2)/2})$, so we change the labeling of two edges $(u_{(n-4)/2}u_{(n-2)/2})$ and $(u_n u_1)$ as follows: $f(u_{(n-4)/2}u_{(n-2)/2}) = 4n\delta$ and $f(u_n u_1) = \delta((7n/2) - 2)$.

Then, $f^*(u_{(n-4)/2}) - (n+1)\delta$, $f^*(u_{(n-2)/2}) = (n+2)\delta$, $f^*(u_1) = (7n\delta/2)$, and $f^*(u_n) = ((n/2) - 1)\delta$.

Case 3. When $n \equiv 6 (\text{mod } 8)$. If $n \equiv 6 (\text{mod } 8) \Rightarrow n = 8r + 6 \Rightarrow \delta q = 4n\delta = 32r\delta + 24\delta$, then

$$f^*(v_0) = \left[32r^2\delta + 52r\delta + 21\delta\right] \text{mod } (32r\delta + 24\delta) \equiv (28r\delta + 21\delta) \text{ mod } (32r\delta + 24\delta) = \frac{7n\delta}{2}. \tag{37}$$

In this case, we have $f^*(v_0)$ will equal to $f^*(v_{(3n-2)/4})$, so we change the labeling of two edges $(v_{(3n-2)/4}v_{(3n+2)/4})$ and $(v_{(3n+2)/4}v_{(3n+6)/4})$ as follows: $f(v_{(3n-2)/4}v_{(3n+2)/4}) = \delta((11n+2)/4)$ and $f(v_{(3n+2)/4}v_{(3n+6)/4}) = \delta((11n-2)/4)$.

Then, $f^*(v_{(3n-2)/4}) = ((7n+2)/2)\delta$, $f^*(v_{(3n+2)/4}) = ((7n+4)/2)\delta$, and $f^*(v_{(3n+6)/4}) = ((7n+6)/2)\delta$.

Case 4. When $n \equiv 1 (\text{mod } 8)$. If $n \equiv 1 (\text{mod } 8) \Rightarrow n = 8r + 1 \Rightarrow \delta q = 4n\delta = 32r\delta + 4\delta$, then $f^*(v_0) = [32r^2\delta + 12r\delta + \delta]$ mod $(32r\delta + 4\delta) \equiv (8r\delta + \delta)$ mod $(32r\delta + 4\delta) = n\delta$.

In this case, we have $f^*(v_0)$ will equal to $f^*(u_{n-1})$, so we change the labeling of two edges $(u_{n-2}u_{n-1})$ and $(u_n u_1)$ as follows: $f(u_{n-2}u_{n-1}) = 4n\delta$ and $f(u_n u_1) = \delta(4n-2)$.

Then, $f^*(u_{n-2}) = (n+1)\delta$, $f^*(u_{n-1}) = (n+2)\delta$, $f^*(u_n) = (n-1)\delta$, and $f^*(u_1) = 0$.

Case 5. When $n \equiv 5 (\text{mod } 8)$. If $n \equiv 5 (\text{mod } 8) \Rightarrow n = 8r + 5 \Rightarrow \delta q = 4n\delta = 32r\delta + 20\delta$, then $f^*(v_0) = [32r^2\delta + 44r\delta + 15\delta]$ mod $(32r\delta + 20\delta) \equiv (24r\delta + 15\delta)$ mod $(32r\delta + 20\delta) = 3n\delta$.

In this case, we have $f^*(v_0)$ will equal to $f^*(v_{(n-1)/2})$, so we change the labeling of two edges $(v_{(n-5)/2}v_{(n-3)/2})$ and $(v_{n-3/2}v_{(n-1)/2})$ as follows: $f(v_{(n-5)/2}v_{(n-3)/2}) = (\delta/2)(5n-3)$ and $f(v_{(n-3)/2}v_{(n-1)/2}) = (\delta/2)(5n-5)$.

Then, $f^*(v_{(n-5)/2}) = (3n-3)\delta$, $f^*(v_{(n-3)/2}) = (3n-2)\delta$ and $f^*(v_{(n-1)/2}) = (3n-1)\delta$.

Clearly, in all cases, $f^*(v_0)$ is not congruent to $f^*(v_i)$ nor $f^*(u_i)$ mod $(4n\delta)$, $i = 2, 3, \cdots, n$, and all the labels of the vertices v_i and u_i are distinct and a multiplies of δ. Thus, the closed helm is an edge $\delta-$ graceful graph.

Illustration 8. In Figure 16, we present CH_{11} with edge 7− graceful labeling, CH_{12} with edge 6− graceful labeling, CH_8 with edge 5− graceful labeling, CH_{14} with edge 3− graceful labeling, CH_9 with edge 5− graceful labeling, and CH_{13} with edge 5− graceful labeling, respectively.

9. Edge $\delta-$ Graceful Labeling of the Friendship Fr_n

Theorem 12. *For any positive integer δ, the friendship graph F_n is an edge $\delta-$ graceful graph.*

Proof. The friendship graph F_n is a planar undirected graph constructed by joining n copies of cycle graph C_3 with common vertex, so $p = |V(Fr_n)| = 2n + 1$ and $q = |E(Fr_n)| = 3n$. Let the friendship graph F_n be given as in Figure 17 with central vertex v_0, and the edges of Fr_n will be $\{v_0v_i, v_0u_i, v_iu_i, i = 1, 2, \cdots, n\}$. There are two cases:

Case 1. $n \equiv 1 (\text{mod } 3)$, or $n \equiv 2 (\text{mod } 3)$.

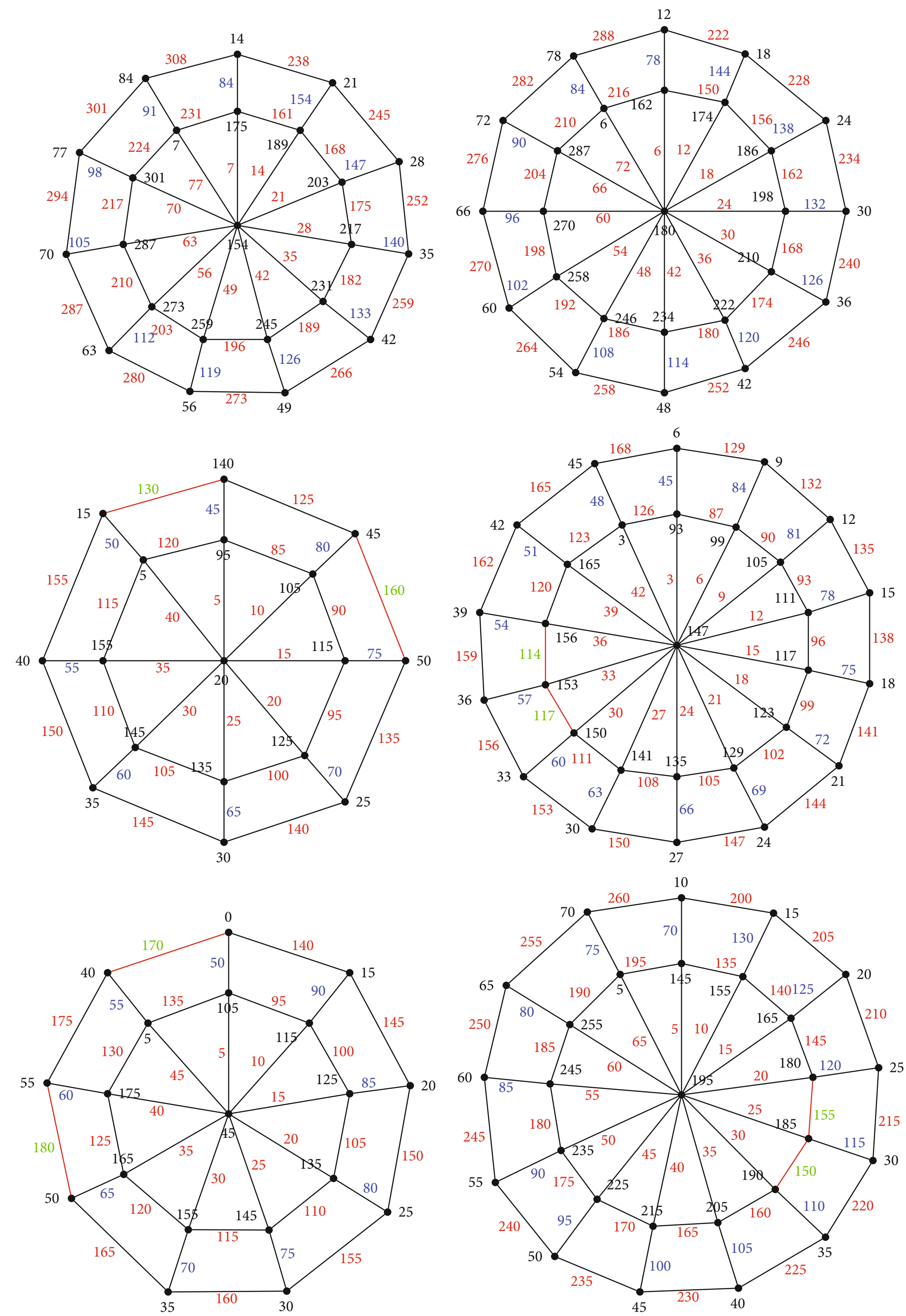

Figure 16

We define the labeling function $f : E(\mathrm{Fr}_n) \to \{\delta, 2\delta, \cdots, 3n\,\delta\}$ as follows:

$$\begin{aligned} &\text{for } i = 1, 2, \cdots, n, f(v_i u_i) = \delta i, \\ &\text{for } i = 1, 2, \cdots, n, f(v_0 v_i) = \delta(n + 2i - 1), \\ &\text{for } i = 1, 2, \cdots, n, f(v_0 u_i) = \delta(n + 2i). \end{aligned} \quad (38)$$

Then, the induced vertex labels are

$$\begin{aligned} f^*(v_i) &= [f(v_0 v_i) + f(v_i u_i)] \bmod (3n\,\delta) \\ &= [\delta(n + 3i - 1)] \bmod (3n\,\delta), \quad i = 1, 2, \cdots n, \\ f^*(u_i) &= [f(v_0 u_i) + f(v_i u_i)] \bmod (3n\,\delta) \\ &= [\delta(n + 3i)] \bmod (3n\,\delta), \quad i = 1, 2, \cdots n. \end{aligned} \quad (39)$$

Hence, the labels of the vertices $v_1, v_2, v_3, \cdots, v_{n-1}, v_n$ are $(n+2)\delta$, $(n+5)\delta$, $(n+8)\delta, \cdots, (n-4)\delta$, $(n-1)\delta$, respectively, and the labels of the vertices $u_1, u_2, u_3, \cdots, u_{n-1}, u_n$ will be $(n+3)\delta$, $(n+6)\delta$, $(n+9)\delta, \cdots, (n-3)\delta$, $n\delta$, respectively, which are all distinct numbers. Also,

$$\begin{aligned} f^*(v_0) &= \left[\sum_{i=1}^{n} f(v_0 v_i) + \sum_{i=1}^{n} f(v_0 u_i)\right] \bmod (3n\,\delta) \\ &= [\delta(n+1) + \delta(n+2) + \delta(n+3) \\ &\quad + \cdots + \delta(n+2n)] \bmod (3n\,\delta) \\ &= \left[\delta\left(4n^2 + n\right)\right] \bmod (3n\,\delta). \end{aligned} \quad (40)$$

(i) If $n \equiv 1 (\bmod 3) \Rightarrow n = 3r + 1 \Rightarrow q\delta = 3n\,\delta = 9r\delta + 3\delta$, then

$$\begin{aligned} f^*(v_0) &= \left[36r^2\delta + 27r\delta + 5\delta\right] \bmod (9r\delta + 3\delta) \\ &\equiv (6r\delta + 2\delta) \bmod (9r\delta + 3\delta) = 2n\,\delta \end{aligned} \quad (41)$$

(ii) If $n \equiv 2 (\bmod\ 3)$, then $f^*(v_0) = 0$

Case 2. $n \equiv 0 (\bmod\ 3)$. We define the labeling function f as follows:

$$\begin{aligned} &\text{for } i = 1, 2, \cdots, n-1, f(v_i u_i) = \delta(2n + i + 1), f(v_n u_n) = \delta, \\ &\text{for } i = 1, 2, \cdots, n, f(v_0 v_i) = \delta(2i), \\ &\text{for } i = 1, 2, \cdots, n, f(v_0 u_i) = \delta(2i + 1). \end{aligned} \quad (42)$$

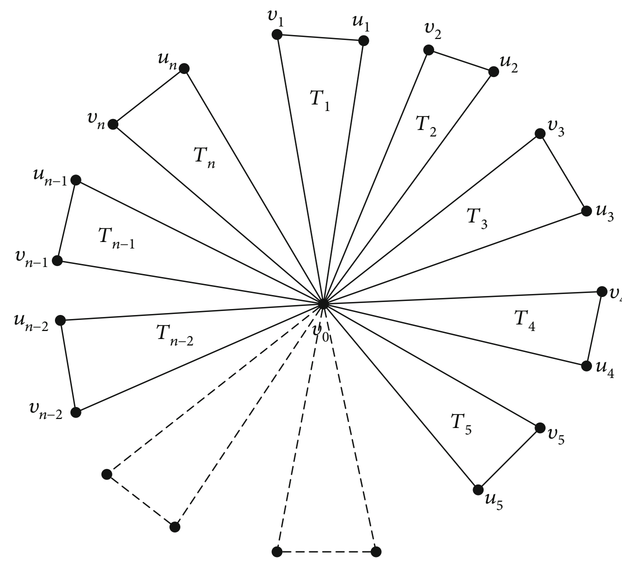

Figure 17: The friendship graph Fr_n with ordinary labeling.

In view of the above labeling pattern, then the induced vertex labels are

$$\begin{aligned} f^*(v_i) &= [f(v_0 v_i) + f(v_i u_i)] \bmod (3n\,\delta) \\ &= \delta[2n + 3i + 1] \bmod (3n\,\delta), \quad i = 1, 2, \cdots, n, \\ f^*(u_i) &= [f(v_0 u_i) + f(v_i u_i)] \bmod (3n\,\delta) \\ &= \delta[2n + 3i + 2] \bmod (3n\,\delta), \quad i = 1, 2, \cdots, n. \end{aligned} \quad (43)$$

Hence, the labels of the vertices $v_1, v_2, v_3, \cdots, v_{n-1}, v_n$ are $(2n+4)\delta$, $(2n+7)\delta$, $(2n+10)\delta, \cdots, (n-2)\delta$, $(2n+1)\delta$, respectively, and the labels of the vertices $u_1, u_2, u_3, \cdots, u_{n-1}$, u_n are $(2n+5)\delta$, $(2n+8)\delta$, $(2n+11)\delta, \cdots, (2n-1)\delta$, $(2n+2)\delta$, respectively, which are all distinct numbers.

$$\begin{aligned} f^*(v_0) &= \left[\sum_{i=1}^{n} f(v_0 v_i) + \sum_{i=1}^{n} f(v_0 u_i)\right] \bmod (3n\,\delta) \\ &= [2\delta + 3\delta + 4\delta + \cdots + \delta(1 + 2n)] \bmod (3n\,\delta) \\ &= \left[\delta\left(2n^2 + 3n\right)\right] \bmod (3n\,\delta). \end{aligned} \quad (44)$$

(i) If $n \equiv 0 (\bmod 3) \Rightarrow n = 3r \Rightarrow q\delta = 3n\,\delta = 9r\,\delta$, then $f^*(v_0) = [18r^2\delta + 9r\delta] \bmod (9r\delta) \equiv 0$

Overall, all the labels of the vertices v_i and u_i are all distinct, multiple of δ, and different from $f^*(v_0)$ which completes the proof.

Illustration 9. In Figure 18, we present the friendship graphs Fr_9 with edge 5– graceful labeling and Fr_{10} with edge 6– graceful labeling.

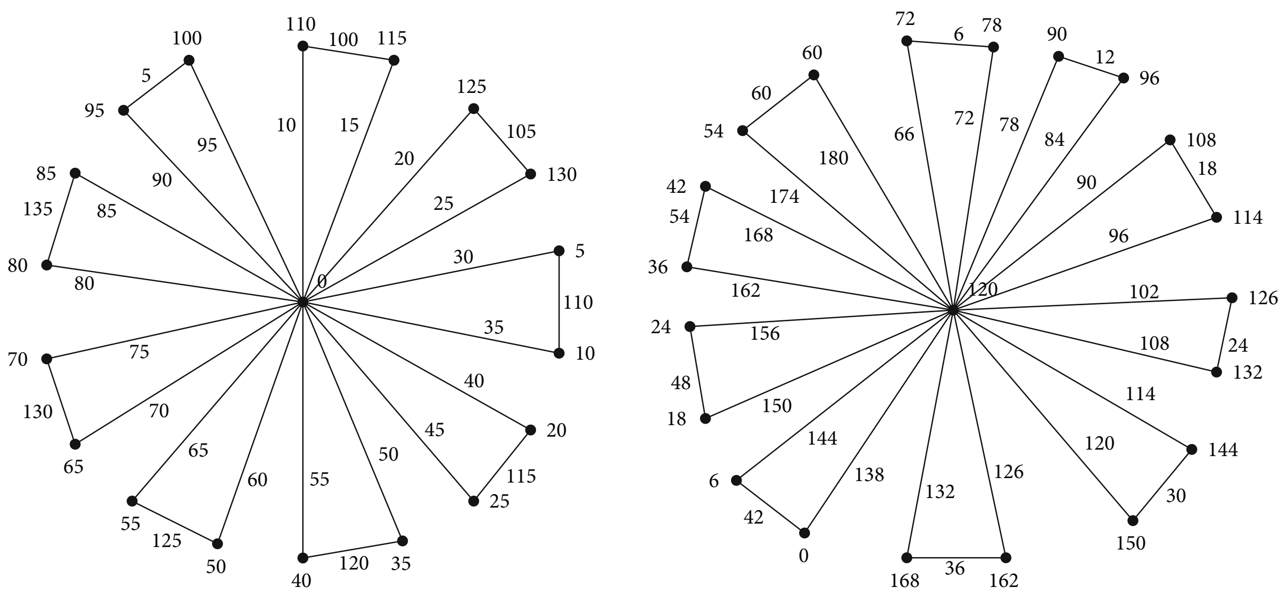

FIGURE 18: The friendship graphs Fr_9 with edge 5− graceful labeling and Fr_{10} with edge 6− graceful labeling.

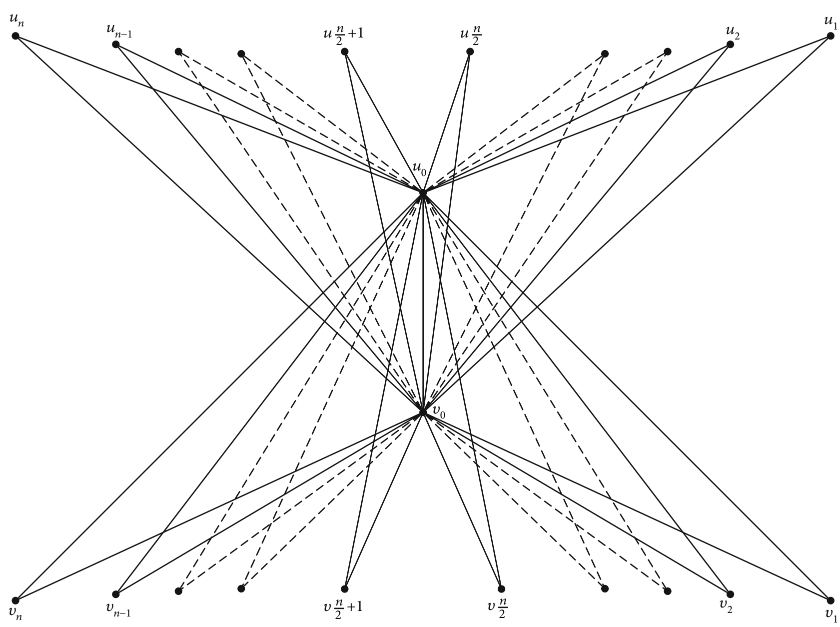

FIGURE 19: The butterfly graph B_n with ordinary labeling.

10. Edge $\delta-$ Graceful Labeling of the Butterfly Graph B_n

Theorem 13. *For any positive integer δ, the butterfly graph B_n is an edge $\delta-$ graceful graph.*

Proof. The butterfly graph B_n is a planer graph constructed by joining $2n$ copies of cycle graph C_3 with a common edge, a butterfly graph B_n has $p = 2n + 2$ vertices and $q = 4n + 1$ edges, let the common edge in all cycles be (v_0u_0) and the vertices of one copy be $v_1, v_2, \cdots, v_n$ and the vertices of the other copy be $u_1, u_2, \cdots, u_n$; the edges will be $\{v_0v_i, u_0v_i, u_0u_i, v_0u_i, v_0u_0, \quad i = 1, 2, \cdots, n\}$, see Figure 19. There are two cases:

Case 1. When n is even. Define the labeling function $f : E(B_n) \to \{\delta, 2\delta, 3\delta, \cdots, (4n+1)\delta\}$ as follows:

$$\begin{aligned}
&f(u_0v_i) = \delta\, i, \quad \text{for } i = 1, 2, \cdots, n,\\
&f(v_0v_i) = \delta(2n+i), \quad \text{for } i = 1, 2, \cdots, n,\\
&f(u_0u_i) = \delta(3n+i), \quad \text{for } i = 1, 2, \cdots, n,\\
&f(v_0u_0) = 2n\,\delta, \quad f(v_0u_1) = (4n+1)\delta, f(v_0u_i)\\
&\qquad = \delta(n+i-1), \text{if } i = 2, 3, \cdots, n.
\end{aligned} \tag{45}$$

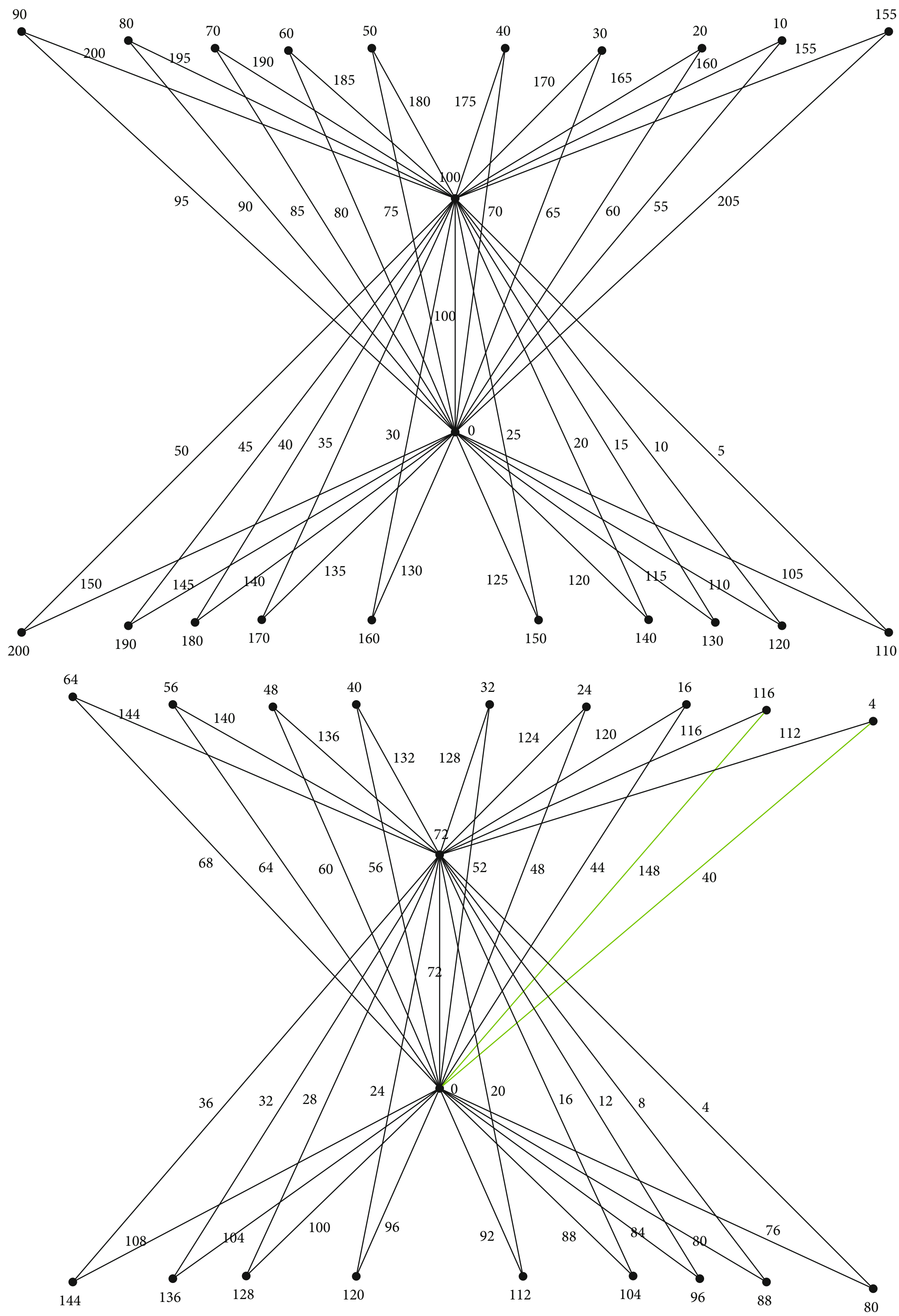

Figure 20: B_{10} with edge 5− graceful labeling and B_9 with edge 4− graceful labeling.

In view of the above labeling pattern, then the induced vertex labels are

$$\begin{aligned} f^*(v_i) &= [f(v_0v_i)+f(u_0v_i)] \bmod [(4n+1)\delta] \\ &= 2\delta(n+i), \quad i=1,2,\cdots n, \\ f^*(u_1) &= \delta(3n+1), \quad f^*(u_i) \\ &= 2\delta(i-1), i=2,3,\cdots n, \end{aligned} \tag{46}$$

In this labeling, we can check that

$$\begin{aligned} &[f(v_0v_i)+f(v_0u_{n+2-1})] \bmod [(4n+1)\delta] \\ &\quad \equiv 0 \bmod [(4n+1)\delta], \quad i=2,3,\cdots,n, \\ &[f(u_0u_i)+f(u_0u_{n+1-i})] \bmod [(4n+1)\delta] \\ &\quad \equiv 0 \bmod [(4n+1)\delta], \quad i=1,2,\cdots,n. \end{aligned} \tag{47}$$

Then

$$\begin{aligned} f^*(v_0) &= \left[\sum_{i=1}^{n} f(v_0v_i)+\sum_{i=1}^{n} f(v_0u_i) \right. \\ &\quad \left. + f(v_0u_0)\right] \bmod [(4n+1)\delta) = 0, \\ f^*(u_0) &= \left[\sum_{i=1}^{n} f(u_0v_i)+\sum_{i=1}^{n} f(u_0u_i) \right. \\ &\quad \left. + f(v_0u_0)\right] \bmod [(4n+1)\delta) = 2n\delta. \end{aligned} \tag{48}$$

Case 2. When n is odd. By using the same labeling as in the first case, in this case, we have $f^*(u_0)$ will equal to $f^*(v_{(n+1)/2})$, so we change the labeling of two edges $(v_0\, u_0)$ and $(v_0\, u_1)$ as follows: $f(v_0u_0)=\delta(n+1)$ and $f(v_0u_1)=\delta(4n+1)$, then $f^*(u_1)=\delta$, and $f^*(u_2)=(3n+2)\delta$.

We can see that the labels of the vertices v_i, and u_i, $i=1, 2, \cdots, n$ are all a multiple of δ and distinct numbers and different from the labels of the vertices v_0 and u_0. Hence, the butterfly graph B_n is an edge $\delta-$ graceful graph, for any positive integer δ.

Illustration 10. In Figure 20, we present B_{10} with edge 5– graceful labeling and B_9 with edge 4– graceful labeling.

11. Conclusion

In the past few years, edge graceful labeling of graphs has been studied heavily and these topics continue to be attractive in the field of graph theory and discrete mathematics. A great number of published papers and results exist. So far, many graphs are unknown if it is edge graceful or not.

In this work, a new type of labeling called edge $\delta-$ graceful labeling is defined; the graph which satisfies the edge $\delta-$ graceful labeling is called an edge $\delta-$ graceful graph. Edge $\delta-$ graceful labeling of some special classes of graphs like wheel graph, double wheel, prism graph, gear graph, closed helm, friendship graph, and butterfly graph is investigated. In future work, we will study the necessary and sufficient conditions for some path-related graph to be an edge $\delta-$ graceful graph.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there is no conflict of interests.

Authors' Contributions

The author read and approved the final manuscript.

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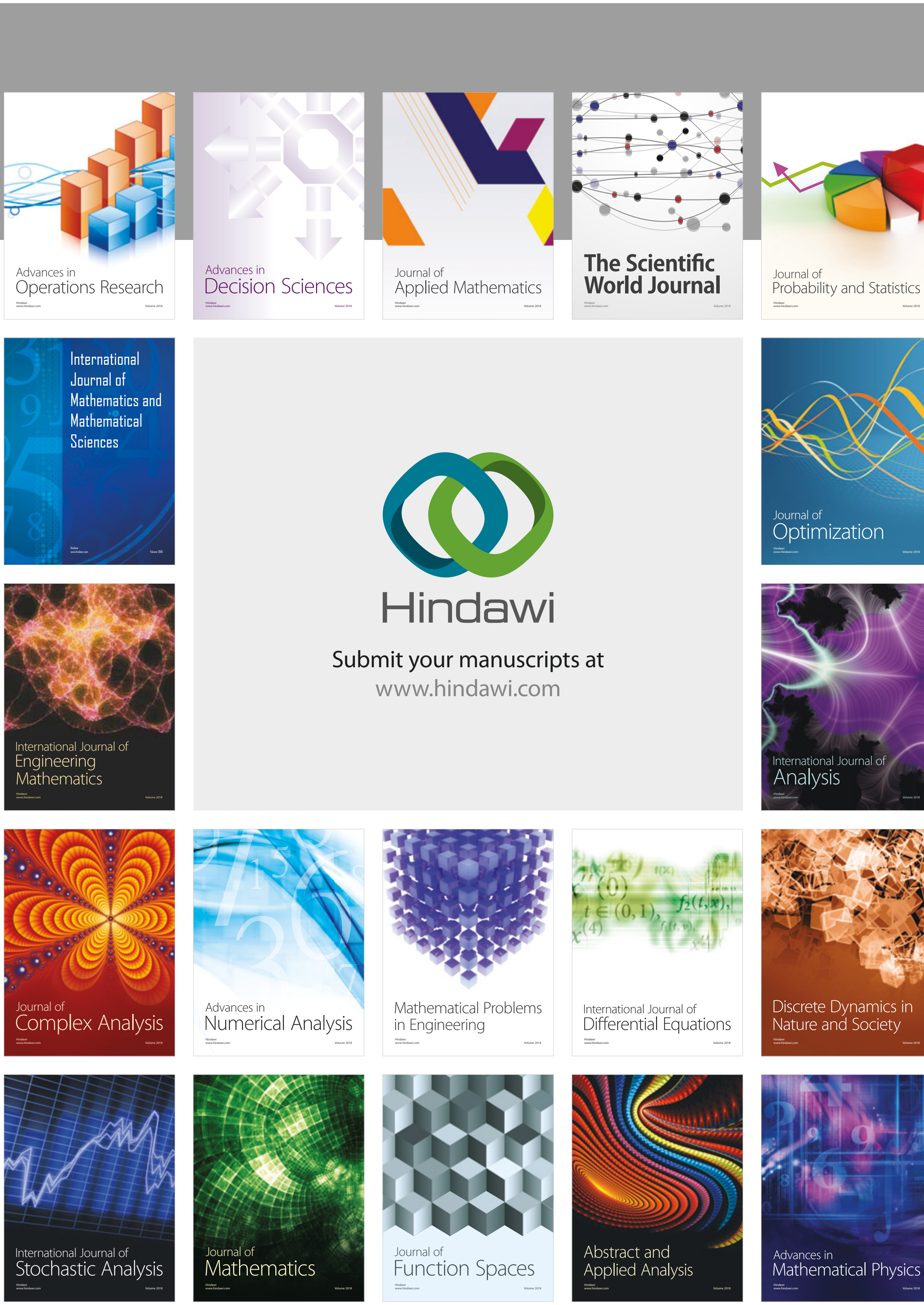
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