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Research Article

Existence of Solution for Double-Phase Problem with Singular Weights

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The aim of this paper is to establish the existence of solutions for singular double-phase problems depending on one parameter. This work improves and complements the existing ones in the literature. There seems to be no results on the existence of solutions for singular double-phase problems.

1. Introduction and Main Results

The study of various mathematical problems involving the double-phase operator has become very attractive in recent decades. The existence and multiplicity of solutions of double-phase Dirichlet problems has been studied by several authors (see, e.g., [1–8]); in particular, for the eigenvalues of the double-phase operator, see [7]. For other double-phase problems with variable exponents, there are the works of Zhang and Radulescu [9], Shi et al. [10], and Cencelj et al. [11].

But up to now, to the best of our knowledge, no paper discussing the existence of solutions for singular double-phase problems via critical point theory can be found in the existing literature. In order to fill in this gap, we study double-phase problems from a more extensive viewpoint. More precisely, we are going to prove that problem (P_λ) has at least one solution. To the best of our knowledge, this is one of the first works which combines a singular term and indefinite term in one problem.

This paper is concerned with the existence of solutions to the following singular double-phase problem:

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) = b(x)|u|^{-\theta-1}u + \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (P_\lambda), \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $0 < \theta < 1$, $\lambda \in \mathbb{R}$, $a : \bar{\Omega} \mapsto [0, +\infty)$ is Lipschitz continuous, and b is a given measurable function. The precise conditions on the data will be presented later.

Problems of the above type arise for instance in nonlinear elasticity. The main reasons are to describe the behavior of Lavrentiev's phenomenon; we refer to [12–14]. In fact, Zhikov intended to provide models for strongly anisotropic materials in the context of homogenization. In particular, he considered the following functional:

$$\int_\Omega \left(|\nabla u|^p + a(x)|\nabla u|^q\right)dx, \quad 0 \leq a(x) \leq L, 1 < p < q, \tag{2}$$

where the modulating coefficient $a(x)$ dictates the geometry of the composite made of two differential materials, with hardening exponents p and q, respectively. Recently, there is a wide literature on the regularity theory for minimizers of variational problems and solutions of differential equations with the double-phase operator; far from being complete, we refer the readers to [15–21], respectively, and references therein.

In the entire paper, we suppose the following assumptions:

$H(a)$: $a : \bar{\Omega} \mapsto [0, +\infty)$ is Lipschitz continuous and $1 < p < q < N$ are chosen such that $(q/p) < 1 + (1/N)$.

$H(b)$: $b \in L^{p/(p+\theta-1)}(\Omega)$ such that $b(x) > 0$ in Ω.

$H(f)_1$: $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function such that for a.a. $x \in \Omega$, $f(x, 0) = 0$, and

(i) there exists positive measurable subsets $\Omega_1 \subset \Omega$ and $c \in L^{s_1}(\Omega)$ such that $c(x) \geq 0$ on Ω_1, and

$$\lim_{t \to 0} \frac{f(x,t)}{c(x)|t|^{r_1-2}t} = 0 \quad \text{uniformly for a.a.} x \in \Omega, \tag{3}$$

where $1 < r_1 < p < N < s_1$;

(ii) there exists $d \in L^{s_2}(\Omega)$ such that $d(x) > 0$ on Ω, and

$$\lim_{t \to +\infty} \frac{f(x,t)}{d(x)|t|^{r_2-2}t} = 0 \quad \text{uniformly for a.a.} x \in \Omega, \tag{4}$$

where $1 < r_2 < p < N < s_2$;

(iii) there exists $M > 0$ such that

$$\int_\Omega F(x,t)dx > 0, \quad \forall |t| > M, \tag{5}$$

where $F(x,t) = \int_0^t f(x,s)ds$.

Example 1. The following function satisfies hypotheses $H(f)_1$:

$$f(x,t) = \begin{cases} c(x)\left(|t|^{p-2}t - |t|^{q-2}t\right), & \text{if } |t| \leq 1, \\ d(x)\left(|t|^{\alpha-2}t - \dfrac{t}{|t|}\right), & \text{if } |t| \geq 1, \end{cases} \tag{6}$$

with $1 < \alpha < r_2$.

We are now in the position to state our main results. Firstly, problem (P_λ) has a solution when $\lambda \leq 0$.

Theorem 1. *Assume that $H(a)$, $H(b)$, and $H(f)_1$ hold. Then for all $\lambda \leq 0$, problem (P_λ) has at least one nontrivial weak solution with negative energy.*

Moreover, we also show that problem (P_λ) has a solution when $\lambda > 0$. In order to do this task, the following conditions are needed:

$H(f)_2$: $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that for a.a. $x \in \Omega$, $f(x, 0) = 0$, and

(i) there exists $C > 0$ and $d \in L^{s_2}(\Omega)$ such that

$$f(x,t) \leq Cd(x)|t|^{r_2-2}t, \quad \forall (x,t) \in \Omega \times \mathbb{R}, \tag{7}$$

where $1 < r_2 < p < s_2$;

(ii) there exists a positive measurable subset $\Omega_1 \subset \Omega$ such that

$$f(x,t), \quad d(x) > 0, \forall x \in \Omega_1, \forall t > 0. \tag{8}$$

Example 2. The following functions satisfy hypotheses $H(f)_2$:

$$f_1(x,t) = Cd(x)|t|^{r_2-2}t,$$
$$f_2(x,t) = \begin{cases} Cd(x)t^{p-1}, & \text{if } t \in [0,1], \\ Cd(x)t^{r_2-1}, & \text{if } t \in [1,+\infty), \\ -Cd(x)(-t)^{r_2-1}, & \text{if } t \in [-1,0], \\ -Cd(x)(-t)^{p-1}, & \text{if } t \in (-\infty,-1]. \end{cases} \tag{9}$$

Theorem 2. *Assume that $H(a)$, $H(b)$, and $H(f)_2$ hold. Then for all $\lambda \geq 0$, problem (P_λ) has at least one nontrivial weak solution with negative energy.*

The rest of this paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on space $W_0^{1,H}(\Omega)$. In Section 3, the proof of the main results is given.

2. Preliminaries

In order to discuss problem (P), we need some facts on space $W_0^{1,H}(\Omega)$ which are called Musielak-Orlicz-Sobolev spaces. For this reason, we will recall some properties involving the Musielak-Orlicz spaces, which can be found in [7, 22–24] and references therein.

Denote by $N(\Omega)$ the set of all generalized N-function. For $1 < p < q$ and $0 \leq a(\cdot) \in L^1(\Omega)$, we define $H(x,t) = t^p + a(x) t^q, \forall (x,t) \in \Omega \times [0, +\infty)$. It is clear that $H \in N(\Omega)$ is a locally integrable function and $H(x, 2t) \leq 2^q H(x,t), \forall (x,t) \in \Omega \times [0, +\infty)$ which is called condition (Δ_2).

The Musielak-Orlicz space $L^H(\Omega)$ is defined by

$$L^H(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ measurable} : \int_\Omega H(x,|u|)dx < +\infty \right\}, \tag{10}$$

endowed with the Luxemburg norm $|u|_H = \inf \{\lambda > 0 : \int_\Omega H(x, |u/\lambda|)dx \leq 1\}$. The Musielak-Orlicz-Sobolev space $W^{1,H}(\Omega)$ is defined by

$$W^{1,H}(\Omega) = \left\{ u \in L^H(\Omega) \colon |\nabla u| \in L^H(\Omega) \right\}, \tag{11}$$

and it is equipped with the norm $\|u\| = |u|_H + |\nabla u|_H$. We denote by $W_0^{1,H}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in $W^{1,H}(\Omega)$. With these norms, the spaces $L^H(\Omega)$, $W_0^{1,H}(\Omega)$, and $W^{1,H}$

(Ω) are separable reflexive Banach spaces (see [7] for the details).

Proposition 3. *([1], Proposition 2.1).*

Set $\rho_H(u) = \int_\Omega (|u|^p + a(x)|u|^q)dx$. *For* $u \in L^H(\Omega)$, *we have*

(i) For $u \neq 0$, $|u|_H = \lambda \Leftrightarrow \rho_H(u/\lambda) = 1$

(ii) $|u|_H < 1(=1;>1) \Leftrightarrow \rho_H(u) < 1(=1;>1)$

(iii) If $|u|_H \geq 1$, *then* $|u|_H^p \leq \rho_H(u) \leq |u|_H^q$

(iv) If $|u|_H \leq 1$, *then* $|u|_H^q \leq \rho_H(u) \leq |u|_H^p$

Proposition 4. *([7], Proposition 2.15, Proposition 2.18).*

(1) *If* $1 \leq s \leq Np/(N-p)$, *then the embedding from* $W_0^{1,H}(\Omega)$ *to* $L^s(\Omega)$ *is continuous. In particular, if* $s \in [1, Np/(N-p))$, *then the embedding* $W_0^{1,H}(\Omega) \hookrightarrow L^s(\Omega)$ *is compact*

(2) *Assume that* $H(a)$ *holds. Then, Poincare's inequality holds; that is, there exists a positive constant* C_0 *such that* $|u|_H \leq C_0\|u\|, \forall u \in W_0^{1,H}(\Omega)$

By the above Proposition, there exists $c_\tau > 0$ such that $|u|_\tau \leq c_\tau\|u\|, \forall u \in W_0^H(\Omega)$, where $|u|_\tau$ denotes the usual norm in $L^\tau(\Omega)$ for all $1 \leq \tau < Np/(N-p)$. It follows from (2) of Proposition 4 that $|\nabla u|_H$ is an equivalent norm in $W_0^{1,H}(\Omega)$. We will use the equivalent norm in the following discussion and write $\|u\| = |\nabla u|_H$ for simplicity.

In order to discuss the problem (P), we need to define a functional in $W_0^{1,H}(\Omega)$:

$$J(u) = \int_\Omega \left(\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q\right)dx. \qquad (12)$$

We know that (see [25], P63, example) $J \in C^1(W_0^{1,H}(\Omega), \mathbb{R})$ and the double-phase operator

$$-\text{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right), \qquad (13)$$

is the derivative operator of J in the weak sense. Moreover, similar to the proof of Theorem 3.1 in [25], we know that the energy functional J is sequentially weakly lower semicontinuous.

3. Variational Setting and Proof of the Main Results

For any $\lambda \in \mathbb{R}$ and each $u \in E$, we define

$$\varphi_\lambda(u) = J(u) - \int_\Omega \frac{b(x)}{1-\theta}|u|^{1-\theta}dx - \lambda\int_\Omega F(x,u)dx, \qquad (14)$$

where $J(u) = \int_\Omega ((1/p)|\nabla u|^p + (a(x)/q)|\nabla u|^q)dx$. By using $H(f)_1$, we get $r_1 < s_1r_1/(s_1-1) < Np/(N-p)$ and $r_2 < s_2r_2/(s_2-1) < Np/(N-p)$. Also, by Proposition 4 (1) we deduce that embeddings $E \hookrightarrow L^{s_1r_1/(s_1-1)}(\Omega)$ and $E \hookrightarrow L^{s_2r_2/(s_2-1)}(\Omega)$ are compact and continuous. Furthermore, there exists a constant $C_0 > 0$ such that

$$\max\left\{|u|_{s_1r_1/(s_1-1)}, |u|_p, |u|_{s_2r_2/(s_2-1)}\right\} \leq C_0\|u\|, \quad \forall u \in E. \qquad (15)$$

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. To complete the proof of the main result, we need to consider the following three steps.

Step 1. We first show that for every $\lambda \leq 0$, the functional φ_λ is coercive on E.

Let $\lambda \leq 0$ be fixed. Put $\Omega_M = \bar{\Omega} \times [-M, M]$. Clearly, from the continuity of f, there exists $K_M > 0$ such that

$$K_M = \max_{(x,t)\in\Omega_M} |f(x,t)|. \qquad (16)$$

Thus, we deduce that for any $x \in \Omega$ and $|t| \leq M$,

$$|F(x,t)| = \left|\int_0^t f(x,s)ds\right| \leq \int_0^t |f(x,s)|ds \leq MK_M. \qquad (17)$$

By virtue of assumption $H(f)_1$ (iii), (15), (17), and Proposition 3, one has for any $u \in E$ with $\|u\| > 1$

$$\begin{aligned}\varphi_\lambda(u) &\geq \frac{1}{q}\|u\|^p - \frac{1}{1-\theta}|b|_{p/(p+\theta-1)}\left||u|^{1-\theta}\right|_{p/(1-\theta)} \\ &\quad - \lambda\int_{|u(x)|\leq M} F(x,u)dx - \lambda\int_{|u(x)|>M} F(x,u)dx \\ &\geq \frac{1}{q}\|u\|^p - \frac{1}{1-\theta}|b|_{p/(p+\theta-1)}|u|_p^{1-\theta} + \lambda MK_M|\Omega| \\ &\geq \frac{1}{q}\|u\|^p - \frac{1}{1-\theta}|b|_{p/(p+\theta-1)}C_0^{1-\theta}\|u\|^{1-\theta} + \lambda MK_M|\Omega|.\end{aligned} \qquad (18)$$

Since $0 < \theta < 1$ and $1-\theta < p$, so this implies $\varphi_\lambda(u) \longrightarrow +\infty$ as $\|u\| \longrightarrow +\infty$. The proof of Step 1 is now completed.

Step 2. We show that there exists $\xi \in E$ with $\xi > 0$, $\varphi_\lambda(t\xi) < 0$ for $t > 0$ small enough.

Let $\xi \in C_0^\infty(\Omega)$ such that supp $(\xi) \subset \Omega_1 \subset \Omega$, $\xi = 1$ in a subset $\Omega_0 \subset$ supp (ξ), and $0 \leq \xi \leq 1$ in Ω_1. Thus, by condition $H(f)_1$ (i), it follows that there exists $t_0 \in (0,1)$ such that

$$F(x, t\xi(x)) \leq \frac{c(x)}{r_1}|t\xi(x)|^{r_1} \leq c(x)|t\xi(x)|^{r_1}, \quad \forall t \in (0, t_0), x \in \Omega. \qquad (19)$$

Hence, for any $t \in (0, t_0)$, from $H(b)$ and $H(f)_1$ (i), we deduce that

$$\begin{aligned}\varphi_\lambda(t\xi) &\leq \frac{t^p}{p}\max\left\{\|\xi\|^p, \|\xi\|^q\right\} - \frac{t^{1-\theta}}{1-\theta}\int_\Omega b(x)|\xi|^{1-\theta}dx - \lambda\int_\Omega F(x,t\xi)dx \\ &\leq \frac{t^p}{p}\max\left\{\|\xi\|^p, \|\xi\|^q\right\} - \frac{1}{1-\theta}\int_\Omega b(x)|t\xi|^{1-\theta}dx \\ &\quad - \lambda|t|^{r_1}\int_{\Omega_1} c(x)|\xi|^{r_1}dx \\ &\leq t^{r_1}\left[\max\left\{\|\xi\|^p, \|\xi\|^q\right\} - \lambda\int_{\Omega_1} c(x)|\xi|^{r_1}dx\right] \\ &\quad - \frac{t^{1-\theta}}{1-\theta}\int_\Omega b(x)|\xi|^{1-\theta}dx.\end{aligned} \tag{20}$$

Since $r_1 > 1-\theta$, we have $\varphi_\lambda(t\xi) < 0$ for $t < t_1$ with

$$0 < t_1 < \min\left\{t_0, \left(\frac{(1/(1-\theta))\int_\Omega b(x)|\xi|^{1-\theta}dx}{\max\left\{\|\xi\|^p, \|\xi\|^q\right\} - \lambda\int_{\Omega_1} c(x)|\xi|^{r_1}dx}\right)^{1/(r_1+\theta-1)}\right\}. \tag{21}$$

The proof of Step 2 is now complete.

Step 3. We show that there exists $u_\lambda \in E$ such that $\varphi_\lambda(u_\lambda) = \inf_{u\in E}\varphi_\lambda(u)$ for any $\lambda \leq 0$.

Let $\{u_\lambda^n\} \subset E$ be a minimizing sequence of φ_λ. Then, using Step 1, we get that $\{u_\lambda^n\}$ is a bounded sequence. So, there exists $u_\lambda \in E$ such that, up to a subsequence,

$$\begin{aligned} u_\lambda^n &\rightharpoonup u_\lambda \quad \text{in } E, \\ u_\lambda^n &\longrightarrow u_\lambda \quad \text{in } L^p(\Omega), \\ u_\lambda^n(x) &\rightharpoonup u_\lambda(x) \quad \text{a.e.in } \Omega. \end{aligned} \tag{22}$$

Recall that J is sequentially weakly lower semicontinuous, and so we deduce that

$$J(u_\lambda) \leq \liminf_{n\to+\infty} J(u_\lambda^n). \tag{23}$$

Now, using Hölder's inequality, we get that, as $n \longrightarrow +\infty$,

$$\begin{aligned}\int_\Omega b(x)|u_\lambda^n|^{1-\theta}dx &\leq \int_\Omega b(x)|u_\lambda|^{1-\theta}dx + \int_\Omega b(x)|u_\lambda^n - u_\lambda|^{1-\theta}dx \\ &\leq \int_\Omega b(x)|u_\lambda|^{1-\theta}dx \\ &\quad + |b|_{p/(p+\theta-1)}\left||u_\lambda^n - u_\lambda|^{1-\theta}\right|_{p/(1-\theta)} \\ &= \int_\Omega b(x)|u_\lambda|^{1-\theta}dx + |b|_{p/(p+\theta-1)}|u_\lambda^n - u_\lambda|_p^{1-\theta} \\ &= \int_\Omega b(x)|u_\lambda|^{1-\theta}dx + o_n(1).\end{aligned} \tag{24}$$

Analogously,

$$\begin{aligned}\int_\Omega b(x)|u_\lambda|^{1-\theta}dx &\leq \int_\Omega b(x)|u_\lambda^n|^{1-\theta}dx + \int_\Omega b(x)|u_\lambda - u_\lambda^n|^{1-\theta}dx \\ &\leq \int_\Omega b(x)|u_\lambda^n|^{1-\theta}dx \\ &\quad + |b|_{p/(p+\theta-1)}\left||u_\lambda^n - u_\lambda|^{1-\theta}\right|_{p/(1-\theta)} \\ &= \int_\Omega b(x)|u_\lambda^n|^{1-\theta}dx + |b|_{p/(p+\theta-1)}|u_\lambda^n - u_\lambda|_p^{1-\theta} \\ &= \int_\Omega b(x)|u_\lambda^n|^{1-\theta}dx + o_n(1).\end{aligned} \tag{25}$$

Hence, by (24) and (25), one yields

$$\lim_{n\to+\infty}\int_\Omega b(x)|u_\lambda^n|^{1-\theta}dx = \int_\Omega b(x)|u_\lambda|^{1-\theta}dx. \tag{26}$$

Moreover, using assumptions $H(f)_1$ (i) and $H(f)_1$ (ii), for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(x,u_\lambda)| \leq \varepsilon c(x)|u_\lambda|^{r_1} + C_\varepsilon d(x)|u_\lambda|^{r_2}. \tag{27}$$

The above information and Hölder's inequality imply

$$\begin{aligned}\int_\Omega |F(x,u_\lambda)|dx &\leq \varepsilon|C|_{s_1}\left||u_\lambda|^{r_1}\right|_{s_1/(s_1-1)} + C_\varepsilon|d|_{s_2}\left||u_\lambda|^{r_2}\right|_{s_2/(s_2-1)}d \\ &= \varepsilon|C|_{s_1}|u_\lambda|^{r_1}_{s_1r_1/(s_1-1)} + C_\varepsilon|d|_{s_2}|u_\lambda|^{r_2}_{s_2r_2/(s_2-1)}.\end{aligned} \tag{28}$$

Again, by Proposition 4 (1), we deduce that

$$\begin{aligned} &E \hookrightarrow L^{s_1r_1/(s_1-1)}(\Omega), \quad u_\lambda^n \longrightarrow u \text{ in } L^{s_1r_1/(s_1-1)}(\Omega), \\ &E \hookrightarrow L^{s_2r_2/(s_2-1)}(\Omega), \quad u_\lambda^n \longrightarrow u \text{ in } L^{s_2r_2/(s_2-1)}(\Omega), \\ &u_\lambda^n \longrightarrow u \quad \text{for a.a.} x \in \Omega, \\ &F(x, u_\lambda^n(x)) \longrightarrow F(x, u_\lambda(x)) \quad \text{for a.a.} x \in \Omega. \end{aligned} \tag{29}$$

Thus, using the fact that $\{u_\lambda^n\}$ is bounded in E and the dominated convergence theorem, we can infer that

$$\lim_{n\to+\infty}\int_\Omega F(x, u_\lambda^n(x)) = \int_\Omega F(x, u_\lambda(x))dx. \tag{30}$$

Hence, for every $\lambda < 0$, by (26) and (30), one yields

$$\begin{aligned}\liminf_{n\to\infty}\varphi_\lambda(u_\lambda^n) &= \liminf_{n\to\infty}\left(J(u_\lambda^n) - \int_\Omega \frac{b(x)}{1-\theta}|u_\lambda^n|^{1-\theta}dx - \lambda\int_\Omega F(x,u_\lambda^n)dx\right)\\ &= \liminf_{n\to\infty} J(u_\lambda^n) - \lim_{n\to\infty}\int_\Omega \frac{b(x)}{1-\theta}|u_\lambda^n|^{1-\theta}dx - \lambda\lim_{n\to\infty}\int_\Omega F(x,u_\lambda^n)dx\\ &\geq J(u_\lambda) - \int_\Omega \frac{b(x)}{1-\theta}|u_\lambda|^{1-\theta}dx - \lambda\int_\Omega F(x,u_\lambda)dx\\ &= J(u_\lambda) - \Phi(u_\lambda) - \lambda\Psi(u_\lambda) = \varphi_\lambda(u_\lambda),\end{aligned} \tag{31}$$

which implies that φ_λ is weakly lower semicontinuous, and consequently,

$$\varphi_\lambda(u_\lambda) \leq \liminf_{u\in E}\varphi_\lambda(u_\lambda^n) = \inf_{v\in E}\varphi_\lambda(v) \leq \varphi_\lambda(u_\lambda), \tag{32}$$

which implies that

$$\varphi(u_\lambda) = \inf_{u\in E}\varphi_\lambda(u_\lambda^n). \tag{33}$$

So, we complete Step 3.

Therefore, combining the above Steps 2 and 3, we deduce that u_λ is the required nontrivial solution of problem (P_λ). Therefore, we complete the Proof of Theorem 1.

Now, we are ready to prove Theorem 2.

Proof of Theorem 2. To complete the proof of the main result, we need to consider the following three steps.

Step 1. We first show that for every $\lambda \geq 0$, the functional φ_λ is coercive on E.

Firstly, due to condition $H(f)_2$ (i), one has

$$F(x,t) \leq \frac{Cd(x)}{r_2}|t|^{r_2}, \quad \forall (x,t)\in\Omega\times\mathbb{R}. \tag{34}$$

Again, using the condition $H(f)_2$ (i), Hölder's inequality, Proposition 3, and relation (34), we deduce that for any $u\in E$ with $\|u\| > 1$, the following inequality holds true:

$$\begin{aligned}\varphi_\lambda(u) &\geq \frac{1}{q}\|u\|^p - \frac{1}{1-\theta}|b|_{p/(p+\theta-1)}\left||u|^{1-\theta}\right|_{p/(1-\theta)} - \frac{\lambda C}{r_2}\int_\Omega |d(x)||u|^{r_2}dx\\ &\geq \frac{1}{q}\|u\|^p - \frac{1}{1-\theta}|b|_{p/(p+\theta-1)}|u|_p^{1-\theta} - \frac{\lambda C}{r_2}|d|_{s_2}\left||u|^{r_2}\right|_{s_2/(s_2-1)}\\ &= \frac{1}{q}\|u\|^p - \frac{1}{1-\theta}|b|_{p/(p+\theta-1)}|u|_p^{1-\theta} - \frac{\lambda C C_0^{r_2}}{r_2}|d|_{s_2}\|u\|^{r_2}.\end{aligned} \tag{35}$$

Since $1-\theta < 1 < r_2 < p$, we infer that $\varphi_\lambda(u) \longrightarrow +\infty$ as $\|u\| \longrightarrow +\infty$. The proof of Step 1 is now complete.

Step 2. We show that there exists $\xi\in E$ with $\xi > 0$, $\varphi_\lambda(t\xi) < 0$ for $t > 0$ small enough.

Let $\xi \in C_0^\infty(\Omega)$ such that supp $(\xi)\subset\Omega_1\subset\Omega$, $\xi = 1$ in a subset $\Omega_0 \subset$ supp (ξ), and $0\leq\xi\leq 1$ in Ω_1. Thus, by condition $H(f)_2$ (ii), it follows that

$$F(x,t\xi(x)) \geq 0, \quad \forall t\in(0,1), x\in\Omega. \tag{36}$$

Hence, for any $t\in(0,1)$, from $H(b)$ and $H(f)_2$ (ii), we deduce that

$$\begin{aligned}\varphi_\lambda(t\xi) &= \int_\Omega\left(\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q\right)dx - \int_\Omega \frac{b(x)}{1-\theta}|t\xi|^{1-\theta}dx - \lambda\int_\Omega F(x,t\xi)dx\\ &\leq \frac{t^p}{p}\max\left\{\|\xi\|^p, \|\xi\|^q\right\} - \frac{t^{1-\theta}}{1-\theta}\int_{\Omega_1} b(x)|\xi|^{1-\theta}dx.\end{aligned} \tag{37}$$

Since $p > 1 > 1-\theta$, we have $\varphi_\lambda(t\xi) < 0$ for $t < t_2$ with

$$0 < t_2 < \min\left\{1, \left(\frac{p\int_\Omega b(x)|\xi|^{1-\theta}dx}{(1-\theta)\max\left\{\|\xi\|^p,\|\xi\|^q\right\}}\right)^{1/(p+\theta-1)}\right\}. \tag{38}$$

The proof of Step 2 is now complete.

Step 3. We show that there exists $u_\lambda\in E$ such that $\varphi_\lambda(u_\lambda) = \inf_{u\in E}\varphi_\lambda(u)$ for any $\lambda\geq 0$.

Let $\{u_\lambda^n\}\subset E$ be a minimizing sequence of φ_λ. Then, using Step 1, we get that $\{u_\lambda^n\}$ is a bounded sequence. So, there exists $u_\lambda\in E$ such that, up to a subsequence,

$$\begin{aligned} u_\lambda^n &\rightharpoonup u_\lambda \quad \text{in } E,\\ u_\lambda^n &\longrightarrow u_\lambda \quad \text{in } L^p(\Omega),\\ u_\lambda^n(x) &\rightharpoonup u_\lambda(x) \quad \text{a.e.in } \Omega.\end{aligned} \tag{39}$$

Thus, as the proof of Step 3 in Theorem 1, we also obtain that

$$\lim_{n\to+\infty}\int_\Omega b(x)|u_\lambda^n|^{1-\theta}dx = \int_\Omega b(x)|u_\lambda|^{1-\theta}dx,$$
$$\lim_{n\to+\infty}\int_\Omega F(x,u_\lambda^n(x)) = \int_\Omega F(x,u_\lambda(x))dx, \tag{40}$$

and φ_λ is weakly lower semicontinuous, and consequently,

$$\varphi_\lambda(u_\lambda) \le \lim_{u\in E}\inf \varphi_\lambda(u_\lambda^n) = \inf_{v\in E}\varphi_\lambda(v) \le \varphi_\lambda(u_\lambda), \tag{41}$$

which implies that

$$\varphi(u_\lambda) = \inf_{u\in E}\varphi_\lambda(u_\lambda^n). \tag{42}$$

The proof of Step 3 is complete.

Therefore, combining the above Steps 2 and 3, we deduce that u_λ is the required nontrivial solution of problem (P_λ). Thus, we complete the Proof of Theorem 2.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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