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Research Article

Lie Symmetry Analysis and Invariant Solutions of a Nonlinear Fokker-Planck Equation Describing Cell Population Growth

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In this paper, a complete Lie symmetry analysis is performed for a nonlinear Fokker-Planck equation for growing cell populations. Moreover, an optimal system of one-dimensional subalgebras is constructed and used to find similarity reductions and invariant solutions. A new power series solution is constructed via the reduced equation, and its convergence is proved.

1. Introduction

During the last few decades, Lie symmetry group theory has been developed considerably and plays an increasingly important role in many scientific fields such as constructing similarity solutions, conservation laws, and symmetry-preserving difference schemes [1–5]. For the partial differential equations (PDEs), the Lie symmetry analysis method provides similar variables which are used to construct new differential equations with lower dimension, then group-invariant solutions of the studied PDEs is constructed via the reduced differential equations. With the benefit of the Lie symmetry analysis method, many differential equations were studied successfully [6–15].

To construct inequivalent invariant solutions which means that it is impossible to connect them with some group transformation, one needs to seek a minimal list of group generators in the simplest form that span these inequivalent group-invariant solutions. Such a scenario motivates emergence of the definition of an optimal system of subalgebra. From an algorithmic perspective of an optimal system, we need to simplify a general element of the infinitesimal operators to several simple and inequivalent forms by using adjoint transformations; refer to [1, 3] for details.

In this paper, we use the Lie symmetry method to study a nonlinear diffusion-type PDE describing cell population growth:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial \mu} = e^{pv} f^n \frac{\partial^2 f}{\partial v^2}, \tag{1}$$

where p and n are nonzero constants. This model describes the changes of cell population density $f = f(\mu, v, t)$ with the maturation of the cell populations $\mu \in (0, 1)$, the maturation velocity $v \in (0,\infty)$, and time $t \in (0,\infty)$. Equation (1) incorporates an exponential function in v and power function in f in the diffusion coefficient and extends the model proposed by Rotenberg [16]:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial \mu} = D\frac{\partial^2 f}{\partial v^2}, \tag{2}$$

where D is a diffusion constant. This equation is analyzed numerically in [17] and closed-form solutions under different reproduction rules of it are constructed. Equation (2) without a diffusion term is considered in [18]. A stationary modified version of Rotenberg's model with a nonlinear transition rate is studied in [19].

The remainder of this article is organized as follows. In Section 2, we first determine the symmetry group of equation (1) and derive an optimal system of one-dimensional subalgebras. Consequently, similarity reductions for equation (1) are performed and an explicit power series solution of equation (1) is presented. Finally, the last section summarizes our work.

2. Main Results

2.1. Determination of Lie Symmetry. Consider a local one-parameter Lie group of point transformation:

$$\begin{aligned} \mu^* &= \mu + \varepsilon\xi(\mu, v, t, f) + O(\varepsilon^2), \\ v^* &= v + \varepsilon\zeta(\mu, v, t, f) + O(\varepsilon^2), \\ t^* &= t + \varepsilon\tau(\mu, v, t, f) + O(\varepsilon^2), \\ f^* &= f + \varepsilon\eta(\mu, v, t, f) + O(\varepsilon^2), \end{aligned} \tag{3}$$

where $\varepsilon \in \mathbb{R}$ is the group parameter.

In Lie's framework, the following infinitesimal operator characterises the one-parameter Lie group (3) completely [2]:

$$\begin{aligned} X = {} & \xi(\mu, v, t, f)\frac{\partial}{\partial \mu} + \zeta(\mu, v, t, f)\frac{\partial}{\partial v} \\ & + \tau(\mu, v, t, f)\frac{\partial}{\partial t} + \eta(\mu, v, t, f)\frac{\partial}{\partial f}. \end{aligned} \tag{4}$$

Thus, if Lie group (3) leaves equation (1) invariant, then on the solution space of equation (1), operator (4) must satisfy the infinitesimal invariance criterion below:

$$Pr^{(2)}X\left[\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial \mu} - e^{pv}f^n\frac{\partial^2 f}{\partial v^2}\right] = 0, \tag{5}$$

where the second prolongation $Pr^{(2)}X$ is given by

$$Pr^{(2)}X = X + \eta_t^{(1)}\frac{\partial}{\partial f_t} + \eta_\mu^{(1)}\frac{\partial}{\partial f_\mu} + \eta_{vv}^{(2)}\frac{\partial}{\partial f_{vv}}, \tag{6}$$

in which $\eta_\mu^{(1)}$ and $\eta_{vv}^{(2)}$ are determined by classical formulae [1–3].

For the sake of determining Lie group (3) admitted by equation (1), inserting (6) into condition (5) and making the coefficients of different order derivatives of f equal to zero, we obtain a linear overdetermined system of PDEs about $\xi = \xi(\mu, v, t, f)$, $\zeta = \zeta(\mu, v, t, f)$, $\tau = \tau(\mu, v, t, f)$, and $\eta = \eta(\mu, v, t, f)$:

$$\begin{aligned} \frac{\partial \zeta}{\partial \mu} = \frac{\partial \zeta}{\partial v} = \frac{\partial \zeta}{\partial t} = \frac{\partial \zeta}{\partial f} = 0, \frac{\partial \eta}{\partial \mu} = \frac{\partial \eta}{\partial v} = \frac{\partial \eta}{\partial t} = 0, \\ \frac{\partial \xi}{\partial f} = \frac{\partial \xi}{\partial v} = 0, \frac{\partial \tau}{\partial \mu} = \frac{\partial \tau}{\partial v} = \frac{\partial \tau}{\partial f} = 0, \\ \eta - f\frac{\partial \eta}{\partial f} = 0, \zeta - \frac{\partial \xi}{\partial t} = 0, \\ pf\zeta + n\eta + f\frac{\partial \xi}{\partial \mu} = 0, pf\zeta + n\eta + f\frac{\partial \tau}{\partial t} = 0. \end{aligned} \tag{7}$$

By solving equation (7), we have

$$\begin{aligned} \xi = c_2(-p\mu + t) - c_1 n\mu + c_4, \zeta = c_2, \\ \tau = -c_1 nt - c_2 pt + c_3, \eta = c_1 f, \end{aligned} \tag{8}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

Consequently, the infinitesimal generators admitted by equation (1) are given by

$$\begin{aligned} X_1 &= (-p\mu + t)\frac{\partial}{\partial \mu} + \frac{\partial}{\partial v} - pt\frac{\partial}{\partial t}, \\ X_2 &= -n\mu\frac{\partial}{\partial \mu} - nt\frac{\partial}{\partial t} + f\frac{\partial}{\partial f}, \\ X_3 &= \frac{\partial}{\partial t}, \\ X_4 &= \frac{\partial}{\partial \mu}. \end{aligned} \tag{9}$$

In the wake of these infinitesimal generators X_i $(i = 1, 2, 3, 4)$, we obtain four Lie groups of point transformation admitted by equation (1):

$$\begin{aligned} &\mathrm{G1}: (\mu, v, t, f) \longrightarrow (e^{-p\varepsilon}(\mu + t\varepsilon), v + \varepsilon, e^{-p\varepsilon}t, f), \\ &\mathrm{G2}: (\mu, v, t, f) \longrightarrow (e^{-n\varepsilon}\mu, v, e^{-n\varepsilon}t, e^{\varepsilon}f), \\ &\mathrm{G3}: (\mu, v, t, f) \longrightarrow (\mu, v, t + \varepsilon, f), \\ &\mathrm{G4}: (\mu, v, t, f) \longrightarrow (\mu + \varepsilon, v, t, f). \end{aligned} \tag{10}$$

where $\varepsilon \in \mathbb{R}$ is the group parameter.

That is to say, if $f = \theta(\mu, v, t)$ satisfies equation (1), then f_i $(i = 1, 2, 3, 4)$ are also solutions of equation (1):

$$\begin{aligned} f_1 &= \theta(e^{p\varepsilon}(\mu - t\varepsilon), v - \varepsilon, e^{p\varepsilon}t), \\ f_2 &= e^{\varepsilon}\theta(e^{-n\varepsilon}\mu, v, e^{-n\varepsilon}t), \\ f_3 &= \theta(\mu, v, t - \varepsilon), \\ f_4 &= \theta(\mu - \varepsilon, v, t). \end{aligned} \tag{11}$$

2.2. Optimal System of One-Dimensional Subalgebras. In this subsection, we will find a one-dimensional optimal system of Lie subalgebras admitted by equation (1) up to adjoint representation. First of all, the commutator table of X_i $(i = 1, \cdots, 4)$

Table 1: Commutator table.

	X_1	X_2	X_3	X_4
X_1	0	0	$pX_3 - X_4$	pX_4
X_2	0	0	nX_3	nX_4
X_3	$-pX_3 + X_4$	$-nX_3$	0	0
X_4	$-pX_4$	$-nX_4$	0	0

Table 2: The adjoint representation of X_1, X_2, X_3, and X_4.

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	$e^{-p\varepsilon}(X_3 + \varepsilon X_4)$	$e^{-p\varepsilon}X_4$
X_2	X_1	X_2	$e^{-n\varepsilon}X_3$	$e^{-n\varepsilon}X_4$
X_3	$X_1 - \varepsilon(X_4 - pX_3)$	$X_2 + nX_3$	X_3	X_4
X_4	$X_1 + p\varepsilon X_4$	$X_2 + nX_4$	X_3	X_4

is shown in Table 1 where the (i, j) entry means $[X_i, X_j] = X_iX_j - X_jX_i$.

Each X_i generates an adjoint representation $\mathrm{Ad}(e^{\varepsilon X_i})X_j$ defined by [3]

$$\mathrm{Ad}\left(e^{\varepsilon X_i}\right)X_j = X_j - \varepsilon\left[X_i, X_j\right] + \frac{\varepsilon^2}{2!}\left[X_i, \left[X_i, X_j\right]\right] - \cdots. \quad (12)$$

Using (12) in conjunction with Table 1, we get Table 2 where the (i, j)-th entry represents $\mathrm{Ad}(e^{\varepsilon X_i})X_j$.

Proposition 1. *An optimal system of one-dimensional subalgebras spanned by X_1, X_2, X_3, and X_4 admitted by equation (1) is*

$$\{X_1 + aX_2, X_2, X_3 + X_4, X_3, X_4\}, \quad (13)$$

where a is an arbitrary constant.

Proof. Consider an arbitrary element spanned by X_1~X_4:

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4, \quad (14)$$

where, hereinafter, $a_i\ (i = 1, \cdots, 4)$ are arbitrary constants. Our target is to simplify as many of the coefficient a_i as possible through the adjoint maps to X. We start with the coefficient of X_1 and consider two cases about a_1.

Case 1. $a_1 \neq 0$.

Without loss of generality, we assume that $a_1 = 1$. According to Table 2, we act X by $\mathrm{Ad}(e^{a_3/(a_2n+p)}X_3)$:

$$X' = \mathrm{Ad}\left(e^{-a3/^{a}2^{n}+p}X_3\right)X = X_1 + a_2X_2 + a_4'X_3, \quad (15)$$

where $a_4' = a_4 + a_3/(a_2n + p)$. With the adopted adjoint action, the coefficient of X_4 disappears.

In order to cancel coefficient a_4', we further act on X' by $\mathrm{Ad}(e^{-a_4'/(a_2'n+p)}X_4)$, then we get $X'' = \mathrm{Ad}(e^{-a_4'/(a_2'n+p)}X_4)$ $X' = X_1 + a_2X_2$,

To summarize, X with $a_1 \neq 0$ is equivalent to $X_1 + aX_2$, where a is an arbitrary constant.

Case 2. $a_1 = 0$.

Consider $a_2 \neq 0$. Following the above procedure, we act on X by $\mathrm{Ad}(e^{a_3/n}X_3)$ and $\mathrm{Ad}(e^{a_4'/n}X_4)$ successively to make the coefficient a_3 and a_4 zero. Thus, every one-dimensional subalgebra generated by X with $a_1 = 0$ and $a_2 \neq 0$ is equivalent to the subalgebra spanned by X_2.

For $a_1 = 0$, $a_2 = 0$, and $a_3 \neq 0$, acting $\mathrm{Ad}(e^{\varepsilon X_1})$ on X, we obtain

$$X' = \mathrm{Ad}\left(e^{\varepsilon X_1}\right)X = e^{-n\varepsilon}(X_3 + a_4X_4), \quad (16)$$

which depends on the sign of a_4. In fact, we can simplify the coefficient of X_4 to either +1, −1, or 0. Moreover, we adopt the discrete symmetry $(\mu, \nu, t, f) \longrightarrow (-\mu, \nu, t, f)$, which maps $X_3 - X_4$ to $X_3 + X_4$. Thus, the one-dimensional subalgebra spanned by $X_3 + a_4X_4$ is equivalent to one spanned by either $X_3 + X_4$ or X_3.

Therefore, an optimal system of one-dimensional subalgebras admitted by equation (1) is determined by

$$\{X_1 + aX_2, X_2, X_3 + X_4, X_3, X_4\}, \quad (17)$$

where a is an arbitrary constant. It completes the proof.

2.3. Similarity Reductions. In this subsection, we perform similarity reductions and construct invariant solutions for equation (1) based on the optimal system calculated in the preceding subsection.

Case 3. Reduction by $X_1 + aX_2$.

For $a \neq 0$, the characteristic equation for the generator a $X_1 + X_2$, is

$$\frac{\mathrm{d}\mu}{-an\mu - p\mu + t} = \frac{\mathrm{d}\nu}{1} = \frac{\mathrm{d}t}{(-\mathrm{an} - p)t} = \frac{\mathrm{d}f}{af}, \quad (18)$$

which gives the similarity variables $x = e^{(\mathrm{an}+p)\nu}t$, $y = \nu - \mu/t$, and $F(x, y) = e^{a\nu}f(\mu, \nu, t)$. Substituting them into equation (1), we obtain

$$(\mathrm{an} + p)^2x^3F^n\frac{\partial^2 F}{\partial x^2} + (\mathrm{an} + p)x^2F^n\left[(\mathrm{an} + 2a + p)\frac{\partial F}{\partial x} + 2\frac{\partial^2 F}{\partial x\partial y}\right] + xF^n\left(a^2F + 2a\frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial y^2}\right) - x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = 0. \quad (19)$$

For $a = 0$, repeating the above procedure, the reduced equation corresponds to equation (19) with $a = 0$.

Case 4. Reduction by X_2.

Similarly, the similarity variables for X_2 are $y = \mu/t$ and $F(v, y) = \mu^{1/n} f(\mu, v, t)$. Then, the corresponding reduced equation is

$$\frac{v}{n} F + \left(y^2 - vy\right) \frac{\partial F}{\partial y} + e^{pv} F^n \frac{\partial^2 F}{\partial v^2} = 0. \quad (20)$$

Next, we use its symmetry to perform further symmetry reductions for equation (20). The infinitesimal operator of symmetry admitted by equation (20) is

$$Y = \frac{\partial}{\partial v} + \frac{\partial}{\partial y} + \frac{F}{ny}(1 - py)\frac{\partial}{\partial F}, \quad (21)$$

whose similarity variables are $x = y - v$ and $F(v, y) = g(x) y^{1/n} e^{py/n}$. In fact, we can transform X_1 to Y via $y = \mu/t$ and $F(v, y) = \mu^{1/n} f(\mu, v, t)$, that is to say, Y inherited symmetries of equation (1). Substituting the similarity variables into equation (20) leads to $g = g(x)$, satisfying the following reduced ordinary differential equation (ODE):

$$2npg^n \frac{\mathrm{d}g}{\mathrm{d}x} + n^2 g^n \frac{\mathrm{d}^2 g}{\mathrm{d}x^2} + p^2 g^{n+1} + n^2 x \frac{\mathrm{d}g}{\mathrm{d}x} + ng = 0. \quad (22)$$

Obviously, it is difficult to directly solve equation (22) by integration; thus, in the next subsection, we will construct power series solutions.

Case 5. Reduction by $X_3 + X_4$.

Via the symmetry $X_3 + X_4$, we get the group-invariant solution of the form $f(\mu, v, t) = F(v, y)$ in which $y = \mu - t$ and $F = F(v, y)$ satisfy

$$(v - 1)\frac{\partial F}{\partial y} = e^{pv} F^n \frac{\partial^2 F}{\partial v^2}. \quad (23)$$

Again, using the symmetry $Y = -ny(\partial/\partial y) + F(\partial/\partial F)$ for equation (23), which actually is the inherited symmetry of equation (1) ($X_2 \longrightarrow Y$), we obtain the group-invariant solution $F(v, y) = y^{-1/n} g(v)$, where $g = g(v)$ satisfies

$$e^{pv} g^{n-1} \frac{\mathrm{d}^2 g}{\mathrm{d}v^2} = \frac{1}{n}(1 - v). \quad (24)$$

In particular, for $n = 1$, we obtain a solution of equation (1):

$$f(\mu, v, t) = \left[\frac{p - pv - 2}{p^3} e^{-pv} + c_1 v + c_2\right](\mu - t)^{-1}, \quad (25)$$

where c_1 and c_2 are arbitrary constants.

The evolutionary procedure of solution (25) is shown in Figure 1 by choosing appropriate parameters from different perspectives.

Case 6. Reduction by X_3.

For the generator X_3, we have $f(\mu, v, t) = F(\mu, v)$, where $F = F(\mu, v)$ satisfies a $(1 + 1)$-dimensional PDE:

$$v\frac{\partial F}{\partial \mu} = e^{pv} F^n \frac{\partial^2 F}{\partial v^2}. \quad (26)$$

Similarly, by means of the corresponding infinitesimal operator $Y = -n\mu(\partial/\partial \mu) + F(\partial/\partial F)$ of equation (26), we reduce equation (26) into an ODE:

$$v = e^{pv} g(v)^{n-1} \frac{\mathrm{d}^2 g(v)}{\mathrm{d}v^2}, \quad (27)$$

where $F(\mu, v) = \mu^{-1/n} g(v)$.

In particular, for $n = 1$, we get a solution of equation (1):

$$f(\mu, v, t) = \mu^{-1/n}\left(-\frac{pv + 2}{p^3} e^{-pv} + c_1 v + c_2\right), \quad (28)$$

where c_1 and c_2 are arbitrary constants.

Case 7. Reduction by X_4.

For X_4, the group-invariant solution is $f(\mu, v, t) = F(\mu, v)$, where $F = F(\mu, v)$ satisfies

$$\frac{\partial F}{\partial t} = e^{pv} F^n \frac{\partial^2 F}{\partial v^2}. \quad (29)$$

Equation (29) admits two Lie symmetries with infinitesimal operators $Y_1 = -nt(\partial/\partial t) + F(\partial/\partial F)$ and $Y_2 = \partial/\partial v - pt(\partial/\partial t)$. The symmetry Y_1 produces the group-invariant solution with the form $F(v, t) = t^{-1/n} g(v)$, where $g = g(v)$ satisfies

$$-\frac{1}{n} g = e^{pv} g^n \frac{\mathrm{d}^2 g}{\mathrm{d}v^2}. \quad (30)$$

In particular, for $n = 1$, equation (1) has a solution:

$$f(\mu, v, t) = t^{1/n}\left[-\frac{e^{-pv}}{p^2} + c_1 v + c_2\right]. \quad (31)$$

On the other hand, the similarity variables of Y_2 are $x = t \exp(pv)$ and $g = F$. Then, the group-invariant solution is $F(v, t) = g(x)$, where $g = g(x)$ satisfies

$$\frac{\mathrm{d}g}{\mathrm{d}x} = p^2 x g^n \frac{\mathrm{d}^2 g}{\mathrm{d}x}. \quad (32)$$

2.4. Power Series Solutions via the Reduced Equation by X_2. In this subsection, we will seek a power series solution of equation (22). Suppose that equation (22) has a power series solution of the form

$$F(x) = \sum_{i=0}^{\infty} C_i x^i, \quad (33)$$

where c_i are undetermined constants.

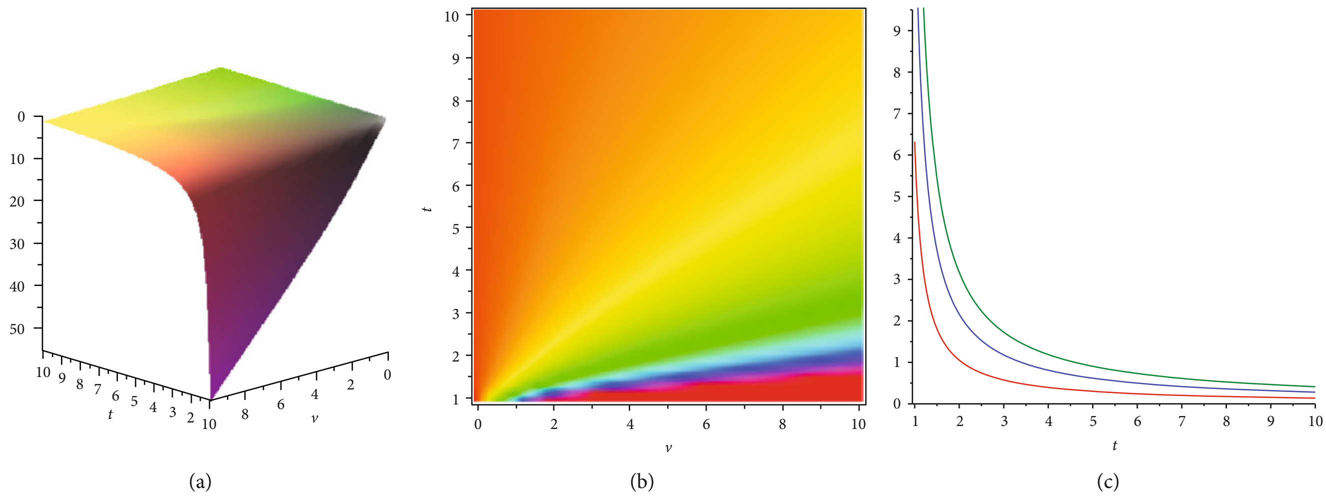

FIGURE 1: Solution (25) of Equation (1) by choosing suitable parameters: $p = 1$, $n = 1$, and $\mu = 0.8$. (a) Perspective view. (b) Overhead view. (c) Propagation pattern of the wave along the t axis.

Observe that

$$F'(x) = \sum_{i=0}^{\infty}(i+1)c_{i+1}x^i,\ F''(x) = \sum_{i=0}^{\infty}(i+2)(i+1)c_{i+2}x^i. \tag{34}$$

Substituting expressions (33) and (34) into equation (22), we have

$$\left(\sum_{i=0}^{\infty}c_i x^i\right)^n\left[2np\sum_{i=0}^{\infty}(i+1)c_{i+1}x^i + n^2\sum_{i=0}^{\infty}(i+2)(i+1)c_{i+2}x^i + p^2\sum_{i=0}^{\infty}c_i x^i\right] + n^2 x\sum_{i=0}^{\infty}(i+1)c_{i+1}x^i + n\sum_{i=0}^{\infty}c_i x_i = 0. \tag{35}$$

Equating the coefficients of different powers of x gives rise to the explicit expressions of c_i. For $i = 0$, one has

$$c_0^n\left(2npc_1 + 2n^2c_2 + p^2c_0\right) + nc_0 = 0, \tag{36}$$

which leads to

$$c_2 = -\frac{p}{n}c_1 - \frac{p^2}{2n^2}c_0 - \frac{1}{2nc_0^{n-1}}. \tag{37}$$

Generally for $i \geq 1$, we have

$$\begin{aligned} c_{i+2} = &-\frac{1}{n^2(i+2)(i+1)c_0^n}\sum_{k=0}^{i-1}\sum_{j1+j2+\cdots+j_n=i-k}c_{j1}c_{j2}\cdots c_{j_n} \\ &\times\left[2np(k+1)c_{k+1} + n^2(k+2)(k+1)c_{k+2} + p^2c_k\right] \\ &-\frac{1}{n^2(i+2)(i+1)c_0^n}\left[n^2ic_i + nc_i + c_0^n 2np(i+1)c_{i+1} + p^2c_ic_0^n\right]. \end{aligned} \tag{38}$$

Therefore, for the chosen constants c_0 and c_1, the sequence $\{c_i\}_{i=0}^{\infty}$ can be determined by equations (37) and (38) successively.

Now, we show that the power series solution (33) with c_i given by (37) and (38) is convergent. As a matter of fact,

$$|c_2| \leq M(2\mid c_0| + |c_1\mid), \tag{39}$$

and for $i \geq 1$,

$$\begin{aligned} |c_{i+2}| \leq M\Bigg[&\sum_{k=0}^{i-1}\sum_{j1+\cdots j_n=i-k}|c_{j1}||c_{j2}|\cdots|c_{j_n}|\left(|c_{k+1}| + |c_{k+2}|\right. \\ &\left.+ |c_k|\right) + 3|c_i| + |c_{i+1}|\Bigg], \end{aligned} \tag{40}$$

where $M = \max\{2p/nc_0^n, 1/nc_0^n, p^2/n^2c_0^n, 2p/n, p^2/n^2\}$.

Consider another power series $Q(z) = \sum_{i=0}^{\infty}Q_iz^i$, where

$$\begin{aligned} Q_0 &= |c_0|, Q_1 = |c_1|, Q_2 = M(2|c_0| + |c_1|) \\ &= M(Q_0 + 2Q_1), Q_{i+2} \\ &= M\Bigg[\sum_{k=0}^{i-1}\sum_{j1+\cdots+j_n=i-k}Q_{j1}Q_{j2}\cdots Q_{j_n}(Q_{k+1} \\ &+ Q_{k+2} + Q_k) + 3Q_i + Q_{i+1}\Bigg], \quad i \geq 1. \end{aligned} \tag{41}$$

It is easy to find that $|c_i| \leq Q_i (i = 0, 1, \cdots)$. Thus, the series $Q = Q(z) = \sum_{i=0}^{\infty}Q_iz^i$ is the majorant series of (33). Next, we show that the series $Q = Q(z)$ has positive radius of convergence:

$$\begin{aligned} Q(z) &= Q_0 + Q_1 z + Q_2 z^2 + \sum_{i=1}^{\infty} Q_{i+2} z^{i+2} = Q_0 + Q_1 z + Q_2 z^2 \\ &\quad + M\left(\sum_{i=1}^{\infty} \sum_{k=0}^{i-1} \sum_{j_1+\cdots+j_n=n} Q_{j_1} \cdots Q_{j_n} Q_{k+1} z^{i+2} \right. \\ &\quad + \sum_{i=1}^{\infty} \sum_{k=0}^{i-1} \sum_{j_1+\cdots+j_n=n} Q_{j_1} \cdots Q_{j_n} Q_{k+2} z^{i+2} \\ &\quad + \sum_{i=1}^{\infty} \sum_{k=0}^{i-1} \sum_{j_1+\cdots+j_n=n} Q_{j_1} \cdots Q_{j_n} Q_{k} z^{i+2} + 3\sum_{i=1}^{\infty} Q_i z^{i+1} \\ &\quad \left. + \sum_{i=1}^{\infty} Q_{i+1} z^{i+2} \right) = \left[\left(-Q - Q^2\right) Q_0^2 + 4Q_0 \right. \\ &\quad \left. + Q^3 + Q^2 Q_2 - 3Q + Q_1 \right] M z^2 \\ &\quad + \left[\left(Q - Q_0\right)\left(Q^2 - Q_0^2 + 1\right) M + Q_1 \right] z + Q_0. \end{aligned} \tag{42}$$

Now, we construct the implicit function with respect to the independent variable z:

$$\begin{aligned} \lambda(z, \varphi) &= \varphi - M\left[\left(-Q - Q^2\right) Q_0^2 + 4Q_0 + Q^3 \right. \\ &\quad \left. + Q^2 Q_2 - 3Q + Q_1 \right] z^2 \\ &\quad - \left[\left(Q - Q_0\right)\left(Q^2 - Q_0^2 + 1\right) M + Q_1 \right] z - Q_0, \end{aligned} \tag{43}$$

and we verify that $\lambda(z, \varphi)$ is analytic in the neighborhood of $(0, Q_0)$ where $\lambda(0, Q_0) = 0$ and $\lambda'_{\varphi}(0, Q_0) = 1 \neq 0$. By the implicit function theorem [20, 21], we see that $Q = Q(z)$ is analytic in a neighborhood of the point $(0, Q_0)$ and with a positive radius. This implies that the power series $Q(z) = \sum_{i=0}^{\infty} Q_i z^i$ converges in a neighborhood of the point $(0, Q_0)$.

Hence, an explicit power series solution of equation (1) is given by

$$f(\mu, \nu, t) = \sum_{i=0}^{\infty} c_i \left(-\nu + \frac{\mu}{t} \right)^i t^{-1/n} e^{p\mu/nt}, \tag{44}$$

where the coefficients $c_i (i \geq 2)$ depend on (37) and (38) completely.

3. Conclusion

In this paper, Lie symmetry analysis is employed to study a new nonlinear equation describing the growing cell populations. An optimal system of one-dimensional subalgebra is constructed and used to construct reduced equations and invariant solutions. Moreover, we obtain a new power series solution of equation (1). Such results provide positive potential roles for analyzing cell population growth with equation (1).

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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