Hindawi
Advances in Mathematical Physics
Volume 2020, Article ID 4378691, 6 pages
https://doi.org/10.1155/2020/4378691

Research Article

Semiclassical Solutions for a Kind of Coupled Schrödinger Equations

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Received 11 June 2020; Accepted 27 July 2020; Published 17 August 2020

Guest Editor: Xin Yu

In this paper, we are concerned with the following coupled Schrödinger equations $\begin{cases} -\lambda^2\Delta u + a_1(x)u = c(x)v + a_2(x)|u|^{p-2}u + a_3(x)|u|^{2^*-2}u, & x\in\mathbb{R}^N, \\ -\lambda^2\Delta v + b_1(x)v = c(x)u + b_2(x)|v|^{p-2}v + b_3(x)|v|^{2^*-2}v, & x\in\mathbb{R}^N, \end{cases}$ where $2<p<2^*, 2<q<2^*, 2^*=2N/(N-2)$, and$N\geq 3$; $\lambda>0$ is a parameter; and $a_1, a_2, a_3, b_1, b_2, b_3, c \in C(\mathbb{R}^N, \mathbb{R})$ and $u, v\in H^1(\mathbb{R}^N)$. Under some suitable conditions that $a_1^0=\inf a_1=0$ or $b_1^0=\inf b_1=0$ and $|c(x)|^2\leq \vartheta a_1(x)b_1(x)$ with $\vartheta\in(0,1)$, the above coupled Schrödinger system possesses nontrivial solutions if $\lambda \in (0,\lambda_0)$, where λ_0 is related to $a_1, a_2, a_3, b_1, b_2, b_3$, and N.

1. Introduction

We consider the following coupled Schrödinger equations in this paper:

$$\begin{cases} -\lambda^2\Delta u + a_1(x)u = c(x)v + a_2(x)|u|^{p-2}u + a_3(x)|u|^{2^*-2}u, & x\in\mathbb{R}^N, \\ -\lambda^2\Delta v + b_1(x)v = c(x)u + b_2(x)|v|^{p-2}v + b_3(x)|v|^{2^*-2}v, & x\in\mathbb{R}^N, \end{cases} \tag{1}$$

where $2<p<2^*$, $2<q<2^*$, $N\geq 3$, and $2^*=2N/(N-2)$ are the Sobolev critical exponent;$\lambda>0$ is a parameter; and a_1, $a_2, a_3, b_1, b_2, b_3, c\in C(\mathbb{R}^N,\mathbb{R})$ and $u, v\in H^1(\mathbb{R}^N)$.

As it is known in [1], this type of systems arises in nonlinear optics. In the past years, under different kinds of assumptions on the potential V and the nonlinearity f, many authors [2–8] focus on the following kind of Schrödinger equation:

$$-\lambda^2\Delta u + V(x)u = f(x,u), \quad x\in\mathbb{R}^N. \tag{2}$$

As one knows, single-mode optical fibers are not really "single mode" but actually bimodal because of the presence of birefringence. So recently, the coupled Schrödinger systems are investigated by the authors [9–12]. For more related results and physical background on Schrödinger systems, please see [13–23] and references therein.

In [11], the authors investigated standing waves for the following kind of coupled Schrödinger equations:

$$\begin{cases} -\lambda^2\Delta u + a_1(x)u = cv + |u|^{p-2}u, & x\in\mathbb{R}^N, \\ -\lambda^2\Delta v + b_1(x)v = cu + |v|^{2^*-2}v, & x\in\mathbb{R}^N, \end{cases} \tag{3}$$

where $a_1, b_1\in C(\mathbb{R}^N,\mathbb{R})$, $N\geq 3$, $u,v>0$, $u,v\in H^1(\mathbb{R}^N)$, and $u(x), v(x)\to 0$ as $|x|\to\infty$. Under the following conditions,

(A0) there exist positive constants $a_1^0>0$ and $b_1^0>0$ such that $a_1(x)\geq a_1^0$, $b_1(x)\geq a_1^0$, and $0<c\leq\sqrt{a_1^0 b_1^0}$; they obtained the existence of a positive solution for (3) if λ is sufficiently small. But, if $a_1^0=\inf a_1=0$ or $b_1^0=\inf b_1=0$, then $0<c\leq \sqrt{a_1^0 b_1^0}$ cannot hold. So in the very recent paper [12], Peng et al. investigated the following coupled Schrödinger equations and generalize the result in [11]:

$$\begin{cases} -\lambda^2 \Delta u + a_1(x)u = c(x)v + |u|^{p-2}u, & x \in \mathbb{R}^N, \\ -\lambda^2 \Delta v + b_1(x)v = c(x)u + |v|^{q-2}v, & x \in \mathbb{R}^N, \end{cases} \tag{4}$$

where a_1, b_1 are the same as in (3), $N \geq 3$. Under the following conditions,

(A1) $a_1(x) \geq a_1(0) = 0$ and $b_1(x) \geq 0$, and there exist constants $a_1^0 > 0$ and $b_1^0 > 0$ such that the measure of the sets $A_{a_1^0} := \{x : a_1(x) < a_1^0\}$ and $B_{b_1^0} := \{x : b_1(x) < b_1^0\}$ are finite

(A2) there exists a constant $\vartheta \in (0,1)$ such that $|c(x)|^2 \leq \vartheta a_1(x) b_1(x)$ for all $x \in \mathbb{R}^N$; Peng et al. proved that system (4) has at least one nontrivial solution. An interesting question is what will happen if the nonlinearity is also critical growth in system (4)? Motivated mainly by the above-mentioned results, we will answer this question and prove that system (1), under conditions (A1) and (A2), and

(A3) there exist constants $a_2^0, a_2^1, a_3^0, a_3^1, b_2^0, b_2^1, b_3^0, b_3^1 > 0$ such that

$$\begin{gathered} a_2^0 \leq a_2(x) \leq a_2^1, a_3^0 \leq a_3(x) \leq a_3^1, b_2^0 \leq b_2(x) \leq b_2^1, \\ b_3^0 \leq b_3(x) \leq b_3^1, \quad \forall x \in \mathbb{R}^N, \end{gathered} \tag{5}$$

possesses nontrivial solutions if $\lambda \in (0, \lambda_0)$, where λ_0 is related to $a_1, a_2, a_3, b_1, b_2, b_3$, and N. As far as we know, similar results for system (1) with a critical exponent have not been investigated by variational methods in the literature. The following condition is similar to condition (A1):

(A1') $b_1(x) \geq b_1(0) = 0$ and $a_1(x) \geq 0$, and there exist constants $a_1^0 > 0$ and $b_1^0 > 0$ such that the measure of the sets $A_{a_1^0} := \{x \in \mathbb{R}^N : a_1(x) < a_1^0\}$ and $B_{b_1^0} := \{x \in \mathbb{R}^N : b_1(x) < b_1^0\}$ are finite.

Since $(q-2)N - 2q < 0$ and $(p-2)N - 2p < 0$, one can choose $d_0 \geq 1$ such that

$$C_1 a_2^1 \alpha^{(p-2)/p} + C_2 b_2^1 \beta^{(q-2)/q} + C_3 a_3^1 \alpha^{2/N} + C_4 b_3^1 \beta^{2/N} \leq \frac{1}{2}(1-\vartheta), \tag{6}$$

where

$$\begin{aligned} \alpha &= \frac{\omega_N a_2^0 (p-2)}{2Np} \left\{ \frac{N^2 + 2(N+2)}{(N+2)(1-2^{-N})^2} \right\}^{p/(p-2)} d_0^{[(p-2)N-2p]/(p-2)}, \\ \beta &= \frac{\omega_N b_2^0 (q-2)}{2Nq} \left\{ \frac{N^2 + 2(N+2)}{(N+2)(1-2^{-N})^2} \right\}^{q/(q-2)} d_0^{[(q-2)N-2q]/(q-2)}, \\ C_1 &= \left[\frac{2p(\eta_0 \eta_{2^*})^N}{(p-2)a_2^0} \right]^{(p-2)/p} (a_1^0)^{[p(N-2)-2N]/2p}, \\ C_2 &= \left[\frac{2q(\eta_0 \eta_{2^*})^N}{(q-2)b_2^0} \right]^{(q-2)/q} (b_1^0)^{[q(N-2)-2N)]/2q}, \\ C_3 &= \left[\frac{22^*(\eta_0 \eta_{2^*})^N}{(2^*-2)a_3^0} \right]^{2/N}, \\ C_4 &= \left[\frac{22^*(\eta_0 \eta_{2^*})^N}{(2^*-2)b_3^0} \right]^{2/N}, \end{aligned} \tag{7}$$

η_0 and η_{2^*} are embedding constants and ω_N is the volume of the unit ball in $\mathbb{R}^N$. From (A1') and (A1), using $b_1(0) = 0$ and $a_1(0) = 0$, one can let $\mu_0 > 1$ such that

$$\sup_{\mu^{1/2}|x| \leq 2d_0} |b_1(x)| \leq d_0^{-2}, \quad \sup_{\mu^{1/2}|x| \leq 2d_0} |a_1(x)| \leq d_0^{-2}, \quad \forall \mu \geq \mu_0. \tag{8}$$

Let $w = (u, v)$ and $\lambda^{-2} = \mu$, then system (1) can be rewritten as

$$\begin{cases} -\Delta u + \mu a_1(x)u = \mu c(x)v + \mu a_2(x)|u|^{p-2}u + \mu a_3(x)|u|^{2^*-2}u, & x \in \mathbb{R}^N, \\ -\Delta v + \mu b_1(x)v = \mu c(x)u + \mu b_2(x)|v|^{p-2}v + \mu b_3(x)|v|^{2^*-2}v, & x \in \mathbb{R}^N, \end{cases} \tag{9}$$

and the functional of (9) is given by

$$\begin{aligned} S_\mu(w) = {} & \frac{1}{2} \int_{\mathbb{R}^N} \left[|\Delta u|^2 + |\Delta v|^2 + \mu a_1(x)|u|^2 + \mu b_1(x)|v|^2 \right] dx \\ & - \frac{\mu}{p} \int_{\mathbb{R}^N} a_2(x)|u|^p dx - \frac{\mu}{q} \int_{\mathbb{R}^N} b_2(x)|v|^q dx \\ & - \frac{\mu}{2^*} \int_{\mathbb{R}^N} a_3(x)|u|^{2^*} dx - \frac{\mu}{2^*} \int_{\mathbb{R}^N} b_3(x)|v|^{2^*} dx \\ & - \mu \int_{\mathbb{R}^N} c(x)uv dx. \end{aligned} \tag{10}$$

As is known, the solutions of (1) are the critical points of $S_{\lambda^{-1/2}}(w)$. The main results are the following.

Theorem 1. *Suppose that (A1)–(A3) or (A1')–(A3) hold. Then, (9) possesses at least one nontrivial solution $w_\mu = (u_\mu, v_\mu)$ such that $0 < S_\mu(w_\mu) \leq \beta \mu^{1-N/2}$ for $\mu \geq \mu_0$.*

Theorem 2. *Suppose that (A1)–(A3) or (A1')–(A3) hold. Then, (1) possesses at least one nontrivial solution $w_\lambda = (u_\lambda, v_\lambda)$ such that $0 < S_{\lambda^{-1/2}}(w_\lambda) \leq \beta \lambda^{N-2}$ for $0 < \lambda < \mu_0^{-1/2}$.*

Remark 3. Since the presence of the terms $a_2(x)|u|^{p-2}u$, $a_3(x)|u|^{2^*-2}u$, $b_2(x)|v|^{p-2}v$, and $b_3(x)|v|^{2^*-2}v$, system (1) is more general than (4), and it is more difficult to deal with the nontrivial solutions. In order to prove that system (1) has nontrivial solutions, we need to find some conditions to restrict $a_2(x)$, $a_3(x)$, $b_2(x)$, and $b_3(x)$. It seems that there is no literature considering system (1).

2. Preliminaries

Let

$$E = \left\{ (u, v) : \int_{\mathbb{R}^N} \left[a_1(x)|u|^2 + b_1(x)|v|^2 \right] dx < \infty, u, v \in H^1(\mathbb{R}^N) \right\}, \tag{11}$$

$$\|w\|_{\mu^\dagger} = \left\{\int_{\mathbb{R}^N} \left[|\Delta u|^2 + |\Delta v|^2 + \mu a_1(x)|u|^2 + \mu b_1(x)|v|^2\right] dx\right\}^{1/2}, \quad \forall \quad w = (u, v) \in E. \tag{12}$$

From Lemma 1 of [17], by (A1) or (A1') and the Sobolev inequality, there exists a positive constant $\eta_0 > 0$ independent of μ such that

$$\|w\|_{H^1} := \left\{\int_{\mathbb{R}^N} \left[|\Delta u|^2 + |\Delta v|^2 + |u|^2 + |v|^2\right] dx\right\}^{1/2} \le \eta_0 \|w\|_{\mu^\dagger}, \quad \forall w = (u, v) \in E, \mu \ge 1, \tag{13}$$

where $H^1 := H^1(\mathbb{R}^N)$. Then, $(E, \|\cdot\|_{\mu^\dagger})$ is a Banach space for $\mu \ge 1$ equipped with the norm given by (12). Moreover, for $s \in [2, 2^*]$, one has

$$\|w\|_s \le \eta_s \|w\|_{H^1} \le \eta_s \eta_0 \|w\|_{\mu^\dagger}, \quad \forall w \in E, \mu \ge 1, \tag{14}$$

where $\|w\|_s$ is the usual norm in space $L^s(\mathbb{R}^N)$. From (12), we rewrite S_μ as

$$\begin{aligned} S_\mu(w) = {} & \frac{1}{2}\|w\|_{\mu^\dagger}^2 - \frac{\mu}{p}\int_{\mathbb{R}^N} a_2(x)|u|^p dx - \frac{\mu}{q}\int_{\mathbb{R}^N} b_2(x)|v|^q dx \\ & - \frac{\mu}{2^*}\int_{\mathbb{R}^N} a_3(x)|u|^{2^*} dx - \frac{\mu}{2^*}\int_{\mathbb{R}^N} b_3(x)|v|^{2^*} dx \\ & - \mu\int_{\mathbb{R}^N} c(x)uv dx, \quad \forall w \in E. \end{aligned} \tag{15}$$

It is not difficult to see that $S_\mu \in C^1(E, \mathbb{R})$ and

$$\begin{aligned} \left\langle S'_\mu(w), \bar{w}\right\rangle = {} & \int_{\mathbb{R}^N} [\Delta u \cdot \bar{u} + \Delta v \cdot \bar{v} + \mu a_1(x) u\bar{u} + \mu b_1(x) v\bar{v}] dx \\ & - \mu\int_{\mathbb{R}^N} c(x)(u\bar{v} + v\bar{u}) dx - \mu\int_{\mathbb{R}^N} a_2(x)|u|^{p-2}u\bar{u} dx \\ & - \mu\int_{\mathbb{R}^N} b_2(x)|v|^{q-2}v\bar{v} dx - \mu\int_{\mathbb{R}^N} a_3(x)|u|^{2^*-2}u\bar{u} dx \\ & - \mu\int_{\mathbb{R}^N} b_3(x)|v|^{2^*-2}v\bar{v} dx, \quad \forall w = (u, v), \bar{w} = (\bar{u}, \bar{v}) \in E. \end{aligned} \tag{16}$$

As in [12, 22], let

$$\theta(x) = \begin{cases} \dfrac{1}{d_0}, & |x| \le d_0, \\ \dfrac{d_0^{N-1}}{1 - 2^{-N}}\left[|x|^{-N} - (2d_0)^{-N}\right], & d_0 < |x| \le 2d_0, \\ 0, & |x| > 2d_0. \end{cases} \tag{17}$$

Then, $\theta \in H^1(\mathbb{R}^N)$; moreover,

$$\|\nabla\theta\|_2^2 = \int_{\mathbb{R}^N} |\nabla\theta(x)|^2 dx \le \frac{N\omega_N d_0^{N-4}}{(N+2)\left(1 - 2^{-N}\right)^2}, \tag{18}$$

$$\|\theta\|_2^2 = \int_{\mathbb{R}^N} |\theta(x)|^2 dx \le \frac{2\omega_N d_0^{N-2}}{N\left(1 - 2^{-N}\right)^2}. \tag{19}$$

In the next section, we will prove the main results.

3. Proof of the Main Results

Proof of Theorem 1. The proof of Theorem 1 is divided into four steps.

Step 1. We first prove that for any $\mu \ge \mu_0 > 1$, one has

$$\sup\left\{S_\mu\left(0, te_\mu\right) : t \ge 0\right\} \le \beta\mu^{1-N/2}, \quad \sup\left\{S_\mu\left(te_\mu, 0\right) : t \ge 0\right\} \le \alpha\mu^{1-N/2}, \tag{20}$$

where $e_\mu(x) = \theta(\mu^{1/2}x)$. From (8), (9), (17), (18), (19), and (A3), we have

$$\begin{aligned} S_\mu\left(0, te_\mu\right) = {} & \frac{t^2}{2}\int_{\mathbb{R}^N} \left[|\nabla e_\mu|^2 + \mu b_1(x)|e_\mu|^2\right] dx - \frac{\mu}{q}\int_{\mathbb{R}^N} b_2(x)|te_\mu|^q dx \\ & - \frac{\mu}{2^*}\int_{\mathbb{R}^N} b_3(x)|te_\mu|^{2^*} dx = \mu^{1-N/2}\left[\frac{t^2}{2}\int_{\mathbb{R}^N} \left(|\nabla\theta|^2 + b_1\left(\mu^{-1/2}x\right)|\theta|^2\right) dx \right. \\ & \left. - \frac{1}{q}\int_{\mathbb{R}^N} b_2\left(\mu^{-1/2}x\right)|t\theta|^q dx - \frac{1}{2^*}\int_{\mathbb{R}^N} b_3\left(\mu^{-1/2}x\right)|t\theta|^{2^*} dx\right] \\ \le {} & \mu^{1-N/2}\left[\frac{t^2}{2}\left(\|\nabla\theta\|_2^2 + \|\theta\|_2^2 \sup_{|x|\le 2d_0}\left|b_1\left(\mu^{-1/2}x\right)\right|\right)\right. \\ & \left. - \frac{1}{q}\int_{|x|\le d_0} b_2\left(\mu^{-1/2}x\right)\left|\frac{t}{d_0}\right|^q dx - \frac{1}{2^*}\int_{|x|\le d_0} b_3\left(\mu^{-1/2}x\right)\left|\frac{t}{d_0}\right|^{2^*} dx\right] \\ \le {} & \mu^{1-N/2}\left[\frac{t^2}{2}\left(\|\nabla\theta\|_2^2 + d_0^{-2}\|\theta\|_2^2\right) - \frac{\omega_N b_2^0}{qN} t^q d_0^{N-q} - \frac{\omega_N b_3^0}{2^* N} t^{2^*} d_0^{N-2^*}\right] \\ \le {} & \mu^{1-N/2}\left[\frac{t^2}{2}\left(\|\nabla\theta\|_2^2 + d_0^{-2}\|\theta\|_2^2\right) - \frac{\omega_N b_2^0}{qN} t^q d_0^{N-q}\right] \\ \le {} & \mu^{1-N/2}(q-2)\left(\|\nabla\theta\|_2^2 + d_0^{-2}\|\theta\|_2^2\right)^{q/(q-2)}(2q)^{-1}\left(\frac{\omega_N b_2^0 d_0^{N-q}}{N}\right)^{-2/(q-2)} \\ \le {} & \mu^{1-N/2}\frac{\omega_N b_2^0(q-2)}{2Nq}\left\{\frac{N^2 + 2(N+2)}{(N+2)\left(1 - 2^{-N}\right)^2}\right\}^{q/(q-2)} \\ & \cdot d_0^{[(q-2)N-2q]/(q-2)} := \beta\mu^{1-N/2}. \end{aligned} \tag{21}$$

Similarly, from (8), (9), (17), (18), (19), and (A3), we have

$$\begin{aligned} S_\mu\left(te_\mu, 0\right) \le {} & \mu^{1-N/2}\frac{\omega_N a_2^0(p-2)}{2Np}\left\{\frac{N^2 + 2(N+2)}{(N+2)\left(1 - 2^{-N}\right)^2}\right\}^{p/(p-2)} \\ & \cdot d_0^{[(p-2)N-2p]/(p-2)} := \alpha\mu^{1-N/2}, \end{aligned} \tag{22}$$

which together with (21) implies that (20) holds.

Step 2. Let $c_\mu^* = \min\{S_\mu(te_\mu, 0), S_\mu(0, te_\mu)\}$, we should prove that there exists a constant $c_\mu \in (0, c_\mu^*]$ and a sequence $\{w_n\} \subset E$ satisfying

$$S_\mu(w_n) \to c_\mu, \ \left\| S'_\mu(w_n) \right\|_{E^*} \left(1 + \|w_n\|_{\mu^\dagger}\right), \quad \text{as } n \to \infty. \tag{23}$$

By a standard argument, one can obtain (23) by employing the mountain-pass lemma without the (PS) condition, so we omit the details here.

Step 3. We prove that any sequence $\{w_n\} \subset E$ satisfying (23) is bounded in E. From (A2) and Young's inequality, we have

$$\begin{aligned} \mu \int_{\mathbb{R}^3} |c(x) u_n v_n| dx &\le \mu \vartheta \int_{\mathbb{R}^3} \sqrt{a_1(x) b_1(x)} |u_n v_n| dx \\ &\le \frac{\vartheta}{2} \int_{\mathbb{R}^3} \left[\mu a_1(x) u_n^2 + \mu b_1(x) v_n^2\right] dx \\ &\le \frac{\vartheta}{2} \|w_n\|_{\mu^\dagger}^2. \end{aligned} \tag{24}$$

For $2 < p \le q < 2^*$, from (15), (16), (23), and (24), we have

$$\begin{aligned} c_\mu + o(1) &= S_\mu(w_n) - \frac{1}{p} \left\langle S'_\mu(w_n), w_n \right\rangle = \left(\frac{1}{2} - \frac{1}{p}\right) \|w_n\|_{\mu^\dagger}^2 \\ &\quad + \left(\frac{1}{p} - \frac{1}{q}\right) \mu \int_{\mathbb{R}^N} b_2(x) |v_n|^q dx \\ &\quad + \left(\frac{1}{p} - \frac{1}{2^*}\right) \mu \int_{\mathbb{R}^N} a_3(x) |u_n|^{2^*} dx \\ &\quad - \left(1 - \frac{2}{p}\right) \mu \int_{\mathbb{R}^N} c(x) u_n v_n dx \\ &\quad + \left(\frac{1}{p} - \frac{1}{2^*}\right) \mu \int_{\mathbb{R}^N} b_3(x) |v_n|^{2^*} dx \\ &\ge \left(\frac{1}{2} - \frac{1}{p}\right) (1 - \vartheta) \|w_n\|_{\mu^\dagger}^2. \end{aligned} \tag{25}$$

For $2 < q \le p < 2^*$, from (15), (16), (23), and (24), we obtain

$$\begin{aligned} c_\mu + o(1) &= S_\mu(w_n) - \frac{1}{q} \left\langle S'_\mu(w_n), w_n \right\rangle = \left(\frac{1}{2} - \frac{1}{q}\right) \|w_n\|_{\mu^\dagger}^2 \\ &\quad + \left(\frac{1}{q} - \frac{1}{p}\right) \mu \int_{\mathbb{R}^N} a_2(x) |v_n|^p dx + \left(\frac{1}{q} - \frac{1}{2^*}\right) \\ &\quad \cdot \mu \int_{\mathbb{R}^N} a_3(x) |u_n|^{2^*} dx - \left(1 - \frac{2}{q}\right) \mu \int_{\mathbb{R}^N} c(x) u_n v_n dx \\ &\quad + \left(\frac{1}{q} - \frac{1}{2^*}\right) \mu \int_{\mathbb{R}^N} b_3(x) |v_n|^{2^*} dx \\ &\ge \left(\frac{1}{2} - \frac{1}{q}\right) (1 - \vartheta) \|w_n\|_{\mu^\dagger}^2. \end{aligned} \tag{26}$$

It follows from (25) and (26) that $\{w_n\}$ is bounded in E.

Step 4. We show that there exists a nontrivial solution. By Steps 1–3, we know that there exists a bounded sequence $\{w_n\} \subset E$ satisfying (23) with

$$c_\mu \le c_\mu^*, \quad \forall \mu \ge \mu_0. \tag{27}$$

Passing to a subsequence, one can suppose that $w_n = (u_n, v_n) \rightharpoonup w_\mu = (u_\mu, v_\mu)$ in $(E, \|\cdot\|_{\mu^\dagger})$ and $S'_\mu(w_n) \to 0$, as $n \to \infty$. Now, we verify that $w_\mu \ne (0, 0)$. Arguing by contradiction, assume that $w_\mu = (0, 0)$, that is, $w_n \rightharpoonup (0, 0)$ in E, so by [24], we have $w_n \to (0, 0)$ in $L^s_{loc}(\mathbb{R}^N)$, $s \in [2, 2^*]$, and $w_n \to (0, 0)$ a.e. on $\mathbb{R}^N$. Since $A_{a_1^0}$ and $B_{b_1^0}$ are sets with finite measure, we have

$$\begin{aligned} \|u_n\|_2^2 &= \int_{\mathbb{R}^N \setminus A_{a_1^0}} |u_n|^2 dx + \int_{A_{a_1^0}} |u_n|^2 dx = \int_{\mathbb{R}^N \setminus A_{a_1^0}} |u_n|^2 dx \\ &\quad + \int_{A_{a_1^0}} |u_n|^2 dx \le \int_{\mathbb{R}^N \setminus A_{a_1^0}} \frac{1}{\mu a_1^0} \mu a_1(x) |u_n|^2 dx \\ &\quad + \int_{A_{a_1^0}} |u_n|^2 dx \le \frac{1}{\mu a_1^0} \|w_n\|_{\mu^\dagger}^2 + o(1), \end{aligned} \tag{28}$$

$$\begin{aligned} \|v_n\|_2^2 &= \int_{\mathbb{R}^N \setminus B_{b_1^0}} |v_n|^2 dx + \int_{B_{b_1^0}} |v_n|^2 dx = \int_{\mathbb{R}^N \setminus B_{b_1^0}} |v_n|^2 dx \\ &\quad + \int_{B_{b_1^0}} |v_n|^2 dx \le \int_{\mathbb{R}^N \setminus B_{b_1^0}} \frac{1}{\mu b_1^0} \mu b_1(x) |v_n|^2 dx \\ &\quad + \int_{B_{b_1^0}} |v_n|^2 dx \le \frac{1}{\mu b_1^0} \|w_n\|_{\mu^\dagger}^2 + o(1). \end{aligned} \tag{29}$$

Similar to [12], from (14), (28), (29), and the Hölder inequality, we obtain

$$\begin{aligned} \|u_n\|_s^s &= \int_{\mathbb{R}^N} |u_n|^s dx \le \left(\int_{\mathbb{R}^N} |u_n|^{2(2^*-s)/(2^*-2)} dx\right) \\ &\quad \cdot \left(\int_{\mathbb{R}^N} |u_n|^{2^*(s-2)/(2^*-2)} dx\right) \le (\eta_0 \eta_{2^*})^{2^*(s-2)/(2^*-2)} \\ &\quad \cdot (\mu a_1^0)^{-(2^*-s)/(2^*-2)} \|w_n\|_{\mu^\dagger}^s + o(1), \quad s \in (2, 2^*) \end{aligned} \tag{30}$$

$$\begin{aligned} \|v_n\|_s^s &= \int_{\mathbb{R}^N} |v_n|^s dx \le \left(\int_{\mathbb{R}^N} |v_n|^{2(2^*-s)/(2^*-2)} dx\right) \\ &\quad \cdot \left(\int_{\mathbb{R}^N} |v_n|^{2^*(s-2)/(2^*-2)} dx\right) \le (\eta_0 \eta_{2^*})^{2^*(s-2)/(2^*-2)} \\ &\quad \cdot (\mu b_1^0)^{-(2^*-s)/(2^*-2)} \|w_n\|_{\mu^\dagger}^s + o(1), \quad s \in (2, 2^*). \end{aligned} \tag{31}$$

It follows from (15), (16), (23), and (A3) that

$$\begin{aligned} c_\mu + o(1) &= S_\mu(w_n) - \frac{1}{2}\left\langle S'_\mu(w_n), w_n \right\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\mu\int_{\mathbb{R}^N} a_2(x)|u_n|^p dx + \left(\frac{1}{2} - \frac{1}{q}\right)\mu\int_{\mathbb{R}^N} b_2(x)|v_n|^q dx \\ &\quad + \left(\frac{1}{2} - \frac{1}{2^*}\right)\mu\int_{\mathbb{R}^N} a_3(x)|u_n|^{2^*} dx + \left(\frac{1}{2} - \frac{1}{2^*}\right) \\ &\quad \cdot \mu\int_{\mathbb{R}^N} b_3(x)|v_n|^{2^*} dx \geq \frac{(p-2)\mu a_2^0}{2p}\|u_n\|_p^p + \frac{(q-2)\mu b_2^0}{2q}\|v_n\|_q^q \\ &\quad + \frac{(2^*-2)\mu a_3^0}{22^*}\|u_n\|_{2^*}^{2^*} + \frac{(2^*-2)\mu b_3^0}{22^*}\|v_n\|_{2^*}^{2^*}. \end{aligned} \tag{32}$$

From (14), (30), (31), and (32), we have

$$\begin{aligned} \mu\|u_n\|_p^p &= \mu\|u_n\|_p^{p-2}\|u_n\|_p^2 \leq \mu\left(\mu a_1^0\right)^{-2(2^*-p)/[p(2^*-2)]} \\ &\quad \cdot (\eta_0\eta_{2^*})^{22^*(p-2)/[p(2^*-2)]}\left[\frac{2pc_\mu}{\mu a_2^0(p-2)}\right]^{(p-2)/p}\|w_n\|_{\mu^\dagger}^2 \\ &\quad + o(1) \coloneqq C_1\left[\mu^{(N-2)/2}c_\mu\right]^{(p-2)/p}\|w_n\|_{\mu^\dagger}^2 + o(1), \end{aligned} \tag{33}$$

$$\begin{aligned} \mu\|v_n\|_q^q &= \mu\|u_n\|_q^2\|v_n\|_q^{q-2} \leq \mu\left(\mu b_1^0\right)^{-2(2^*-q)/[q(2^*-2)]} \\ &\quad \cdot (\eta_0\eta_{2^*})^{22^*(q-2)/[q(2^*-2)]}\left[\frac{2qc_\mu}{\mu b_2^0(q-2)}\right]^{(q-2)/q}\|w_n\|_{\mu^\dagger}^2 \\ &\quad + o(1) \coloneqq C_2\left[\mu^{(N-2)/2}c_\mu\right]^{(q-2)/q}\|w_n\|_{\mu^\dagger}^2 + o(1). \end{aligned} \tag{34}$$

From (14) and (32), we have

$$\begin{aligned} \mu\|u_n\|_{2^*}^{2^*} &= \mu\|u_n\|_{2^*}^2\|u_n\|_{2^*}^{2^*-2} \leq \mu(\eta_0\eta_2)^2\left[\frac{22^*c_\mu}{(2^*-2)\mu a_3^0}\right]^{(2^*-2)/2^*} \\ &\quad \cdot \|w_n\|_{\mu^\dagger}^2 + o(1) = (\eta_0\eta_2)^2\left[\frac{22^*c_\mu}{(2^*-2)a_3^0}\right]^{(2^*-2)/2^*} \\ &\quad \cdot \left[\mu^{(N-2)/N}c_\mu\right]^{2/N}\|w_n\|_{\mu^\dagger}^2 + o(1) \\ &\coloneqq C_3\left[\mu^{(N-2)/N}c_\mu\right]^{2/N}\|w_n\|_{\mu^\dagger}^2 + o(1), \end{aligned} \tag{35}$$

$$\begin{aligned} \mu\|v_n\|_{2^*}^{2^*} &= \mu\|v_n\|_{2^*}^2\|v_n\|_{2^*}^{2^*-2} \leq \mu(\eta_0\eta_2)^2\left[\frac{22^*c_\mu}{(2^*-2)\mu b_3^0}\right]^{\frac{2^*-2}{2^*}} \\ &\quad \cdot \|w_n\|_{\mu^\dagger}^2 + o(1) = (\eta_0\eta_2)^2\left[\frac{22^*c_\mu}{(2^*-2)b_3^0}\right]^{\frac{2^*-2}{2^*}} \\ &\quad \cdot \left[\mu^{\frac{N-2}{N}}c_\mu\right]^{\frac{2}{N}}\|w_n\|_{\mu^\dagger}^2 + o(1) \coloneqq C_4\left[\mu^{\frac{N-2}{N}}c_\mu\right]^{\frac{2}{N}}\|w_n\|_{\mu^\dagger}^2 + o(1). \end{aligned} \tag{36}$$

It follows from (6), (16), (20), (33), (34), (35), and (36) that

$$\begin{aligned} o(1) &= \left\langle S'_\mu(w_n), w_n \right\rangle = \|w_n\|_{\mu^\dagger}^2 - \mu\int_{\mathbb{R}^N} a_2(x)|u_n|^p dx \\ &\quad - \mu\int_{\mathbb{R}^N} b_2(x)|v_n|^q dx - \mu\int_{\mathbb{R}^N} a_3(x)|u_n|^{2^*} dx \\ &\quad - \mu\int_{\mathbb{R}^N} b_3(x)|v_n|^{2^*} dx - 2\mu\int_{\mathbb{R}^N} c(x)u_nv_n dx \\ &\geq (1-\vartheta)\|w_n\|_{\mu^\dagger}^2 - \left\{C_1a_2^1\left[\mu^{(N-2)/2}c_\mu\right]^{(p-2)/p}\right. \\ &\quad + C_2b_2^1\left[\mu^{(N-2)/2}c_\mu\right]^{(q-2)/q} + C_3a_3^1\left[\mu^{(N-2)/N}c_\mu\right]^{2/N} \\ &\quad \left. + C_4b_3^1\left[\mu^{(N-2)/N}c_\mu\right]^{2/N}\right\}\|w_n\|_{\mu^\dagger}^2 + o(1) \\ &\geq (1-\vartheta)\|w_n\|_{\mu^\dagger}^2 - \left[C_1a_2^1\alpha^{(p-2)/p} + C_2b_2^1\beta^{(q-2)/q}\right. \\ &\quad \left. + C_3a_3^1\alpha^{2/N} + C_4b_3^1\beta^{2/N}\right]\|w_n\|_{\mu^\dagger}^2 + o(1) \\ &\geq \frac{1-\vartheta}{2}\|w_n\|_{\mu^\dagger}^2 + o(1). \end{aligned} \tag{37}$$

Hence, we obtain

$$\lim_{n\to\infty}\|w_n\|_{\mu^\dagger}^2 = 0. \tag{38}$$

From (15), (23), and (38), we have

$$0 < c_\mu = \lim_{n\to\infty} S_\mu(w_n) \leq \frac{1}{2}\|w_n\|_{\mu^\dagger}^2 = 0, \tag{39}$$

a contradiction, which implies that $w_\mu \neq (0,0)$. We can easily check that $S'_\mu(w_n) = 0$ and $S_\mu(w_n) \leq c_\mu$ by a standard argument. Hence, w_μ is a nontrivial solution for (9).

It is easy to see that Theorem 2 is a direct consequence of Theorem 1.

Data Availability

No data were used to support this study.

Conflicts of Interest

No potential conflict of interest was reported by the authors.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (No. 11961014 and No. 61563013) and Guangxi Natural Science Foundation (2016GXNSFAA380082, 2018GXNSFBA281019, and 2018GXNSFAA281021).

References

[1] N. Akhmediev and A. Ankiewicz, "Novel soliton states and bifurcation phenomena in nonlinear fiber couplers," *Physical Review Letters*, vol. 70, no. 16, pp. 2395–2398, 1993.

[2] Y. Y. Li, "On a singularly perturbed elliptic equation," *Advances in Differential Equations*, vol. 2, pp. 955–980, 1997.

[3] A. Ambrosetti, A. Malchiodi, and W. M. Ni, "Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, part I," *Communications in Mathematical Physics*, vol. 235, no. 3, pp. 427–466, 2003.

[4] A. Ambrosetti, A. Malchiodi, and W. M. Ni, "Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, part II," *Indiana University Mathematics Journal*, vol. 53, no. 2, pp. 297–330, 2004.

[5] A. Pomponio, "Coupled nonlinear Schrödinger systems with potentials," *Journal of Differential Equations*, vol. 227, no. 1, pp. 258–281, 2006.

[6] Y. H. Ding and F. H. Lin, "Solutions of perturbed Schrödinger equations with critical nonlinearity," *Calculus of Variations and Partial Differential Equations*, vol. 30, no. 2, pp. 231–249, 2007.

[7] Y. H. Ding and J. C. Wei, "Semiclassical states for nonlinear Schrodinger equations with sign-changing potentials," *Journal of Functional Analysis*, vol. 251, no. 2, pp. 546–572, 2007.

[8] W. N. Huang and X. H. Tang, "Semi-classical solutions for the nonlinear Schrödinger-Maxwell equations," *Journal of Mathematical Analysis and Applications*, vol. 415, no. 2, pp. 791–802, 2014.

[9] Z. J. Chen and W. M. Zou, "On coupled systems of Schrödinger equations," *Advances in Differential Equations*, vol. 16, pp. 755–800, 2011.

[10] Z. J. Chen and W. M. Zou, "Ground states for a system of Schrödinger equations with critical exponent," *Journal of Functional Analysis*, vol. 262, no. 7, pp. 3091–3107, 2012.

[11] Z. J. Chen and W. M. Zou, "Standing waves for linearly coupled Schrodinger equations with critical exponent," *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, vol. 31, no. 3, pp. 429–447, 2014.

[12] J. Y. Peng, S. T. Chen, and X. H. Tang, "Semiclassical solutions for linearly coupled Schrödinger equations without compactness," *Complex Variables and Elliptic Equations*, vol. 64, pp. 548–556, 2018.

[13] X. H. Tang, "New super-quadratic conditions for asymptotically periodic Schrödinger equations," *Canadian Mathematical Bulletin*, vol. 60, no. 2, pp. 422–435, 2017.

[14] X. H. Tang, X. Y. Lin, and J. S. Yu, "Nontrivial solutions for Schrödinger equation with local super-quadratic conditions," *Journal of Dynamics and Differential Equations*, vol. 31, no. 1, pp. 369–383, 2019.

[15] X. H. Tang, "Non-Nehari manifold method for superlinear Schrödinger equation," *Taiwanese Journal of Mathematics*, vol. 18, no. 6, pp. 1957–1979, 2014.

[16] X. H. Tang, "Non-Nehari manifold method for asymptotically linear Schrödinger equation," *Journal of the Australian Mathematical Society*, vol. 98, no. 1, pp. 104–116, 2015.

[17] B. Sirakov, "Standing wave solutions of the nonlinear Schrödinger equations in $\mathbb{R}^N$," *Annali di Matematica Pura ed Applicata*, vol. 183, no. 4, pp. 73–83, 2002.

[18] G. F. Che and H. B. Chen, "Existence of multiple nontrivial solutions for a class of quasilinear Schrödinger equations on $\mathbb{R}^N$," *Bulletin of the Belgian Mathematical Society - Simon Stevin*, vol. 25, no. 1, pp. 39–53, 2018.

[19] X. Y. Lin, Y. B. He, and X. H. Tang, "Existence and asymptotic behavior of ground state solutions for asymptotically linear Schrödinger equation with inverse square potential," *Communications on Pure and Applied Analysis*, vol. 18, no. 3, pp. 1547–1565, 2019.

[20] J. H. Chen, X. H. Tang, and B. T. Cheng, "Existence of ground state solutions for a class of quasilinear Schrödinger equations with general critical nonlinearity," *Communications on Pure and Applied Analysis*, vol. 18, no. 1, pp. 493–517, 2019.

[21] D. D. Din, X. H. Tang, and Q. F. Wu, "Ground states of nonlinear Schrödinger systems with periodic or non-periodic potentials," *Communications on Pure and Applied Analysis*, vol. 18, no. 3, pp. 1261–1280, 2019.

[22] X. Y. Lin and X. H. Tang, "Semiclassical solutions of perturbed p-Laplacian equations with critical nonlinearity," *Journal of Mathematical Analysis and Applications*, vol. 413, no. 1, pp. 438–449, 2014.

[23] L. B. Wang, X. Y. Zhang, and H. Fang, "Existence of ground state solutions for a class of quasilinear elliptic systems in Orlicz-Sobolev spaces," *Boundary Value Problems*, vol. 2017, no. 1, p. 106, 2017.

[24] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.