Research Article

Periodic Coulomb Dynamics of Three Equal Negative Charges in the Field of Equal Positive Charges Fixed in Octagon Vertices

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Periodic solutions of the Coulomb equation motion for three equal negative point charges in the field of six equal positive point charges fixed in vertices of a convex octagon are found. The system possesses an equilibrium configuration. The Lyapunov center theorem is applied.

1. Introduction

In this paper, we find an equilibrium in the Coulomb system of three equal negative point charges in the field of six equal positive point charges fixed in vertices of a convex symmetric octagon. This gives us a possibility to find periodic solutions for one-dimensional (line), two-dimensional (planar), and three-dimensional (spacial) systems. Earlier, we found periodic and quasiperiodic solutions for the system of two and three negative charges in the field of two equal positive charges [1–4].

To obtain these results, we found explicit expressions for eigenvalues of the matrix U^0 of second partial derivatives of the potential energy at the equilibrium of the systems. The existence of the periodic solutions followed from the Lyapunov center theorem [5–9] whenever there is no zero eigenvalue among the eigenvalues. The existence of the quasiperiodic solutions was proved with the help of the procedure of elimination of a node [8] and an application of the Lyapunov center theorem taking into account that zero eigenvalue of U^0 results from a rotation invariance of the potential energy. U^0 turns out to be the direct sum of two and three three-dimensional matrices for the considered here plane and spacial systems, respectively. Their simple structure permits to find their eigenvalues. U^0 of the considered here space system does not possess zero eigenvalue, and the system potential energy does not have a rotation invariance.

To find periodic solutions in Coulomb systems, it is not necessary to exploit the existence of their equilibria. In [10], we found the solutions for the system of n equal negative charges in the field of n equal positive charges fixed on a line (a coordinate axis). Our technique was inspired by the Siegel advanced majorant technique [11] which permits finding solutions of the Newton equation for three gravitating bodies.

The Lyapunov center theorem concerns periodic solutions of the Hamiltonian systems with an equilibrium in the origin, which belong to its neighborhood, and is formulated precisely as follows.

Theorem 1. *Let an n-dimensional Hamiltonian system have real analytic Hamiltonian whose Taylor power expansion converges absolutely and uniformly at a neighborhood of the origin and begins from quadratic terms. Let also $\lambda_1, \cdots, \lambda_{2n}$ be nonzero eigenvalues of the matrix determining the linear term of the Hamiltonian vector field such that the following nonresonance relation hold for purely imaginary $\lambda_s, s = 1, \cdots, k: \lambda_j \neq n'\lambda_s, s \neq j = 1, \cdots, 2n$ for an arbitrary integer n'. Then, the Hamiltonian equation possesses k periodic solutions in a neighborhood of the origin such that each of them depends on a different real-valued parameter c_j for some $j = 1, \cdots, k$. These solutions and their periods $\tau_1(c_1), \cdots, \tau_k(c_k)$ are real analytic functions in the parameters in a neighborhood of the origin and $\tau_j(0) = 2\pi/|\lambda_j|$.*

The periodic solutions from this theorem take values in a neighborhood of the origin due to the fact that their expansion in the parameters c_j does not contain a constant term.

The equation of motion of mechanical systems of N d-dimensional particles (bodies, charges) with masses m_j and the potential energy U looks like

$$m_j \frac{d^2 x_j}{dt^2} = -\frac{\partial U\left(x_{(N)}\right)}{\partial x_j}, j = 1, \cdots, N, x_{(N)}$$
$$= (x_1, \cdots, x_N) \in \mathbb{R}^{dN}, x_j = \left(x_j^1, \cdots, x_j^d\right). \tag{1}$$

If a potential energy possess an equilibrium $x_j^0, j = 1, \cdots, N$, then the potential energy with the new variables $x_j - x_j^0$, $j = 1, \cdots, N$ will have the equilibrium at the origin, and it is possible to apply the Lyapunov center theorem to (1).

The Coulomb potential energy U is expressed through the pair Coulomb potential. It is well known [12] that for (1) with $m_j = m$, the eigenvalues from Theorem 1 coincide with $\lambda_j = \pm\sqrt{-m^{-1}\sigma_j}$, where $\sigma_j, j = 1, \cdots, Nd$ are eigenvalues of U^0.

We believe that the obtained results may be useful for the semiclassical and Born-Oppenheimer approximations for quantum models of ionized molecules. The fixed positive charges and the equal positive and equal negative charges are associated with heavy nuclei and light electrons, respectively.

Our paper is organized as follows. In Sections 2, 3, and 4, we formulate our results concerning periodic solutions of the considered Coulomb equations of motion for the line, planar, and spacial systems, respectively. They are formulated in the theorems in the end of the sections.

2. Line Coulomb Dynamics

We consider the line dynamics of three point equal negative charges $-e_0$ in the field of six equal positive point charges with the same value $e' > 0$ fixed in octagon vertices with the first coordinates $-a, 0, a$ and second coordinates $\pm b, \pm \sqrt{3a^2 + b^2}, \pm b$, respectively (see the next section). The three negative point charges move along the first axis which is an invariant manifold of the planar and spacial dynamics.

The potential energy for the system is given by

$$U\left(x_{(3)}\right) = \frac{1}{2} \sum_{j\neq k=1}^{3} \frac{e_j e_k}{|x_j - x_k|} - 2e_0 e' \sum_{j=1}^{3} \left[\left(\sqrt{(x_j - a)^2 + b^2} \right)^{-1} \right.$$
$$+ \left(\sqrt{(x_j + a)^2 + b^2} \right)^{-1}$$
$$\left. + \left(\sqrt{x_j^2 + 3a^2 + b^2} \right)^{-1} \right], x_j \in \mathbb{R}, a, b > 0. \tag{2}$$

The equilibrium equations are given by $(\partial/\partial x_j) U(x_{(3)}) = 0, j = 1, 2, 3..$ Let us insert the equalities into it for $k = 1$:

$$\frac{\partial}{\partial x_1} |x_1 - x_2|^{-k} = -k \frac{x_1 - x_2}{|x_1 - x_2|^{k+2}},$$
$$\frac{\partial}{\partial x_1} \left(\sqrt{(x_1 - a)^2 + b^2} \right)^{-k} = -k \frac{x_1 - a}{\left(\sqrt{x_1^2 + b^2} \right)^{k+2}}. \tag{3}$$

That is,

$$\frac{\partial}{\partial x_j} U\left(x_{(3)}\right) = -e_0^2 \sum_{j=k, k=1}^{3} \frac{x_j - x_k}{|x_j - x_k|^3} + 2e_0 e' \left[\frac{x_j - a}{\left(\sqrt{(x_j - a)^2 + b^2} \right)^3} \right.$$
$$\left. + \frac{x_j + a}{\left(\sqrt{(x_j + a)^2 + b^2} \right)^3} + \frac{x_j}{\left(\sqrt{x_j^2 + 3a^2 + b^2} \right)^3} \right]. \tag{4}$$

As a result, we obtain the equilibrium relation for $j = 1, 3$ putting $x_1 = x_1^0 = -a, x_2 = x_2^0 = 0, x_3 = x_3^0 = a$:

$$\frac{e_0^2}{(2a)^2} + \frac{e_0^2}{a^2} = \frac{3e'(2a)e_0}{\left(\sqrt{(2a)^2 + b^2} \right)^3},$$
$$\frac{5e_0}{(2a)^3} = \frac{3e'}{\left(\sqrt{(2a)^2 + b^2} \right)^3}. \tag{5}$$

Equality $(\partial/\partial x_2) U(x_{(3)}) = 0$ is true since $|x_2^0 - x_1^0| = |x_3^0 - x_2^0|$.

The most important information is a spectrum of the matrix of second derivatives of U. Its nondiagonal elements are easily calculated as

$$\frac{\partial U\left(x_{(3)}\right)}{\partial x_l \partial x_j} = \frac{\partial U\left(x_{(3)}\right)}{\partial x_j \partial x_l} = -2e_0^2 |x_j - x_l|^{-3}. \tag{6}$$

Let $U_{j,l}^0$ be this function at the equilibrium. Then,

$$U_{1,3}^0 = U_{3,1}^0 = -\frac{e_0^2}{4a^3} = -u',$$
$$U_{1,2}^0 = U_{2,1}^0 = U_{2,3}^0 = U_{3,2}^0 = -8u'. \tag{7}$$

Further,

$$\frac{\partial^2}{\partial x_j^2} U\left(x_{(2)}\right) = \sum_{j=k,k=1}^{3} \frac{2e_0^2}{|x_j - x_k|^3} + 2e_0 e' \left[\frac{1}{\left(\sqrt{(x_j - a)^2 + b^2}\right)^3} \right.$$

$$- \frac{3(x_j - a)^2}{\left(\sqrt{(x_j - a)^2 + b^2}\right)^5} + \frac{1}{\left(\sqrt{(x_j + a)^2 + b^2}\right)^3}$$

$$- \frac{3(x_j + a)^2}{\left(\sqrt{(x_j + a)^2 + b^2}\right)^5} + \frac{1}{\left(\sqrt{x_j^2 + 3a^2 + b^2}\right)^3}$$

$$\left. - \frac{3x_j^2}{\left(\sqrt{x_j^2 + 3a^2 + b^2}\right)^5} \right].$$

$$(8)$$

Let $U_{j,j}^0$ be this function at the equilibrium. Then,

$$U_{1,1}^0 = U_{3,3}^0 = \frac{9e_0^2}{4a^3}$$

$$+ 2e_0 e' \left[b^{-3} + \frac{2}{\left(\sqrt{(2a)^2 + b^2}\right)^3} - \frac{15a^2}{\left(\sqrt{(2a)^2 + b^2}\right)^5} \right],$$

$$U_{2,2}^0 = \frac{16e_0^2}{4a^3}$$

$$+ 2e_0 e' \left[\frac{2}{\left(\sqrt{a^2 + b^2}\right)^3} - \frac{6a^2}{\left(\sqrt{a^2 + b^2}\right)^5} + \frac{1}{\left(\sqrt{3a^2 + b^2}\right)^3} \right].$$

$$(9)$$

From the equilibrium relation, it follows that

$$\left(\frac{5e_0}{3e'}\right)^{1/3} \frac{1}{2a} = \frac{1}{\sqrt{(2a)^2 + b^2}}, \qquad (10)$$

As a result,

$$a^2 + b^2 = a^2 \left[4\left(\eta^{-1} - 1\right) + 1\right] = a^2 \eta^{-1}(4 - 3\eta),$$

$$3a^2 + b^2 = a^2 \left[4\left(\eta^{-1} - 1\right) + 3\right] = a^2 \eta^{-1}(4 - \eta),$$

$$2e_0 e' b^{-3} = 5\frac{2}{3}(2a)^{-3} e_0^2 (1 - \eta)^{-3/2} = \frac{5u'}{3}(1 - \eta)^{-3/2},$$

$$\frac{4e_0 e'}{\left(\sqrt{(2a)^2 + b^2}\right)^3} = 4e_0 e' \frac{5e_0}{3e'}\left(\frac{1}{2a}\right)^3 = \frac{10u'}{3},$$

$$\frac{30e_0 e' a^2}{\left(\sqrt{(2a)^2 + b^2}\right)^5} = 4^{-1} 30 e_0 e' \left(\frac{5e_0}{3e'}\right)^{5/3} (2a)^{-3} = \frac{25u'}{4}\eta,$$

$$\frac{4e_0 e'}{\left(\sqrt{a^2 + b^2}\right)^3} = 4e_0 e' a^{-3} \eta^{3/2}(4 - 3\eta)^{-3/2} = \frac{80}{3}u'(4 - 3\eta)^{-3/2},$$

$$\frac{12e_0 e' a^2}{\left(\sqrt{a^2 + b^2}\right)^5} = 12 e_0 e' a^{-3} \eta^{5/2}(4 - 3\eta)^{-5/2} = 80u'\eta(4 - 3\eta)^{-5/2},$$

$$\frac{2e_0 e'}{\left(\sqrt{3a^2 + b^2}\right)^3} = 2e_0 e' a^{-3} \eta^{3/2}(4 - \eta)^{-3/2} = \frac{40}{3}u'(4 - \eta)^{-3/2}.$$

$$(11)$$

These equalities allow to represent $u'^{-1} U_{j,k}^0$ as simple functions of η.

$$U_{1,1}^0 = U_{3,3}^0 = u' v, \quad v = \frac{37}{3} + \frac{5}{3}(1 - \eta)^{-3/2} - \frac{25}{4}\eta,$$

$$U_{2,2} = u' g = u' \left[\left(16 + \frac{80}{3}\right)(4 - 3\eta)^{-3/2} - 80\eta(4 - 3\eta)^{-5/2} \right.$$

$$+ \frac{40}{3}\eta(4 - \eta)^{-3/2} = u' \left[\left(16 + \frac{80}{3}\right)(4 - 3\eta)^{-5/2}(4 - 3\eta \right.$$

$$- 3\eta) + \frac{40}{3}\eta(4 - \eta)^{-3/2} \right], g = 3^{-1} 8 \left[6 + 20(2 \right.$$

$$\left. - 3\eta)(4 - 3\eta)^{-5/2} + 5\eta(4 - \eta)^{-3/2} \right].$$

$$(12)$$

Let U^0 be the matrix with the elements $U_{j,l}^0$, $j, l = 1, 2, 3$.

$$U^0 = u' U'_1, \quad U'_1 = \begin{pmatrix} v & -8 & -1 \\ -8 & g & -8 \\ -1 & -8 & v \end{pmatrix} = -2U_{*1} + (v + 1)I,$$

$$U_*(g_1) = U_{*1} = 2^{-1} \begin{pmatrix} 1 & 8 & 1 \\ 8 & 2g_1 & 8 \\ 1 & 8 & 1 \end{pmatrix}, \quad 2g_1 = -g + v + 1,$$

$$(13)$$

where I is unity matrix. U_{*1} has identical first and third rows and this means that $Det U_{*1} = 0$. This allows one to find roots of the characteristic polynomials p_{*1} of U_{*1} and p'_1 of U'_1:

$$p_*(\lambda, q) = Det(\lambda I - U_*(q)) = \left[\lambda^2 - (q + 1)\lambda + q - 32\right]\lambda$$

$$(14)$$

Here, we subtracted the third row of $-U_*(q) + \lambda I$ from the first row. The determinant does not change after

the subtraction.

$$\begin{pmatrix} \lambda - 2^{-1} & -4 & 2^{-1} \\ -4 & \lambda - q & -4 \\ -2^{-1} & -4 & \lambda - 2^{-1} \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda & 0 & -\lambda \\ -4 & \lambda - q & -4 \\ -2^{-1} & -4 & \lambda - 2^{-1} \end{pmatrix}. \tag{15}$$

After that, we decomposed the determinant in the elements of the first row:

$$\begin{aligned} p_*(\lambda, q) &= \lambda \left[(\lambda - q)(\lambda - 2^{-1}) - 16 - 16 - 2^{-1}(\lambda - q) \right] \\ &= \lambda [(\lambda - q)(\lambda - 1) - 32]. \end{aligned} \tag{16}$$

The roots of $p_*(q)$ are given by

$$2\lambda = q + 1 \pm \sqrt{(q-1)^2 + 128}, \lambda = 0. \tag{17}$$

The roots p_1' of U_1' are given by

$$p_1'(\lambda) = -2^3 p_* \left(-\frac{\lambda}{2} + \frac{v+1}{2}, g_1 \right), \tag{18}$$

$$\lambda = v - g_1 \pm \sqrt{(g_1 - 1)^2 + 128}, \quad \lambda = v + 1 = \zeta_1'.$$

Let ζ_2', ζ_3' be the roots corresponding to the plus and minus before the sign of the square root, respectively:

$$\begin{aligned} \zeta_2' &= \frac{g+v-1}{2} + \sqrt{\left(\frac{g-v+1}{2}\right)^2 + 128}, \\ \zeta_3' &= \frac{g+v-1}{2} - \sqrt{\left(\frac{g-v+1}{2}\right)^2 + 128}. \end{aligned} \tag{19}$$

We shall always use $\eta < 1$. Let $0 < \eta \le 3^{-1}$, then

$$\begin{aligned} g &\ge 3^{-1} 8 \left(6 + \frac{20}{32} \right) = 16 + 3^{-1} 5 = 17\frac{2}{3}, \\ v &< \frac{37}{3} + \frac{5}{3} \left(\frac{27}{8}\right)^{1/2} < 12\frac{1}{3} + 3^{-1} 10 \left(\frac{27}{32}\right)^{1/2} < 15\frac{2}{3}. \end{aligned} \tag{20}$$

Here, we applied $5/3 < \sqrt{3} < 7/4$. This leads to $\zeta_2' > g > v + 2 > \zeta_1' + 1$ and there is no resonance in ζ_2. Here, we used also

$$\sqrt{\left(\frac{g-v+1}{2}\right)^2 + 128} \ge \frac{g-v+1}{2}. \tag{21}$$

If $\eta \ge 3^{-1} 2$, then

$$v + 5 > g, \tag{22}$$

since

$$\begin{aligned} v &> \frac{37}{3} + \frac{5}{3} 3\sqrt{3} - \frac{25}{4} > \frac{37}{3} + \frac{25}{3} - \frac{25}{4} > \frac{37}{3} + \frac{25}{12} > 14, \\ g &< 3^{-1} 8 \left[6 + 3^{-3/2} 5 \right] = 16 + \frac{8}{3} < 19. \end{aligned} \tag{23}$$

Here, we applied $5/3 < \sqrt{3} < 7/4$. From

$$\begin{aligned} v > g \longrightarrow \left(\frac{g-v+1}{2}\right)^2 &= 4^{-1}(v-g+5)^2 - 6(v-g+5) \\ &+ 4^{-1} 6^2 < 4^{-1}(v-g+5)^2 \\ &+ 9 \longrightarrow \sqrt{\left(\frac{g-v+1}{2}\right)^2} \\ &+ 128 < 2^{-1}(v-g+5) \\ &+ 12 \longrightarrow \zeta_2' < \zeta_1' + 13, \end{aligned} \tag{24}$$

it follows

$$v > \frac{10}{3} \longrightarrow \zeta_1' > \frac{13}{3} \longrightarrow \frac{\zeta_2'}{\zeta_1'} < 1 + \frac{13}{\zeta_1'} < 4. \tag{25}$$

We have to exclude the equality $\zeta_2' = \zeta_1'$ with the help of

$$\zeta_2' - \zeta_1' = \frac{g-v-3}{2} + \sqrt{\left(\frac{g-v+1}{2}\right)^2 + 128}. \tag{26}$$

Equality $\zeta_2' - \zeta_1' = 0$ leads to

$$\left(\frac{g-v-3}{2}\right)^2 = \left(\frac{g-v+1}{2}\right)^2 + 128 \longrightarrow v - g = 63. \tag{27}$$

We proved the following proposition.

Proposition 2. *There is no resonance in $\zeta_2' > 0$ if $0 < \eta \le 1/3$. Moreover if $v - g \ne 63$ and $\eta \ge 3^{-1} 2$, then there is no quadratic resonance in ζ_1', i.e., $\zeta_s' \zeta_1'^{-1} \ne k^2, s = 2, 3$, where k is an integer.*

Let's prove the next proposition.

Proposition 3. $\zeta_2' > 0$ *and there is no quadratic resonance in ζ_1 if $0 < \eta \le 2/3$.*

Proof. We shall prove $\zeta_2' \zeta_1'^{-1} < 4, \zeta_1' \ne \zeta_2'$ and take into account $\zeta_3' < \zeta_2'$. Let $0 < \eta \le 1/3$.

Then,

$$v > \frac{37}{3} + \frac{5}{3} - \frac{25}{12} = \frac{37}{3} - \frac{5}{12} > 11,$$

$$g < 3^{-1}8\left[6 + 3^{-5/2}40 + 3^{-5/2}5\right] = 16 + 3^{-7/2}320 + 3^{-7/2}40$$

$$= 16 + \frac{320}{45} + \frac{8}{9}$$

$$= 16 + \frac{64}{9} + \frac{8}{9} = 24.$$

(28)

Here, we used $5/3 < \sqrt{3}$. From $v - g + 13 > 0$ and

$$\left(\frac{g - v + 1}{2}\right)^2 = 4^{-1}(v - g + 13)^2 - 7(v - g + 13) + 49$$

$$< 4^{-1}(v - g + 13)^2 + 49$$

$$\longrightarrow \sqrt{\left(\frac{g - v + 1}{2}\right)^2 + 128} < 2^{-1}(v - g + 13)$$

$$+ 14 \rightarrow \zeta_2{}' < \zeta_1{}' + 19,$$

(29)

it follows

$$v > \frac{16}{3} \rightarrow \zeta_1{}' > \frac{19}{3} \rightarrow \frac{\zeta_2{}'}{\zeta_1{}'} < 1 + \frac{19}{\zeta_1{}'} < 4.$$

(30)

Let $1/3 \le \eta \le 2/3$. Then,

$$v > \frac{37}{3} + \frac{5}{3}\left(\frac{3}{2}\right)^{3/2} - \frac{25}{6} = \frac{49}{6} + \frac{5}{3}\left(\frac{\sqrt{27}}{\sqrt{8}}\right) > \frac{49}{6} + \frac{5}{3}\sqrt{3}$$

$$> \frac{49}{6} + \frac{25}{9} > \frac{49}{6} + \frac{8}{3} = 10\frac{5}{6},$$

$$g < 3^{-1}8\left[6 + \frac{20}{4\sqrt{2}} + \frac{10}{3}\left(\frac{3}{10}\right)^{3/2}\right]$$

$$= 3^{-1}8\left[6 + \frac{5}{\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{10}}\right] < \frac{8}{3}\left[6 + \frac{5}{\sqrt{2}} + \frac{1}{\sqrt{3}}\right]$$

$$< 3^{-1}8\left[6 + \frac{5}{\sqrt{2}} + \frac{3}{5}\right] < 16 + \frac{20}{3}\sqrt{2} + \frac{8}{5} = 27 + \frac{3}{5}, v$$ (31)

$$< \frac{37}{3} + \frac{5}{3}3^{3/2} - \frac{25}{12} < \frac{37}{3} + 5\sqrt{3} - 2$$

$$= 12\frac{1}{3} + \frac{15}{\sqrt{3}} < 15 + 9 + 24,$$

$$g > 3^{-1}8\left[6 + \frac{5}{3}\left(4 - 3^{-1}\right)^{-3/2}\right] = 16 + \frac{5}{9} > 16.$$

(32)

Here, we used $5/3 < \sqrt{3}, \sqrt{2} < 3/2$. That is, $\zeta_2{}' > 0$. From $v - g + 17 > 0$ and $(g - v + 1/2)^2 = 4^{-1}(v - g + 17)^2 - 18/2(v - g + 17) + (18)^2/4 < 4^{-1}(v - g + 17)^2 + 81 \longrightarrow$

$$\sqrt{(g - v + 1/2)^2 + 128} < 2^{-1}(v - \quad g + 17) + 15 \longrightarrow \zeta_2{}' < \zeta_1{}'$$

$+ 22$, it follows

$$v > \frac{19}{3} \longrightarrow \zeta_1{}' > \frac{22}{3} \longrightarrow \frac{\zeta_2{}'}{\zeta_1{}'} < 1 + \frac{22}{\zeta_1{}'} < 4.$$ (33)

We have also $\zeta_1 \ne \zeta_2$ since $v - g < 8$ for $1/3 \le \eta \le 2/3$ and $\zeta_2 > \zeta_1$, if $0 < \eta \le 1/3$.

Now, we have to estimate $\zeta_2{}'$ on the interval $2/3 \le \eta' < 1$. Obviously $g > 16$ if $\eta = 2/3$. Let $\eta = 6^{-1}5$, then

$$g = 3^{-1}8\left[6 + 20\left(4 - \frac{5}{2}\right)^{-5/2}\left(2 - \frac{5}{2}\right) + \frac{25}{6}\left(4 - 6^{-1}5\right)^{-3/2}\right]$$

$$> 3^{-1}8\left(6 - 10\frac{4\sqrt{2}}{9\sqrt{3}} + \frac{25}{6}\frac{6\sqrt{6}}{20\sqrt{20}}\right)$$

$$> 3^{-1}8\left(6 - 5\frac{\sqrt{2}}{\sqrt{3}} + \frac{5\sqrt{6}}{4\sqrt{20}}\right) > 3^{-1}8\left(6 - 3\sqrt{2} + \frac{5}{8}\right)$$

$$> 3^{-1}8\left(6 - \frac{9}{2} + \frac{5}{8}\right) > 16 - 12 + 1$$

$$= 5, v = \frac{37}{3} + \frac{5}{3}6\sqrt{6} - \frac{125}{24} > \frac{37}{3} + \frac{200}{9} - \frac{125}{24}$$

$$> \frac{37}{3} + 22 - 5\left(1 + \frac{1}{24}\right) > 29.$$

(34)

Let $\eta = 9^{-1}8$, then

$$g = 3^{-1}8\left(6 + 20\left(4 - \frac{8}{3}\right)^{-5/2}\left(4 - \frac{8}{3}\right)^{-5/2}\left(\left(2 - \frac{8}{3}\right)2 - \frac{8}{3}\right)\right.$$

$$\left. + \frac{40}{9}\frac{9\sqrt{9}}{28\sqrt{28}}\right) < 3^{-1}8\left(6 - \frac{40}{3}\frac{9\sqrt{3}}{32} + \frac{40}{9}\frac{1}{3\sqrt{3}}\right)$$

$$< 3^{-1}8\left(6 - 5\frac{3\sqrt{3}}{4} + \frac{8}{9}\right) < 3^{-1}8\left(6 - \frac{25}{4} + \frac{8}{9}\right)$$

$$= -\frac{2}{3} + 3^{-1}8\frac{8}{9} < 2, g > 3^{-1}8\left(6 - 5\frac{3\sqrt{3}}{4} + \frac{40}{9}\frac{1}{4\sqrt{4}}\right)$$

$$> 3^{-1}8\left(6 - 5\frac{21}{16} + \frac{5}{9}\right) > 3^{-1}8\left(6 - 7 + \frac{5}{9}\right)$$

$$= -\frac{32}{27} > -2, v = \frac{37}{3} + \frac{5}{3}9\sqrt{9} - \frac{50}{9} > \frac{37}{3} + 45 - \frac{54}{9} > 51.$$

(35)

Here, we used $4/3 < \sqrt{2} < 3/2, 7/4 > \sqrt{3} > 5/3$. From $v > 8$,

$$g > 3^{-1}8(6 - 20) > 3^{-1}8(-15) > -40.$$ (36)

and these inequalities follows that $v + g - 1 > 0$ and $\zeta'_2 > 0$ since v and g are monotonically increasing and decreasing functions, respectively, on the interval $2/3 \le \eta < 1$. Let us

prove this, i.e., that the derivatives $\partial v, -\partial g$ are positive on the interval.

$$\partial v = \frac{5}{2}(1-\eta)^{-5/2} - \frac{25}{4} > \frac{5}{2}9\sqrt{3} - \frac{25}{4} > 3\frac{25}{2} - \frac{25}{4} > 0, \partial g$$

$$= 3^{-1}8\left[150(2-3\eta)(4-3\eta)^{-7/2} + \frac{15}{2}\eta(4-\eta)^{-5/2}\right.$$

$$\left. - 60(4-3\eta)^{-5/2}\right] < 3^{-1}8\left[\frac{15}{2}\left(9\sqrt{3}\right)^{-1} - \frac{60}{\sqrt{32}}\right]$$

$$< 3^{-1}8\left[\frac{1}{2} - \frac{60}{\sqrt{32}}\right] < 0.$$

(37)

Taking into account Proposition 2, we see that $\zeta'_2 > 0$. Thus, we proved the proposition.

To apply the Lyapunov center theorem, we have to guarantee that $\zeta'_3 \neq 0$. Taking to the second power both terms in the expression for ζ'_3, we see that this condition is satisfied if $g(v-1) \neq 128$. This condition is true if $0 < \eta \leq 3^{-1}2$ since $v > 9$ and $g > 16$.

The order of charges is preserved due to the infinite repulsion and we can substitute the holomorphic functions $(x_j - x_k)^{-1}$ instead of $|x_j - x_k|^{-1}$ in the expression for their potential energies.

Since the eigenvalues of U^0 coincide with $\zeta_j = u'\zeta'_j$, the following theorem follows from the Lyapunov center theorem [5–7].

Theorem 4. *If $0 < \eta \leq 1/3$, then the line Coulomb equation of motion (1) with $m_j = m, d = 1,$ and $N = 3$ and the potential energy (2) possesses the equilibrium $x_1^0 = -a, x_2^0 = 0, x_3^0 = a,$ $j = 1, 2, 3,$ and two periodic solutions in its neighborhood such that each of them depends on its own real parameter c_j for $j = 1, 2$. These solutions and their periods $\tau_j(c), j = 1, 2$ are real analytic functions in a neighborhood of the origin in these parameters and $\tau_j(0) = 2\pi\sqrt{m/\zeta_j}$. If $3^{-1} < \eta \leq 3^{-1}2$ or $3^{-1}2 < \eta < 1$ and $g(v-1) \neq 128, v - g \neq 63$; then, the equation possesses this equilibrium and a periodic solution in its neighborhood that depends on a real parameter c. This solution and its period $\tau(c)$ are real analytic functions in a neighborhood of the origin in this parameter and $\tau(0) = 2\pi\sqrt{m/\zeta_1}$.*

3. Planar Coulomb Dynamics

In this section, we consider the planar system of three equal charges: $-e_0$ in the field of six equal positive charges e' fixed at the octagon vertices with the coordinates $b_j, 1 \leq j \leq 6,$ $b_j = (b_j^1, b_j^2) \in \mathbb{R}^2$.

$$b_1 = (a, b), b_2 = (a, -b), b_3 = (-a, b), b_4 = (-a, -b), b_5$$
$$= \left(0, \sqrt{3a^2 + b^2}\right), b_6 = \left(0, -\sqrt{3a^2 + b^2}\right), a, b > 0,$$

(38)

with the potential energy

$$U\left(x_{(3)}\right) = \frac{1}{2}\sum_{j\neq k=1}^{3}\frac{e_j e_k}{|x_j - x_k|} - e_0 e'\sum_{j=1}^{3}\sum_{k=1}^{6}|x_j - b_k|^{-1}, \quad (39)$$

where

$$x_j = \left(x_j^1, x_j^2\right) \in \mathbb{R}^2, |x_j|^2 = \left(x_j^1\right)^2 + \left(x_j^2\right)^2, e_j = -e_0 < 0.$$

(40)

The equilibrium is given by $x_1^1 = x_1^{01} = -a, x_2^1 = x_2^{01} = 0,$ $x_3^1 = x_3^{01} = a, x_j^\alpha = x_j^{0\alpha} = 0, \alpha = 2$. The first partial derivatives of U look like

$$\frac{\partial}{\partial x_j^\alpha}U\left(x_{(3)}\right) = -e_0^2\sum_{k=1,k\neq j}^{3}\frac{x_j^\alpha - x_k^\alpha}{|x_j - x_k|^3} + e_0 e'\sum_{k=1}^{6}\frac{x_1^\alpha - b_k^\alpha}{|x_j - b_k|^3}.$$

(41)

This gives the equilibrium relation between $e_0, e', a,$ and b the same as in the previous section equating to zero the right-hand sides of these equalities for $j = 1, 3$ (the result is the same), taking into account $x_1^{01} - x_3^{01} = -2a,$

$$|x_3^0 - b_1|^2 = |x_3^0 - b_2|^2 = b^2, |x_3^0 - b_3|^2 = |x_3^0 - b_4|^2$$
$$= |x_3^0 - b_5|^2 = |x_3^0 - b_6|^2 = (2a)^2 + b^2, |x_1^0 - b_1|^2$$
$$= |x_1^0 - b_2|^2 = |x_1^0 - b_5|^2 = |x_1^0 - b_6|^2$$
$$= (2a)^2 + b^2, |x_1^0 - b_3|^2 = |x_1^0 - b_4|^2 = b^2, |x_2^0 - b_j|^2$$
$$= a^2 + b^2, j = 1, 2, 3, 4; |x_2^0 - b_k|^2 = 3a^2 + b^2, k = 5, 6$$

$$\sum_{k=1}^{6}\frac{x_1^{01} - b_k^1}{|x_1^0 - b_k|^3} = -\frac{6a}{\left(\sqrt{(2a)^2 + b^2}\right)^3},$$

$$\sum_{k=1}^{6}\frac{x_1^{02} - b_k^2}{|x_1^0 - b_k|^3} = -\sum_{k=1}^{6}\frac{b_k^2}{|x_1^0 - b_k|^3} = 0.$$

(42)

The right-hand of (41) is zero for $\alpha = 2$ and for $j = 2$ since $|x_2^0 - x_1^0| = |x_3^0 - x_2^0|, x_1^{01} = -x_3^{01}, x_2^0 = 0$.

The equilibrium relation is given by

$$\frac{e_0^2}{(2a)^2} + \frac{e_0^2}{a^2} = \frac{3e'(2a)e_0}{\left(\sqrt{(2a)^2 + b^2}\right)^3}, \frac{5e_0}{(2a)^3} = \frac{3e'}{\left(\sqrt{(2a)^2 + b^2}\right)^3}.$$

(43)

The second derivatives of the potential energy (39) are given by

$$\frac{\partial U\left(x_{(3)}\right)}{\partial x_j^\alpha \partial x_k^\beta} = \frac{\partial U\left(x_{(3)}\right)}{\partial x_k^\beta \partial x_j^\alpha}$$

$$= e_0^2 \left[\frac{\delta_{\alpha,\beta}}{\left|x_j - x_k\right|^3} - 3\frac{\left(x_j^\alpha - x_k^\alpha\right)\left(x_j^\beta - x_k^\beta\right)}{\left|x_j - x_k\right|^5}\right], \qquad (44)$$

$$\alpha, \beta = 1, 2,$$

$$\frac{\partial^2 U\left(x_{(3)}\right)}{\partial x_j^\beta \partial x_j^\alpha} = e_0^2 \sum_{k=1, k\neq j}^{3} \left[-\frac{\delta_{\alpha,\beta}}{\left|x_j - x_k\right|^3} + 3\frac{\left(x_j^\alpha - x_k^\alpha\right)\left(x_j^\beta - x_k^\beta\right)}{\left|x_j - x_k\right|^5}\right]$$

$$+ e_0 e' \sum_{k=1}^{6} \left[\frac{\delta_{\alpha,\beta}}{\left|x_j - b_k\right|^3} - 3\frac{\left(x_j^\alpha - b_k^\alpha\right)\left(x_j^\beta - b_k^\beta\right)}{\left|x_j - b_k\right|^5}\right]. \qquad (45)$$

Now, we shall find the equilibrium value for all the terms in these equalities. Let η, u' be as in the previous section. Then, we derive the following equalities:

$$e_0^2 \sum_{k=1, k\neq j}^{3} \frac{1}{\left|x_j^0 - x_k^0\right|^3} = e_0^2 \left((2a)^{-3} + a^{-3}\right)\left(1 - \delta_{j,2}\right) + 2a^{-3}\delta_{j,2}$$

$$= \frac{9u'}{2}\left(1 - \delta_{j,2}\right) + 8u'\delta_{j,2}, j = 1, 2, 3, \qquad (46)$$

$$e_0 e' \sum_{k=1}^{6} \frac{\delta_{\alpha,\beta}}{\left|x_j^0 - b_k\right|^3} = \delta_{\alpha,\beta} 2e_0 e'\left(b^{-3} + 2\left((2a)^2 + b^2\right)^{-3/2}\right)$$

$$= \frac{5}{3}\delta_{\alpha,\beta} u'\left[(1 - \eta)^{-3/2} + 2\right], j = 1, 3, \qquad (47)$$

$$e_0 e' \sum_{k=1}^{6} \frac{\delta_{\alpha,\beta}}{\left|x_2^0 - b_k\right|^3} = \delta_{\alpha,\beta} e_0 e'\left[4\left(a^2 + b^2\right)^{-3/2} + 2\left(3a^2 + b^2\right)^{-3/2}\right]$$

$$= \delta_{\alpha,\beta} \frac{40}{3} u'\left[2(4 - 3\eta)^{-3/2} + (4 - \eta)^{-3/2}\right] \qquad (48)$$

relying on equalities below (10) from the previous section. Let

$$T_j(\alpha, \beta) = \sum_{k=1}^{6} \frac{\left(x_j^\alpha - b_k^\alpha\right)\left(x_j^\beta - b_k^\beta\right)}{\left|x_j - b_k\right|^5}. \qquad (49)$$

Let also $T_j^0(\alpha, \beta)$ be the equilibrium value of $T_j(\alpha, \beta)$. Now, we will prove the equalities

$$T_j^0(\alpha, \beta) = \delta_{\alpha,\beta}\left[10a^2(2a)^2 + b^2\right)^{-5/2}\delta_{\alpha,1}$$

$$+ 2\left(b^{-3} + \left(3a^2 + 2b^2\right)\left((2a)^2 + b^2\right)^{-5/2}\right)\delta_{\alpha,2}], j = 1, 3, \qquad (50)$$

$$T_2^0(\alpha, \beta) = \delta_{\alpha,\beta}\left\{4a^2\left(a^2 + b^2\right)^{-5/2}\delta_{\alpha,1} + 2\left[2b^2\left(a^2 + b^2\right)^{-5/2}\right.\right.$$

$$\left.\left. + \left(3a^2 + b^2\right)^{-3/2}\right]\delta_{\alpha,2}\right\}. \qquad (51)$$

with the help of the following equalities:

$$T_3^0(1, 2) = -\left[b^{-5}\left(\left(a - b_1^1\right)b_1^2 + \left(a - b_2^1\right)b_2^2\right)\right.$$

$$+ \left((2a)^2 + b^2\right)^{-5/2}\left(\left(a - b_3^1\right)b_3^2 + \left(a - b_4^1\right)b_4^2\right)$$

$$\left. + \left(a - b_5^1\right)b_5^2 + \left(a - b_6^1\right)b_6^2\right)\right] = 0,$$

$$T_1^0(1, 2) = -\left[\left((2a)^2 + b^2\right)^{-5/2}\left(-a - b_1^1\right)b_1^2 + \left(-a - b_2^1\right)b_2^2\right.$$

$$+ \left(-a - b_5^1\right)b_5^2 + \left(-a - b_6^1\right)b_6^2\right) + b^{-5}\left(\left(-a - b_3^1\right)b_3^2\right.$$

$$\left.\left. + \left(-a - b_4^1\right)b_4^2\right)\right] = 0,$$

$$T_3^0(1, 1) = b^{-5}\left[\left(a - b_1^1\right)^2 + \left(a - b_2^1\right)^2\right] + \left((2a)^2 + b^2\right)^{-5/2}\left[\left(a - b_3^1\right)^2 + \left(a - b_4^1\right)^2 + \left(a - b_5^1\right)^2 + \left(a - b_6^1\right)^2\right]$$

$$= 10a^2\left((2a)^2 + b^2\right)^{-5/2},$$

$$T_1^0(1, 1) = \left((2a)^2 + b^2\right)^{-5/2}\left[\left(-a - b_1^1\right)^2 + \left(-a - b_2^1\right)^2\right.$$

$$+ \left(-a - b_5^1\right)^2 + \left(-a - b_6^1\right)^2] + b^{-5}\left[\left(-a - b_3^1\right)^2\right.$$

$$\left. + \left(-a - b_4^1\right)^2\right] = 10a^2\left((2a)^2 + b^2\right)^{-5/2},$$

$$T_j^0(2, 2) = \sum_{k=1}^{6} \frac{\left(b_k^2\right)^2}{\left|x_j^0 - b_k\right|^5}, j = 1, 3,$$

$$T_1^0(2, 2) = T_3^0(2, 2) = 2b^2\left[b^{-5} + \left((2a)^2 + b^2\right)^{-5/2}\right]$$

$$+ 2\left(3a^2 + b^2\right)\left((2a)^2 + b^2\right)^{-5/2}$$

$$= 2\left[b^{-3} + \left(3a^2 + 2b^2\right)\left((2a)^2 + b^2\right)^{-5/2},$$

$$T_2^0(\alpha, \beta) = \sum_{k=1}^{6} \frac{b_k^\alpha b_k^\beta}{\left|x_2^0 - b_k\right|^5} = \left(a^2 + b^2\right)^{-5/2}\sum_{k=1}^{4} b_k^\alpha b_k^\beta$$

$$+ \left(3a^2 + b^2\right)^{-5/2}\sum_{k=5}^{6} b_k^\alpha b_k^\beta = \delta_{\alpha,\beta}\left\{4a^2\left(a^2 + b^2\right)^{-5/2}\delta_{\alpha,1}\right.$$

$$\left. + \left[4b^2\left(a^2 + b^2\right)^{-5/2} + 2\left(3a^2 + b^2\right)^{-3/2}\right]\delta_{\alpha,2}\right\}. \qquad (52)$$

Here, we took into account

$$\sum_{k=1}^{2} b_k^1 b_k^2 = 0,$$

$$\sum_{k=3}^{4} b_k^1 b_k^2 = 0. \qquad (53)$$

(50) and (51) are proved. Taking into account

$$e_0^2 \sum_{k=1,k\neq j}^3 \frac{\left(x_j^{0\alpha} - x_k^{0\alpha}\right)\left(x_j^{0\beta} - x_k^{0\beta}\right)}{\left|x_j^0 - x_k^0\right|^5} = e_0^2\left(a^{-3} + (2a)^{-3}\right)\delta_{\alpha,\beta}\delta_{\alpha,1}$$
$$= \frac{9u'}{2}\delta_{\alpha,\beta}\delta_{\alpha,1}, j = 1, 3 \tag{54}$$

we derive from (47), (46), and (50)

$$U_{1,\alpha;1,\beta}^0 = U_{3,\alpha;3,\beta}^0 = u'\delta_{\alpha,\beta}\left(v' - \delta_{\alpha,1}u'_* - \delta_{\alpha,2}u''_*\right), \tag{55}$$

where

$$v' = \frac{1}{3}\left[5(1 - \eta)^{-3/2} - \frac{7}{2}\right], u'u'_* + \frac{27u'}{2}$$
$$= 6e_0e'5a^2\left((2a)^2 + b^2\right)^{-5/2}, u'u''_*$$
$$= 6e_0e'\left(b^{-3} + (3a^2 + b^2)\left((2a)^2 + b^2\right)^{-5/2}\right), u'_*$$
$$= -\frac{27}{2} + \frac{25}{4}\eta, u''_* = \frac{5u'}{4}\left[4 - \eta + 4(1 - \eta)^{-3/2}\right]. \tag{56}$$

Here, we took into account

$$6e_0e5a^2\left((2a)^2 + b^2\right)^{-5/2} = 6e_0e'5a^2\left(\frac{5e_0}{3e'}\right)^{5/3}\left(\frac{1}{2a}\right)^5$$
$$= \frac{25u'}{4}\eta, 6e_0e'b^{-3}$$
$$= 5u'(1 - \eta)^{-3/2}, 6e_0e'b^2\left((2a)^2 + b^2\right)^{-5/2}$$
$$= 6e_0e'(2a)^2(1 - \eta)\eta^{-1}\left(\frac{5e_0}{3e'}\right)^{5/3}\left(\frac{1}{2a}\right)^5$$
$$= 5u'(1 - \eta), \frac{18e_0e'a^2}{\left(\sqrt{(2a)^2 + b^2}\right)^5}$$
$$= 2^{-1}9e_0e'\left(\frac{5e_0}{3e'}\right)^{5/3}(2a)^{-3} = \frac{15}{4}u'\eta. \tag{57}$$

From the equality

$$e_0^2 \sum_{k=1,k\neq 2}^3 \frac{\left(x_2^{0\alpha} - x_k^{0\alpha}\right)\left(x_2^{0\beta} - x_k^{0\beta}\right)}{\left|x_2^0 - x_k^0\right|^5} = e_0^2 2a^{-3}\delta_{\alpha,\beta}\delta_{\alpha,1} = 8u'\delta_{\alpha,\beta}\delta_{\alpha,1}, \tag{58}$$

(48), (51), and (46), it follows

$$U_{2,\alpha;2,\beta}^0 = u'\delta_{\alpha,\beta}\left(v_1 - u_1\delta_{\alpha,1} - \delta_{\alpha,2}u_1'\right), \tag{59}$$

where

$$v_1 = -8u' + \frac{40}{3}u'\left[2(4 - 3\eta)^{-3/2} + (4 - \eta)^{-3/2}\right], u_1$$
$$= 3e_0e'4a^2\left(a^2 + b^2\right)^{-5/2} - 24u' = 80u'\eta(4 - 3\eta)^{-5/2}$$
$$- 24u', u_1' = 3e_0e'\left[4b^2\left(a^2 + b^2\right)^{-5/2} + 2\left(3a^2 + b^2\right)^{-3/2}\right]$$
$$= 40u'\left[8(1 - \eta)(4 - 3\eta)^{-5/2} + (4 - \eta)^{-3/2}\right]. \tag{60}$$

Here, we used

$$2a = (1 - \eta)^{-1/2}\sqrt{\eta}b, a^2 + b^2$$
$$= a^2\eta^{-1}(4 - 3\eta), e_0e'\left(a^2 + b^2\right)^{-5/2}a^2$$
$$= e_0e'a^{-3}\eta^{5/2}(4 - 3\eta)^{-5/2}$$
$$= \frac{20}{3}u'\eta(4 - 3\eta)^{-5/2}, e_0e'\left(a^2 + b^2\right)^{-5/2}b^2$$
$$= e_0e'a^{-5}\eta^{3/2}(1 - \eta)(4 - 3\eta)^{-5/2}(2a)^2$$
$$= \frac{80}{3}u'(1 - \eta)(4 - 3\eta)^{-5/2}, e_0e'\left(3a^2 + b^2\right)^{-3/2}$$
$$= e_0e'a^{-3}\eta^{3/2}(4 - \eta)^{-3/2} = \frac{20}{3}u'(4 - \eta)^{-3/2}. \tag{61}$$

We have also

$$e_0^2 \frac{\left(x_2^{0\alpha} - x_k^{0\alpha}\right)\left(x_2^{0\beta} - x_k^{0\beta}\right)}{\left|x_2^0 - x_k^0\right|^5} = e_0^2 \frac{x_k^{0\alpha}x_k^{0\beta}}{a^5} = e_0^2 a^{-3}\delta_{\alpha,\beta}\delta_{\alpha,1}$$
$$= 4u'\delta_{\alpha,\beta}\delta_{\alpha,1}, k$$
$$= 1, 3, e_0^2 \frac{\left(x_1^{0\alpha} - x_3^{0\alpha}\right)\left(x_1^{0\beta} - x_3^{0\beta}\right)}{\left|x_1^0 - x_3^0\right|^5}$$
$$= e_0^2(2a)^{-3}\delta_{\alpha,\beta}\delta_{\alpha,1} = \frac{u'}{2}\delta_{\alpha,\beta}\delta_{\alpha,1}. \tag{62}$$

From these two equalities, one derives

$$U_{1,\alpha;3,\beta}^0 = U_{3,\alpha;1,\beta}^0 = \frac{u'}{2}\delta_{\alpha,\beta}(1 - 3\delta_{\alpha,1}), U_{2,\alpha;3,\beta}^0 = U_{3,\alpha;2,\beta}^0$$
$$= U_{2,\alpha;1,\beta}^0 = U_{1,\alpha;2,\beta}^0 = 4u'\delta_{\alpha,\beta}(1 - 3\delta_{\alpha,1}). \tag{63}$$

Now, we determine two matrices $U_\alpha^0, \alpha = 1, 2$ by the rule

$$U_{j,\alpha;k,\beta}^0 = \delta_{\alpha,\beta}U_{\alpha;j,k}^0 \tag{64}$$

and renumerate indexes of coordinates in the following way:

$$(1, 1) = 1 ; (2, 1) = 2 ; (3, 1) = 3 ; (1, 2)$$
$$= 4 ; (2, 2) = 5 ; (3.2) = 6. \tag{65}$$

where the first and second indexes under the round brackets corresponds to the lower and upper indexes of coordinates. This gives

$$U^0 = U_1^0 \oplus U_2^0. \tag{66}$$

The elements of the symmetric matrices U_j^0 are defined as follows:

$$
\begin{aligned}
U_{1;1,1}^0 &= U_{1,1;1,1}^0 = U_{1;3,3}^0 = U_{3,1;3,1}^0 = u'\left(v' - u_*{}'\right), U_{1;2,2}^0 \\
&= U_{2,1;2,1}^0 = u'(v_1 - u_1), U_{1;2,1}^0 = U_{1;1,2}^0 = U_{1,1;2,1}^0 \\
&= -8u', U_{1;3,1}^0 = U_{1;1,3}^0 = U_{1,1;3,1}^0 = -u', U_{1;3,2}^0 = U_{1;2,3}^0 \\
&= U_{1,1;3,1}^0 = -8u', U_{2;1,1}^0 = U_{1,2;1,2}^0 = U_{2;3,3}^0 = U_{3,2;3,2}^0 \\
&= u'\left(v' - u''_*\right) = -u'u'' + \frac{u'}{2}, U_{2;2,2}^0 = U_{2,2;2,2}^0 \\
&= u'\left(v_1 - u'_1\right) = u'g', U_{2;2,1}^0 = U_{2;1,2}^0 = U_{1,2;2,2}^0 \\
&= 4u', U_{2;3,1}^0 = U_{2;1,3}^0 = U_{1,2;3,2}^0 = \frac{u'}{2}, U_{2;3,2}^0 = U_{2;2,3}^0 \\
&= U_{2,2;3,2}^0 = 4u'.
\end{aligned} \tag{67}
$$

The parameters of the matrix elements are defined as follows:

$$
u'\left(v' - u'_*\right) = \frac{5u'}{3}(1-\eta)^{-3/2} - \frac{7u'}{6} - \frac{25u'}{4}\eta + \frac{27u'}{2} = u'v,
$$

$$
u'(v_1 - u_1) = -8u' + 24u' + \frac{40}{3}u'\left[2(4-3\eta)^{-3/2} + (4-\eta)^{-3/2}\right] \\
- 80u'\eta(4-3\eta)^{-5/2} = u'g,
$$

$$
-u''u' = v'u' - u''_* - \frac{u'}{2} = u'\left[\frac{1}{3}5(1-\eta)^{-3/2} - \frac{7}{2}\right] \\
- \frac{5u'}{4}\left[4 - \eta + 4(1-\eta)^{-3/2}\right] - \frac{u'}{2},
$$

$$
u'' = 9 - \frac{5\eta}{4} + \frac{10}{3}(1-\eta)^{-3/2}, u'' > 11,
$$

$$
u'g' = u'\left(v_1 - u'_1\right),
$$

$$
g' = -8 + \frac{40}{3}\left[2(4-3\eta)^{-3/2} + (4-\eta)^{-3/2}\right] \\
- 40\left[8(1-\eta)(4-3\eta)^{-5/2} + (4-\eta)^{-3/2}\right] \\
= -8 - \frac{80}{3}(4-\eta)^{-3/2} + \frac{80}{3}(4-3\eta)^{-5/2}[4 - 3\eta \\
- 12(1-\eta)],
$$

$$
g' = -8 - \frac{80}{3}\left[(4-\eta)^{-3/2} + (4-3\eta)^{-5/2}(8 - 9\eta)\right], \tag{68}
$$

where v, g are the same as in the previous section. As a result,

$$
U_1^0 = u' \begin{pmatrix} v & -8 & -1 \\ -8 & g & -8 \\ -1 & -8 & v \end{pmatrix} = u'U'_1, U_1{}' \\
= -2U_*(g_1) + (v+1)I, 2g_1 = -g + v + 1,
$$

$$
U_2^0 = u'2^{-1} \begin{pmatrix} -2u'' + 1 & 8 & 1 \\ 8 & 2g' & 8 \\ 1 & 8 & -2u'' + 1 \end{pmatrix} = u'U'_2, U'_2 \\
= U_*(g_2) - u''I, g_2 = g' + u'',
$$
$$\tag{69}$$

where matrix U_* is defined in the previous section where we found its eigenvalues as the roots of $p_*(q)$. Now, it is not difficult to find eigenvalues of U'_2 as roots of polynomial p'_2. They are given by

$$
p'_2(\lambda) = p_*\left(\lambda + u'', g_2\right).2\lambda = g_2 - 2u'' \\
+ 1 \pm \sqrt{(g_2 - 1)^2 + 128}, \lambda = -u'' = \zeta'_4 < 0. \tag{70}
$$

Let ζ'_5, ζ'_6 be the roots corresponding to the plus and minus before the sign of the square root, respectively. Then,

$$
2\zeta'_5 = g' - u'' + 1 + \sqrt{\left(g' + u'' - 1\right)^2 + 128}, \\
2\zeta'_6 = g' - u'' + 1 - \sqrt{\left(g' + u'' - 1\right)^2 + 128}. \tag{71}
$$

If $0 < \eta \le 1/3$, then

$$
-g' > 8 + \frac{80}{24} + 5\frac{80}{96} > 8 + \frac{10}{3} + 4 > 15, |g'| > 15. \tag{72}
$$

$\zeta'_5 < 0$ if $g' < 0$ and

$$
\left(|g'| - u'' + 1\right)^2 + 128 < \left(|g'| + u'' - 1\right)^2, \left(u'' - 1\right)|g'| > 32. \tag{73}
$$

This is true if $0 < \eta \le 1/3$. In this case, $\zeta'_6 < \zeta'_5 < 0$. This means that there is no resonance in ζ_2 and quadratic resonance in ζ'_1 for the eigenvalues $\zeta_j, 1 \le j \le 6$ of U^0. This and the Lyapunov center theorem imply the following theorem.

Theorem 5. *If $0 < \eta \le 1/3$, then the planar Coulomb equation of motion (1) with $m_j = m, d = 2$, and $N = 3$ and the potential energy (39) possesses the equilibrium $x_1^{01} = -a, x_2^{01} = 0, x_3^{01} = a$, $x_j^{02} = 0, j = 1, 2, 3$, and two periodic solutions in its neighborhood such that each of them depends on its own real*

parameter c_j for $j = 1, 2$. These solutions and their periods τ_j $(c), j = 1, 2$ are real analytic functions in a neighborhood of the origin in these parameters and $\tau_j(0) = 2\pi\sqrt{m/\zeta_j}$.

4. Spacial Coulomb Dynamics

In this section, we consider the spacial Coulomb dynamics of three equal negative charges $-e_0$ in the field of six equal positive charges e' fixed at vertices of the octagon with vertices $b_j, 1 \le j \le 6, b_j = (b_j^1, b_j^2, b_j^3 = 0) \in \mathbb{R}^3$.

$$b_1 = (a, b, 0), b_2 = (a, -b, 0), b_3 = (-a, b, 0), b_4 = (-a, -b, 0), b_5$$
$$= \left(0, \sqrt{3a^2 + b^2}, 0\right), b_6 = \left(0, -\sqrt{3a^2 + b^2}, 0\right), a, b > 0,$$
$$\tag{74}$$

with the potential energy

$$U\left(x_{(3)}\right) = \frac{1}{2}\sum_{j \ne k=1}^{3}\frac{e_j e_k}{|x_j - x_k|} - e_0 e' \sum_{j=1}^{3}\sum_{k=1}^{6}|x_j - b_k|^{-1}, \quad \tag{75}$$

where

$$|x_j|^2 = \left(x_j^1\right)^2 + \left(x_j^2\right)^2 + \left(x_j^3\right)^2. \tag{76}$$

It is not difficult to repeat calculations of the previous section to see that the elements of the matrix of second partial derivatives at the equilibrium $x_1^1 = x_1^{01} = -a, x_2^1 = x_2^{01} = 0, x_3^1 = x_3^{01} = a, x_j^\alpha = x_j^{0\alpha} = 0, \alpha = 2, 3$ look like for $\alpha, \beta = 1, 2, 3$ as follows:

$$U_{1,\alpha;3,\beta}^0 = U_{3,\alpha;1,\beta}^0 = \frac{u'}{2}\delta_{\alpha,\beta}(1 - 3\delta_{\alpha,1}),$$

$$U_{2,\alpha;3,\beta}^0 = U_{3,\alpha;2,\beta}^0 = U_{2,\alpha;1,\beta}^0 = U_{1,\alpha;2,\beta}^0 = 4u'\delta_{\alpha,\beta}(1 - 3\delta_{\alpha,1}),$$

$$U_{1,\alpha;1,\beta}^0 = U_{3,\alpha;3,\beta}^0 = \delta_{\alpha,\beta}\left(u'v' - \delta_{\alpha,1}u_*' - \delta_{\alpha,2}u''_*\right),$$

$$U_{2,\alpha;2,\beta}^0 = u'\delta_{\alpha,\beta}\left(v_1 - u_1\delta_{\alpha,1} - \delta_{\alpha,2}u_1'\right).$$
$$\tag{77}$$

Now, we determine three matrices $U_\alpha^0, \alpha = 1, 2, 3$ by the rule

$$U_{j,\alpha;k,\beta}^0 = \delta_{\alpha,\beta}U_{\alpha;j,k}^0 \tag{78}$$

and renumerate indexes of coordinates in the following way:

$$(1, 1) = 1 ; (2, 1) = 2 ; (3, 1) = 3 ; (1, 2) = 4 ; (2, 2) = 5 ; (3.2)$$
$$= 6 ; (1, 3) = 7 ; (2, 3) = 8 ; (3.3) = 9.$$
$$\tag{79}$$

where the first and second indexes under the round brackets correspond to the lower and upper indexes of coordinates. This gives

$$U^0 = U_1^0 \oplus U_2^0 \oplus U_3^0. \tag{80}$$

The elements of the symmetric matrices U_1^0, U_2^0 are defined in the previous section and

$$U_{3;1,3}^0 = U_{3;3,1}^0 = U_{1,\alpha;3,\beta}^0 = U_{3,\alpha;1,\beta}^0 = \frac{u'}{2}\delta_{\alpha,\beta},$$

$$U_{3;1,2}^0 = U_{3;2,1}^0 = U_{3;2,3}^0 = U_{3;3,2}^0 = U_{2,3;3,3}^0$$
$$= U_{3,3;2,3}^0 = U_{2,3;1,3}^0 = U_{1,3;2,3}^0 = 4u',$$

$$U_{3;1,1}^0 = U_{1,3;1,3}^0 = U_{3;3,3}^0 = U_{3,3;3,3}^0 = u'v',$$

$$U_{3;2,2}^0 = U_{2,3;2,3}^0 = u'v_1, U_3^0 = u'2^{-1}\begin{pmatrix} 2v' & 8 & 1 \\ 8 & 2v_1 & 8 \\ 1 & 8 & 2v' \end{pmatrix}$$

$$= u'U'_3, U'_3 = U_*(g_3) + \left(v' - 1\right)I, g_3 = v_1 - v' + 1.$$
$$\tag{81}$$

The roots p'_3 of U'_3 are given by

$$p'_3(\lambda) = p_*\left(\lambda + 1 - v', g_3\right),$$

$$2\lambda = g_3 - 2\left(1 - v'\right) + 1 \pm \sqrt{(g_3 - 1)^2 + 128}, \lambda$$
$$= v' - 1 = \zeta'_7.$$
$$\tag{82}$$

Let ζ'_8, ζ'_9 be the roots corresponding to the plus and minus before the sign of the square root, respectively. Then,

$$2\zeta'_8 = v_1 + v' - 1 + \sqrt{\left(v_1 - v'\right)^2 + 128},$$

$$2\zeta'_9 = v_1 + v' - 1 - \sqrt{\left(v_1 - v'\right)^2 + 128}.$$
$$\tag{83}$$

We remind that

$$v_1 = -8 + \frac{40}{3}\left[2(4 - 3\eta)^{-3/2} + (4 - \eta)^{-3/2}\right],$$

$$v' = \frac{1}{3}\left[5(1 - \eta)^{-3/2} - \frac{7}{2}\right].$$
$$\tag{84}$$

Let us assume $0 < \eta \le 1/3$. Then,

$$\frac{1}{2} < v' \le \frac{1}{3}\left(\frac{5\sqrt{27}}{\sqrt{8}} - \frac{7}{2}\right) < \frac{1}{3}\left(10 - \frac{7}{2}\right) = 2\frac{1}{6},$$

$$-3 = -8 + \frac{40}{8} < v_1 < -8 + \frac{40}{\sqrt{27}} < 0,$$

$$-5\frac{1}{6} < v_1 - v' < -\frac{1}{2},$$

$$-2\frac{1}{2} < v_1 + v' < 2\frac{1}{6},$$

$$|v_1 + v' - 1| < 3\frac{1}{2},$$

$$\left(v_1 - v'\right)^2 + 128 > (11)^2. \tag{85}$$

As a result,

$$2\zeta'_8 \ge \sqrt{\left(v_1 - v'\right)^2 + 128} - |v_1 + v' - 1| > 11 - 3\frac{1}{2} = 7\frac{1}{2},$$

$$2\zeta'_8 = v_1 + v' - 1 + \sqrt{\left(v_1 - v'\right)^2 + 128} < 1\frac{1}{6} + 13 = 14\frac{1}{6},$$

$$2\zeta'_9 < -11 + \frac{1}{6} < -10. \tag{86}$$

The inequality for v' and last three inequalities imply

$$-\frac{1}{2} < \zeta'_7 < 1\frac{1}{6}, 3\frac{3}{4} < \zeta'_8 < 7\frac{1}{12}, \zeta'_9 < -5. \tag{87}$$

From $\zeta'_1 < \zeta'_2$, $\zeta'_1 > 12$, it follows

$$\zeta'_7 < \zeta'_1 < \zeta'_2, 0 < \zeta'_8 < \zeta'_1 < \zeta'_2. \tag{88}$$

This means that there is no resonance in $\zeta_2 = u'\zeta'_2$ and quadratic resonance in $\zeta_1 = u'\zeta'_1$ for eigenvalues of U^0 in the spacial Coulomb system if $0 < \eta \le 1/3$ and $\zeta'_7 \ne 0$, i.e., $\eta \ne 1 - (10/13)^{2/3} < 1/3$.

The last inequality follows from

$$\left(\frac{2}{3}\right)^{3/2} = \frac{2\sqrt{2}}{3\sqrt{3}} < \frac{3}{5} < \frac{10}{13}, \frac{5}{3} < \sqrt{3}, \frac{3}{2} > \sqrt{2}. \tag{89}$$

We proved the following theorem with the help of the Lyapunov center theorem.

Theorem 6. *If $0 < \eta \le 1/3$ and $\eta \ne 1 - (10/13)^{2/3}$, then the spacial Coulomb equation of motion (1) with $m_j = m, d = 3$, and $N = 3$ and the potential energy (75) possesses the equilibrium $x_1^{01} = -a, x_2^{01} = 0, x_3^{01} = a, x_j^{0\alpha} = 0, j = 1, 2, 3, \alpha = 2, 3,$ and two periodic solutions in its neighborhood such that each of them depends on its own real parameter c_j for $j = 1, 2$. These solu-*

tions and their periods $\tau_j(c_j), j = 1, 2$ are real analytic functions in a neighborhood of the origin in these parameters and $\tau_j(0) = 2\pi\sqrt{m/\zeta_j}$.

5. Conclusion

We have shown that the matrix U^0 of second partial derivatives of the potential energy of our spacial system at the equilibrium is the direct sum of the three dimensional matrices $U_j^0, j = 1, 2, 3$ such that U_1^0 and the direct sum of U_1^0 with U_2^0 coincide with matrices of the second partial derivatives of the potential energy at the equilibrium of one-dimensional and planar systems, respectively. We have shown also that U_1^0 possesses two positive eigenvalues ζ_1, ζ_2 and that $\sqrt{\zeta_j}$ is not in resonance with square roots of other eigenvalues for $j = 1, 2$ if $0 < \eta \le 1/3$, $\eta = (5e_0/3e')^{2/3}$. The Lyapunov center theorem implies that these eigenvalues generate periodic solutions mentioned in Theorem 6 if $0 < \eta \le 1/3$ and $\eta \ne 1 - (10/13)^{2/3}$. The last condition guarantees that neither of the eigenvalues are zero.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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