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Research Article

Multiple Lump Solutions of the (4 + 1)-Dimensional Fokas Equation

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In this paper, we investigate multiple lump wave solutions of the new $(4+1)$-dimensional Fokas equation by adopting a symbolic computation method. We get its 1-lump solutions, 3-lump solutions, and 6-lump solutions by using its bilinear form. Moreover, some basic characters and structural features of multiple lump waves are explained by depicting the three-dimensional plots.

1. Introduction

Nonlinear evolution equations can be used to simulate many nonlinear phenomena in the real world, which appear in many areas, especially in physical [1], engineering sciences [2], applied mathematics [3], chemistry, and biology [4]. Recently, it is well known that rogue waves play an essential role in helping us apprehending the qualitative properties of many phenomena; it is interesting that lump functions can provide approximate fitting prototypes to model rogue waves. In addition to appearing in the ocean [5], lump waves also actually appear in many other fields, such as atmosphere [6], superfluids [7], and capillary waves [8]. Further study of lump waves will help us interpret some unknown fields more deeply. Certain ways have been arranged to solve the lump wave solutions of some equations; they are inclusive of the Hirota bilinear method [9, 10], the inverse scattering transformation [11], the Darboux transformation [12], the Bäcklund transformation [13], the functional variable method [14], the reduced differential transform method [15], and so on. Many integrable equations which have lump wave solutions are enumerated here, for example, the $(3+1)$-dimensional KPI equation [16], the Davey-Stewartson I equation [17], the $(3+1)$-dimensional nonlinear evolution equation [18, 19], and the nonlinear Schrödinger equation [20]. Generally speaking, it is easier to solve the lower order rational solutions than to solve the multiple lump waves of the nonlinear evolution equation. In this paper, we mainly work on a $(4+1)$-dimensional Fokas equation.

$$4u_{tx} - u_{xxxy} + u_{xyyy} + 12u_x u_y + 12uu_{xy} - 6u_{zw} = 0, \qquad (1)$$

which was first derived by Fokas by the generalization of two critical nonlinear evolution equations, which are the integrable KP equation and DS equation [21]. The $(4+1)$-dimensional Fokas equation could be applied to portray nonelastic and elastic interactions [21, 22]. In nonlinear wave theory, KP and DS equations can be used to characterize the surface waves and internal waves in straits or channels of varying depth and width, respectively [23–26]. The significance of the $(4+1)$-dimensional Fokas equation follows naturally from the physical applications of the KP and DS equations. Therefore, the $(4+1)$-dimensional Fokas equation could be adopted to represent a number of phenomena in fluid mechanics, optical fiber communications, ocean engineering, and many others. More recently, $(4+1)$-dimensional Fokas equation has been discussed by some scholars. Demiray et al. obtained the exact solutions of Equation (1) by applying the generalized Kudryashov method (GKM) [27]. Based on the Hirota bilinear form, two classes of lump-type solutions of Equation (1) are studied by Cheng and Zhang [28]. In 2017, two distinct methods, namely, the modified simple equation method (MSEM) and the extended simplest equation method (ESEM), are employed to look for exact traveling wave solutions of Equation (1) [29]. Zhang et al. used the fractional subequation method to obtain the exact analytical solutions of Equation (1) [30]. The lump-bell solutions of Equation (1) are obtained based on the bilinear equation and different test functions in [31]. Particularly, El-Ganaini and Al-Amr discussed the space-time fractional $(4+1)$-dimensional Fokas equation via the functional variable, the generalized Kudryashov, the Jacobi elliptic function expansion, and the generalized Riccati equation mapping methods and got abundant distinct types of new exact solutions [32]. Zhang and Xia obtained soliton solutions, fissionable wave solutions, M-lump solutions, and interaction solutions of the $(4+1)$-dimensional Fokas equation based on the Hirota bilinear method [33]. However, Zhaqilao proposed a novel method to construct the multiple rogue wave solutions of nonlinear partial differential equation; the multiple rogue wave solutions of Equation (1) have not been extracted by this new method. This paper is constructed as follows. In Section 2, the bilinear equation of Equation (1) is acquired. The 1-lump waves are also gained by employing a new ansatz. In Section 3, 3-lump waves of Equation ((1)) are researched when the subscript of f_n is equal to 1 in Equation (10). In Section 4, the 6-lump waves of Equation (1) are studied when the subscript of f_n is equal to 2 in Equation (10). Section 5 is devoted to a short conclusion and discussion.

2. 1-Lump Solutions

The fundamental desire of this section is to investigate the 1-lump solutions of the new $(4+1)$-dimensional Fokas equation. Firstly, setting $X = x + my + nt$ and $Z = z + cw$ in Equation (1) yields

$$4nu_{XX} + \left(m^3 - m\right)u_{XXXX} + 12mu_X^2 + 12muu_{XX} - 6cu_{ZZ} = 0, \tag{2}$$

where m, n, c are all real parameters. With the help of variable transformation

$$u = u_0 + \left(m^2 - 1\right)(\ln f)_{XX}, \tag{3}$$

we can convert the Equation (2) into a bilinear form that reads

$$\left(4nD_X^2 + \left(m^3 - m\right)D_X^4 - 6cD_Z^2\right)f \cdot f = 0, \tag{4}$$

where $D_X^2 f \cdot f = ff_{XX} - f_X^2$, $D_X^4 f \cdot f = 3f_{XX}^2 - 4f_X f_{XXX} + f f_{XXXX}$, and $D_Z^2 f \cdot f = ff_{ZZ} - f_Z^2$, where f is a real function with regard to variable X, Z. D_X^2, D_X^4, and D_Z^2 are called Hirota bilinear D operators. By applying the symbolic computation approach, assuming

$$f = (X - \beta)^2 + a_1(Z - \alpha)^2 + a_0, \tag{5}$$

where α, β, a_1, and a_0 are constants to be determined.

Substituting (5) into (4) and equating the coefficients of all powers of $X^{i_1}Z^{i_2}$ to 0, one has

$$\begin{aligned} 12ca_1^2 + 8ma_1 &= 0, \\ -24c\alpha a_1^2 - 16m\alpha a_1 &= 0, \\ -12ca_1 - 8m &= 0, \\ 24\beta ca_1 + 16\beta m &= 0, \\ 12a_1^2\alpha^2 c + \left(\left(-12\beta^2 - 12a_0\right)c + 8m\alpha^2\right)a_1 & \\ + \left(-8\beta^2 + 8a_0\right)m + 12n^3 - 12n &= 0. \end{aligned} \tag{6}$$

Solving these equations, one has

$$\begin{aligned} a_0 &= -\frac{3n\left(n^2 - 1\right)}{4m}, \\ a_1 &= -\frac{2m}{3c}. \end{aligned} \tag{7}$$

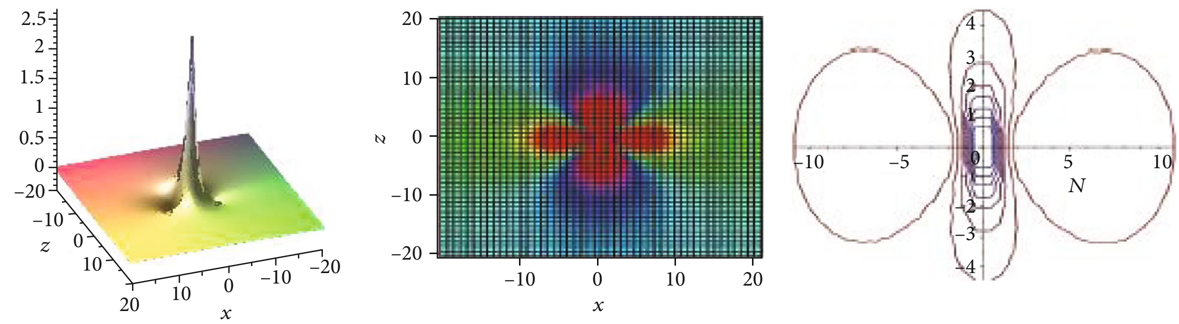
Therefore, we can get a solution of Equation (4) as

$$f = (X - \beta)^2 - \frac{2m(Z - \alpha)^2}{3c} - \frac{3n\left(n^2 - 1\right)}{4m}. \tag{8}$$

By using variable transformation (3), the 1-lump wave solutions of Equation (1) read

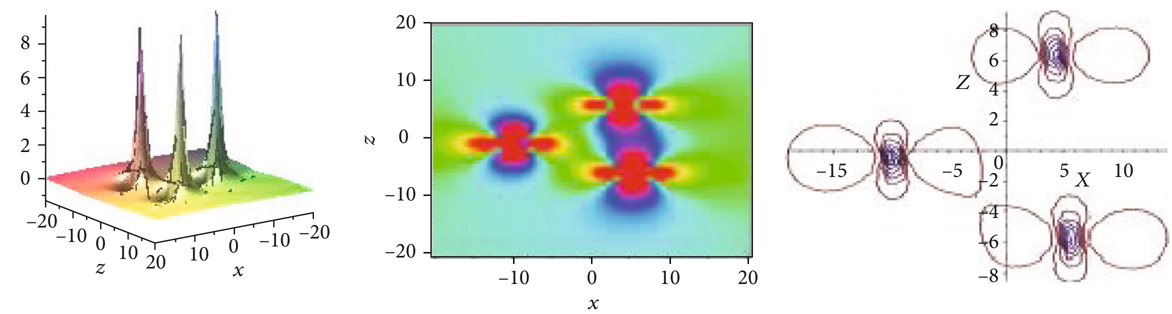
$$u(x, y, z, t, w) = \left(m^2 - 1\right)\left(\frac{2}{(X - \beta)^2 - \left(\left(2m(Z - \alpha)^2\right)/3c\right) - \left(\left(3n(n^2 - 1)\right)/4m\right)} - \frac{(2X - 2\beta)^2}{\left((X - \beta)^2 - \left(\left(2m(Z - \alpha)^2\right)/3c\right) - \left(\left(3n(n^2 - 1)\right)/4m\right)\right)^2}\right), \tag{9}$$

where $X = x + mt$, $Z = z + cw$, m, n, c, α, β are arbitrary real constants. The 1-lump wave (9) has the structure for three wave peaks. One peak is higher than the water level, and the other two are opposite. Figure 1 presents the three dimensional plot, the density plot, and the corresponding contour plot of the 1-lump wave solution of Equation (1). From



(a) (b) (c)

Figure 1: The 1-lump solution u of Equation (1) with the parameter selections $m = -2, n = 2, c = 1, u_0 = 0, \alpha = 0, \beta = 0$. (a) Perspective view of the wave $u(X, Z)$, (b) overhead view of the wave $u(X, Z)$, and (c) the corresponding contour plot $u(X, Z)$.



(a) (b) (c)

Figure 2: The 3-lump solution u of Equation (1) with the parameter selections $m = -3, n = 2, c = 1, u_0 = 0, \alpha = 1000, \beta = 1000, c_{2,0} = 1, b_{0,2} = 1$. (a) Perspective view of the wave $u(X, Z)$, (b) overhead view of the wave $u(X, Z)$, and (c) the corresponding contour plot $u(X, Z)$.

Figure 1, we can see that the 1-lump wave has one center (β, α). Furthermore, at the point $(((\sqrt{-3mn(n^2-1)} + 2m\beta)/2m), \alpha)$ in the plane (X, Z), the maximum amplitude of the 1-lump wave is $(u_0 - ((2\sqrt{-3mn(n^2-1)})/(3n(n^2-1))))$.

3. 3-Lump Solutions

In the section, in order to construct the multiple lump solutions, we propose the notation just like this:

$$\begin{aligned}
F_n(X, Z) &= \sum_{k=0}^{n(n+1)/2} \sum_{i=0}^{k} a_{n(n+1)-2k,2i} Z^{2i} X^{n(n+1)-2k}, \\
P_n(X, Z) &= \sum_{k=0}^{n(n+1)/2} \sum_{i=0}^{k} b_{n(n+1)-2k,2i} X^{2i} Z^{n(n+1)-2k}, \\
Q_n(X, Z) &= \sum_{k=0}^{n(n+1)/2} \sum_{i=0}^{k} c_{n(n+1)-2k,2i} Z^{2i} X^{n(n+1)-2k}, \\
f(X, Z) &= f_{n+1}(X, Z) = F_{n+1}(X, Z) + 2\alpha Z P_n(X, Z) \\
&\quad + 2\beta X Q_n(X, Z) + (\alpha^2 + \beta^2),
\end{aligned} \tag{10}$$

where $a_{i,j}$, $b_{i,j}$, and $c_{i,j}$ are arbitrary constants. Then, we take $n = 1$,

$$\begin{aligned}
f(X, Z) &= F_2(X, Z) + 2\alpha Z P_1(X, Z) \\
&\quad + 2\beta X Q_1(X, Z) + (\alpha^2 + \beta^2),
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
F_2(X, Z) &= X^6 + a_{4,0} X^4 + a_{4,2} Z^2 X^4 \\
&\quad + (a_{2,0} + a_{2,2} Z^2 + a_{2,4} Z^4) X^2 \\
&\quad + a_{0,0} + a_{0,2} Z^2 + a_{0,4} Z^4 + a_{0,6} Z^6, \\
P_1(X, Z) &= b_{0,0} + b_{0,2} X^2 + b_{2,0} Z^2, \\
Q_1(X, Z) &= c_{0,0} + c_{0,2} Z^2 + c_{2,0} X^2.
\end{aligned} \tag{12}$$

Substituting Equation (11) into Equation (4) and collecting all the coefficients of $X^{i_1} Z^{i_2}$, we can get a group of constraining relationships for the parameters. Dealing with these equations, one gets

$$
\begin{aligned}
a_{0,0} &= -\frac{5625n^9 - 192\beta^2 c_{2,0}^2 m^3 - 16875n^7 + 32\alpha^2 cm^2 b_{0,2}^2 + 192\alpha^2 m^3 + 192\beta^2 m^3 + 16875n^5 - 5625n^3}{192m^3},\\
a_{0,2} &= -\frac{475n^2\left(n^4 - 2n^2 + 1\right)}{24cm},\\
a_{0,4} &= -\frac{17n\left(n^2 - 1\right)m}{9c^2},\\
a_{0,6} &= -\frac{8m^3}{27c^3},\\
a_{2,0} &= -\frac{125n^2\left(n^4 - 2n^2 + 1\right)}{16m^2},\\
a_{2,2} &= \frac{15n\left(n^2 - 1\right)}{c},\\
a_{2,4} &= \frac{4m^2}{3c^2},\\
a_{4,0} &= -\frac{-25n\left(n^2 - 1\right)}{4m},\\
a_{4,2} &= -\frac{2m}{c},\\
b_{0,0} &= -\frac{5b_{0,2}n\left(n^2 - 1\right)}{12m},\\
b_{2,0} &= \frac{2mb_{0,2}}{9c},\\
c_{0,0} &= \frac{c_{2,0}n\left(n^2 - 1\right)}{4m},\\
c_{0,2} &= \frac{2mc_{2,0}}{c},\\
c_{2,0} &= c_{2,0},\\
b_{0,2} &= b_{0,2},
\end{aligned}
\tag{13}
$$

where $b_{0,2}$ and $c_{2,0}$ are arbitrary constants. Thus, the 3-lump solution of Equation (1) is shown by

$$
u = u_0 + \left(m^2 - 1\right)(lnf)_{XX}, \tag{14}
$$

where f is given in Equation (11), in which $X = x + my + nt$, $Z = z + cw$. By increasing the value of α and β, 3-lump wave merge and their centers form a triangle (see Figure 2). The 3-lump wave is the arrangement of three 1-lump waves in the plane (X, Z).

4. 6-Lump Solutions

To get the 6-lump solution of Equation (1), the 6-lump solution waves of a $(4+1)$-dimensional Fokas equation can be presented if we choose $n = 2$,

$$
\begin{aligned}
f(X, Z) = {} & F_3(X, Z) + 2\alpha Z P_2(X, Z) + 2\beta X Q_2(X, Z)\\
& + \left(\alpha^2 + \beta^2\right) F_1(X, Z),
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
F_3(X, Z) = {} & X^{12} + \left(a_{10,0} + a_{10,2}Z^2\right)X^{10}\\
& + \left(a_{8,0} + a_{8,2}Z^2 + a_{8,4}Z^4\right)X^8\\
& + \left(a_{6,0} + a_{6,2}Z^2 + a_{6,4}Z^4 + a_{6,6}Z^6\right)X^6\\
& + \left(a_{4,0} + a_{4,2}Z^2 + a_{4,4}Z^4 + a_{4,6}Z^6 + a_{4,8}Z^8\right)X^4\\
& + \left(a_{2,0} + a_{2,2}Z^2 + a_{2,4}Z^4 + a_{2,6}Z^6 + a_{2,8}Z^8 + a_{2,10}Z^{10}\right)X^2\\
& + a_{0,0} + a_{0,2}Z^2 + a_{0,4}Z^4 + a_{0,6}Z^6\\
& + a_{0,8}Z^8 + a_{0,10}Z^{10} + a_{0,12}Z^{12},
\end{aligned}
$$

$$P_2(X,Z) = b_{0,0} + \left(b_{2,0} + b_{2,2}X^2 + b_{2,4}X^4\right)Z^2 + \left(b_{4,0} + b_{4,2}X^2\right)Z^4 + Z^6 + b_{0,2}X^2 + b_{0,4}X^4 + b_{0,6}X^6,$$

$$Q_2(X,Z) = c_{0,0} + c_{0,2}Z^2 + c_{0,4}Z^4 + c_{0,6}Z^6 + \left(c_{2,0} + c_{2,2}Z^2 + c_{2,4}Z^4\right)X^2 + \left(c_{4,0} + c_{4,2}Z^2\right)X^4 + X^6,$$

$$F_1(X,Z) = X^2 + a_{0,2}Z^2 + a_{0,0}. \tag{16}$$

Substitute (15) into (4) and let all the coefficients of the different powers of $X^{i_1}Z^{i_2}$ equal to zero, we can get a set of constraining relations for the parameters. Figuring out these equations, one has

$$a_{0,0} = \frac{1}{36864m^8\left(\alpha^2 + \beta^2 + 1\right)}\left(\left(878826025m^2n^{17} - 5272956150m^2n^{15} + 13182390375m^2n^{13} - 17576520500m^2n^{11} + 472392\alpha^2c^7n^2 - 27648\beta^2m^7n^2 + 13182390375m^2n^9 - 472392\alpha^2c^7 + 27648m^7\beta^2 - 5272956150m^2n^7 + 878826025m^2n^5\right)n\right),$$

$$a_{0,10} = \frac{464m^4n\left(n^2-1\right)}{243c^5},$$

$$a_{0,12} = \frac{64m^6}{729c^6},$$

$$a_{0,2} = \frac{1}{2304m^6c\left(\alpha^2+\beta^2+1\right)} \cdot \left(150448375m^2n^{15} - 752241875m^2n^{13} + 1504483750m^2n^{11} - 1504483750n^9m^2 + 26244\alpha^2c^7 - 1536m^7\beta^2 + 752241875m^2n^7 - 150448375m^2n^5\right),$$

$$a_{0,4} = \frac{16391725n^4\left(n^8 - 4n^6 + 6n^4 - 4n^2 + 1\right)}{1728m^2c^2},$$

$$a_{0,6} = \frac{199745n^3\left(n^6 - 3n^4 + 3n^2 - 1\right)}{162c^3},$$

$$a_{0,8} = \frac{1445n^2\left(n^4 - 2n^2 + 1\right)m^2}{27c^4},$$

$$a_{10,0} = -\frac{49n\left(n^2-1\right)}{2m},$$

$$a_{10,2} = -\frac{4m}{c},$$

$$a_{2,0} = -\frac{1}{1536m^7}\left(79893275m^2n^{15} - 399466375m^2n^{13} + 798932750m^2n^{11} - 798932750n^9m^2 + 26244\alpha^2c^7 + 1536m^7\alpha^2 + 399466375m^2n^7 - 79893275m^2n^5\right),$$

$$a_{2,10} = -\frac{64m^5}{81c^5},$$

$$a_{2,2} = -\frac{94325n^4\left(n^8 - 4n^6 + 6n^4 - 4n^2 + 1\right)}{64cm^3},$$

$$a_{2,4} = \frac{1225n^3\left(n^6 - 3n^4 + 3n^2 - 1\right)}{12c^2m},$$

$$a_{2,6} = -\frac{17710n^2\left(n^4 - 2n^2 + 1\right)m}{27c^3},$$

$$a_{2,8} = -\frac{760m^3n\left(n^2-1\right)}{27c^4},$$

$$a_{4,0} = -\frac{5187875n^4\left(n^8 - 4n^6 + 6n^4 - 4n^2 + 1\right)}{768m^4},$$

$$a_{4,2} = \frac{18375n^3\left(n^6 - 3n^4 + 3n^2 - 1\right)}{8cm^2},$$

$$a_{4,4} = \frac{18725n^2\left(n^4 - 2n^2 + 1\right)}{18c^2},$$

$$a_{4,6} = \frac{2920m^2n\left(n^2-1\right)}{27c^3},$$

$$a_{4,8} = \frac{80m^4}{27c^4},$$

$$a_{6,0} = -\frac{18865n^3\left(n^6 - 3n^4 + 3n^2 - 1\right)}{48m^3},$$

$$a_{6,2} = -\frac{4655n^2\left(n^4 - 2n^2 + 1\right)}{6cm},$$

$$a_{6,4} = -\frac{1540mn\left(n^2-1\right)}{9c^2},$$

$$a_{6,6} = -\frac{160m^3}{27c^3},$$

$$a_{8,0} = \frac{735n^2\left(n^4 - 2n^2 + 1\right)}{16m^2},$$

$$a_{8,2} = \frac{115n\left(n^2-1\right)}{c},$$

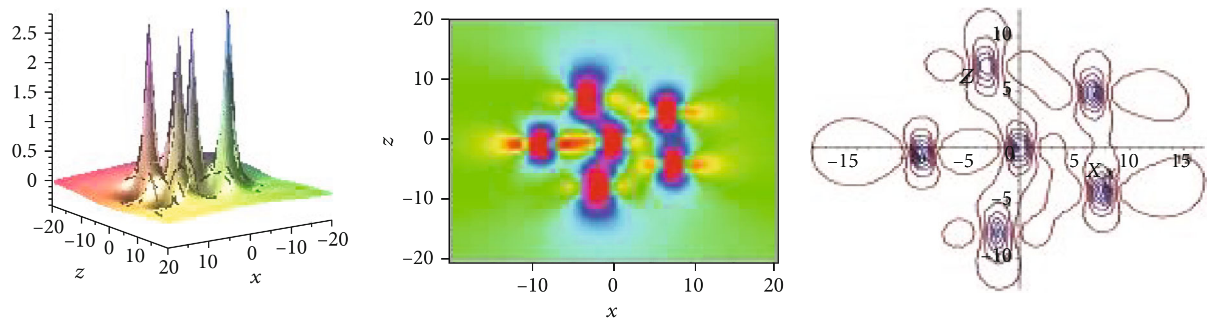
$$a_{8,4} = \frac{20m^2}{3c^2},$$

$$b_{0,0} = \frac{169785n^3\left(n^6 - 3n^4 + 3n^2 - 1\right)c^3}{512m^6},$$

$$b_{0,2} = \frac{17955c^3n^2\left(n^4 - 2n^2 + 1\right)}{128m^5},$$

$$b_{0,4} = \frac{2835c^3n\left(n^2-1\right)}{32m^4},$$

$$b_{0,6} = -\frac{135c^3}{8m^3},$$



(a) (b) (c)

Figure 3: The 6-lump solution u of Equation (1) with the parameter selections $m = -2, n = 2, c = 1, u_0 = 0, \alpha = 80000, \beta = 80000$. (a) Perspective view of the wave $u(X, Z)$, (b) overhead view of the wave $u(X, Z)$, and (c) the corresponding contour plot $u(X, Z)$.

$$
\begin{aligned}
b_{2,0} &= -\frac{2205c^2 n^2 (n^4 - 2n^2 + 1)}{64m^4},\\
b_{2,2} &= \frac{855c^2 n (n^2 - 1)}{8m^3},\\
b_{2,4} &= -\frac{45c^2}{4m^2},\\
b_{4,0} &= -\frac{21cn (n^2 - 1)}{8m^2},\\
b_{4,2} &= \frac{27c}{2m},\\
c_{0,0} &= -\frac{12005n^3 (n^6 - 3n^4 + 3n^2 - 1)}{192m^3},\\
c_{0,2} &= -\frac{535n^2 (n^4 - 2n^2 + 1)}{24cm},\\
c_{0,4} &= -\frac{5mn (n^2 - 1)}{c^2},\\
c_{0,6} &= -\frac{40m^3}{27c^3},\\
c_{2,0} &= -\frac{245n^2 (n^4 - 2n^2 + 1)}{16m^2},\\
c_{2,2} &= -\frac{115n (n^2 - 1)}{3c},\\
c_{2,4} &= -\frac{20m^2}{9c^2},\\
c_{4,0} &= -\frac{13n (n^2 - 1)}{4m},\\
c_{4,2} &= \frac{6m}{c}.
\end{aligned} \tag{17}
$$

By this mean, we arrive at the 6-lump solution of Equation (1) represented by

$$u(x, y, z, t, w) = (lnf)_{XX}, \tag{18}$$

where f is given in Equation (15). Figure 3 shows that the 3D plot of the 6-lump wave consists of a central peak and five 1-lump waves in a ring. One can observe that the six peaks tend to the same height as $|\alpha|$ and $|\beta|$ increase.

5. Conclusions

In this work, we have analytically established and analyzed novel multiple lump solutions of a $(4 + 1)$-dimensional Fokas equation based on the bilinear equation and a new ansatz. A series of rational solutions including the 1-lump wave solutions, the 3-lump wave solutions, and the 6-lump wave solutions are obtained. The 1-lump wave has one positive peak and two negative peaks. In order to search the 3-lump and the 6-lump solutions, three polynomial functions F_n, P_n, and Q_n are utilized. It is notable that these lump waves all have the properties $\lim_{x\to\pm\infty} u = u_0$, $\lim_{y\to\pm\infty} u = u_0$, and $\lim_{z\to\pm\infty} u = u_0$. The 3-lump and 6-lump waves consist of three and six independent single 1-lump waves, respectively. All the peaks of the multiple lump waves tend to the same height when α and β are large enough. The results of this paper enrich the types of solutions of the $(4 + 1)$-dimensional Fokas equation. Comparing with the existing results in the literature, our results are new. We expect these results to provide some values for researching the dynamics of multiple waves in the deep ocean and nonlinear optical fibers. And it is very helpful for us to obtain the soliton molecules in the future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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