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Research Article

The Existence of Normalized Solutions for a Nonlocal Problem in $\mathbb{R}^3$

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In this paper, we study the following fractional Schrödinger equation in $\mathbb{R}^3(-\Delta)^\sigma u - \lambda u = |u|^{p-2}u$, in $\mathbb{R}^3$ with $\sigma \in (0,1)$, $\lambda \in \mathbb{R}$ and $p \in (2+\sigma, 2+(4/3)\sigma)$. By using the constrained variational method, we show the existence of solutions with prescribed L^2 norm for this problem.

1. Introduction

This paper concerns with the following fractional Schrödinger problem:

$$(-\Delta)^\sigma u - \lambda u = |u|^{p-2}u, \text{ in } \mathbb{R}^3, \tag{1}$$

where $\sigma \in (0,1)$, $p \in (2+\sigma, 2+(4/3)\sigma)$, and $\lambda \in \mathbb{R}$. Here, the fractional Laplacian $(-\Delta)^\sigma$ in $\mathbb{R}^n$ is defined by

$$(-\Delta)^\sigma u = C_{n,\sigma} PV \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2\sigma}} dy, \tag{2}$$

where PV stands for the Cauchy principal value and $C_{n,\sigma}$ is a normalization constant.

In the present paper, the motivation for studying such equations comes from mathematical physics: searching for the form of standing wave $\psi = e^{-iht}u$ of the evolution equation

$$i\frac{\partial \psi}{\partial t} + (-\Delta)^\sigma \psi + (\lambda - h)\psi = |\psi|^{p-2}\psi \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \tag{3}$$

leads to looking for solutions of (1). Here, i is the imaginary unit and $h \in \mathbb{R}$. This class of Schrödinger-type equations is of particular interest in fractional quantum mechanics for the study of particles on stochastic fields modelled by Lévy processes. A path integral over the Lévy flight paths and a fractional Schrödinger equation of fractional quantum mechanics are formulated by Laskin [1, 2] from the idea of Feynman and Hibbs's path integrals.

On the other hand, problem (1) has attracted considerable attention in the recent period. Part of the motivation is to consider $h \in \mathbb{R}$ as a fixed parameter and then to search for a solution $u \in H^\sigma(\mathbb{R}^3)$ solving (1). In this direction, mainly by variational methods, many researches have been devoted to the study of the existence, multiplicity, uniqueness, regularity, and asymptotic decay properties of the solutions to fractional Schrödinger equation (1). For this information, we can refer to [3–11] and the references therein. Besides, some more complicated fractional equations and systems were also studied, and indeed, some interesting results were obtained. Nearly, Mingqi et al. [12] investigated a critical Schrödinger-Kirchhoff type systems driven by nonlocal integrodifferential operators and by applying the mountain pass theorem and Ekeland's variational principle; the authors obtained the existence and asymptotic behavior of solutions for this system under some suitable assumptions. Later, in [13], the same authors as in [12] studied a diffusion model of Kirchhoff-type. Under some appropriate conditions, by employing the Galerkin method, the local existence of nonnegative solutions was obtained, and then by virtue of a differential inequality technique, they proved that the local nonnegative solutions blow up in finite time with arbitrary negative initial energy and suitable initial values. Moreover, in [14], Mingqi et al. concerned with a class of fractional

Kirchhoff-type problems with the Trudinger-Moser nonlinearity. By applying minimax techniques combined with the fractional the Trudinger-Moser inequality, they found the existence of a ground state solution with positive energy and the existence of nonnegative solutions with negative energy by using Ekeland's variational principle. In [15], the three authors considered a fractional Choquard-Kirchhoff-type problem involving an external magnetic potential and a critical nonlinearity and established a fractional version of the concentration-compactness principle with magnetic field, and then together with the mountain pass theorem, they verified the existence of nontrivial radial solutions in nondegenerate and degenerate cases. Furthermore, Mingqi et al. [16] concerned the Schrödinger-Kirchhoff-type problems involving the fractional p-Laplacian and critical exponent. By using the concentration-compactness principle in fractional Sobolev spaces, they showed the existence of m pairs of solutions for any $m \in \mathbb{N}$, and by applying Krasnoselskii's genus theory, they also got the existence of infinitely many solutions under some suitable conditions for the parameter. For more information on this direction, one can refer to [17–24] and the references therein.

In the present paper, inspired by the fact that physicists are often interested in normalized solutions, we look for solutions in $H^\sigma(\mathbb{R}^3)$ having a prescribed L^2 norm to equation (1). Such types of problems were studied extensively in recent years for the classical Schrödinger equations with the standard Laplacian operator. We refer the interested reader to [25–31] and to the references therein. But up to our knowledge, not much is obtained for the existence of normalized solutions of equation (1) in $H^\sigma(\mathbb{R}^3)$ with a fractional Laplacian operator. So, in this paper, the aim is to get the normalized solutions of equation (1). Here, we give the definition of prescribed ρ-L^2 norm solutions. For fixed $\rho > 0$, if $u_\rho \in H^\sigma(\mathbb{R}^3)$ is a solution of problem (1) such that

$$\|u_\rho\|_2 := \left(\int_{\mathbb{R}^3} |u_\rho|^2\right)^{1/2} = \rho, \tag{4}$$

we call it a prescribed ρ-L^2 norm solution. Naturally, a prescribed ρ-L^2 norm solution $u_\rho \in H^\sigma\mathbb{R}$ of (1) can be a constrained critical point of the functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(\left|(-\Delta)^{\sigma/2}u\right|^2 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p, \tag{5}$$

on the L^2 sphere B_ρ in $H^\sigma(\mathbb{R}^3)$, where

$$B_\rho = \left\{u \in H^\sigma\left(\mathbb{R}^3\right) : \|u\|_2 = \rho, \rho > 0\right\}. \tag{6}$$

Note that for any $p \in (2, 6/(3-2\sigma))$, $I(u)$ is a well-defined and C^1 functional. Set

$$m_\rho = \inf_{B_\rho} I(u). \tag{7}$$

It is standard that if u_ρ is a minimizer of (7), then u_ρ is a solution of (1) with prescribed ρ-L^2 norm with the constraint $\lambda_\rho \in \mathbb{R}$ being the Lagrange multiplier. However, it is worth mentioning that dealing with this kind of problem, one has to face the main difficulty concerning with the lack of compactness of the minimizing sequence $\{u_n\} \subset B_\rho$. In fact, we will encounter two possible bad scenarios that $u_n \rightharpoonup 0$ and $u_n \rightharpoonup \bar{u} \neq 0$ with $0 < \|\bar{u}\|_2 < \rho$. In order to avoid the possible cases and to get that the infimum is obtained, we prove an important lemma (Lemma 6) that guarantees the compactness of minimizing sequence. As a consequence of this lemma, setting

$$\begin{aligned}
\rho_1^* &= \inf\left\{\rho > 0 : m_\rho < 0\right\},\\
\rho_2^* &= \inf\left\{\rho > 0 : \exists u \in B_\rho \text{ such that } I(u) \leq 0\right\},
\end{aligned} \tag{8}$$

we can get our main result as follows:

Theorem 1. *If $p \in (2+\sigma, 2+(4/3)\sigma)$, then m_ρ has a minimizer if and only if $\rho \in [\rho_1^*, +\infty]$.*

In particular, there is a prescribed ρ-L^2 norm solution $u_\rho \in H^\sigma(\mathbb{R}^3)$ of (1) with the constraint $\lambda_\rho \in \mathbb{R}$. But, when $p = 2+(4/3)\sigma$, m_ρ has no minimizer for any $\rho \in (\rho_2^*, +\infty)$.

Remark 2. In fact, for any $\rho > 0$, we can infer that $m_\rho \leq 0$. To see this, letting $u \in B_{\theta^{-1}\rho}$ be arbitrary and considering $u^\theta(y) = \theta^{5/2}u(\theta y)$ for any $\theta > 0$, we find $u^\theta \in B_\rho$ and

$$I\left(u^\theta\right) = \frac{1}{2}\theta^{2+2\sigma}\int_{\mathbb{R}^3} \left|(-\Delta)^{\sigma/2}u\right|^2 - \frac{1}{p}\theta^{(5/2)p-3}\int_{\mathbb{R}^3} |u|^p. \tag{9}$$

Hence, $I(u^\theta) \longrightarrow 0$ as $\theta \longrightarrow 0$ and the conclusion is as follows:

Finally, we give the following notations which can be used in this paper:

(i) $H^\sigma(\mathbb{R}^3)$ is the usual Sobolev space endowed with the standard norm

$$\|u\|_{H^\sigma}^2 = \int_{\mathbb{R}^3} \left(\left|(-\Delta)^{\sigma/2}u\right|^2 + u^2\right) \tag{10}$$

(ii) Denote $\|u\|_\sigma^2 := \int_{\mathbb{R}^3} \left|(-\Delta)^{\sigma/2}u\right|^2$

(iii) $\|u\|_q$ is the norm of the Lebesgue space $L^q(\mathbb{R}^3)$ for $1 < q < \infty$

(iv) Denote $C > 0$ by various positive constants which may vary from one line to another and which are not important for the analysis of the problem

This paper is organized as follows: In Section 2, we will give some preliminary results which are crucial to prove our main result. And then the proof of our main result is given in Section 3.

2. Preliminaries

In this part, we give some important results. First, similar to the classical Gagliardo-Nirenberg inequality to the Laplacian operator, we introduce the fractional version of Gagliardo-Nirenberg inequality as follows:

Lemma 3 (see [32]). *Let $1 \le q, p_2 < \infty, 0 < \alpha < q < \infty$, $0 < \sigma < n$, and $1 < p_1 < (n/\sigma)$. We have*

$$\|u\|_{L^q(\mathbb{R}^n)} \le C_0^{\alpha/q} \left\|(-\Delta)^{\sigma/2} u\right\|_{L^{p_1}(\mathbb{R}^n)}^{\alpha/q} \|u\|_{L^{p_2}(\mathbb{R}^n)}^{(q-\alpha)/q}, \quad (11)$$

with $\alpha((1/p_1) - (\sigma/n)) + ((q-\alpha)/p_2) = 1$and

$$C_0 = 2^{-\sigma}\pi^{-\sigma/2} \frac{\Gamma((n-\sigma)/2)}{\Gamma((n+\sigma)/2)} \left(\frac{\Gamma(n)}{\Gamma(n/2)}\right)^{\sigma/n}. \quad (12)$$

In particular, when $n = 3$, one has

$$\|u\|_q \le C_0^{\vartheta} \left\|(-\Delta)^{\sigma/2} u\right\|_2^{\vartheta} \|u\|_2^{1-\vartheta}, \quad (13)$$

with $\vartheta = (3(q-2))/2q\sigma$.

In [33], the authors have established the Pohozaev identity for the fractional Laplacian operator.

Applying the Pohozaev identity, we have the following:

Lemma 4. *If u^* is a critical point of $I(u)$ on B_ρ, then $\eta(u^*) = 0$, where*

$$\eta(u) = 2\sigma \int_{\mathbb{R}^3} \left|(-\Delta)^{\sigma/2} u\right|^2 - \frac{3(p-2)}{p} \int_{\mathbb{R}^3} |u|^p. \quad (14)$$

Proof. Define the functional energy corresponding to (1) as

$$F_\lambda(u) := \frac{1}{2} \int_{\mathbb{R}^3} \left(\left|(-\Delta)^{\sigma/2} u\right|^2 - \frac{\lambda}{2} \int_{\mathbb{R}^3} |u|^2 - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p. \quad (15)$$

Then any critical point u of $F_\lambda(u)$ satisfies the Pohozaev identity for (1) (see [33]), that is,

$$D_\lambda(u) := (3-2\sigma) \int_{\mathbb{R}^3} \left(\left|(-\Delta)^{\sigma/2} u\right|^2 - 3\lambda \int_{\mathbb{R}^3} |u|^2 - \frac{6}{p} \int_{\mathbb{R}^3} |u|^p = 0. \quad (16)$$

On the other hand, if u is a critical point of $I(u)$ restricted to B_ρ, there is a Lagrange multiplier $\lambda^* \in \mathbb{R}$ such that

$$I'(u^*) = \lambda^* u^*. \quad (17)$$

So, for any $\psi \in H^\sigma(\mathbb{R}^3)$, we have

$$\langle F_{\lambda^*}(u^*), \psi\rangle = \left\langle I'(u^*) - \lambda^* u^*, \psi \right\rangle = 0. \quad (18)$$

Furthermore, if we now know that

$$A_\lambda(u) := \langle F_\lambda(u), u\rangle = \int_{\mathbb{R}^3} \left(\left|(-\Delta)^{\sigma/2} u\right|^2 - \lambda \int_{\mathbb{R}^3} |u|^2 - \int_{\mathbb{R}^3} |u|^p, \quad (19)$$

by (18), we find $A_{\lambda^*}(u^*) = 0$. As a result,

$$\eta(u^*) = \left(3A_{\lambda^*}(u) - D_{\lambda^*}(u)\right)\big|_{u=u^*} = 0. \quad (20)$$

Using Lemma 3, the estimate (13) leads to the following fact:

Lemma 5. *If $p \in (2+\sigma, 2+(4/3)\sigma)$, then for any $\rho > 0$, functional I is bounded from below and coercive on B_ρ.*

Proof. If $\|u\|_2 = \rho$, from Lemma 3, we have

$$\|u\|_p \le C\|u\|_\sigma^{(3(p-2))/2p\sigma}. \quad (21)$$

So,

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left|(-\Delta)^{\sigma/2} u\right|^2 - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \ge \frac{1}{2} \|u\|_\sigma^2 - C\|u\|_\sigma^{(3(p-2))/2\sigma}. \quad (22)$$

Since $p < 2+(4/3)\sigma$ and $((3(p-2))/2\sigma) < 2$, our result follows.

From the above Lemma, we can prove that

Lemma 6. *If $p \in (2+\sigma, 2+(4/3)\sigma)$, then*

(i) there exists $\rho_1 > 0$ such that for all $\rho \in (\rho_1, \infty)$, $m_\rho < 0$

(ii) for any $\rho > 0$ such that $m_\rho < 0$, m_ρ admits a minimizer

(iii) the functional $\rho \mapsto m_\rho$ is continuous about each $\rho > 0$

Proof.

(i) Letting $u^\beta(y) = \beta^{1-(3/2)\theta} u(\beta^{-\theta} y)$ such that $\|u^\beta\|_2 = \beta \|u\|_2$ with $\beta, \theta \in \mathbb{R}$, we see

$$\begin{aligned} \int_{\mathbb{R}^3} \left|(-\Delta)^{\sigma/2} u^\beta\right|^2 &= \beta^{2-2\sigma\theta} \int_{\mathbb{R}^3} \left|(-\Delta)^{\sigma/2} u\right|^2, \\ \int_{\mathbb{R}^3} \left|u^\beta\right|^p &= \beta^{p(1-(3/2)\theta)+3\theta} \int_{\mathbb{R}^3} |u|^p. \end{aligned} \quad (23)$$

If we take $\theta = -2$, one has

$$I\left(u^\beta\right) = \frac{1}{2}\beta^{2(1+2\sigma)} \int_{\mathbb{R}^3} \left|(-\Delta)^{\sigma/2} u\right|^2 - \frac{1}{p}\beta^{4p-6} \int_{\mathbb{R}^3} |u|^p. \quad (24)$$

Since $p > 2 + \sigma$ and $4p - 6 > 2 + 4\sigma$, for β is large enough, we get $I(u^\beta) < 0$ and then (i) has been proved.

(ii) We will divide it into three steps to show (ii).

Step 1. We claim that for any $0 < \nu < \rho$, $m_\rho < m_\nu + m_{\sqrt{\rho^2-\nu^2}}$. To see this, let $\{u_n\}$ be a minimizing sequence on B_ρ for m_ρ. Since $m_\rho < 0$, from (22), we can get

$$\begin{aligned} &0 < C_1 < \|u_n\|_\sigma^2 < C_2, \\ &0 < C_3 < \|u_n\|_p^p < C_4. \end{aligned} \tag{25}$$

On the other hand, applying (24), we obtain that for $\theta = -2$,

$$\begin{aligned} I\left(u_n^\beta\right) &= \frac{1}{2}\beta^{2(1+2\sigma)}\int_{\mathbb{R}^3}\left|(-\Delta)^{\sigma/2}u_n\right|^2 - \frac{1}{p}\beta^{4p-6}\int_{\mathbb{R}^3}|u_n|^p \\ &= \beta^2(I(u_n) + f(\beta, u_n)), \end{aligned} \tag{26}$$

where

$$f(\beta, u_n) = \frac{1}{2}\left(\beta^{4\sigma} - 1\right)\int_{\mathbb{R}^3}\left|(-\Delta)^{\sigma/2}u_n\right|^2 - \frac{1}{p}\left(\beta^{4p-8} - 1\right)\int_{\mathbb{R}^3}|u_n|^p, \tag{27}$$

with $4p - 8 > 4\sigma$. Moreover, using $I(u_n) < 0$, we can deduce that

$$\frac{df(\beta, u_n)}{d\beta}\left|_{\beta=1}\right| = 2\sigma\int_{\mathbb{R}^3}\left|(-\Delta)^{\sigma/2}u_n\right|^2 - \frac{1}{p}(4p - 8) < 0, \tag{28}$$

and for all $\beta > 1$,

$$\begin{aligned} \frac{d^2 f(\beta, u_n)}{d\beta^2} &= 2\sigma(4\sigma - 1)\beta^{4\sigma-2}\int_{\mathbb{R}^3}\left|(-\Delta)^{\sigma/2}u_n\right|^2 \\ &\quad - \frac{1}{p}(4p - 8)(4p - 9)\beta^{4p-10}\int_{\mathbb{R}^3}|u_n|^p < 0. \end{aligned} \tag{29}$$

Thus, combining (28) and (29), for all $\beta > 1$, we find $f(\beta, u_n) < 0$ and from (26),

$$m_{\beta\rho} < \beta^2 m_\rho. \tag{30}$$

As a result, if $\sqrt{\rho^2 - \nu^2} < \nu < \rho$,

$$\begin{aligned} m_\rho &= m_{(\rho/\nu)\nu} < \frac{\rho^2}{\nu^2}m_\nu = \frac{\rho^2 - \nu^2 + \nu^2}{\nu^2}m_\nu \\ &= \frac{\rho^2 - \nu^2}{\nu^2}m_{\left(\nu/\sqrt{\rho^2-\nu^2}\right)\sqrt{\rho^2-\nu^2}} + m_\nu < m_{\sqrt{\rho^2-\nu^2}} + m_\nu, \end{aligned} \tag{31}$$

and if $\nu < \sqrt{\rho^2 - \nu^2} < \rho$,

$$\begin{aligned} m_\rho &= m_{\left(\rho/\sqrt{\rho^2-\nu^2}\right)^{\sqrt{\rho^2-\nu^2}}} < \frac{\rho^2}{\rho^2 - \nu^2}m_{\sqrt{\rho^2-\nu^2}} \\ &= \frac{\rho^2 - \nu^2 + \nu^2}{\rho^2 - \nu^2}m_{\sqrt{\rho^2-\nu^2}} = m_{\sqrt{\rho^2-\nu^2}+}\frac{\nu^2}{\rho^2 - \nu^2}m_{\nu\left(\sqrt{\rho^2-\nu^2}/\nu\right)} \\ &< m_{\sqrt{\rho^2-\nu^2}} + m_\nu. \end{aligned} \tag{32}$$

It follows from (31) and (32) that the claim holds.

Step 2. We will show that all the minimizing sequences $\{u_n\}$ for m_ρ have a weak limit, up to translations, different from zero. Let $\{u_n\}$ be a minimizing sequence on B_ρ for m_ρ. Note that for any sequence $\{y_n\} \subset \mathbb{R}^3$, $u_n(\cdot + y_n)$ is still a minimizing sequence for m_ρ. So the proof of this step can be finished if we can prove the existence of a sequence $\{y_n\} \subset \mathbb{R}^3$ such that the weak limit of $u_n(\cdot + y_n)$ is different from zero.

Applying Lion's lemma, we know that if

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}^3}\int_{B(x,1)}|u_n|^2 dy = 0, \tag{33}$$

then $u_n \longrightarrow 0$ in $L^q(\mathbb{R}^3)$ for any $q \in (2, 6/(3 - 2\sigma))$, where $B(a, r) = \{x \in \mathbb{R}^3 : |x - a| \le r\}$. Since $m_\rho < 0$, we see that

$$\int_{B(0,1)}|u_n(\cdot + y_n)|^2 dy \ge \delta > 0. \tag{34}$$

Therefore, it follows the compactness of the embedding $H^\sigma(B(0,1)) \hookrightarrow L^2(B(0,1))$ that the weak limit of the sequence $u_n(\cdot + y_n)$ is not the trivial function.

Step 3. Finally, we verify that m_ρ has a minimizer for $m_\rho < 0$. Suppose $\{u_n\}$ to be a minimizing sequence on B_ρ for m_ρ with $m_\rho < 0$. Then by Lemma 5, $\{u_n\}$ is bounded in $H^\sigma(\mathbb{R}^3)$ and $L^q(\mathbb{R}^3)$ for any $q \in [2, 6/(3 - 2\sigma)]$. So there exists $\bar{u} \in H^\sigma(\mathbb{R}^3)$ such that $u_n \rightharpoonup \bar{u}$ in $H^\sigma(\mathbb{R}^3)$ and then we can get

$$\frac{1}{2}\int_{\mathbb{R}^3}\left|(-\Delta)^{\sigma/2}u_n\right|^2 - \frac{1}{p}\int_{\mathbb{R}^3}|u_n|^p = m_\rho + o(1), \tag{35}$$

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3}\left|(-\Delta)^{\sigma/2}(u_n - \bar{u})\right|^2 - \frac{1}{p}\int_{\mathbb{R}^3}|u_n - \bar{u}|^p + \frac{1}{2}\int_{\mathbb{R}^3}\left|(-\Delta)^{\sigma/2}\bar{u}\right|^2 \\ &\quad - \frac{1}{p}\int_{\mathbb{R}^3}|\bar{u}|^p = m_\rho + o(1). \end{aligned} \tag{36}$$

If we set $\nu := \|\bar{u}\|_2$ and $\epsilon_n := \left(\sqrt{\rho^2 - \nu^2}\right)/\|u_n - \bar{u}\|_2$, then by the Step 2, $0 < \nu \le \rho$. Now, we want to prove that

$\nu = \rho$. To see this, we assume that $0 < \nu < \rho$. From (36), we find

$$\frac{1}{2}\|\epsilon_n(u_n - \bar{u})\|_\sigma^2 - \frac{1}{p}\|\epsilon_n(u_n - \bar{u})\|_p^p + I(\bar{u}) = m_\rho + o(1), \tag{37}$$

since $\epsilon_n \longrightarrow 1$. Noting that $\|\epsilon_n(u_n - \bar{u})\|_2 = \sqrt{\rho^2 - \nu^2}$, (37) tells that

$$m_{\sqrt{\rho^2-\nu^2}} + m_\nu \le m_\rho + o(1), \tag{38}$$

which contradicts to (31) and (32) and then $\nu = \rho$.

Since $\bar{u} \in B_\rho$, we have $\|u_n - \bar{u}\|_2 = o(1)$. Hence, if we would verify that $u_n \longrightarrow \bar{u}$ in $H^\sigma(\mathbb{R}^3)$, it remains to show that $\|u_n - \bar{u}\|_\sigma^2 = o(1)$ up to a subsequence. First, by assumption, there is $\{\lambda_n\} \subset \mathbb{R}$ such that

$$\left\langle I'(u_n) - \lambda_n u_n, \psi \right\rangle = o(1), \quad \forall \psi \in H^\sigma(\mathbb{R}^3). \tag{39}$$

So,

$$\left\langle I'(u_n) - \lambda_n u_n, u_n \right\rangle = o(1), \tag{40}$$

which implies $\{\lambda_n\}$ is bounded and up to a subsequence; there exists $\lambda \in \mathbb{R}$ with $\lambda_n \longrightarrow \lambda$.

On the other hand, we find $n, m \longrightarrow \infty$,

$$\begin{aligned} &\left\langle I'(u_n) - I'(u_m) - \lambda_n u_n + \lambda_m u_m, u_n - u_m \right\rangle = o(1), \\ &\qquad (\lambda_n - \lambda_m)\langle u_m, u_n - u_m \rangle = o(1). \end{aligned} \tag{41}$$

Hence,

$$\begin{aligned} &\|u_n - u_m\|_\sigma^2 - \int_{\mathbb{R}^3} \left(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m\right)(u_n - u_m) - \lambda_n \|u_n \\ &\quad - u_m\|_2^2 = o(1). \end{aligned} \tag{42}$$

Notice that by the interpolation inequality, we get

$$\|u_n - u_m\|_p \le C\|u_n - u_m\|_2^\gamma \|u_n - u_m\|_\sigma^{1-\gamma}, \tag{43}$$

with $(\gamma/2) + ((1-\gamma)/2_\sigma^*) = 1/p$ and $2_\sigma^* = 6/(3 - 2\sigma)$, and then

$$\|u_n - u_m\|_p = o(1). \tag{44}$$

As a result,

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} \left(|u_n|^{p-2}u_n |u_m|^{p-2}u_m\right)(u_n - u_m) \right| \\ &\quad \le \left[\left(\int_{\mathbb{R}^3} |u_n|^p \right)^{(p-1)/p} + \left(\int_{\mathbb{R}^3} |u_m|^p \right)^{(p-1)/p} \right] \\ &\quad \cdot \|u_n - u_m\|_p = o(1). \end{aligned} \tag{45}$$

Using $\|u_n - u_m\|_2 = o(1)$, $\lambda_n \longrightarrow \lambda$, (42) and (45), one can get that $\{u_n\}$ is a Cauchy sequence in $H^\sigma(\mathbb{R}^3)$ and hence $\|u_n - \bar{u}\|_{H^\sigma(\mathbb{R}^3)} \longrightarrow 0$ as $n \longrightarrow \infty$.

(iii) Now, we come to prove that if $\rho_n \longrightarrow \rho$, then $\lim_{n\to\infty} m_{\rho_n} = m_\rho$. For any $n \in \mathbb{N}^+$, let $w_n \in B_{\rho_n}$ such that $I(w_n) < m_{\rho_n} + (1/n)$. Using Lemma 5, we deduce that $\{w_n\}$ is bounded in $H^\sigma(\mathbb{R}^3)$ and then $\|w_n\|_\sigma$ and $\|w_n\|_p$ are bounded. So it is easy to find that

$$\begin{aligned} m_\rho &\le I\left(\frac{\rho}{\rho_n} w_n\right) = \frac{1}{2}\left(\frac{\rho}{\rho_n}\right)^2 \|w_n\|_\sigma^2 - \frac{1}{p}\left(\frac{\rho}{\rho_n}\right)^p \|w_n\|_p^p \\ &= I(w_n) + o(I) < m_{\rho_n} + o(1). \end{aligned} \tag{46}$$

On the other hand, letting $\{v_n\} \subset B_\rho$ be a minimizing sequence for m_ρ, we have

$$m_{\rho_n} \le I\left(\frac{\rho_n}{\rho} v_n\right) = I(v_n) + o(1) = m_\rho + o(1). \tag{47}$$

Hence, from (46) and (47), $\lim_{n\to\infty} I_{\rho n} = m_\rho$ follows.

3. Proof of the Main Result

To prove our main theorem, we first give the following important results:

Lemma 7. *When $p \in (2 + \sigma, 2 + (4/3)\sigma)$, there exists $\rho_3 > 0$ such that m_ρ has no minimizer for all $\rho \in (0, \rho_3)$.*

Proof. We prove it by contradiction and suppose that there exist $\{\rho_n\} \subset \mathbb{R}^+$ with $\rho_n \longrightarrow 0^+$ as $n \longrightarrow \infty$ and $\{u_n\} \subset B_{\rho_n}$ such that $I(u_n) = m_{\rho_n}$.

Since for any $\rho > 0$, $m_\rho \le 0$, we have $m_{\rho_n} \le 0$ and then by Lemma 3,

$$\frac{1}{2}\|u_n\|_\sigma^2 \le \frac{1}{p}\|u_n\|_p^p \le C\|u_n\|_\sigma^{(3(p-2))/2\sigma} \rho_n^{p(1-\vartheta)}, \tag{48}$$

with $\vartheta = (3(p-2))/2p\sigma$. Due to $p < 2 + (4/3)\sigma$ and $(3(p-2))/(2\sigma < 2)$, (48) tells us that

$$\|u_n\|_\sigma \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{49}$$

So, from (48) and Lemma 4, we infer that

$$\eta(u_n) = 2\sigma\|u_n\|_\sigma^2 - \frac{3(p-2)}{p}\|u_n\|_p^p \geq 2\sigma\|u_n\|_\sigma^2 - C\|u_n\|_\sigma^{(3(p-2))/2\sigma}\rho_n^{p(1-\vartheta)} > 0, \tag{50}$$

which is impossible since $\eta(u_n) = 0$ from Lemma 4.

Recall that

$$\rho_1^* = \inf\{\rho > 0 : m_p < 0\}, \quad \rho_2^* = \inf\{\rho > 0 : \exists u \in B_\rho \text{ such that } I(u) \leq 0\}. \tag{51}$$

We have

Lemma 8. *If $p = 2 + (4/2)\sigma$, then $p_2^* \in (0,+\infty)$.*

Proof. First, it follows from Lemma 3 that for any $u \in B_\rho$, if $p = 2 + (4/3)\sigma$

$$\|u\|_p^p \leq C\|u\|_\sigma^2\rho^2. \tag{52}$$

So,

$$I(u) \geq \frac{1}{2}\|u\|_\sigma^2 - \frac{C}{p}|u\|_\sigma^2\rho^2. \tag{53}$$

Thus, $I(u) > 0$ for any $u \in B_\rho$ with $\rho > 0$ small enough and then $\rho_2^* > 0$ follows.

On the other hand, taking $u_1 \in B_1$, then $u^\theta(y) = \theta^{(5/2)} u_1(\theta y) \in B_\theta$ for all $\theta > 0$ and

$$I(u^\theta) = \frac{1}{2}\theta^{2+2\sigma}\|u_1\|_\sigma^2 - \frac{1}{p}\theta^{(5/2)p-3}\|u_1\|_p^p. \tag{54}$$

This implies that if $\theta > 0$ is large enough, $I(u^\theta) < 0$ and our result has been proved.

Lemma 9. *When $p \in (2 + \sigma, 2 + (4/3)\sigma)$, we have the following:*

(i) $\rho_1^* \in (0,\infty)$

(ii) $m_p = 0$ if $\rho \in (0, \rho_1^*)$

(iii) $m_\rho < 0$ and is strictly decreasing about ρ if $\rho \in (\rho_1^*,+\infty)$

(iv) Moreover, when $p = 2 + (4/3)\sigma$, then $\rho_2^* \in (0,\infty)$ and $m_\rho = 0$ as $\rho \in (0, \rho_2^*)$ and $m_\rho = -\infty$ if $\rho \in (\rho_2^*,+\infty)$

Proof.

(i) We prove it by contradiction. Suppose that $\rho_1^* = 0$, and then from the definition of ρ_1^*, for any $\rho > 0$, we can get $m_\rho < 0$. Hence, it follows from Lemma 6 (ii) that m_ρ has a minimizer for any $\rho > 0$. But this gives a contradiction with Lemma 7. On the other hand, using Lemma 6 (i), we can infer that $\rho_1^* < \infty$ and so (i) follows.

(ii) First, from the definition of ρ_1^* and $m_\rho \leq 0$ for any $\rho > 0$, we know that $m_\rho = 0$ for $\rho \in (0, \rho_1^*)$. Furthermore, using the continuity of $\rho \longrightarrow m_\rho$ (see Lemma 6 (iii)), we have $m_{\rho_1^*} = 0$ and (ii) is proved.

(iii) By the definition of ρ_1^*, we have $m_\rho < 0$ for $\rho \in (\rho_1^*,+\infty)$. So, by Lemma 6 (ii), m_ρ admits a minimizer $u_\rho \in B_\rho$. Setting with $u_\rho^\theta = \theta^{(5/2)}u_\rho(\theta y) \in m_{\theta\rho}$ with $\theta > 1$, we can find that

$$\begin{aligned} m_{\theta\rho} \leq I(u_\rho^\theta) &= \frac{1}{2}\theta^{2+2\sigma}\|u_\rho\|_\sigma^2 - \frac{1}{p}\theta^{(5/2)p-3}\|u_\rho\|_p^p \\ &= \theta^{2+2\sigma}\left(\frac{1}{2}\|u_\rho\|_\sigma^2 - \frac{1}{p}\theta^{(5/2)p-5-2\sigma}\|u_\rho\|_p^p\right) \\ &< \theta^{2+2\sigma}\left(\frac{1}{2}\|u_\rho\|_\sigma^2 - \frac{1}{p}\|u_\rho\|_p^p\right) = \theta^{2+2\sigma}I(u_\rho), \end{aligned} \tag{55}$$

which implies that $m_{\theta\rho} < \theta m_\rho < m_\rho$ since $m_\rho < 0$ and $\theta > 1$. This tells that (iii) holds.

(iv) It follows from Lemma 8 that $\rho_2^* \in (0,+\infty)$. On the other hand, from the definition of ρ_2^*, it is direct to see that $m_\rho = 0$ for $\rho \in (0, \rho_2^*)$. Now, it remains to show that $m_\rho = -\infty$ if $\rho \in (\rho_2^*,+\infty)$. Here, we claim that for any $\rho \in (\rho_2^*,+\infty)$, there exists $u^* \in B_\rho$ such that $I(u^*) \leq 0$. In fact, suppose that $I(u) > 0$ for all $\rho \in (\rho_2^*,+\infty)$. Then for an arbitrary $\bar{\rho} \in [\rho_2^*, \rho)$, taking $\bar{v} \in B_{\bar{\rho}}$ and $\bar{v}_\theta = \theta^{(5/2)}\bar{v}(\theta y)$ for $\theta = \rho/\bar{\rho}$, we find $\bar{v}_\theta \in B_\rho$ and from (55),

$$0 < I(\bar{v}_\theta) < \theta^{2+2\sigma}I(\bar{v}), \tag{56}$$

which tells that $I(\bar{v}) > 0$ for $\bar{v} \in B_{\bar{\rho}}$. This contradicts the definition of ρ_2^* since $\bar{\rho} \in [\rho_2^*, \rho)$ and so the claim holds.

Now we consider another scaling $u_\theta^* = \theta^{(3/2)}u^*(\theta y)$ for all $\theta > 0$. Then $u_\theta^* \in B_\rho$ and

$$I(u_\theta^*) = \frac{1}{2}\theta^{2\sigma}\|u^*\|_\sigma^2 - \frac{1}{p}\theta^{(3/2)p-3}\|u^*\|_p^p. \tag{57}$$

This yields $I(u_\theta^*) \longrightarrow -\infty$ as $\theta \longrightarrow +\infty$, and we get our result.

Proof of Theorem 1. We will divide it into three steps to prove Theorem 1.

Step 1. First, we prove that if $p \in (2 + \sigma, 2 + (4/3)\sigma)$, $m_{\rho_1^*}$ has a minimizer. Set $\ell_n = \rho_1^* + (1/n)$ for all $n \in \mathbb{N}^+$. Then

$\ell_n \longrightarrow \rho_1^*$ and using Lemmas 6 (iii) and 9 (ii), we find $m_{\ell n} \longrightarrow m_{\rho_1^*} = 0$.

So applying Lemmas 9 (iii) and 6 (ii), $m_{\ell_n} < 0$ and m_{ℓ_n} have a minimizer, denoted by v_n. By Lemma 5, $\{v_n\}$ is bounded in $H^\sigma(\mathbb{R}^3)$.

Now, we claim that $\|v_n\|_p \nrightarrow 0$ as $n \longrightarrow \infty$. Suppose that $\|v_n\|_p \longrightarrow 0$ as $n \longrightarrow \infty$ by contradiction. Since $I(v_n) < 0$, similar to (49) and (48), we find

$$\|v_n\|_\sigma \longrightarrow 0, \tag{58}$$

$$I(v_n) \geq \frac{1}{2}\|v_n\|_\sigma^2 - \frac{C}{p}\|v_n\|_\sigma^{(3(p-2))/2\sigma}\ell_n^{p(1-\vartheta)}, \tag{59}$$

with $\vartheta = (3(p-2))/2p\sigma$, which, combining (58), implies that $I(v_n) \geq 0$ for n is large enough. This contradicts $I(v_n) = m_{\ell_n} < 0$, and then the claim holds.

With the same argument as the proof of (34), there exists $\delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\int_{B(0,1)} |v_n(\cdot + y_n)|^2 dy \geq \delta > 0. \tag{60}$$

Letting $\bar{v}_n = v_n(\cdot + y_n)$, then $\bar{v}_n$ is bounded in $H^\sigma(\mathbb{R}^3)$ and there exists $v_0 \in H^\sigma(\mathbb{R}^3)$ such that $\bar{v}_n \rightharpoonup v_0$ in $H^\sigma(\mathbb{R}^3)$, and $\bar{v}_n \longrightarrow v_0$ in $L^2_{loc}(\mathbb{R}^3)$.

Thus, by (60), we can check that $v_0 \neq 0$. Finally, we come to show v_0 is a minimizer of $m_{\rho_1^*}$. First, we have

$$\lim_{n\to\infty}\|\bar{v}_n\|_2 = \|v_0\|_2 + \lim_{n\to\infty}\|\bar{v}_n - v_0\|_2 = \rho_1^*, \tag{61}$$

and then by Lemma 6 (iii) and 9 (ii),

$$\lim_{n\to\infty} I(\bar{v}_n - v_0) \geq \lim_{n\to\infty} m_{\|\bar{v}_n - v_0\|_2} = m_{\rho_1^* \|v_0\|_2} = 0. \tag{62}$$

On the other hand, from Lemma 9 (ii), we have $\|v_0\|_2 \leq \rho_1^*$ and $m_{\|v_0\|_2} = 0$ and then $I(v_0) < 0$ is impossible. So by (61) and (62), we can get $I(v_0) = 0 = m_{\rho_1^*}$ and v_0 is a minimizer of $m_{\|v_0\|_2} = 0$. Now if we assume that $\|v_0\|_2 < \rho_1^*$, using (55), one has

$$m_{\rho_1^*} = m_{\rho_1^*/\|v_0\|_2 \|v_0\|_2} < \frac{\rho_1^*}{\|v_0\|_2} m_{\|v_0\|_2} = 0. \tag{63}$$

This yields a contradiction with $m_{\rho_1^*} = 0$, and thus $\|v_0\|_2 = \rho_1^*$.

Step 2. We will prove that m_ρ has a minimizer if and only if $\rho \in [\rho_1^*, +\infty)$ if $p \in (2 + \sigma, 2 + (4/3)\sigma)$.

Suppose that there is $\rho_0 \in (0, \rho_1^*)$ such that m_{ρ_0} has a minimizer. Then from the definition of ρ_1^* and (55), we get that $m_{\rho_0} = 0$ and $m_\rho < 0$ for any $\rho > \rho_0$. This contradicts the definition of ρ_1^*.

On the other hand, if $\rho \in (\rho_1^*, +\infty)$, using Lemmas 9 (iii) and 6 (ii), m_ρ admits a minimizer.

Step 3. Finally, it follows from the definition of $\eta(u)$ that for any $u \in B_\rho$ and if $p = 2 + (4/3)\sigma$,

$$I(u) - \frac{1}{3(p-2)}\eta(u) = \left(\frac{1}{2} - \frac{2\sigma}{3(p-2)}\right)\|(-\Delta)^{\sigma/2}u\|_2^2. \tag{64}$$

Thus, if we assume that m_ρ has a minimizer $u_\rho \in B_\rho$ for some $\rho \in (\rho_2^*, +\infty)$, then from (64) and Lemma 9 (iv), we have

$$-\infty = m_\rho = I(u_\rho) = 0, \tag{65}$$

which completes the Proof of Theorem 1.

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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