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Research Article

Fractional Langevin Equations with Nonseparated Integral Boundary Conditions

Khalid Hilal, Lahcen Ibnelazyz, Karim Guida, and Said Melliani

Sultan Moulay Slimane University, BP 523, 23000 Beni Mellal, Morocco

Correspondence should be addressed to Lahcen Ibnelazyz; ibnelazyzlahcen@gmail.com

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In this paper, we discuss the existence of solutions for nonlinear fractional Langevin equations with nonseparated type integral boundary conditions. The Banach fixed point theorem and Krasnoselskii fixed point theorem are applied to establish the results. Some examples are provided for the illustration of the main work.

1. Introduction

Fractional derivatives give an excellent description of memory and hereditary properties of different processes. Properties of the fractional derivatives make the fractional-order models more useful and practical than the classical integral-order models.

Several researchers in the recent years have employed the fractional calculus as a way of describing natural phenomena in different fields such as physics, biology, finance, economics, and bioengineering (for more details see [1–9] and many other references).

With the recent outstanding development in fractional differential equations, the Langevin equation has been considered a part of fractional calculus, and thus, important results have been elaborated (see [10–14]).

The Langevin equation was first introduced by Langevin in 1908; it is a fundamental theory of the Brownian motion to describe the evolution of physical phenomena in fluctuating environments [15, 16]. The fractional model of the Langevin equation as a generalization of the classical one gives a fractional Gaussian process parametrized by two indices, and this fractional model is more flexible for modeling fractal processes [17, 18].

The fractional Langevin equation is extensively studied in the literature from both the theoretical and numerical point of views (for more details see [19–25]). In [26], the authors studied a nonlinear Langevin equation involving two fractional orders in different intervals. In [27], the authors discussed the existence theory for a nonlinear Langevin equations with nonlocal multipoint and multistrip boundary conditions. In [28], fractional Langevin equations with nonlocal integral boundary conditions have been investigated by Salem et al. In [14], an antiperiodic boundary value problem for the Langevin equation involving two fractional orders has been studied.

Recently, in [29], the authors discussed the nonlinear fractional differential equations with nonseparated type integral boundary conditions; however, the fractional Langevin equations involving nonseparated integral boundary conditions have not been investigated yet; that is why, in this work and motivated by all the works cited above, we study the existence and uniqueness of the fractional Langevin equations with nonseparated integral boundary conditions as follows:

$$\begin{cases} {}^{c}D^{\beta}({}^{c}D^{\alpha}+\lambda)x(t)=f(t,x(t),I^{p}x(t)), \quad t\in[0,1], \\ x(0)+\mu x(1)=\sigma_1\int_0^1 g(s,x(s))ds, \\ {}^{c}D^{\alpha}x(0)+\mu\,{}^{c}D^{\alpha}x(1)=\sigma_2\int_0^1 h(s,x(s))ds, \\ {}^{c}D^{2\alpha}x(0)+\mu\,{}^{c}D^{2\alpha}x(1)=\sigma_3\int_0^1 k(s,x(s))ds, \end{cases} \tag{1}$$

where $0 < \alpha < 1$, $1 < \beta \leq 2$, $p > 0$, $\lambda, \mu, \sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}^*$ with $\mu \neq -1$, and ${}^cD^\beta, {}^cD^\alpha$ are the Caputo fractional derivatives and $f : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g, h, k : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ are given continuous functions.

This paper is divided into four sections, in which the second provides some notations and basic known results, in the third section, we study the existence and uniqueness of solutions to problem (1), and in the fourth section, we give two examples to illustrate our results.

2. Preliminaries and Notations

In this section, we give some notation, definitions, and lemma which are needed throughout this paper.

Definition 1 (see [5]). The fractional integral of order $\alpha > 0$ with the lower limit zero for a function f can be defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds. \tag{2}$$

Definition 2 (see [5]). The Caputo derivative of order $\alpha > 0$ with the lower limit zero for a function f can be defined as

$${}^cD^\alpha f = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \tag{3}$$

where $n \in \mathbb{N}$, $0 \leq n-1 < \alpha < n$, and $t > 0$.

Theorem 3 (see [30]). *Let M be a bounded, closed, convex, and nonempty subset of a Banach space X. Let A and B be operators such that*

(I) $Ax + By \in M$ whenever $x, y \in M$

(II) A is compact and continuous

(III) B is a contraction mapping

Then, there exists $z \in M$ such that $z = Az + Bz$.

Lemma 4 (see [5]). *Let $\alpha, \beta \geq 0$; then, the following relations hold:*

$$\begin{aligned} I^\alpha t^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}, \\ {}^cD^\alpha t^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}. \end{aligned} \tag{4}$$

Lemma 5 (see [5]). *Let $n \in \mathbb{N}$ and $n-1 < \alpha < n$. If f is a continuous function, then, we have*

$$I^\alpha {}^cD^\alpha f(t) = f(t) + \alpha_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1}. \tag{5}$$

Lemma 6. *Let $y \in C([0,1], \mathbb{R})$. Then, a unique solution of the bondary value problem*

$$\begin{cases} {}^cD^\beta({}^cD^\alpha + \lambda)x(t) = y(t), \quad t \in [0,1], \\ x(0) + \mu x(1) = \sigma_1 \displaystyle\int_0^1 g(s; x(s)) ds, \\ {}^cD^\alpha x(0) + \mu {}^cD^\alpha x(1) = \sigma_2 \displaystyle\int_0^1 h(s; x(s)) ds, \\ {}^cD^{2\alpha} x(0) + \mu {}^cD^{2\alpha} x(1) = \sigma_3 \displaystyle\int_0^1 k(s; x(s)) ds \end{cases} \tag{6}$$

is given by

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} y(s) ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds \\ & + A_1(t) \int_0^1 h(s; x(s)) ds + A_2(t) \int_0^1 g(s; x(s)) ds \\ & + A_3(t) \int_0^1 k(s; x(s)) ds + \frac{A_4(t)}{\Gamma(\beta-\alpha)} \int_0^1 (1-s)^{\beta-\alpha-1} y(s) ds \\ & + \frac{A_5(t)}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} y(s) ds + \frac{\mu\lambda}{(1+\mu)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} x(s) ds \\ & - \frac{\mu}{(1+\mu)\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} y(s) ds, \end{aligned} \tag{7}$$

where

$$\begin{aligned} A_1(t) = {} & \frac{t^\alpha \sigma_2 (1 - \lambda\Gamma(2-\sigma))}{\Gamma(\alpha+1)(1+\mu)} + \frac{t^{\alpha+1} \lambda_{\sigma_2} \Gamma(2-\alpha)}{\Gamma(2+\alpha)} \\ & + \left(\frac{\mu}{(1+\mu)^2 \Gamma(\alpha+1)} - \frac{1}{(1+\mu)\Gamma(\alpha+2)} \right) \Gamma(2-\alpha) \lambda\sigma_2 \\ & - \frac{\mu\sigma_2}{(1+\mu)^2 \Gamma(\alpha+1)}, \end{aligned}$$

$$A_2(t) = \frac{t^\alpha \lambda\sigma_1}{\Gamma(\alpha+1)(1+\mu)} - \frac{\sigma_1}{1+\mu} + \frac{\mu\lambda\sigma_1}{(1+\mu)^2 \Gamma(\alpha+1)},$$

$$\begin{aligned} A_3(t) = {} & \frac{t^\alpha \Gamma(2-\alpha)\sigma_3}{\Gamma(\alpha+1)(1+\mu)} + \frac{t^{\alpha+1} \Gamma(2-\alpha)\sigma_3}{\Gamma(\alpha+2)\mu} + \frac{\mu\Gamma(2-\alpha)\sigma_3}{(1+\mu)^2 \Gamma(\alpha+1)} \\ & - \frac{\Gamma(2-\alpha)\sigma_3}{(1+\mu)\Gamma(\alpha+2)}, \end{aligned}$$

$$\begin{aligned} A_4(t) = {} & \frac{t^\alpha \Gamma(2-\alpha)\mu}{\Gamma(\alpha+1)(1+\mu)} - \frac{t^{\alpha+1} \Gamma(2-\alpha)}{\Gamma(\alpha+2)} + \Gamma(2-\alpha) \\ & \cdot \left(\frac{\mu}{(1+\mu)\Gamma(\alpha+2)} - \frac{\mu^2}{(1+\mu)^2 \Gamma(\alpha+1)} \right), \end{aligned}$$

$$A_5(t) = \frac{t^\alpha \mu}{\Gamma(\alpha+1)(1+\mu)} - \frac{\mu^2}{(1+\mu)^2 \Gamma(\alpha+1)}. \tag{8}$$

Proof. By applying Lemma 5, we have

$$\begin{aligned}({}^cD^{\alpha}+\lambda)x(t) &= I^{\beta}y(t)+a_0+a_1t,\\ {}^cD^{a}x(t) &= I^{\beta}y(t)+a0+a_1t-\lambda x(t),\\ x(t) &= I^{\alpha+\beta}y(t)+I^{\alpha}a_0+I^{\alpha}a_1t-I^{\alpha}\lambda x(t)+a_2,\end{aligned} \tag{9}$$

where $a_0, a_1, a_2 \in \mathbb{R}$.

Using the condition ${}^cD^{2\alpha}x(0)+\mu{}^cD^{2\alpha}x(1)=\sigma_3\int_0^1 k(s;x(s))ds$, we obtain

$$\begin{aligned}a_1 = \Gamma(2-\alpha)\Bigg(&\frac{\sigma_3}{\mu}\int_0^1 k(s;x(s)ds+\frac{\lambda\sigma_2}{\mu}\int_0^1 h(s;x(s))ds\\ &-\frac{1}{\Gamma(\beta-\alpha)}\int_0^1(1-s)^{\beta-\alpha}y(s)ds\Bigg).\end{aligned} \tag{10}$$

By conditions ${}^cD^{\alpha}x(0)+\mu{}^cD^{\alpha}x(1)=\sigma_2\int_0^1 k(s;x(s))ds$ and $x(0)+\mu x(1)=\sigma_1\int_0^1 g(s;x(s))ds$, we have

$$\begin{aligned}a_0 = &\frac{-\Gamma(2-\alpha)\sigma_3}{1+\mu}\int_0^1 k(s,x(s))ds+\frac{(1-\lambda\Gamma(2-\alpha))\sigma_2}{1+\mu}\int_0^1 h(s,x(s))ds\\ &+d\frac{\lambda\sigma_1}{1+\mu}\int_0^1 g(s,x(s))ds+\frac{\Gamma(2-\alpha)\mu}{(1+\mu)\Gamma(\beta-\alpha)}\int_0^1(1-s)^{\beta-\alpha-1}y(s)ds\\ &-\frac{\mu}{(1+\mu)\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}y(s)ds,\end{aligned}$$

$$\begin{aligned}a_2 = &\frac{\mu\lambda}{(1+\mu)\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}x(s)ds+\left(\frac{\sigma_1}{1+\mu}-\frac{\mu\lambda\sigma_1}{\Gamma(\alpha+1)(1+\mu)^2}\right)\\ &\cdot\int_0^1 g(s,x(s))ds-\frac{\mu}{(1+\mu)\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}y(s)ds\\ &+\frac{\mu^2}{(1+\mu)^2\Gamma(\beta)\Gamma(\alpha+1)}\int_0^1(1-s)^{\beta-1}y(s)ds+\frac{\Gamma(2-\alpha)}{\Gamma(\beta-\alpha)}\\ &\cdot\left(\frac{\mu}{(1+\mu)\Gamma(\alpha+2)}-\frac{\mu^2}{(1+\mu)^2\Gamma(\alpha+1)}\right)\int_0^1(1-s)^{\beta-\alpha-1}y(s)ds\\ &+\left[\left(\frac{\mu^2}{(1+\mu)^2\Gamma(\alpha+1)}-\frac{\mu}{(1+\mu)\Gamma(\alpha+2)}\right)\frac{\Gamma(2-\alpha)\lambda\sigma_2}{\mu}\right.\\ &\left.-\frac{\mu\sigma_2}{(1+\mu)^2\Gamma(\alpha+1)}\right]\int_0^1 h(s,x(s))ds.\end{aligned} \tag{11}$$

Substituting the value of a_0, a_1, and a_2, we obtain the desired results. And by direct computation, one can obtain the converse of the lemma.

3. Main Results

Denote by X the Banach space of all continuous functions from $[0,1] \longrightarrow \mathbb{R}$ endowed with norm $\|x\| = \sup\{|x(t)|: t \in [0,1]\}$.

By Lemma 6, we transform problem (1) into a fixed point problem as $x = Px$, where $P : X \longrightarrow X$ is given by

$$\begin{aligned}Px(t) = &\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}f(s,x(s),I^px(s))ds\\ &-\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}x(s)ds+A_1(t)\int_0^1 h(s;x(s))ds\\ &+A_2(t)\int_0^1 g(s;x(s))ds+A_3(t)\int_0^1 k(s;x(s))ds\\ &+\frac{A_4(t)}{\Gamma(\beta-\alpha)}\int_0^1(1-s)^{\beta-\alpha-1}f(s,x(s),I^px(s))ds\\ &+\frac{A_5(t)}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}f(s,x(s),I^px(s))ds\\ &+\frac{\mu\lambda}{(1+\mu)\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}x(s)ds\\ &-\frac{\mu}{(1+\mu)\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}f(s,x(s),I^px(s))ds.\end{aligned} \tag{12}$$

Theorem 2. *Suppose that $f : [0,1]\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ and $h, g, k : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions satisfying*

(H_1)—there exist positive constants q_1, q_2 such that

$$|f(t,x_1,y_1)-f(t,x_2,y_2)| \le q_1|x_1-x_2|+q_2|y_1-y_2|, \tag{13}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$, $t \in [0,1]$.

(H_2)—there exist positive constants q_3, q_4, q_5 such that

$$\begin{aligned}&|g(t,x)-g(t,y)| \le q_3|x-y|,\ |h(t,x)-h(t,y)| \le q_4|x-y|,\\ &|k(t,x)-k(t,y)| \le q_5|x-y|, \quad \forall x,y \in \mathbb{R}.\end{aligned} \tag{14}$$

Then there exist a unique solution for boundary value problem (1) provided that $r_1 < 1$, where

$$\begin{aligned}r_1 = &\left[\left(q_1+\frac{q_2}{\Gamma(p+1)}\right)\left(\frac{1}{\Gamma(\alpha+\beta+1)}+\frac{A_4}{\Gamma(\beta-\alpha+1)}\right.\right.\\ &\left.+\frac{A_5}{\Gamma(\beta+1)}+\left|\frac{\mu}{(1+\mu)\Gamma(\alpha+\beta+1)}\right|\right)+\frac{|\lambda|}{\Gamma(\alpha+1)}\\ &\left.+\left|\frac{\mu\lambda}{(1+\mu)\Gamma(\alpha+1)}\right|+A_1q_4+A_2q_3+A_3q_5\right]\end{aligned} \tag{15}$$

and $A_i = \max_{t\in[0,1]}|A_i(t)|$ for $i = 1, 2, .., 5$.

Proof. We set $\sup_{0\le t\le 1}|f(t,0,0)| = M_0$, $\sup_{0\le t\le 1}|g(t,0)| = M_1$, $\sup_{0\le t\le 1}|h(t,0)| = M_2$, $\sup_{0\le t\le 1}|k(t,0)| = M_3$.

Let $B_r = \{x \in X : \|x\| \le r\}$ the ball with radius r, where $r \ge (r_2/(1-r_1))$, with

$$r_2 = \frac{M_0}{\Gamma(\alpha+\beta+1)} + \frac{M_0A_4}{\Gamma(\beta-\alpha+1)} + A_1M_1 + A_2M_2 + A_3M_3 + \frac{A_5M_0}{\Gamma(\beta+1)} + \left|\frac{\mu M_0}{(1+\mu)\Gamma(\alpha+\beta+1)}\right|. \tag{16}$$

Then, B_r is a closed, convex, and nonempty subset of the Banach space X.

Our aim is to prove that the operator P has a unique fixed point on B_r.We show that $PB_r \subseteq B_r$.

For $x \in B_r$, $t \in [0,1]$, we have

$$\begin{aligned}
|Px(t)| &\le \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|f(s,x(s),I^px(s))|ds \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x(s)|ds + |A_1(t)|\int_0^1 |h(s;x(s))|ds \\
&\quad + |A_2(t)|\int_0^1 |g(s;x(s))|ds + |A_3(t)|\int_0^1 |k(s;x(s))|ds \\
&\quad + \frac{|A_4(t)|}{\Gamma(\beta-\alpha)}\int_0^1 (1-s)^{\beta-\alpha-1}|f(s,x(s),I^px(s))|ds \\
&\quad + \frac{|A_5(t)|}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}|f(s,x(s),I^px(s))|ds \\
&\quad + \left|\frac{\mu\lambda}{(1+\mu)\Gamma(\alpha)}\right|\int_0^1 (1-s)^{\alpha-1}|x(s)|ds \\
&\quad + \left|\frac{\mu}{(1+\mu)\Gamma(\alpha+\beta)}\right|\int_0^1 (1-s)^{\alpha+\beta-1}|f(s,x(s),I^px(s))|ds \\
&\le \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}[|f(s,x(s),I^px(s)) - f(s,0,0)| \\
&\quad + |f(s,0,0)|]ds + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x(s)|ds \\
&\quad + |A_1(t)|\int_0^1 |h(s;x(s)) - h(s;0)| + |h(s;0)|ds \\
&\quad + |A_2(t)|\int_0^1 |g(s;x(s)) - g(s;0)| + |g(s;0)|ds \\
&\quad + |A_3(t)|\int_0^1 |k(s;x(s)) - k(s;0)| + |k(s;0)|ds \\
&\quad + \frac{|A_4(t)|}{\Gamma(\beta-\alpha)}\int_0^1 (1-s)^{\beta-\alpha-1}[|f(s,x(s),I^px(s)) \\
&\quad - f(s,0,0)| + |f(s,0,0)|]ds + \frac{|A_5(t)|}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} \\
&\quad \cdot [|f(s,x(s),I^px(s)) - f(s,0,0)| + |f(s,0,0)|]ds \\
&\quad + \left|\frac{\mu\lambda}{(1+\mu)\Gamma(\alpha)}\right|\int_0^1 (1-s)^{\alpha-1}|x(s)|ds \\
&\quad + \left|\frac{\mu}{(1+\mu)\Gamma(\alpha+\beta)}\right|\int_0^1 (1-s)^{\alpha+\beta-1} \\
&\quad \times [|f(s,x(s),I^px(s)) - f(s,0,0)| + |f(s,0,0)|]ds,
\end{aligned} \tag{17}$$

which implies that

$$\begin{aligned}
\|Px\| &\le \frac{1}{\Gamma(\alpha+\beta+1)}\left[\left(q_1 + \frac{q_2}{\Gamma(p+1)}\right)\|x\| + M_0\right] \\
&\quad + \frac{\lambda}{\Gamma(\alpha+1)}\|x\| + A_1[q_4\|x\| + M_2] + A_2[q_3\|x\| + M_1] \\
&\quad + A_3[q_5\|x\| + M_3] + \frac{A_4}{\Gamma(\beta-\alpha+1)} \\
&\quad \cdot \left[\left(q_1 + \frac{q_2}{\Gamma(p+1)}\right)\|x\| + M_0\right] + \frac{A_5}{\Gamma(\beta+1)} \\
&\quad \cdot \left[\left(q_1 + \frac{q_2}{\Gamma(p+1)}\right)\|x\| + M_0\right] + \left|\frac{\mu\lambda}{(1+\mu)\Gamma(\alpha+1)}\right|\|x\| \\
&\quad + \left|\frac{\mu}{(1+\mu)\Gamma(\alpha+\beta+1)}\right|\left[\left(q_1 + \frac{q_2}{\Gamma(p+1)}\right)\|x\| + M_0\right] \\
&\le \left[\left(q_1 + \frac{q_2}{\Gamma(p+1)}\right)\left(\frac{1}{\Gamma(\alpha+\beta+1)} + \frac{A_4}{\Gamma(\beta-\alpha+1)}\right.\right. \\
&\quad \left.\left. + \frac{A_5}{\Gamma(\beta+1)} + \left|\frac{\mu}{(1+\mu)\Gamma(\alpha+\beta+1)}\right|\right) + \frac{\lambda}{\Gamma(\alpha+1)} \\
&\quad + \left|\frac{\mu\lambda}{(1+\mu)\Gamma(\alpha+1)}\right| + A_1q_4 + A_2q_3 + A_3q_5\right]\|x\| \\
&\quad + \frac{M_0}{\Gamma(\alpha+\beta+1)} + \frac{M_0A_4}{\Gamma(\beta-\alpha+1)} + A_1M_2 + A_2M_1 \\
&\quad + A_3M_3 + \frac{A_5M_0}{\Gamma(\beta+1)} + \left|\frac{\mu M_0}{(1+\mu)\Gamma(\alpha+\beta+1)}\right| \\
&\le r_1r + r_2 \le r.
\end{aligned} \tag{18}$$

Now, for $x, y \in B_r$ and for $t \in [0;1]$,

$$\begin{aligned}
&|Px(t) - Py(t)| \\
&\quad \le \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|f(s,x(s),I^px(s)) \\
&\qquad - f(s,y(s),I^py(s))|ds + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x(s) \\
&\qquad - y(s)|ds + |A_1(t)|\int_0^1 |h(s;x(s)) - h(s;y(s))|ds \\
&\qquad + |A_2(t)|\int_0^1 |g(s;x(s)) - g(s;y(s))|ds + |A_3(t)| \\
&\qquad \cdot \int_0^1 |k(s;x(s)) - k(s;y(s))|ds + \frac{|A_4(t)|}{\Gamma(\beta-\alpha)}\int_0^1 \\
&\qquad \cdot (1-s)^{\alpha-\beta-1}|f(s,x(s),I^px(s)) - f(s,y(s),I^py(s))|ds \\
&\qquad + \frac{|A_5(t)|}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}|f(s,x(s),I^px(s)) \\
&\qquad - f(s,y(s),I^py(s))|ds + \left|\frac{\mu\lambda}{(1+\mu)\Gamma(\alpha)}\right|\int_0^1 (1-s)^{\alpha-1} \\
&\qquad \cdot |x(s) - y(s)|ds + \left|\frac{\mu}{(1+\mu)\Gamma(\alpha+\beta)}\right|\int_0^1 (1-s)^{\alpha+\beta-1} \\
&\qquad \cdot |f(s,x(s),I^px(s)) - f(s,y(s),I^py(s))|ds.
\end{aligned} \tag{19}$$

Thus,

$$\|Px - Py\| \le r_1 \|x - y\|. \tag{20}$$

Since $r_1 < 1$, then operator P is a contraction mapping. Therefore, boundary value problem (1) has a unique solution.

Theorem 7. *Assume that* (H_1) *and* (H_2) *hold,* $f : [0;1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is a continuous function. Further, we suppose*

(H_3)—$|f(t,x,y)| \le m(t)$; $|h(t,x)| \le \rho(t)$; $|k(t,x)| \le \psi(t)$; $|g(t,x)| \le \varphi(t), \forall (t,x,y) \in [0,1] \times \mathbb{R}^2$ *with* $m, \rho, \varphi, \psi \in C([0,1];\mathbb{R}^+)$. *Then, boundary value problem (1) has at least one solution on* $[0,1]$ *if* $Q < 1$ *and* $r_1' < 1$, *where*

$$\begin{aligned} Q &= \left[A_1 q_4 + A_2 q_3 + A_3 q_5 + \frac{A_4}{\Gamma(\beta-\alpha+1)}\left(q_1 + \frac{q_2}{\Gamma(p+1)}\right) \right. \\ &\quad + \frac{A_5}{\Gamma(\beta+1)}\left(q_1 + \frac{q_2}{\Gamma(p+1)}\right) + \frac{|\mu\lambda|}{|(1+\mu)|\Gamma(\alpha+1)} \\ &\quad \left. + \frac{|\mu|}{|(1+\mu)|\Gamma(\alpha+\beta+1)}\left(q_1 + \frac{q_2}{\Gamma(p+1)}\right)\right], \\ r_1' &= \left|\frac{\lambda}{\Gamma(\alpha+1)}\right| + \frac{|\mu\lambda|}{(|1+\mu|)\Gamma(\alpha+1)}. \end{aligned} \tag{21}$$

Proof. Consider the closed ball $B_r{}' = \{x \in X : \|x\| \le r'\}$ with the radius $r' \ge (r'_2/(1 - r'_1))$, where

$$\begin{aligned} r'_2 &= \frac{\|m\|}{\Gamma(\alpha+\beta+1)} + A_1\|\rho\| + A_2\|\phi\| + A_3\|\psi\| \\ &\quad + \frac{A_4\|m\|}{\Gamma(\beta-\alpha+1)} + \frac{A_5\|m\|}{\Gamma(\beta+1)} + \frac{|\mu|\|m\|}{(|1+\mu|)\Gamma(\alpha+\beta+1)}. \end{aligned} \tag{22}$$

We introduce the decomposition $P = P_1 + P_2$, where

$$\begin{aligned} P_1x(t) &= \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} f(s, x(s), I^p x(s))ds \\ &\quad - \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} x(s)ds, \\ P_2x(t) &= A_1(t)\int_0^1 h(s; x(s))ds + A_2(t)\int_0^1 g(s; x(s))ds \\ &\quad + A_3(t)\int_0^1 k(s; x(s))ds + \frac{A_4(t)}{\Gamma(\beta-\alpha)}\int_0^1 \\ &\quad \cdot (1-s)^{\beta-\alpha-1} f(s, x(s), I^p x(s))ds + \frac{A_5(t)}{\Gamma(\beta)}\int_0^1 \\ &\quad \cdot (1-s)^{\beta-1} f(s, x(s), I^p x(s))ds + \frac{\mu\lambda}{(1+\mu)\Gamma(\alpha)} \\ &\quad \cdot \int_0^1 (1-s)^{\alpha-1} x(s)ds - \frac{\mu}{(1+\mu)\Gamma(\alpha+\beta)} \\ &\quad \cdot \int_0^1 (1-s)^{\alpha+\beta-1} f(s, x(s), I^p x(s))ds. \end{aligned} \tag{23}$$

For $x, y \in B_r$, we have

$$\begin{aligned} &|P_1x(t) + P_2y(t)| \\ &\le \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|f(s, x(s), I^p x(s))|ds \\ &\quad + \left|\frac{\lambda}{\Gamma(\alpha)}\right|\int_0^t (t-s)^{\alpha-1}|x(s)|ds + |A_1(t)|\int_0^1 |h(s; y(s))|ds \\ &\quad + |A_2(t)|\int_0^1 |g(s; y(s))|ds + |A_3(t)|\int_0^1 |k(s; y(s))|ds \\ &\quad + \frac{|A_4(t)|}{\Gamma(\beta-\alpha)}\int_0^1 (1-s)^{\beta-\alpha-1}|f(s, y(s), I^p y(s))|ds \\ &\quad + \frac{|A_5(t)|}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}|f(s, y(s), I^p y(s))|ds \\ &\quad + \frac{|\mu\lambda|}{(|1+\mu|)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}|y(s)|ds \\ &\quad + \frac{|\mu|}{(|1+\mu|)\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}|f(s, y(s), I^p y(s))|ds \\ &\le \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} m(s)ds \\ &\quad + \left|\frac{\lambda}{\Gamma(\alpha)}\right|\int_0^t (t-s)^{\alpha-1}|x(s)|ds + A_1\int_0^1 \rho(s)ds \\ &\quad + A_2\int_0^1 \phi(s)ds + A_3\int_0^1 \psi(s)ds + \frac{A_4}{\Gamma(\beta-\alpha)}\int_0^1 \\ &\quad \cdot (1-s)^{\beta-\alpha-1} m(s)ds + \frac{A_5}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} m(s)ds \\ &\quad + \frac{|\mu\lambda|}{(|1+\mu|)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}|y(s)|ds \\ &\quad + \frac{|\mu|}{(|1+\mu|)\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1} m(s)ds, \end{aligned} \tag{24}$$

which implies that

$$\begin{aligned} \|P_1x + P_2y\| &\le \frac{\|m\|}{\Gamma(\alpha+\beta+1)} + \left|\frac{\lambda}{\Gamma(\alpha+1)}\right| r' + A_1\|\rho\| \\ &\quad + A_2\|\phi\| + A_3\|\psi\| + \frac{A_4\|m\|}{\Gamma(\beta-\alpha+1)} \\ &\quad + \frac{A_5\|m\|}{\Gamma(\beta+1)} + \frac{|\mu\lambda|}{(|1+\mu|)\Gamma(\alpha+1)} r' \\ &\quad + \frac{|\mu|\|m\|}{(|1+\mu|)\Gamma(\alpha+\beta+1)} \le r'. \end{aligned} \tag{25}$$

Thus, $P_1x + P_2y \subset B_r{}'$.

For $x, y \in B_r'$, we have

$$\begin{aligned}
&|P_2x(t) - P_2y(t)| \\
&\quad \le |A_1(t)| \int_0^1 |h(s; x(s))) - h(s; y(s))| ds \\
&\qquad + |A_2(t)| \int_0^1 |g(s; x(s)) - g(s; y(s))| ds \\
&\qquad + |A_3(t)| \int_0^1 |k(s; x(s)) - k(s; y(s))| ds \\
&\qquad + \frac{|A_4(t)|}{\Gamma(\beta - \alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} |f(s, x(s), I^p x(s)) \\
&\qquad - f(s, y(s), I^p y(s))| ds + \frac{|A_5(t)|}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \\
&\qquad \cdot |f(s, x(s), I^p x(s)) - f(s, y(s), I^p y(s))| ds \\
&\qquad + \frac{|\mu\lambda|}{|(1+\mu)|\,\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |x(s) - y(s)| ds \\
&\qquad + \frac{|\mu|}{|(1+\mu)|\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} \\
&\qquad \cdot |f(s, x(s), I^p x(s)) - f(s, y(s), I^p y(s))| ds \\
&\quad \le A_1 \int_0^1 q_4 |x(s) - y(s)| ds + A_2 \int_0^1 q_3 |x(s) - y(s)| ds \\
&\qquad + A_3 \int_0^1 q_5 |x(s) - y(s)| ds + \frac{A_4}{\Gamma(\beta - \alpha)} \int_0^1 \\
&\qquad \cdot (1-s)^{\beta-\alpha-1} (q_1 |x(s) - y(s)| + q_2 |I^p x(s) - I^p y(s)|) ds \\
&\qquad + \frac{A_5}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} (q_1 |x(s) - y(s)| \\
&\qquad + q_2 |I^p x(s) - I^p y(s)|) ds + \frac{|\mu\lambda|}{|(1+\mu)|\,\Gamma(\alpha)} \int_0^1 \\
&\qquad \cdot (1-s)^{\alpha-1} |x(s) - y(s)| ds + \frac{|\mu|}{|(1+\mu)|\Gamma(\alpha+\beta)} \int_0^1 \\
&\qquad \cdot (1-s)^{\alpha+\beta-1} (q_1 |x(s) - y(s)| + q_2 |I^p x(s) - I^p y(s)|) ds,
\end{aligned} \tag{26}$$

which implies that

$$\begin{aligned}
\|P_2x - P_2y\| &\le A_1 q_4 \|x - y\| + A_2 q_3 \|x - y\| + A_3 q_5 \|x - y\| \\
&\quad + \frac{A_4}{\Gamma(\beta - \alpha + 1)} \left(q_1 + \frac{q_2}{\Gamma(p+1)} \right) \|x - y\| \\
&\quad + \frac{A_5}{\Gamma(\beta+1)} \left(q_1 + \frac{q_2}{\Gamma(p+1)} \right) \|x - y\| \\
&\quad + \frac{|\mu\lambda|}{|(1+\mu)|\,\Gamma(\alpha+1)} \|x - y\| + \frac{|\mu|}{|(1+\mu)|\,\Gamma(\alpha+\beta+1)} \\
&\quad \cdot \left(q_1 + \frac{q_2}{\Gamma(p+1)} \right) \|x - y\| \le \Bigg[A_1 q_4 + A_2 q_3 + A_3 q_5 \\
&\quad + \frac{A_4}{\Gamma(\beta - \alpha + 1)} \left(q_1 + \frac{q_2}{\Gamma(p+1)} \right) + \frac{A_5}{\Gamma(\beta+1)} \\
&\quad \cdot \left(q_1 + \frac{q_2}{\Gamma(p+1)} \right) + \frac{|\mu\lambda|}{|(1+\mu)|\,\Gamma(\alpha+1)} \\
&\quad + \frac{|\mu|}{|(1+\mu)|\,\Gamma(\alpha+\beta+1)} \left(q_1 + \frac{q_2}{\Gamma(p+1)} \right) \Bigg] \|x - y\| \\
&\le Q \|x - y\|,
\end{aligned} \tag{27}$$

since $Q < 1$, then P_2 is a contraction.

Next, we show that P_1 is compact and continuous. Continuity of f implies that the operator P_1 is continuous.

Since, $\|P_1x\| \le (\|m\|/(\Gamma(\alpha+\beta+1))) + |(\lambda/(\Gamma(\alpha+1)))|\, r'$, therefore, P_1 is uniformly bounded on B_r'. Suppose that $0 \le t_1 < t_2 \le 1$. We have

$$\begin{aligned}
&|P_1x(t_2) - P_1x(t_1)| \\
&\quad \le \frac{1}{\Gamma(\alpha+\beta)} \Bigg| \int_0^{t_1} \left((t_2 - s)^{\alpha+\beta-1} - (t_1 - s)^{\alpha+\beta-1} \right) f \\
&\qquad \cdot (s, x(s), I^p x(s)) ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha+\beta-1} f(s, x(s), I^p x(s)) ds \Bigg| \\
&\quad \le + \frac{|\lambda|}{\Gamma(\alpha)} \Bigg| \int_0^{t_1} \left((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right) x(s) ds \\
&\qquad + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} x(s) ds \Bigg| \frac{\|m\|}{\Gamma(\alpha+\beta+1)} \Big[2(t_2 - t_1)^{\alpha+\beta} \\
&\qquad + \left| t_1^{\alpha+\beta} - t_2^{\alpha+\beta} \right| \Big] + \frac{|\lambda| r}{\Gamma(\alpha+1)} [2(t_2 - t_1)^{\alpha} + |t_1^{\alpha} - t_2^{\alpha}|].
\end{aligned} \tag{28}$$

As $t_1 \to t_2$, the above expression tends to be zero independently from $x \in B_r'$. This implies that P_1 is relatively compact on B_r'. Then, by the Arzela-Ascoli theorem, operator P_1 is compact on B_r'.

Therefore, according to the Krasnoselskii fixed point theorem, problem (1) has at least one solution on B_r'.

4. Example

Example 8. Consider the following boundary value problem:

$$\begin{cases}
{}^cD^{3/4} \left({}^cD^{1/3} + \dfrac{1}{300} \right) x(t) = \dfrac{1}{500 + t^2} \left(\sin x + \dfrac{1}{\Gamma(3/4)} \displaystyle\int_0^t (t-s)^{-1/4} x ds \right), \quad t \in [0, 1], \\
x(0) + x(1) = \dfrac{1}{200} \displaystyle\int_0^1 \dfrac{|x(s)|}{300 + |x(s)|} ds, \\
{}^cD^{1/3} x(0) + {}^cD^{1/3} x(1) = \dfrac{1}{200} \displaystyle\int_0^1 \left(\dfrac{1}{s+2} \right)^3 \dfrac{|x(s)|}{30 + |x(s)|} ds, \\
{}^cD^{2/3} x(0) + {}^cD^{2/3} x(1) = \dfrac{1}{200} \displaystyle\int_0^1 \left(\dfrac{1}{s+4} \right)^2 \dfrac{|x(s)|}{30 + |x(s)|} ds.
\end{cases} \tag{29}$$

We choose $\beta = 4/3$, $\alpha = 1/3$, $\lambda = 1/300$, $\mu = 1$, $\sigma_1 = 1/200$, $\sigma_2 = 1/200$, $\sigma_3 = 1/200$, $p = 1/4$, and the continuous functions

$$f\left(t, x, I^{1/4}x\right) = \frac{1}{500 + t^2}\left(\sin x + \frac{1}{\Gamma(3/4)}\int_0^t (t-s)^{-1/4} x\, ds\right),$$

$$g(t,x) = \frac{|x(t)|}{300 + |x(t)|},$$

$$h(t,x) = \left(\frac{1}{t+2}\right)^3 \frac{|x(t)|}{30 + |x(t)|},$$

$$k(t,x) = \left(\frac{1}{t+4}\right)^2 \frac{|x(t)|}{30 + |x(t)|}. \tag{30}$$

Clearly, $q_1 = q_2 = 1/500$, $q_3 = 1/300$, $q_4 = 1/240$, and $q_5 = 1/480$,

Then, we have $r_1 \approx 0.07 < 1$.

Thus, all the assumptions of Theorem 2 hold. Then, problem (29) has a unique solution.

Example 9. Consider the following boundary value problem:

$$\begin{cases} {}^cD^{3/4}\left({}^cD^{1/3} + \dfrac{1}{300}\right)x(t) = \dfrac{1}{(t+10)^2}\left(\dfrac{1}{5+x^2} + \displaystyle\int_0^t (t-s)^{-1/4}\frac{1}{5+x^2}\,ds\right), \quad t \in [0,1], \\ x(0) + x(1) = \dfrac{1}{200}\displaystyle\int_0^1 \frac{1}{s+10}\frac{1}{30+|x(s)|}\,ds, \\ {}^cD^{1/3}x(0) + {}^cD^{1/3}x(1) = \dfrac{1}{200}\displaystyle\int_0^1 \left(\frac{1}{s+2}\right)^3 \frac{|x(s)|}{30+|x(s)|}\,ds, \\ {}^cD^{2/3}x(0) + {}^cD^{2/3}x(1) = \dfrac{1}{200}\displaystyle\int_0^1 \left(\frac{1}{s+4}\right)^2 \frac{|x(s)|}{30+|x(s)|}\,ds. \end{cases} \tag{31}$$

Here, $\beta = 3/4$, $\alpha = 1/3$, $\lambda = 1/30$, $\mu = 1$, $\sigma_1 = 1/200$, $\sigma_2 = 1/200$, $\sigma_3 = 1/200$, $p = 1/4$, and the continuous functions

$$f\left(t, x, I^{1/4}x\right) = \frac{1}{(t+10)^2}\left(\frac{1}{5+x^2} + \int_0^t (t-s)^{-1/4}\frac{1}{5+x^2}\,ds\right),$$

$$g(t,x) = \frac{1}{t+10}\frac{|x(t)|}{30+|x(t)|},$$

$$h(t,x) = \left(\frac{1}{t+2}\right)^3 \frac{|x(t)|}{30+|x(t)|},$$

$$k(t,x) = \left(\frac{1}{t+4}\right)^2 \frac{|x(t)|}{30+|x(t)|}. \tag{32}$$

Clearly, $q_1 = q_2 = 1/500$, $q_3 = 1/300$, $q_4 = 1/240$, and $q_5 = 1/480$; thus, we have $Q \approx 0.057 < 1$ and $r'_1 \approx 0,0065 < 1$.

Then, problem (31) has a least one solution.

5. Conclusion

In this paper, we studied the existence and uniqueness of fractional Langevin equations with nonseparated integral boundary conditions. First of all, we transformed the problem into an equivalent fixed point problem; second of all, we utilized the Banach contraction principle and the Krasnoselskii fixed point theorem to prove the existence and uniqueness of solutions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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