

Research Article

Iterative Analysis of Nonlinear BBM Equations under Nonsingular Fractional Order Derivative

Gauhar Ali,¹ Israr Ahmad,¹ Kamal Shah ,¹ and Thabet Abdeljawad ^{2,3,4}

¹Department of Mathematics, University of Malakand, Chakdara Dir (Lower), Khyber Pakhtunkhwa, Pakistan

²Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia

³Department of Medical Research, China Medical University, Taichung 40402, Taiwan

⁴Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan ”

Correspondence should be addressed to Thabet Abdeljawad; tabdeljawad@psu.edu.sa

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The present research work is devoted to investigate fractional order Benjamin-Bona-Mahony (FBBM) as well as modified fractional order FBBM (FMBBM) equations under nonlocal and nonsingular derivative of Caputo-Fabrizio (CF). In this regards, some qualitative results including the existence of at least one solution are established via using some fixed point results of Krasnoselskii and Banach. Further on using an iterative method, some semianalytical results are also studied. The concerned tool is formed when the Adomian decomposition method is coupled with some integral transform like Laplace. Graphical presentations are given for various fractional orders. Also, the concerned method is also compared with some variational-type perturbation method to demonstrate the efficiency of the proposed method.

1. Introduction

Fractional calculus is the generalized form of classical calculus. With the rapid change in science and technology, the aforesaid area has attracted the attention of many researchers. The mentioned branch has many applications in different areas of science like modeling, control theory, physics, signal processing, economics, and chemistry [1–4]. Different researchers have studied fractional differential equations (FODEs) in their own way, including the stability aspect, qualitative theory, optimization, and numerical simulations. Many real-world problems are nonlinear in nature, and their investigation is important for fruitful information. Therefore, researchers have studied various problems of FODEs by using different techniques and methods. One of the important aspects is the existence theory of solution which has given proper attention in the last years [5–11]. By using the fixed point theory, the existence theory to numerous problems has been established [12–16]. The

authors in [17–22] also studied different aspects of FODEs using a derivative with nonsingular kernel and Laplace transform. Therefore, we intend to establish the aforementioned theory for the following problem with $[0, \tau] = J$

$$\begin{aligned} {}^{CF}D_t^\gamma v(t, y) - h(t, v(t, y)) &= 0, \\ v(y, 0) &= f(y), \end{aligned} \quad (1)$$

where $h : J \times R \rightarrow R$ and $f \in C(J)$. The existence of at least one solution of (1) has been studied with the help of a fixed point approach, since the differential operator involving fractional order have a great degree of freedom. Therefore, it comprehensively describes many dynamical properties and characteristic of various processes/phenomena [23, 24]. Then, we establish an algorithm to compute the approximate analytical solutions for the following cases of BBM equations with $y, t \in J, \gamma \in (0, 1]$ as

Case 1.

$${}^{CF}D_t^\gamma v(t, y) - v_{yyt}(t, y) + av(t, y)v_y(t, y) = 0, \tag{2}$$

$$v(0, y) = f(y).$$

Case 2.

$${}^{CF}D_t^\gamma v(t, y) + v_y(t, y) + av^2(t, y)v_y(t, y) + v_{yyy}(t, y) = 0, \tag{3}$$

$$v(y, 0) = f(y).$$

Case 3.

$${}^{CF}D_t^\gamma v(t, y) + v_y(t, y) + av^2(t, y)v_y(t, y) + v_{yyy}(t, y) = 0, \tag{4}$$

$$v(y, 0) = f(y).$$

where a is a real constant. The abovementioned problems are also called regularized long-wave equation which is the improved form of the Korteweg-de Vries equation (KDVE). Such equation has been largely used for modeling of waves of small amplitudes and in the soliton theory of fractals and dynamics. Moreover, KDVE has countless integrals of motion and BBM has only three [25–32]. For generalized n -dimensional BBM equation and its applications, we refer to [25, 33, 34]. The aforementioned equation has been studied in surface waves of a long period of fluid [26]. Also, for the dynamic aspect of the BBM equation, we refer [35]. The mentioned equation is not only suitable for superficial waves but also for acoustic and hydromagnetic waves; because of this, the BBM equation has upper hand on KDVE. We enrich our study by investigating the modified form of BBM equation abbreviated as MBBM [36]. We use the decomposition method coupled with Laplace transform to establish series solution to our proposed problems (2), (3), and (4). The mentioned problems have been studied by the homotopy perturbation method (HPM), variational method (VHPM), wavelet method, etc., but these studies are limited to fractional order derivative involving the usual Caputo and integer order derivative. To the best of authors’ information, no study exists in the present literature to address the investigation of the aforesaid problems under nonsingular CF derivative. The mentioned derivative was introduced in 2016 and has been found suitable in applications of many thermal problems. The concerned nonlocal integral of CF for a function is the average of the function and its Riemann integral which works as a filter, for various applications of the concerned derivative, we refer to [12, 13, 18, 19]. So far, we know that there is no investigation present in the literature which addresses the study of the mentioned problems under nonlocal and nonsingular kernel derivatives with fractional order. We establish some qualitative results of the existence of at least one solution by Krasnoselskii and Banach fixed point results. Further, by the proposed method of Laplace transform coupled with Adomian decomposition (LADM), we compute the series solution whose convergence is also stud-

ied. Also, the results are compared with the results of VHPM. The results reveal that the proposed method can also be used as a powerful tool to find approximate results to many nonlinear problems.

2. Preliminaries

Definition 1 (see [37]). Let $v \in H^1(0, a)$, $a > 0$, $\gamma \in (0, 1)$, then CF derivative is defined below

$${}^{CF}D_t^\gamma v(t) = \frac{N(\gamma)}{(1-\gamma)} \int_0^t \exp\left[-\frac{\gamma(t-s)}{1-\gamma}\right] v'(\mu) d\eta, \tag{5}$$

$$\gamma \in (0, 1), t \geq 0,$$

where the function $N(\gamma)$ is called normalization.

Definition 2 (see [38]). The CF integral with $\gamma \in (0, 1)$ is given below

$${}^{CF}J_t^\gamma v(t) = \frac{(1-\gamma)v(t)}{N(\gamma)} + \frac{\gamma}{N(\gamma)} \int_0^t v(\eta) d\eta. \tag{6}$$

Definition 3 (see [37]). For the CF derivative of order $\gamma \in (0, 1]$ and $n \in \mathbb{N}$, the Laplace transform is given below

$$\begin{aligned} \mathcal{L}\left({}^{CF}D_t^{n+\gamma} v(t)\right)(s) &= \frac{1}{(1-\gamma)} \mathcal{L}\left(v^{(n+1)}(t)\right) \mathcal{L}\left(\exp\left(-\frac{\gamma t}{1-\gamma}\right)\right) \\ &= \frac{s^{n+1} \mathcal{L}[v(t)] - s^n v(0) - s^{n-1} v'(0) - \dots - v^n(0)}{s + \gamma(1-s)}. \end{aligned} \tag{7}$$

Definition 4. The considered method is used to compute the solution in an infinite series form. We consider the solution as

$$v(t, y) = \sum_{n=0}^{\infty} v_n(y, t) \tag{8}$$

and nonlinear term is decompose as

$$\mathcal{N}v = \sum_{n=0}^{\infty} \mathcal{A}_n, \tag{9}$$

where \mathcal{A}_n is given by

$$\mathcal{A}_n = \frac{1}{\Gamma(n+1)} D_\mu^n \left[\mathcal{N} \left(\sum_{j=0}^n \mu^j v_j \right) \right]_{\mu=0}. \tag{10}$$

Theorem 5 (Krasnoselskii’s fixed point theorem [39]). *If $D \subset X$ be a convex and closed nonempty subset, there exist two operators \mathcal{G}_1 and \mathcal{G}_2 such that*

- (i) $\mathcal{G}_1 v_1 + \mathcal{G}_2 v_2 \in D$ for all $v_1, v_2 \in D$
- (ii) \mathcal{G}_1 is a condensing operator

(iii) \mathcal{G}_2 is continuous and compact

then, there exists at least one solution $v \in D$ which satisfies $\mathcal{G}_1(v) + \mathcal{G}_2(v) = v$.

3. Steps for Existence of Results

In the ongoing section, we discuss the existence of the considered problem.

Lemma 6. Under Definitions (1) and (2), we have

$$v(t, y) = f(y) + \frac{(1 - \gamma)}{\mathbb{N}(\gamma)} [h(t, v(t, y)) - h(0, v(0, y))] + \frac{\gamma}{\mathbb{N}(\gamma)} \int_0^t h(\theta, v(\theta, y)) d\theta. \quad (11)$$

The assumptions needed for our work are (B_1) $h(t, v)$ is the nonlinear function satisfy the growth condition as

$$|h(t, v)| \leq b_h + C|v|^p, p \in (0, 1), C \geq 0. \quad (12)$$

(B_2) For all $v_1, v_2 \in R$ there exist a positive constant k_h one can get,

$$|h(t, v_1) - h(t, v_2)| \leq k_h |v_1 - v_2|, \text{ for all } t \in \mathcal{J}. \quad (13)$$

Furthermore, $h(t, 0) = 0$ holds.

$\mathcal{G}_1, \mathcal{G}_2 : X \rightarrow X$ are the operators defined as

$$\begin{cases} \mathcal{G}_1(t, v) = f(y) + \frac{(1 - \gamma)}{\mathbb{N}(\gamma)} [h(t, v(t, y)) - h(0, v(0, y))], \\ \mathcal{G}_2(t, v) = \frac{\gamma}{\mathbb{N}(\gamma)} \int_0^t h(\theta, v(\theta, y)) d\theta. \end{cases} \quad (14)$$

Theorem 7. In light of hypothesis (B_1) and (B_2) , if $(1 - \gamma)/(\mathbb{N}(\gamma))k_h \leq 1$, then (1) has at least one solution.

Proof. Using (2.5), and a bounded set defined as $D = \{v \in X : \|v\|_X \leq R\}$. The continuity of $v(t, y)$ implies that \mathcal{G}_1 and \mathcal{G}_2 are continuous operators. To show that \mathcal{G}_1 is a condensing map, consider $v_1, v_2 \in D$, under the assumption (B_1)

$$\begin{aligned} & \|\mathcal{G}_1(v_1) - \mathcal{G}_1(v_2)\|_X \\ &= \max_{t \in \mathcal{J}} \left| \frac{(1 - \gamma)}{\mathbb{N}(\gamma)} h(t, v_1(t, y)) - \frac{(1 - \gamma)}{\mathbb{N}(\gamma)} h(t, v_2(t, y)) \right| \\ &\leq \frac{(1 - \gamma)}{\mathbb{N}(\gamma)} k_h \|v_2 - v_1\|_X. \end{aligned} \quad (15)$$

This show that G_1 is a condensing map; further, for the continuity and compactness of G_2 for all $v \in D$, consider

$$\begin{aligned} \|\mathcal{G}_2(v)\|_X &= \max_{t \in \mathcal{J}} \left| \frac{\gamma}{\mathbb{N}(\gamma)} \int_0^t h(\theta, v(\theta, y)) d\theta \right| \\ &\leq \frac{\gamma}{\mathbb{N}(\gamma)} \max_{t \in \mathcal{J}} \int_0^t |h(\theta, v(\theta, y))| d\theta \\ &\leq \frac{\gamma}{\mathbb{N}(\gamma)} \int_0^t [b_h + C|v|^p] d\theta \\ &\leq \frac{\gamma}{\mathbb{N}(\gamma)} [b_h + CR^p] \tau. \end{aligned} \quad (16)$$

Therefore, G_2 is bounded on D . For continuity considering $t_1 - t_2 > 0$, one can infer that

$$\begin{aligned} & |\mathcal{G}_2 v_1(t_1) - \mathcal{G}_2 v_2(t_2)| \\ &= \left| \frac{\gamma}{\mathbb{N}(\gamma)} \left[\int_0^{t_1} h(\theta, v(\theta, y)) d\theta - \int_0^{t_2} h(\theta, v(\theta, y)) d\theta \right] \right| \\ &\leq \frac{\gamma}{\mathbb{N}(\gamma)} [b_h + CR^p] (t_1 - t_2). \end{aligned} \quad (17)$$

This implies that $\|\mathcal{G}_2 v_1(t_1) - \mathcal{G}_2 v_2(t_2)\|_X \rightarrow 0$, as t_1 tends to t_2 . So it shows that \mathcal{G}_2 is compact and equicontinuous; by Theorem 1, the problem (1) has no less than one solution in D .

Theorem 8. In view of assumption (B_2) if $((1 + \gamma(\tau - 1))/(\mathbb{N}(\gamma)))k_h$, then problem (1) has a unique solution.

Proof. By using (1), we define the operator G as

$$\begin{aligned} Gv(t, y) &= f(y) + \frac{(1 - \gamma)}{\mathbb{N}(\gamma)} [h(t, v(t, y)) - h(0, v(0, y))] \\ &\quad + \frac{\gamma}{\mathbb{N}(\gamma)} \int_0^t h(\theta, v(\theta, y)) d\theta. \end{aligned} \quad (18)$$

Suppose $v_1, v_2 \in X$, we have

$$\begin{aligned} & \|Gv_1 - Gv_2\| \leq \max_{t \in \mathcal{J}} \left| \frac{(1 - \gamma)}{\mathbb{N}(\gamma)} [h(t, v_1(t, y)) - h(t, v_2(t, y))] \right| \\ &\quad + \max_{t \in \mathcal{J}} \int_0^t [h(\theta, v_1(\theta, y)) - h(\theta, v_2(\theta, y))] d\theta \\ &\leq \frac{(1 - \gamma)}{\mathbb{N}(\gamma)} k_h \|v_1 - v_2\|_X + \frac{\gamma}{\mathbb{N}(\gamma)} \tau k_h \|v_1 - v_2\|_X \\ &= \left(\frac{1 + \gamma(\tau - 1)}{\mathbb{N}(\gamma)} \right) k_h \|v_1 - v_2\|_X. \end{aligned} \quad (19)$$

Therefore, G is a condensing operator which implies the uniqueness of solution.

4. Main Results

To present the iterative solution of our considered problem, we first give a general procedure for the given problem as

$$\begin{cases} {}^{CF}D_t^\gamma v(t, y) = \mathcal{N}v(t, y) + \mathcal{R}v + g(t, y), \\ v(0, y) = f(y) \quad y \in [0, 1], \end{cases} \quad (20)$$

where \mathcal{N} is a nonlinear operator and \mathcal{R} is a linear operator and g is external source function. Further, $f : J \rightarrow R$ is a nonlocal, bounded, and continuous function.

Taking Laplace transform of (14) and using the initial condition, we have

$${}^{CF}D_t^\gamma [v(t, y)] = \frac{f(y)}{s} + \frac{s + \gamma(1-s)}{s} \mathcal{L}[Nv(t, y) + \mathcal{R}v(t, y) + g(t, y)]. \quad (21)$$

Let us consider the solution in terms of a series as

$$v = \sum_{n=0}^{\infty} v_n, \quad (22)$$

and decompose the nonlinear term $\mathcal{N}v(t, y)$ in terms of the Adomian polynomial as

$$\mathcal{N}v = \sum_{n=0}^{\infty} \mathcal{A}_n, \quad (23)$$

where

$$\mathcal{A}_n = \frac{1}{\Gamma(n+1)} D_u^n \left[\mathcal{N} \left(\sum_{j=0}^n \mu^j v_j \right) \right] \Big|_{\mu=0}. \quad (24)$$

Using (15) and comparing the terms on both sides, we have

$$\begin{aligned} v_0 &= f(y), \\ v_1 &= \mathcal{L}^{-1} \left[\frac{\gamma + s(1-\gamma)}{s} \mathcal{L}(\mathcal{A}_0(t, y) \mathcal{R}v_0(t, y) + g(t, y)) \right], \\ &\vdots \\ v_{n+1} &= \mathcal{L}^{-1} \left[\frac{\gamma + s(1-\gamma)}{s} \mathcal{L}(\mathcal{A}_n(t, y) \mathcal{R}v_n(t, y) + g(t, y)) \right], \quad n \geq 0. \end{aligned} \quad (25)$$

After evaluation, the required solution is

$$v(t, y) = \sum_{n=0}^{\infty} v_n(t, y) = v_0(t, y) + v_1(t, y) + v_2(t, y) + \dots \quad (26)$$

Theorem 9. Let T be a nonlinear contractive operator on a Banach space X , such that for all $v, v^* \in X$, one has

$$\|Tv - Tv^*\|_X \leq k \|v - v^*\|_X, \quad 0 < k < 1. \quad (27)$$

Then, the unique fixed point v satisfies the relation $Tv = v$. Let us write the generated series (26) as

$$v_n = T(v_{n-1}), \quad v_{n-1} = \sum_{i=0}^{n-1} v_i, \quad n = 1, 2, 3, \dots, \quad (28)$$

and assume that $v_0 \in S_r(v)$, where $S_r(v) = \{v^* \in X : \|v - v^*\|_X \leq r, r \geq 0\}$. Then, we have

$$\begin{aligned} (A_1) \quad &x_n \in S_r(v). \\ (A_2) \quad &\lim_{n \rightarrow \infty} v_n = v. \end{aligned} \quad (29)$$

Proof. (A₁) By using mathematical induction for $n = 1$, we have

$$\|v_1 - v\|_X = \|Tv_0 - Tv\|_X \leq \|kv_0 - v\|_X. \quad (30)$$

Considering that the result for $n - 1$ is true, then

$$\|v_{n-1} - v\|_X \leq k^{n-1} \|v_0 - v\|_X. \quad (31)$$

Now consider

$$\|v_n - v\|_X = \|Tv_{n-1} - Tv\|_X \leq k \|v_{n-1} - v\|_X \leq k^n \|v_0 - v\|_X. \quad (32)$$

With the help of (A₁), we have

$$\|v_n - v\|_X \leq k^n \|v_0 - v\|_X \leq k^n r \leq r, \quad (33)$$

which gives that $v_n \in S_r(v)$, since

$$\|v_n - v\|_X \leq k^n \|v_0 - v\|_X, \quad (34)$$

and $\lim_{n \rightarrow \infty} k^n = 0$. Therefore, we have $\lim_{n \rightarrow \infty} \|v_n - v\|_X = 0$ which yields $\lim_{n \rightarrow \infty} v_n = v$.

4.1. General Procedure for Case 1. Consider the following FBBM equation under the given condition as

$$\begin{cases} {}^{CF}D_t^\gamma v(t, y) - v_{yyt}(t, y) + av(t, y)v_y(t, y) = 0, \\ v(0, y) = f(y). \end{cases} \quad (35)$$

Taking Laplace transform of (35), one has

$$\mathcal{L}[v(t, y)] = \frac{f(y)}{s} + \frac{\gamma + s(1-\gamma)}{s} \mathcal{L}[v_{yyt}(t, y) - av(t, y)v_y(t, y)]. \quad (36)$$

Let us consider the solution in terms of a series as

$$v = \sum_{n=0}^{\infty} v_n, \quad (37)$$

and the decomposition of the nonlinear term is

$$v v_y = \sum_{n=0}^{\infty} \mathcal{A}_n, \tag{38}$$

where

$$\mathcal{A}_n = \frac{1}{\Gamma(n+1)} D_{\mu}^n \left[\left(\sum_{j=0}^n \mu^j v_j \right) \left(\sum_{j=0}^n \mu^j v_{jy} \right) \right] \Big|_{\mu=0}. \tag{39}$$

\mathcal{A}_n for different values of n are

$$\begin{aligned} \mathcal{A}_0 &= v_0(t, y) v_{0y}(t, y), \\ \mathcal{A}_1 &= v_0(t, y) v_{1y}(t, y) + v_{0y}(t, y) v_1(t, y), \end{aligned} \tag{40}$$

and so on. Putting these values in (36) and comparing the terms on both sides, we have

$$\begin{aligned} v_0(t, y) &= f(y), \\ v_1(t, y) &= \mathcal{L}^{-1} \left[\frac{\gamma + s(1-\gamma)}{s} \mathcal{L}(v_{0yyt}(t, y) - a v_0(t, y) v_{0y}(t, y)) \right], \\ &\vdots \\ v_{n+1}(t, y) &= \mathcal{L}^{-1} \left[\frac{\gamma + s(1-\gamma)}{s} \mathcal{L}(v_{nyyt}(t, y) - a v_n(t, y) v_{ny}(t, y)) \right], \quad n \geq 0. \end{aligned} \tag{41}$$

After calculation, the solution of the considered problem (35) is obtained in the form of a series.

4.2. *General Procedure for Case 2.* Consider the following FBBM equation under the given condition as

$$\begin{cases} {}^{CF}D_t^{\gamma} v(t, y) + v_y(t, y) + a v(t, y) v_y(t, y) + v_{yyy}(t, y) = 0, \\ v(0, y) = f(y). \end{cases} \tag{42}$$

Taking Laplace of (42), one may have

$$\mathcal{L}[v(t, y)] = \frac{f(y)}{s} - \frac{\gamma + s(1-\gamma)}{s} \mathcal{L}[v(t, y) + v_{yyy}(t, y) + a v(t, y) v_y(t, y)]. \tag{43}$$

Here, we consider the unknown solution as

$$v = \sum_{n=0}^{\infty} v_n, \tag{44}$$

and the nonlinear term is decomposed as

$$v(t, y) v_y = \sum_{n=0}^{\infty} \mathcal{A}_n, \tag{45}$$

where \mathcal{A}_n is define as

$$\mathcal{A}_n = \frac{1}{\Gamma(n+1)} D_{\mu}^n \left[\left(\sum_{j=0}^n \mu^j v_j \right) \left(\sum_{j=0}^n \mu^j v_{jy} \right) \right] \Big|_{\mu=0}. \tag{46}$$

\mathcal{A}_n for different values of n are

$$\begin{aligned} \mathcal{A}_0 &= v_0(t, y) v_{0y}(t, y), \\ \mathcal{A}_1 &= v_0(t, y) v_{1y}(t, y) + v_{0y}(t, y) v_1(t, y), \end{aligned} \tag{47}$$

and so on. Using these values in equation (43) and equating the corresponding terms on both sides, we have

$$\begin{aligned} v_0(t, y) &= f(y), \\ v_1 &= -\mathcal{L}^{-1} \left[\frac{\gamma + s(1-\gamma)}{s} \mathcal{L}(v_0(t, y) + v_{0yyy}(t, y) + a \mathcal{A}_0) \right], \\ v_2(t, y) &= -\mathcal{L}^{-1} \left[\frac{\gamma + s(1-\gamma)}{s} \mathcal{L}(v_1(t, y) + v_{1yyy}(t, y) + a \mathcal{A}_1) \right], \\ &\vdots \\ v_{n+1}(t, y) &= -\mathcal{L}^{-1} \left[\frac{\gamma + s(1-\gamma)}{s} \mathcal{L}(v_n(t, y) + v_{nyyy}(t, y) + a \mathcal{A}_n) \right], \quad n \geq 0. \end{aligned} \tag{48}$$

In this way, the series solution of the proposed problem (42) is obtained.

4.3. *Procedure for Case 3.* Consider the following FMBBM equation under the given condition

$$\begin{cases} {}^{CF}D_t^{\gamma} v(t, y) + v_y(t, y) + a v^2(t, y) v_y(t, y) + v_{yyy}(t, y) = 0, \\ v(0, y) = f(y). \end{cases} \tag{49}$$

Taking Laplace of (49) and after rearranging the terms, we have

$$\mathcal{L}[v(t, y)] = \frac{f(y)}{s} - \frac{\gamma + s(1-\gamma)}{s} \mathcal{L}[v(t, y) + v_{yyy}(t, y) + a v^2(t, y) v_y(t, y)]. \tag{50}$$

Here, we consider the unknown $v(t, y)$ as

$$v = \sum_{n=0}^{\infty} v_n, \tag{51}$$

and nonlinear term is decomposed as

$$v^2(t, y) v_y = \sum_{n=0}^{\infty} \mathcal{A}_n, \tag{52}$$

where \mathcal{A}_n is ‘‘Adomian polynomials’’ defined as

$$\mathcal{A}_n = \frac{1}{\Gamma(n+1)} D_\mu^n \left[\left(\sum_{j=0}^n \mu^j v_j \right)^2 \left(\sum_{j=0}^n \mu^j v_{jy} \right) \right]_{\mu=0}. \quad (53)$$

\mathcal{A}_n for different values of n are

$$\begin{aligned} \mathcal{A}_0 &= v_0^2(t, y) v_{0y}(t, y), \\ \mathcal{A}_1 &= v_0^2(t, y) v_{1y}(t, y) + 2v_0(t, y) v_{0y}(t, y) v_1(t, y), \end{aligned} \quad (54)$$

and so on. Putting these values in equation (50) and comparing terms on both sides, we have

$$\begin{aligned} v_0(t, y) &= g(y), \\ v_1(t, y) &= -\mathcal{L}^{-1} \left[\frac{\gamma + s(1-\gamma)}{s} L(v_0(t, y) + v_{0yyy}(t, y) + a\mathcal{A}_0) \right], \\ v_2(t, y) &= -\mathcal{L}^{-1} \left[\frac{\gamma + s(1-\gamma)}{s} L(v_1(t, y) + v_{1yyy}(t, y) + a\mathcal{A}_1) \right], \\ &\vdots \\ v_{n+1}(t, y) &= -\mathcal{L}^{-1} \left[\frac{\gamma + s(1-\gamma)}{s} L(v_n(t, y) + v_{nyyy}(t, y) + a\mathcal{A}_n) \right], \quad n \geq 0. \end{aligned} \quad (55)$$

Hence, in this case, the solution in same way may be computed.

5. Examples

Here, in the ongoing section, we find series solutions for (35), (42), and (49) with the help of LADM using CFFOD.

Example 1. Consider the following FBBM equation [40] as

$$\begin{aligned} {}^{CF}D_t^\gamma v(t, y) &= v_{yyt}(t, y) - v(t, y) v_y(t, y), \\ v(0, y) &= \sec h^2 \left(\frac{y}{4} \right). \end{aligned} \quad (56)$$

With the exact solution given below,

$$v(t, y) = \sec h^2 \left(\frac{y}{4} - \frac{t}{3} \right). \quad (57)$$

With the help of the procedure discussed in Case 1, one has

$$\begin{aligned} v_0 &= \sec h^2 \left(\frac{y}{4} \right), \\ v_1 &= -\frac{1}{2} (1 + \gamma(t-1)) \sec h^4 \left(\frac{y}{4} \right) \tanh \left(\frac{y}{4} \right), \\ v_2 &= -\frac{\gamma}{2} (1 + \gamma t - \gamma) \left[\sec h^6 \left(\frac{y}{4} \right) \tanh^3 \left(\frac{y}{4} \right) \right. \\ &\quad \left. + \frac{3}{2} \sec h^8 \left(\frac{y}{4} \right) \tanh \left(\frac{y}{4} \right) \right] \\ &\quad + \frac{1}{2} \left(1 + \frac{(\gamma t)^2}{2} - 2\gamma^2 t + \gamma^2 \right) \\ &\quad \cdot \left[\sec h^6 \left(\frac{y}{4} \right) \tanh^2 \left(\frac{y}{4} \right) + \frac{1}{4} \sec h^8 \left(\frac{y}{4} \right) \right. \\ &\quad \left. + \frac{1}{2} \sec h^6 \left(\frac{y}{4} \right) \tanh^2 \left(\frac{y}{4} \right) \right]. \end{aligned} \quad (58)$$

And hence, the solution of (56) in the form of a series is given by

$$\begin{aligned} v(t, y) &= \sec h^2 \left(\frac{y}{4} \right) - \frac{1}{2} (1 + \gamma(t, y)) \sec h^4 \left(\frac{y}{4} \right) \tanh \left(\frac{y}{4} \right) \\ &\quad - \frac{\gamma}{2} (1 + \gamma t - \gamma) \left[\sec h^6 \left(\frac{y}{4} \right) \tanh^3 \left(\frac{y}{4} \right) \right. \\ &\quad \left. + \frac{3}{2} \sec h^8 \left(\frac{y}{4} \right) \tanh \left(\frac{y}{4} \right) \right] \\ &\quad + \frac{1}{2} \left(1 + \frac{(\gamma t)^2}{2} - 2\gamma^2 t + \gamma^2 \right) \\ &\quad \cdot \left[\sec h^6 \left(\frac{y}{4} \right) \tanh^2 \left(\frac{y}{4} \right) + \frac{1}{4} \sec h^8 \left(\frac{y}{4} \right) \right. \\ &\quad \left. + \frac{1}{2} \sec h^6 \left(\frac{y}{4} \right) \tanh^2 \left(\frac{y}{4} \right) \right]. \end{aligned} \quad (59)$$

The approximate solution graphs for various fractional orders are given in Figure 1. We see from graphs as the order $\gamma \rightarrow 1$, the behavior of the surfaces of the solution tends to the integer order. If we put $\gamma = 1$ in the approximate solution, we get the solution at the integer order. Now, we compare the four-term LADM solution with the four-term solution of VHPM given in [40] in Table 1 at $\gamma = 1$. From Table 1, we see that the absolute error between exact solutions and four-term LADM solutions at the integer order is slightly good than the absolute error for the mentioned four-term solution by using the VHPM. As compared to VHPM, the LADM is simple and easy to use to handle various nonlinear partial differential equations.

Example 2. Consider the FBBM equation using CFFOD as

$$\begin{aligned} {}^{CF}D_t^\gamma v(t, y) + v_y(t, y) \\ + v(t, y) v_y(t, y) + v_{yyy}(t, y) &= 0, \\ v(0, y) &= e^y. \end{aligned} \quad (60)$$

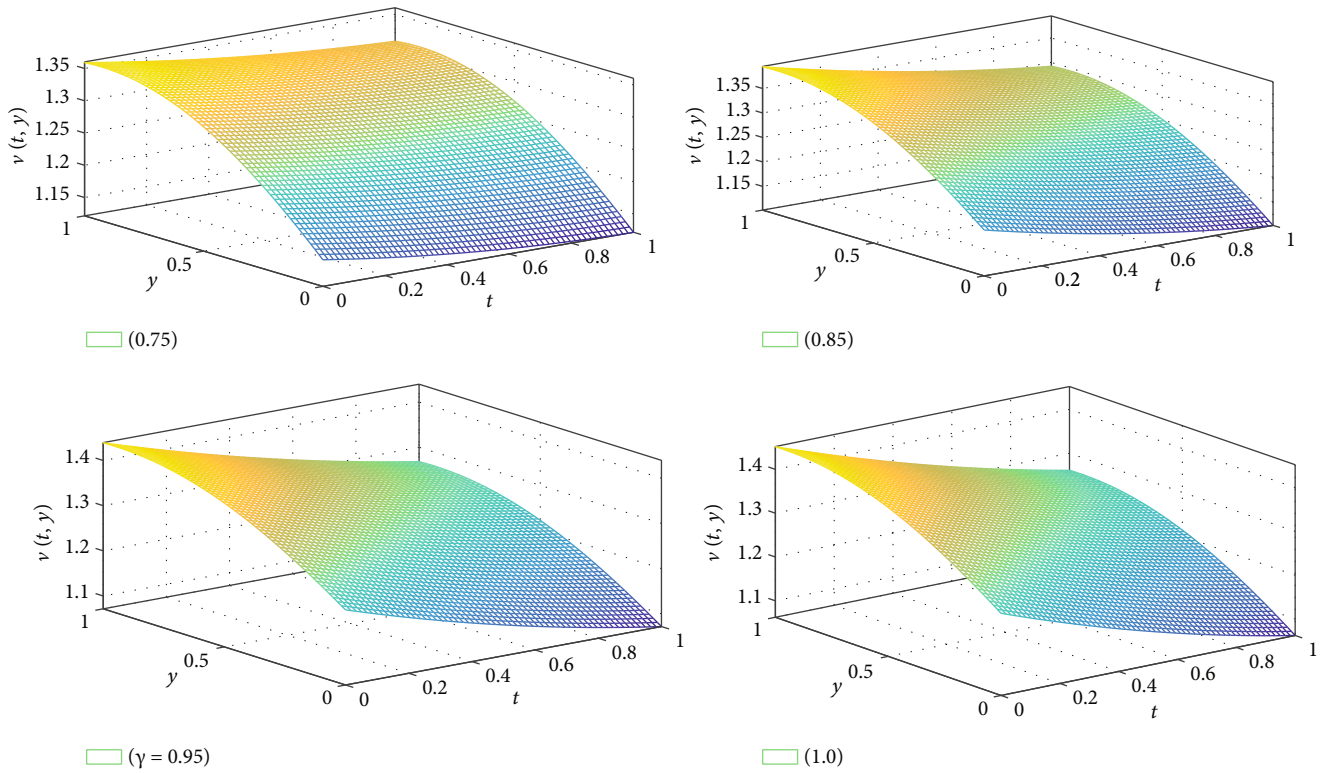


FIGURE 1: Surface plots of the required solution up to four terms at different values of γ for Example 1.

TABLE 1: Comparison between the absolute error at VHPM [40] and the present method (LADM) with the exact solution of Example 1.

$y \rightarrow$	0.03		0.04		0.05	
t	VHPM	LADM	VHPM	LADM	VHPM	LADM
0.01	1.1543×10^{-4}	1.0534×10^{-4}	1.4926×10^{-4}	1.0067×10^{-4}	1.8307×10^{-4}	1.0546×10^{-4}
0.02	2.5862×10^{-4}	1.0987×10^{-4}	3.2626×10^{-4}	2.4321×10^{-4}	3.9387×10^{-4}	2.0088×10^{-4}
0.03	4.2956×10^{-4}	2.2345×10^{-4}	5.3101×10^{-4}	3.7054×10^{-4}	6.3239×10^{-4}	4.1234×10^{-4}
0.04	6.2827×10^{-4}	3.1033×10^{-4}	7.6350×10^{-4}	3.4034×10^{-4}	8.9864×10^{-4}	6.6523×10^{-4}
0.05	8.5474×10^{-4}	4.5643×10^{-4}	1.0237×10^{-3}	3.9876×10^{-4}	1.1926×10^{-4}	1.0195×10^{-4}

With the help of procedure discussed for Case 2, one has

$$\begin{aligned}
 v_0 &= e^y, \\
 v_1 &= -(1 + \gamma t - \gamma)(2e^y + e^{2y}), \\
 v_2 &= \left[(1 - \gamma)^2 + 2\gamma t(1 - \gamma) + \frac{\gamma^2 t^2}{2} \right] (4e^y + 14e^{2y} + 3e^{3y}), \\
 v_3 &= - \left[(1 - \gamma)^3 + 3\gamma t(1 - \gamma^2) + \frac{3}{2}\gamma^2(1 - \gamma)t^2 + \frac{\gamma^3 t^3}{6} \right] \\
 &\quad \cdot (8e^y + 136e^{2y} + 138e^{3y} + 14e^{4y}) \\
 &\quad - \frac{\gamma^2 t^2}{2} \left(1 + \frac{\gamma t}{3} - \gamma \right) (4e^{2y} + 6e^{3y} + 2e^{4y}),
 \end{aligned}
 \tag{61}$$

and in the same way, we can find some more terms; therefore, we have

$$\begin{aligned}
 v(t, y) &= e^y - (1 + \gamma t - \gamma)(2e^y + e^{2y}) \\
 &\quad + \left[(1 - \gamma)^2 + 2\gamma t(1 - \gamma) + \frac{\gamma^2 t^2}{2} \right] \\
 &\quad \cdot (4e^y + 14e^{2y} + 3e^{3y}) - \left[(1 - \gamma)^3 \right. \\
 &\quad \left. + 3\gamma t(1 - \gamma^2) + \frac{3}{2}\gamma^2(1 - \gamma)t^2 + \frac{\gamma^3 t^3}{6} \right] \\
 &\quad \cdot (8e^y + 136e^{2y} + 138e^{3y} + 14e^{4y}) \\
 &\quad - \frac{\gamma^2 t^2}{2} \left(1 + \frac{\gamma t}{3} - \gamma \right) (4e^{2y} + 6e^{3y} + 2e^{4y}) + \dots
 \end{aligned}
 \tag{62}$$

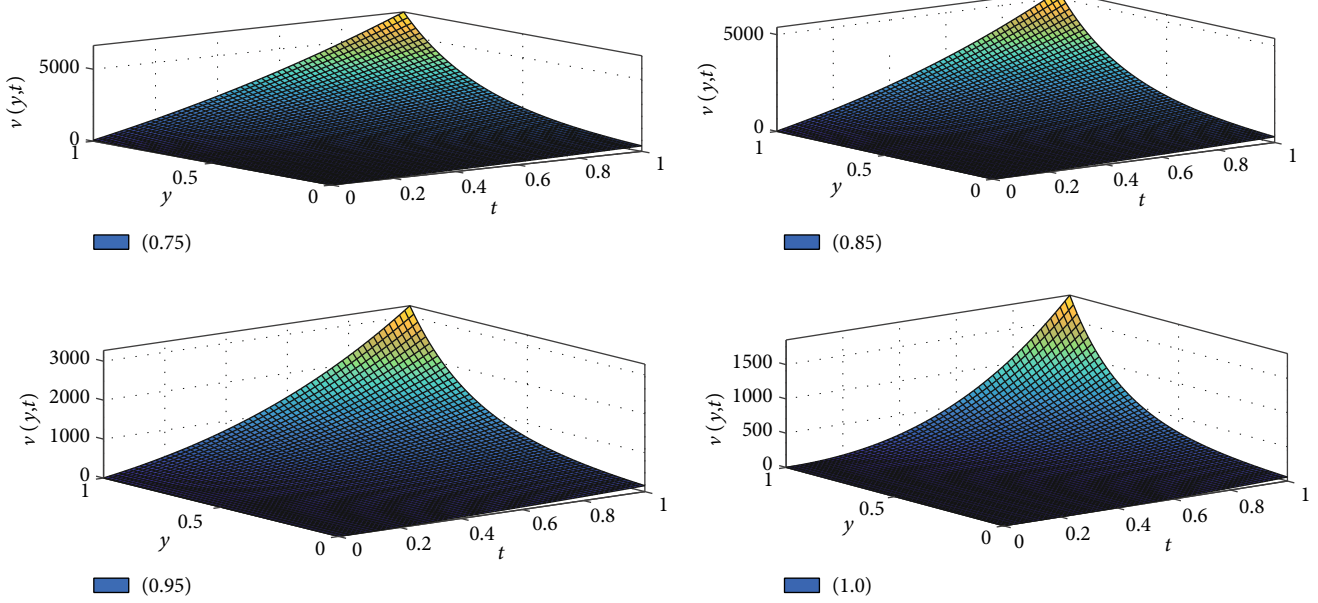


FIGURE 2: Surface plots of the resultant solution up to four terms at different values of γ for Example 2.

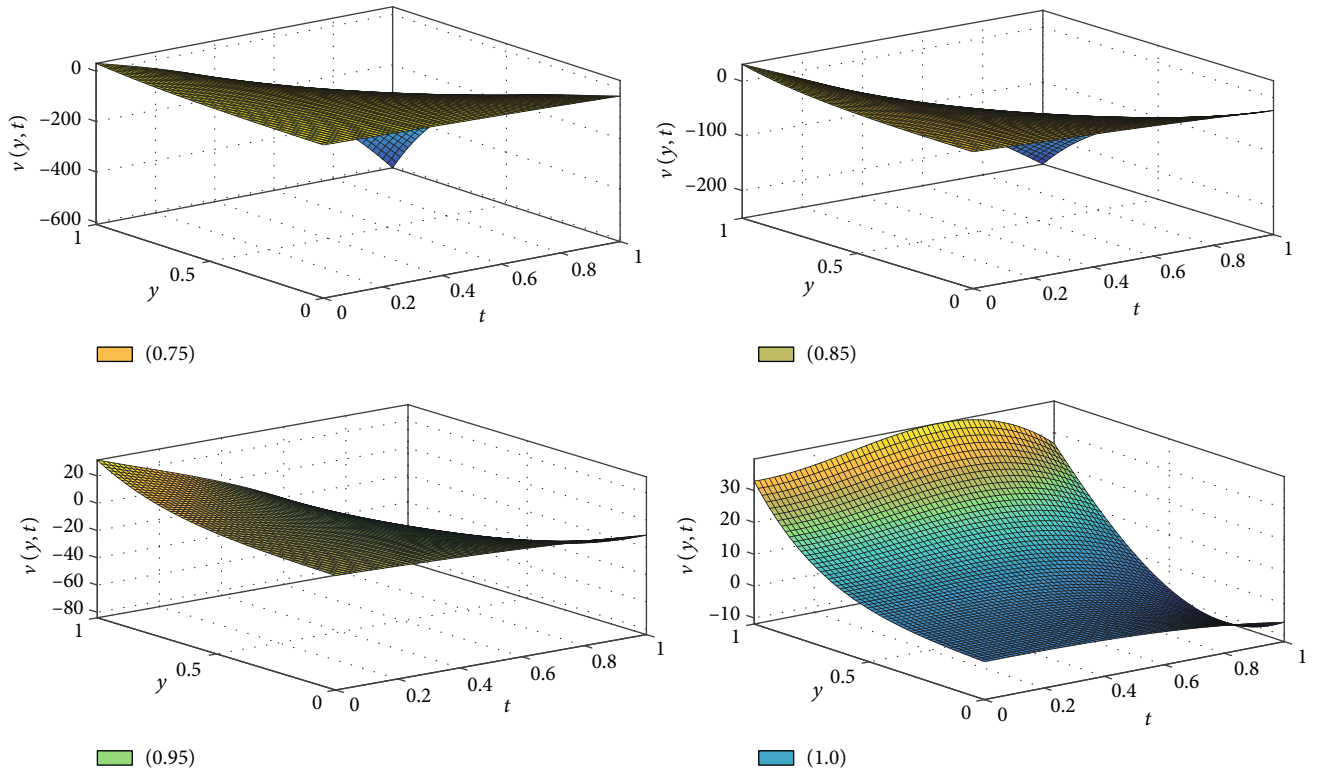


FIGURE 3: Surface plots of the resultant solution up to four terms at different values of γ for Example 3.

Here, we plot the approximate solution of the FBBM equation up to four terms in Figure 2. The approximate solution graphs for various fractional orders are given in Figure 2. We see from graphs as the order $\gamma \rightarrow 1$, the behavior of the surfaces of the solution tends to the integer order. If we put $\gamma = 1$ in the approximate solution, we get the approximate solution at integer order.

Example 3. Consider the FBBM equation using CFFOD as

$$\begin{aligned}
 {}^{CF}D_t^\gamma v(t, y) + v_y(t, y) \\
 + v(t, y)v_y(t, y) + v_{yyy}(t, y) = 0, \quad (63) \\
 v(0, y) = y^2.
 \end{aligned}$$

and in the same way, we can find some more terms; therefore, we have

$$\begin{aligned}
 v_0 &= y^2, \\
 v_1 &= -(1 + \gamma t - \gamma)(2y + 2y^3), \\
 v_2 &= \left[(1 - \gamma)^2 + 2\gamma t(1 - \gamma) + \frac{\gamma^2 t^2}{2} \right] (14^2 y^2 + 4y^3 + 6y^4), \\
 v_3 &= - \left[(1 - \gamma)^3 + 3\gamma t(1 - \gamma^2) + \frac{3}{2} \gamma^2 (1 - \gamma) t^2 + \frac{\gamma^3 t^3}{6} \right] \\
 &\quad \cdot (24 + 200y + 12y^2 + 88y^3 + 20y^4 + 48y^5) \\
 &\quad - \frac{\gamma^2 t^2}{2} \left(1 + \frac{\gamma t}{3} - \gamma \right) (4y + 16y^3 + 12y^5).
 \end{aligned} \tag{64}$$

Here, we plot the approximate solution of FBBM equation up to four terms in Figure 3. The approximate solution graphs for various fractional orders are given in Figure 3. We see from graphs as the order $\gamma \rightarrow 1$, the behavior of the surfaces of the solution tends to the integer order solution. If we put $\gamma = 1$ in the approximate solution, we get the approximate solution at the integer order for the same problem.

Example 4. Consider the modified FBBM equation using CFFOD as

$$\begin{aligned}
 {}^{CF}D_t^\gamma v(t, y) + v_y(t, y) \\
 + v^2(t, y)v_y(t, y) + v_{yyy}(t, y) = 0, \\
 v(0, y) = e^y.
 \end{aligned} \tag{65}$$

With the help of the procedure mentioned in Case 3, we have

$$\begin{aligned}
 v_0 &= e^y, \\
 v_1 &= -(1 + \gamma t - \gamma)(2e^y + e^{3y}), \\
 v_2 &= \left[(1 - \gamma)^2 + 2\gamma t(1 - \gamma) + \frac{\gamma^2 t^2}{2} \right] (4e^y + 36e^{3y} + 5e^{5y}), \\
 v_3 &= - \left[(1 - \gamma)^3 + 3\gamma t(1 - \gamma^2) + \frac{3}{2} \gamma^2 (1 - \gamma) t^2 + \frac{\gamma^3 t^3}{6} \right] \\
 &\quad \cdot (8e^y + 1104e^{3y} + 778e^{5y} + 22e^{7y}) \\
 &\quad - \frac{\gamma^2 t^2}{2} \left(1 + \frac{\gamma t}{3} - \gamma \right) (12e^{3y} + 20e^{5y} + 7e^{7y}),
 \end{aligned} \tag{66}$$

and in the same way, we can find some more terms; therefore, we have

$$\begin{aligned}
 v(t, y) &= e^y - (1 + \gamma t - \gamma)(2e^y + e^{3y}) \\
 &\quad + \left[(1 - \gamma)^2 + 2\gamma t(1 - \gamma) + \frac{\gamma^2 t^2}{2} \right] \\
 &\quad \cdot (4e^y + 36e^{3y} + 5e^{5y}) - \left[(1 - \gamma)^3 \right. \\
 &\quad \left. + 3\gamma t(1 - \gamma^2) + \frac{3}{2} \gamma^2 (1 - \gamma) t^2 + \frac{\gamma^3 t^3}{6} \right] \\
 &\quad \cdot (8e^y + 1104e^{3y} + 778e^{5y} + 22e^{7y}) \\
 &\quad - \frac{\gamma^2 t^2}{2} \left(1 + \frac{\gamma t}{3} - \gamma \right) (12e^{3y} + 20e^{5y} + 7e^{7y}) + \dots
 \end{aligned} \tag{67}$$

Here, we plot the approximate solution of the FBBM equation up to four terms in Figure 4. The approximate solution graphs for various fractional orders are given in Figure 4. We see from graphs as the order $\gamma \rightarrow 1$, the behavior of the surfaces of the solution tends to the integer order solution. Also, if we put $\gamma = 1$ in the approximate solution, we get the approximate solution at integer order for the same problem.

Example 5. Consider the modified FBBM equation using CFFOD as

$$\begin{aligned}
 {}^{CF}D_t^\gamma v(t, y) + v_y(t, y) + v^2(t, y)v_y(t, y) + v_{yyy}(t, y) = 0, \\
 v(0, y) = y^2.
 \end{aligned} \tag{68}$$

With the help of the procedure discussed for Case 3, one may have

$$\begin{aligned}
 v_0 &= y^2, \\
 v_1 &= -(1 + \gamma t - \gamma)(2y + 2y^5), \\
 v_2 &= \left[(1 - \gamma)^2 + 2\gamma t(1 - \gamma) + \frac{\gamma^2 t^2}{2} \right] (2 + 120y^2 + 20y^4 + 18y^8), \\
 v_3 &= - \left[(1 - \gamma)^3 + 3\gamma t(1 - \gamma^2) + \frac{3}{2} \gamma^2 (1 - \gamma) t^2 + \frac{\gamma^3 t^3}{6} \right] \\
 &\quad \cdot (160 + 240y + 100y^3 + 6786y^5 + 344y^7 + 244y^{11}) \\
 &\quad - \frac{\gamma^2 t^2}{2} \left(1 + \frac{\gamma t}{3} - \gamma \right) (12y^3 + 40y^7 + 28y^{11}),
 \end{aligned} \tag{69}$$

and in the same way, we can find the other terms. Therefore, we get

$$\begin{aligned}
 v(t, y) &= y^2 - (1 + \gamma t - \gamma)(2y + 2y^5) \\
 &\quad \cdot \left[(1 - \gamma)^2 + 2\gamma t(1 - \gamma) + \frac{\gamma^2 t^2}{2} \right] \\
 &\quad \cdot (2 + 120y^2 + 20y^4 + 18y^8) \\
 &\quad - \left[(1 - \gamma)^3 + 3\gamma t(1 - \gamma^2) + \frac{3}{2} \gamma^2 (1 - \gamma) t^2 + \frac{\gamma^3 t^3}{6} \right] \\
 &\quad \cdot (160 + 240y + 100y^3 + 6786y^5 + 344y^7 + 244y^{11}) \\
 &\quad - \frac{\gamma^2 t^2}{2} \left(1 + \frac{\gamma t}{3} - \gamma \right) (12y^3 + 40y^7 + 28y^{11}) + \dots
 \end{aligned} \tag{70}$$

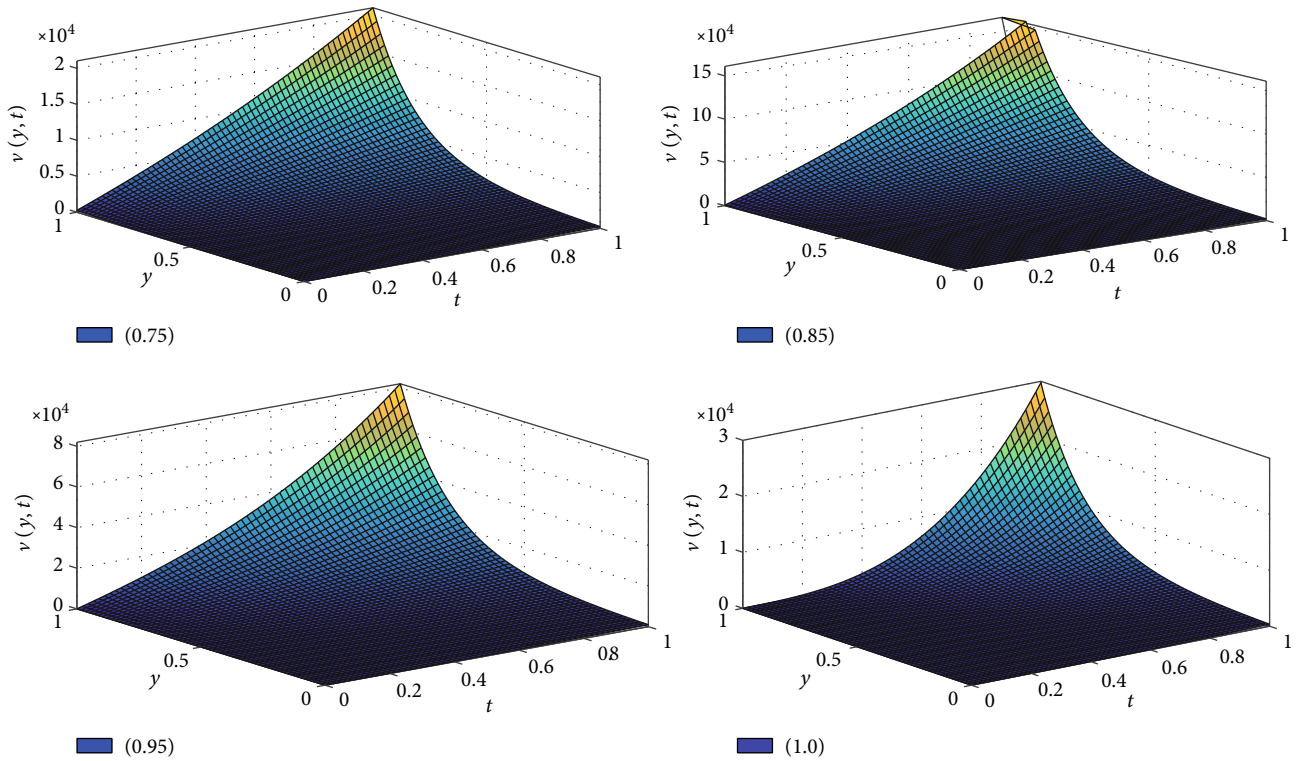


FIGURE 4: Surface plots of the resultant solution up to four terms at different values of γ for Example 4.

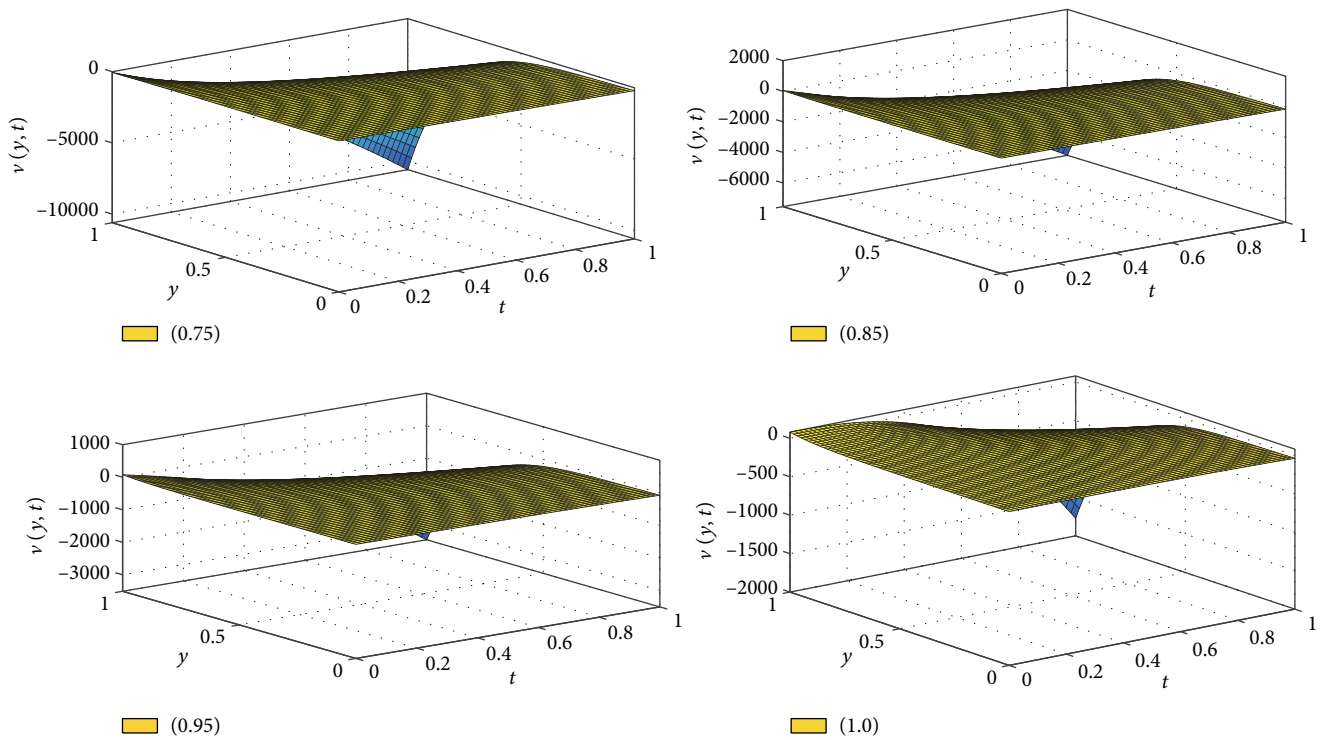


FIGURE 5: Surface plots of the resultant solution up to four terms at different values of γ for Example 5.

Here, we plot the approximate solution of FBBM equation up to four terms in Figure 5. The approximate solution graphs for various fractional orders are given in Figure 5. We see from graphs as the order $\gamma \rightarrow 1$, the behavior of

the surfaces of the solution tends to the integer order solution. Also, if we put $\gamma = 1$ in the approximate solution, we get the approximate solution at integer order for the same problem.

6. Conclusion

In our work, some existence results about the solution to the nonlinear problem of BBM equations under nonsingular kernel-type derivative have been developed successfully. We have discussed different cases of the concerned equations for semianalytical results. For approximate analytical results, a novel iterative method of Laplace transform coupled with Adomian polynomials has been used. Further, by providing an example, we have computed the absolute errors in comparison with VHPM for first four-term solutions at different values of variables t and y against $\gamma = 1$. We observed that the absolute error is slightly good than the mentioned VHPM. Therefore, the concerned method of LADM can be used as a powerful tool to handle many nonlinear problems of FODEs. Since, the aforementioned equations are increasingly used to model numerous phenomena of physics including the propagation of heat or sound waves, fluid flow, elasticity, electrostatics, and electrodynamics, and population dynamics in biology. A large numbers of the aforementioned equations may be used in fluid mechanics and hydrodynamics. Since fractional derivatives have a greater degree of freedom and produce the complete spectrum of the physical phenomenon which include the ordinary derivative as particular case, global dynamics of the aforesaid physical phenomenon may be investigated. Since the BBM equation can also be used to model various physical systems like acoustic-gravity waves in compressible fluids, acoustic waves in enharmonic crystals, the hydromagnetic waves in cold plasma, (see [41]), investigation of the BBM equation and its various cases under different fractional order derivatives may be lead us to investigate some more comprehensive results by using various fractional orders which will include the classical order solution as a special case. The nonlocal behaviors of such problems can be well studied by using nonsingular fractional order derivative. In the future, the concerned BBM equation can be investigated by using more general fractional order derivative with nonsingular kernel of the Mittag-Leffler function.

Data Availability

Data availability is not applicable in this manuscript.

Conflicts of Interest

There is no competing interest regarding this work.

Authors' Contributions

An equal contribution has been done by all the authors.

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