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Research Article

Boundary Effect on Asymptotic Behavior of Solutions to the p-System with Time-Dependent Damping

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In this paper, we consider the asymptotic behavior of solutions to the p-system with time-dependent damping on the half-line $\mathbb{R}^+ = (0, +\infty)$, $v_t - u_x = 0, u_t + p(v)_x = -\left(\alpha/(1+t)^\lambda\right)u$ with the Dirichlet boundary condition $u|_{x=0} = 0$, in particular, including the constant and nonconstant coefficient damping. The initial data $(v_0, u_0)(x)$ have the constant state (v_+, u_+) at $x = +\infty$. We prove that the solutions time-asymptotically converge to $(v_+, 0)$ as t tends to infinity. Compared with previous results about the p-system with constant coefficient damping, we obtain a general result when the initial perturbation belongs to $H^3(\mathbb{R}^+) \times H^2(\mathbb{R}^+)$. Our proof is based on the time-weighted energy method.

1. Introduction

In this paper, we consider the asymptotic behavior and the convergence rates of solutions to the p-system with time-dependent damping

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\frac{\alpha}{(1+t)^\lambda} u, \quad x \in \mathbb{R}^+, \quad t < 0, \end{cases} \tag{1}$$

with the initial data

$$(v(x,0), u(x,0)) = (v_0(x), u_0(x)) \rightarrow (v_+, u_+), \quad v_+ > 0, \quad \text{as} \quad x \rightarrow +\infty, \tag{2}$$

and with the Dirichlet boundary condition

$$u|_{x=0} = 0. \tag{3}$$

Here, $v > 0$ is the specific volume, u is the velocity, the pressure $p(v)$ is a smooth function of v such that $p(v) > 0$, $p'(v) < 0$, the external term $-\left(\alpha/(1+t)^\lambda\right)u$ with physical coefficients $\alpha > 0$ and $\lambda \geq 0$, is called a time-dependent damping. $v_+ > 0$ and u_+ are constant states.

For $\alpha = 0$, the system (1) reduces to the standard compressible Euler equations. There have been many important developments.

For $\alpha > 0, \lambda = 0$, the system (1) becomes the compressible Euler equations with damping which model the compressible flow through porous media. There is a huge literature on the investigations of global existence and large time behaviors of smooth solutions to compressible Euler equations with damping. For the Cauchy problem, the global existence of smooth solutions with small initial data has been studied by many authors, cf. [1–4], and the large time behavior of the solutions was carried out by Hsiao and Liu in [5, 6] firstly. They showed that the solutions of the Cauchy problem to (1) time-asymptotically behave as those of the following system

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha \bar{u}, \end{cases} \tag{4}$$

or

$$\begin{cases} \bar{v}_t = -\frac{1}{\alpha} p(\bar{v})_{xx}, \\ \bar{u} = -\frac{1}{\alpha} p(\bar{v})_x, \end{cases} \tag{5}$$

with the same end states as $v_0(x)$:

$$\bar{v}(\pm\infty, t) = v_\pm. \tag{6}$$

Here, the well-known porous media equation is obtained by Darcy's law, and a better convergence rate and the optimal convergence rate when $v(+\infty,0)=v(-\infty,0)$ were obtained by Nishihara in [7, 8]. For the other related results, we refer to [9–11]. Compared with other results, Zhao in [10] got the asymptotic behavior, and convergence rate lay in the facts that the nonlinear diffusion wave $(\overline{v}(x,t),\overline{u}(x,t))$ which satisfies (4) and (6) need not to be weak, and the initial data can be large properly. For the results such as the generalized Riemann problem for a class of quasilinear hyperbolic systems and nonlinear hyperbolic systems of conservation laws with small BV initial data, we refer to [12, 13]. For other results such as the p-system with linear and nonlinear damping, we refer to [14–22].

For the initial-boundary value problems on $\mathbb{R}^+$ to the equations of viscous conservation laws have been investigated by several authors, cf. [23–27]. For the initial-boundary value problems on $\mathbb{R}^+$ to the p-system with linear damping, see [28, 29], Nishihara and Yang in [29] considered (1)–(3) with $\alpha>0,\lambda=0$, and they got the asymptotic behavior and the convergence rates by perturbing the initial value around the linear diffusion waves $(\overline{v},\overline{u})(x,t)$ which satisfies

$$\begin{cases} \tilde{v}_t-\tilde{u}_x=0, & x\in\mathbb{R}^+, \quad t<0, \\ p'(v_+)\tilde{v}_x=-\alpha\tilde{u}, \\ \tilde{u}|_{x=0}=0, & (\tilde{v},\tilde{u})|_{x=\infty}=(v_+,0). \end{cases} \tag{7}$$

For $\alpha>0,\lambda>0$, the system (1) reduces to compressible Euler equations with time-dependent damping. Hou et al. [30] considered the global existence of smooth solutions for $0\le\lambda\le1$ in three space dimensions; they proved that the solutions will blow up in finite time for $\lambda\ge1$. In [31, 32], Pan obtained the existence and decay rates of solutions near constant states $(1,0)$ for $\lambda=1$. For the Cauchy problem of the system (1), Cui et al. in [33] proved that the solutions time-asymptotically converge to the diffusion wave whose profile is self-similar solution to the corresponding parabolic equation. For other results, we refer to [34].

However, to our knowledge, there are few results for the initial-boundary value problem of the system (1). In this paper, we obtained the asymptotic behavior and the decay rates for the solutions of the system (1) with initial-boundary value data. Because the time-dependent damping will lead to some new phenomena and severe mathematical difficulties, we will use some new techniques here. As usual, we want to get the convergence rates of the solution by the usual energy estimates and some elementary computations. However, the time-dependent damping brings some extra terms, such as in the right hand of (4.19) and (4.61), $C\int(1+t)^{-\kappa}V^2dx$, $C\int(1+t)^{\beta+\lambda-1}(V_x^2+V_t^2)dx$, $C\int(1+t)^{\beta+2\lambda}(V_{xx}^2+V_{xt}^2)dx$. To deal with these terms, we divide the time interval into two regions, $[0,T_0)$ and $[T_0,\infty)$. In the bounded region $[0,T_0)$, we obtain the desired estimates by using Gronwall's inequality and the usual energy estimates. In unbounded region $[T_0,\infty)$, we observe the bad terms could be bounded by the corresponding energy functional when $t\ge T_0$. By using the Gronwall's inequality and the time-weighted energy method, we also obtain the desired decay estimates, and the desired decay rates will not be lost by using the weight function $(1+t)^{\beta}$.

The rest of this paper is organized as follows. In Section 2, we reformulate the problem (1)–(3) and state the main theorems. In Section 3, *a priori* estimates of finite time intervals will be given. In Section 4, decay rates will be given.

1.1. Notation. Hereafter, we denote several generic positive constants depending on T by $C(T)$, C or $O(1)$ denote the generic positive constants depending only on the initial data and the physical coefficients α,λ, but independent of the time without any confusion. ε will always be used to represent sufficiently small positive constants. $L^p=L^p(\mathbb{R}^+)(1\le p\le\infty)$ denotes usual Lebesgue space with the norm

$$\begin{gathered} \|f\|_{L^p}=\left(\int_{\mathbb{R}^+}|f(x)|^p dx\right)^{1/p}, \quad 1\le p<\infty, \\ \|f\|_{L^\infty}=\sup_{\mathbb{R}^+}|f(x)|, \end{gathered} \tag{8}$$

and the integral region $\mathbb{R}^+$ will be omitted without any confusion. $H^l(l\ge0)$ denotes the usual lth-order Sobolev space with the norm

$$\|f\|_l=\left(\sum_{j=0}^{l}\left\|\partial_x^j f\right\|^2\right)^{1/2}, \tag{9}$$

where $\|\cdot\|=\|\cdot\|_0=\|\cdot\|_{L^2}$. For simplicity, $\|f(\cdot,t)\|_{L^p}$ and $\|f(\cdot,t)\|_l$ are denoted by $\|f(t)\|_{L^p}$ and $\|f(t)\|_l$ respectively.

2. Main Theorems

From asymptotic analysis, it is well known that the first term u_t of $(1)_2$ decay to zero, as $t\to\infty$, faster than others. Expecting $(v,u)(x,t)\to(v_+,0)$, $t\to\infty$, we approximate this by the solution $(\overline{v}(x,t),\overline{u}(x,t))$ of

$$\begin{cases} \overline{v}_t-\overline{u}_x=0, \\ p'(v_+)\overline{v}_x=-\frac{\alpha}{(1+t)^\lambda}\overline{u}, \quad x\in\mathbb{R}^+, \quad t<0, \\ (\overline{v}(x,t),\overline{u}(x,t))\to(v_+,0), \ v_+<0, \quad \text{as} \quad x\to+\infty, \\ \overline{u}|_{x=0}=0 \ \left(\text{or} \ \overline{v}_x|_{x=0}=0\right), \end{cases} \tag{10}$$

Explicitly,

$$\overline{v}(x,t)=v_+ +\frac{\delta_0}{\sqrt{4\kappa\pi(1+t)^{\lambda+1}}}e^{-\left(\left((\lambda+1)x^2\right)/\left(4\kappa(1+t)^{\lambda+1}\right)\right)}, \tag{11}$$

where $\kappa:=(-p'(v_+)/\alpha)>0$ and δ_0 is defined by

$$\delta_0=2\sqrt{\lambda+1}\left(\int_0^{+\infty}(v_0(x)-v_+)dx-B(0)u_+\right), \tag{12}$$

$B(0)$ is to be defined later.

Note that $\overline{u}(\infty,t)=0$. Hinted by (1), we suppose

$$u(x,t)\to\beta(t)u_+, \quad \text{as} \quad x\to+\infty, \tag{13}$$

where

$$\beta(t)=\begin{cases} e^{-(\alpha/1-\lambda)\left[(1+t)^{1-\lambda}-1\right]}, & \text{if} \quad \lambda\in[0,1], \\ (1+t)^{-\alpha}, & \text{if} \quad \lambda=1. \end{cases} \tag{14}$$

We define the auxiliary function $(\hat{v},\hat{u})(x,t)$ by

$$\begin{cases} \hat{v}(x,t)=B(t)m_0(x)u_+, \\ \hat{u}(x,t)=u_+\beta(t)\int_0^x m_0(y)dy, \end{cases} \tag{15}$$

where $B(t)=-\int_t^{\infty}\beta(\tau)d\tau$, and $m_0(x)$ is a smooth function with compact support such that

$$\int_0^{\infty} m_0(y)dy=1,\quad \operatorname{supp} m_0(x)\subset\mathbb{R}^+. \tag{16}$$

Then $(\hat{v},\hat{u})(x,t)$ satisfies

$$\begin{cases}\hat{v}_t-\hat{u}_x=0,\\ \hat{u}_t=-\frac{\alpha}{(1+t)^{\lambda}}\hat{u}, \quad x\in\mathbb{R}^+,\quad t<0,\\ \hat{u}|_{x=0}=0,\ (\hat{v},\hat{u})|_{x=\infty}=(0,u_+\beta(t)).\end{cases} \tag{17}$$

By direct calculation, we have

Lemma 2.1. *For $0\le\lambda<1$, there exist constants $0<\vartheta<1-\lambda, c, C>0$, such that for any $t\ge0$, we have*

$$\|\hat{v}\|_{L^1}+\|\hat{v}\|_{L^\infty}+\|\hat{v}_t\|_{L^\infty}\le C|u_+|e^{-ct^{\vartheta}}. \tag{18}$$

Lemma 2.2. *The solution $\bar{v}(x,t)$ of (10) satisfies the following dissipative estimates*

$$\int|\bar{v}_x(x,t)|^2dx\le C|\delta_0|^2(1+t)^{-(3(\lambda+1)/2)}, \tag{19}$$

$$\int|\bar{v}_t(x,t)|^2dx\le C|\delta_0|^2(1+t)^{-(\lambda/2)-(5/2)}, \tag{20}$$

$$\int|\bar{v}_{xx}(x,t)|^2dx\le C|\delta_0|^2(1+t)^{-(5(\lambda+1)/2)}, \tag{21}$$

$$\int|\bar{v}_{xt}(x,t)|^2dx\le C|\delta_0|^2(1+t)^{-(3\lambda/2)-(7/2)}, \tag{22}$$

$$\int|\bar{v}_{xxx}(x,t)|^2dx\le C|\delta_0|^2(1+t)^{-(7(\lambda+1)/2)}, \tag{23}$$

$$\int|\bar{v}_{xxt}(x,t)|^2dx\le C|\delta_0|^2(1+t)^{-(5\lambda/2)-(9/2)}, \tag{24}$$

$$\int|\bar{v}_{tt}(x,t)|^2dx\le C|\delta_0|^2(1+t)^{-(\lambda/2)-(9/2)}, \tag{25}$$

$$\int|(p'(v_+)-p'(\bar{v}))\bar{v}_x(x,t)|^2dx\le C|\delta_0|^2(1+t)^{-(5(\lambda+1)/2)}, \tag{26}$$

$$\int|((p'(v_+)-p'(\bar{v}))\bar{v}_x(x,t))_x|^2dx\le C|\delta_0|^2(1+t)^{-(7(\lambda+1)/2)}. \tag{27}$$

Corollary 2.3. *The solution $\bar{v}(x,t)$ of (10) satisfies the following dissipative estimates*

$$\|\bar{v}_x(\cdot,t)\|_{L^\infty}\le C|\delta_0|(1+t)^{-(\lambda+1)}, \tag{28}$$

$$\|\bar{v}_t(\cdot,t)\|_{L^\infty}\le C|\delta_0|(1+t)^{-(\lambda/2)-(3/2)}, \tag{29}$$

$$\|\bar{v}_{xx}(\cdot,t)\|_{L^\infty}\le C|\delta_0|(1+t)^{-(3(\lambda+1)/2)}, \tag{30}$$

$$\|\bar{v}_{xt}(\cdot,t)\|_{L^\infty}\le C|\delta_0|(1+t)^{-\lambda-2}. \tag{31}$$

Combining (1) with (10) and (17), we have

$$\begin{cases}(v-\bar{v}-\hat{v})_t-(u-\bar{u}-\hat{u})_x=0,\\ (u-\bar{u}-\hat{u})_t+(p(v)-p(\bar{v}))_x=-\frac{\alpha}{(1+t)^{\lambda}}\\ \cdot(u-\bar{u}-\hat{u})-\bar{u}_t+(p'(v_+)-p'(\bar{v}))\bar{v}_x.\end{cases} \tag{32}$$

By virtue of $(32)_1$ and (12),

$$\int_0^{+\infty}(v-\bar{v}-\hat{v})(y,t)dy=\int_0^{+\infty}(v_0(x)-v_+)dx-B(0)u_+-\frac{\delta_0}{2\sqrt{1+\lambda}}=0, \tag{33}$$

and hence we reach the setting of perturbation

$$\begin{cases}V(x,t):=-\int_x^{\infty}(v-\bar{v}-\hat{v})(y,t)dy,\\ U(x,t):=u(x,t)-\bar{u}(x,t)-\hat{u}(x,t).\end{cases} \tag{34}$$

From (32) and (34), we deduce that $(V,U)(x,t)$ solves the following problem

$$\begin{cases}V_t-U=0,\quad x\in\mathbb{R}^+,\quad t<0,\\ U_t+(p(V_x+\bar{v}+\hat{v})-p(\bar{v}))_x+\frac{\alpha}{(1+t)^{\lambda}}U=-\bar{u}_t+(p'(v_+)-p'(\bar{v}))\bar{v}_x,\end{cases} \tag{35}$$

with the initial data

$$\begin{aligned}(V,U)(x,0)=(V_0,U_0)(x):=\Big(&-\int_x^{\infty}(v_0(y)-\bar{v}(y,0)-\hat{v}(y,0))dy,\\ &u_0(x)-\bar{u}(x,0)-\hat{u}(x,0)\Big),\end{aligned} \tag{36}$$

and the boundary condition

$$U|_{x=0}=0, \tag{37}$$

or the linearized problem around $\bar{v}$

$$\begin{cases}V_t-U=0,\quad x\in\mathbb{R}^+,\quad t>0,\\ U_t+(p'(\bar{v})V_x)_x+\frac{\alpha}{(1+t)^{\lambda}}U=F,\end{cases} \tag{38}$$

where

$$\begin{aligned}F=&\frac{(1+t)^{\lambda}}{\alpha}p'(v_+)\bar{v}_{xt}+\frac{\lambda(1+t)^{\lambda-1}}{\alpha}p'(v_+)\bar{v}_x\\ &-(p(V_x+\bar{v}+\hat{v})-p(\bar{v})-p'(\bar{v})V_x)_x+(p'(v_+)-p'(\bar{v}))\bar{v}_x.\end{aligned} \tag{39}$$

In this paper, we do not consider the case $\lambda = 1$. For the case of $0 \le \lambda < 1$, from (12), we have

$$|\delta_0| \le C(\|v_0(x) - v_+\|_{L^1} + |u_+|), \tag{40}$$

we obtain the following theorems.

Theorem 2.4 (the case of $0 \le \lambda < 3/5$). *Suppose that $v_0(x) - v_+ \in L^1$, $(V_0, U_0) \in H^3 \times H^2$, both $\delta = \|v_0(x) - v_+\|_{L^1} + |u_+|$ and $\|V_0\|_3 + \|U_0\|_2$ are sufficiently small. Then there exists a unique time-global solution $(V, U)(x,t)$ of (35)–(37) which satisfies*

$$V \in C^i([0,\infty); H^{3-i}), \quad i = 0,1,2,3, \tag{41}$$

$$U \in C^i([0,\infty); H^{2-i}), \quad i = 0,1,2, \tag{42}$$

and moreover

$$\begin{aligned}
&\sum_{k=0}^{3} (1+t)^{(\lambda+1)k} \|\partial_x^k V(t)\|^2 + \sum_{k=0}^{2} (1+t)^{(\lambda+1)k+2} \|\partial_x^k U(t)\|^2 \\
&\quad + \int_0^t \left(\sum_{j=1}^{3} (1+s)^{(\lambda+1)j-1} \|\partial_x^j V(s)\|^2 + \sum_{j=0}^{2} (1+s)^{(\lambda+1)j+1} \|\partial_x^j U(s)\|^2 \right) ds \\
&\quad \le O(1) \left(\|V_0\|_3^2 + \|U_0\|_2^2 + \delta \right).
\end{aligned} \tag{43}$$

Theorem 2.5 (the case of $3/5 < \lambda < 1$). *Suppose that $v_0(x) - v_+ \in L^1$, $(V_0, U_0) \in H^3 \times H^2$, both $\delta = \|v_0(x) - v_+\|_{L^1} + |u_+|$ and $\|V_0\|_3 + \|U_0\|_2$ are sufficiently small. Then there exists a unique time-global solution $(V, U)(x,t)$ of (35)–(37) which satisfies*

$$V \in C^i([0,\infty); H^{3-i}), \quad i = 0,1,2,3, \tag{44}$$

$$U \in C^i([0,\infty); H^{2-i}), \quad i = 0,1,2, \tag{45}$$

$$\begin{aligned}
&\sum_{k=0}^{3} (1+t)^{(\lambda+1)k+(3/2)-(5\lambda/2)} \|\partial_x^k V(t)\|^2 \\
&\quad + \sum_{k=0}^{2} (1+t)^{(\lambda+1)k+(7/2)-(5\lambda/2)} \|\partial_x^k U(t)\|^2 \\
&\quad \le O(1) \left(\|V_0\|_3^2 + \|U_0\|_2^2 + \delta \right),
\end{aligned} \tag{46}$$

and for any $\beta \in (3/2 - 3\lambda/2, \lambda)$, we have

$$\begin{aligned}
&\int_0^t \left(\sum_{j=0}^{3} (1+s)^{(\lambda+1)(j-1)+\beta} \|\partial_x^j V(s)\|^2 \right. \\
&\quad \left. + \sum_{j=0}^{2} (1+s)^{(\lambda+1)j+\beta-\lambda+1} \|\partial_x^j U(s)\|^2 \right) ds \\
&\quad \le O(1)(1+t)^{\beta+(3\lambda/2)-(3/2)} \left(\|V_0\|_3^2 + \|U_0\|_2^2 + \delta \right).
\end{aligned} \tag{47}$$

Theorem 2.6 (the case of $\lambda = 3/5$). *Suppose that $v_0(x) - v_+ \in L^1$, $(V_0, U_0) \in H^3 \times H^2$, both $\delta = \|v_0(x) - v_+\|_{L^1} + |u_+|$ and $\|V_0\|_3 + \|U_0\|_2$ are sufficiently small. Then there exists a unique time-global solution $(V, U)(x,t)$ of (35)–(37) which satisfies*

$$V \in C^i([0,\infty); H^{3-i}), \quad i = 0,1,2,3, \tag{48}$$

$$U \in C^i([0,\infty); H^{2-i}), \quad i = 0,1,2, \tag{49}$$

furthermore, we have for any sufficiently small $\varepsilon > 0$

$$\begin{aligned}
&\sum_{k=0}^{3} (1+t)^{8k/5} \|\partial_x^k V(t)\|^2 + \sum_{k=0}^{2} (1+t)^{(8k/5)+2} \|\partial_x^k U(t)\|^2 \\
&\quad + \int_0^t \left(\sum_{j=1}^{3} (1+s)^{(8j/5)-1} \|\partial_x^j V(s)\|^2 + \sum_{j=0}^{2} (1+s)^{(8j/5)+1} \|\partial_x^j U(s)\|^2 \right) ds \\
&\le O(1)(1+t)^{\varepsilon} \left(\|V_0\|_3^2 + \|U_0\|_2^2 + \delta \right).
\end{aligned} \tag{50}$$

Remark 1. For the case of $\lambda = 0$, the convergence rate shown in (43) is the same as that in Nishihara and Yang in [29]. In other words, our estimates give a general result by the elementary method.

Remark 2. All results are obtained under the condition that any data are small. For large data, the asymptotic behavior of the solutions of (1) with initial data or initial-boundary data will be very difficult, and we will consider them later.

3. A Priori Estimates on Finite Time Intervals

Compared with the constant damping, the time-dependent damping may bring some extra terms. To deal with the terms, we divide the time interval into two regions, $[0, T]$ and $[T, \infty]$, then apply the L^2-energy method and the Gronwall's inequality to obtain the desired estimates. Much of this section is based on the paper [33].

We now devote ourselves to the *a priori* estimates of the solution $(V, U)(x,t)$ under the *a priori* assumption

$$N(T) := \sup_{0<t<T} \left\{ \sum_{k=0}^{3} \|\partial_x^k V(t)\|^2 + \sum_{k=0}^{2} \|\partial_x^k U(t)\|^2 \right\} \le \varepsilon^2, \tag{51}$$

where $0 < \varepsilon \ll 1$.

By Sobolev inequality and the equation (35), we have

$$\begin{cases} \|\partial_x^k V(t)\|_{L^\infty} \le \varepsilon, & k = 0,1,2, \\ \|\partial_x^k V_t(t)\|_{L^\infty} \le \varepsilon, & k = 0,1, \end{cases} \tag{52}$$

which will be used later.

Proposition 3.1. *Under the conditions of Theorem 2.4 for any given $T > 0$, $0 \le \lambda < 1$, $\alpha > 0$, the solution $(V, V_t)(x,t)$ to the initial-boundary problem (35), (36), (37) on $[0, T]$ satisfying*

$$\|V\|_1^2 + \|V_t\|^2 + \int_0^t \left(\|V_x\|^2 + \|V_t\|^2 \right) ds \le C(T) \left(\|V_0\|_1^2 + \|U_0\|^2 + \delta \right), \tag{53}$$

$$\|V_{xx}\|^2 + \|V_{xt}\|^2 + \int_0^t \left(\|V_{xx}\|^2 + \|V_{xt}\|^2 \right) ds \le C(T) \left(\|V_0\|_2^2 + \|U_0\|_1^2 + \delta \right), \tag{54}$$

and

$$\|V_{xxx}\|^2 + \|V_{xxt}\|^2 + \int_0^t \left(\|V_{xxx}\|^2 + \|V_{xxt}\|^2\right) ds \le C(T)\left(\|V_0\|_3^2 + \|U_0\|_2^2 + \delta\right). \tag{55}$$

Since it suffices to establish the estimates for sufficiently smooth solution, the equations in (35), (33) and $U|_{x=0} = 0$ give the following boundary conditions for higher order derivatives

$$V(0,t) = V_{xx}(0,t) = V_t(0,t) = V_{txx}(0,t) = 0, etc. \tag{56}$$

(39) can be rewritten as the problem to the second order wave of V

$$\begin{cases} V_{tt} + (p'(\overline{v})V_x)_x + \frac{\alpha}{(1+t)^\lambda} V_t = F, \\ |(V,V_t)|_{t=0} = (V_0, U_0)(x), \\ |V|_{x=0} = 0. \end{cases} \tag{57}$$

Step 1. First, multiplying $(57)_1$ by V and integrating the resulting equation with respect to x over $\mathbb{R}^+$, we can get

$$\frac{d}{dt}\int\left[VV_t + \frac{\alpha}{2}(1+t)^{-\lambda}V^2\right]dx + \frac{\alpha\lambda}{2}\int(1+t)^{-\lambda-1}V^2dx - \int p'(\overline{v})V_x^2 dx = \int V_t^2 dx + \int FV dx. \tag{58}$$

We now estimate the last term on the right hand of (58), from (39), we have

$$\int FV\,dx = \int\Big[\frac{(1+t)^\lambda}{\alpha}p'(v_+)\overline{v}_{xt} + \frac{\lambda(1+t)^{\lambda-1}}{\alpha}p'(v_+)\overline{v}_x - (p(V_x+\overline{v}+\hat{v}) - p(\overline{v}) - p'(\overline{v})V_x)_x + (p'(v_+) - p'(\overline{v}))\overline{v}_x\Big]V\,dx. \tag{59}$$

From Lemma 2.2, the Cauchy–Schwarz's inequality and noticing $p'(v) \le -C_0 < 0$, we have

$$\int\frac{(1+t)^\lambda}{\alpha}p'(v_+)\overline{v}_{xt}Vdx = -\int\frac{(1+t)^\lambda}{\alpha}p'(v_+)\overline{v}_tV_xdx \le \frac{C_0}{8}\int V_x^2dx + C\int(1+t)^{2\lambda}|\overline{v}_t|^2dx \le \frac{C_0}{8}\int V_x^2dx + C\delta^2, \tag{60}$$

and

$$\int\frac{\lambda(1+t)^{\lambda-1}}{\alpha}p'(v_+)\overline{v}_xVdx \le \frac{\alpha\lambda}{8}\int(1+t)^{-\lambda-1}V^2dx + C\int(1+t)^{3\lambda-1}|\overline{v}_x|^2dx \le \frac{\alpha\lambda}{8}\int(1+t)^{-\lambda-1}V^2dx + C\delta^2. \tag{61}$$

By using Lemma 2.1, Lemma 2.2 and *a priori* assumption (52), we have

$$\begin{aligned} &-\int\left(p(V_x+\overline{v}+\hat{v}) - p(\overline{v}) - p'(\overline{v})V_x\right)_xVdx \\ &= \int\left(p(Vx+\overline{v}+\hat{v}) - p(\overline{v}) - p'(\overline{v})V_x\right)V_xdx \\ &\le \frac{C_0}{8}\int V_x^2dx + C\int\left(|\hat{v}|^2 + |V_x|^4 + |\hat{v}|^4\right)dx \\ &\le \frac{C_0}{8}\int V_x^2dx + C\varepsilon^2\int V_x^2dx + C\delta^2, \end{aligned} \tag{62}$$

and

$$\int(p'(v_+) - p'(\overline{v}))\overline{v}_xVdx \le \frac{\alpha\lambda}{8}\int(1+t)^{-\lambda-1}V^2dx + C\int\left((p'(v_+) - p'(\overline{v}))\overline{v}_x\right)^2(1+t)^{\lambda+1}dx \le \frac{\alpha\lambda}{8}\int(1+t)^{-\lambda-1}V^2dx + C\delta^2. \tag{63}$$

Substituting (60)–(63) into (58), and noting the smallness of ε, we have

$$\frac{d}{dt}\int\left[VV_t + \frac{\alpha}{2}(1+t)^{-\lambda}V^2\right]dx + \frac{\alpha\lambda}{4}\int(1+t)^{-\lambda-1}V^2dx + \frac{C_0}{4}\int V_x^2dx \le \int V_t^2dx + C\delta^2. \tag{64}$$

Next, multiplying $(57)_1$ by $(1+t)^\lambda V_t$ and integrating the resulting equation with respect to x over $\mathbb{R}^+$ and using Lemma 2.2, we can get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\left[(1+t)^\lambda V_t^2 - (1+t)^\lambda p'(\overline{v})V_x^2\right]dx \\ &+ \alpha\int V_t^2dx = -\frac{1}{2}\int(1+t)^\lambda p''(\overline{v})\overline{v}_tV_x^2dx \\ &- \frac{\lambda}{2}\int(1+t)^{\lambda-1}p'(\overline{v})V_x^2dx + \frac{\lambda}{2}\int(1+t)^{\lambda-1}V_t^2dx \\ &+ \int(1+t)^\lambda FV_tdx \le C\int(1+t)^{\lambda-1}\left(V_x^2 + V_t^2\right)dx \\ &+ \int(1+t)^\lambda FV_tdx. \end{aligned} \tag{65}$$

Now, we estimate the last term in the right hand of (65) as follows:

$$\int(1+t)^\lambda FV_tdx = \int\Big[\frac{(1+t)^{2\lambda}}{\alpha}p'(v_+)\overline{v}_{xt} + \frac{\lambda(1+t)^{2\lambda-1}}{\alpha}p'(v_+)\overline{v}_x - (1+t)^\lambda\left(p(V_x+\overline{v}+\hat{v}) - p(\overline{v}) - p'(\overline{v})V_x\right)_x + (1+t)^\lambda(p'(v+) - p'(\overline{v}))\overline{v}x\Big]V_tdx. \tag{66}$$

By using Lemma 2.1, Lemma 2.2 and *a priori* assumption (52), we have

$$\int\frac{(1+t)^{2\lambda}}{\alpha}p'(v_+)\overline{v}_{xt}V_tdx \le \frac{\alpha}{4}\int V_t^2dx + C\int(1+t)^{4\lambda}|\overline{v}_{xt}|^2dx \le \frac{\alpha}{4}\int V_t^2dx + C\delta^2, \tag{67}$$

$$\int\frac{\lambda(1+t)^{2\lambda-1}}{\alpha}p'(v_+)\overline{v}_xV_tdx \le \frac{\alpha}{4}\int V_t^2dx + C\int(1+t)^{4\lambda-2}|\overline{v}_x|^2dx \le \frac{\alpha}{4}\int V_t^2dx + C\delta^2, \tag{68}$$

$$\begin{aligned}
&-\int (1+t)^{\lambda}\left(p(V_x+\bar{v}+\hat{v})-p(\bar{v})-p'(\bar{v})V_x\right)_x V_t dx \\
&=\int (1+t)^{\lambda}\left(p(V_x+\bar{v}+\hat{v})-p(\bar{v})-p'(\bar{v})V_x\right)V_{xt} dx \\
&=(1+t)^{\lambda}\frac{d}{dt}\int\left(\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}} p(s)ds-p(\bar{v})V_x-\frac{1}{2}p'(\bar{v})V_x^2\right)dx \\
&\quad-\int (1+t)^{\lambda}\left(p(V_x+\bar{v}+\hat{v})-p(\bar{v})-p'(\bar{v})V_x-\frac{1}{2}p''(\bar{v})V_x^2\right)\bar{v}_t dx \\
&\quad-\int (1+t)^{\lambda}p(V_x+\bar{v}+\hat{v})\hat{v}_t dx \\
&\le \frac{d}{dt}\int (1+t)^{\lambda}\left(\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}} p(s)ds-p(\bar{v})V_x-\frac{1}{2}p'(\bar{v})V_x^2\right)dx \\
&\quad-\lambda\int (1+t)^{\lambda-1}\left(\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}} p(s)ds-p(\bar{v})V_x-\frac{1}{2}p'(\bar{v})V_x^2\right)dx \\
&\quad+C\delta\int (1+t)^{\lambda-1}\left(|\hat{v}|+|V_x|^3\right)dx+C\int (1+t)^{\lambda}|\hat{v}_t|dx \\
&\le \frac{d}{dt}\int (1+t)^{\lambda}\left(\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}} p(s)ds-p(\bar{v})V_x-\frac{1}{2}p'(\bar{v})V_x^2\right)dx \\
&\quad+C\varepsilon\int (1+t)^{\lambda-1}V_x^2 dx+C\delta^2,
\end{aligned} \tag{69}$$

and

$$\begin{aligned}
&\int (1+t)^{\lambda}\left(p'(v+)-p'(\bar{v})\right)\bar{v}_x V_t dx \\
&\quad\le \frac{\alpha}{4}\int V_t^2 dx+C\int (1+t)^{2\lambda}\left(\left(p'(v_+)-p'(\bar{v})\right)\bar{v}_x\right)^2 dx \\
&\quad\le \frac{\alpha}{4}\int V_t^2 dx+C\delta^2.
\end{aligned} \tag{70}$$

Substituting (67)–(70) into (65), and noting the smallness of ε, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left[(1+t)^{\lambda}V_t^2-(1+t)^{\lambda}p'(\bar{v})V_x^2\right]dx+\frac{\alpha}{4}\int V_t^2 dx \\
&\quad\le \frac{d}{dt}\int (1+t)^{\lambda}\left(\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}} p(s)ds-p(\bar{v})V_x-\frac{1}{2}p'(\bar{v})V_x^2\right)dx \\
&\qquad+C\int (1+t)^{\lambda-1}\left(V_x^2+V_t^2\right)dx+C\delta^2.
\end{aligned} \tag{71}$$

Thus we have (71) $\times\, k$ + (64), and we have

$$\begin{aligned}
&\frac{d}{dt}\int\left[VV_t+\frac{\alpha}{2}(1+t)^{-\lambda}V^2+\frac{k}{2}(1+t)^{\lambda}V_t^2-\frac{k}{2}(1+t)^{\lambda}p'(\bar{v})V_x^2\right]dx \\
&\qquad+\frac{\alpha\lambda}{4}\int (1+t)^{-\lambda-1}V^2 dx+\frac{C_0}{4}\int V_x^2 dx+\frac{\alpha k-1}{4}\int V_t^2 dx \\
&\quad\le k\frac{d}{dt}\int (1+t)^{\lambda}\left(\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}} p(s)ds-p(\bar{v})V_x-\frac{1}{2}p'(\bar{v})V_x^2\right)dx \\
&\qquad+Ck\int (1+t)^{\lambda-1}\left(V_x^2+V_t^2\right)dx+C\delta^2.
\end{aligned} \tag{72}$$

Choosing $k=6/\alpha$, using Gronwall's inequality on $[0,t]$ and the smallness of ε, we have

$$\|V\|_1^2+\|V_t\|^2+\int_0^t\left(\|V_x\|^2+\|V_t\|^2\right)ds\le C(T)\left(\|V_0\|_1^2+\|U_0\|^2+\delta\right). \tag{73}$$

This proves (53).

Step 2. Differentiating $(57)_1$ with respect to x, one gets

$$V_{xtt}+\left(p'(\bar{v})V_x\right)_{xx}+\frac{\alpha}{(1+t)^{\lambda}}V_{xt}=F_x. \tag{74}$$

Multiplying (74) by $(1+t)^{\lambda}V_{xt}$ and integrating the resulting equation with respect to x over $\mathbb{R}^+$, we can get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left[(1+t)^{\lambda}V_{xt}^2-(1+t)^{\lambda}p'(\bar{v})V_{xx}^2\right]dx+\alpha\int V_{xt}^2 dx \\
&\quad=\frac{\lambda}{2}\int (1+t)^{\lambda-1}V_{xt}^2 dx-\frac{\lambda}{2}\int (1+t)^{\lambda-1}p'(\bar{v})V_{xx}^2 dx \\
&\qquad-\frac{1}{2}\int (1+t)^{\lambda}p''(\bar{v})\bar{v}_t V_{xx}^2 dx-\int (1+t)^{\lambda}p'''(\bar{v})|\bar{v}_x|^2 V_x V_{xt} dx \\
&\qquad-\int (1+t)^{\lambda}p''(\bar{v})\bar{v}_{xx}V_x V_{xt} dx-\int (1+t)^{\lambda}p''(\bar{v})\bar{v}_x V_{xx}V_{xt} dx \\
&\qquad+\int (1+t)^{\lambda}F_x V_{xt} dx:=\sum_{k=1}^{7} I_k.
\end{aligned} \tag{75}$$

By using Cauchy–Schwarz's inequality, Lemma 2.2, Corollary 2.3 and *a priori* assumption (51) to address the following estimates

$$I_1+I_2\le C\int (1+t)^{\lambda-1}\left(V_{xt}^2+V_{xx}^2\right)dx, \tag{76}$$

$$I_3\le C\delta\int (1+t)^{\lambda-1}V_{xx}^2 dx, \tag{77}$$

$$\begin{aligned}
I_4&\le \tfrac{\alpha}{16}\int V_{xt}^2 dx+C\int (1+t)^{2\lambda}|\bar{v}_x|^4 V_x^2 dx \\
&\le \tfrac{\alpha}{16}\int V_{xt}^2 dx+C\varepsilon^2\delta^2,
\end{aligned} \tag{78}$$

$$\begin{aligned}
I_5&\le \tfrac{\alpha}{16}\int V_{xt}^2 dx+C\int (1+t)^{2\lambda}|\bar{v}_{xx}|^2 V_x^2 dx, \\
&\le \tfrac{\alpha}{16}\int V_{xt}^2 dx+C\varepsilon^2\delta^2,
\end{aligned} \tag{79}$$

and

$$\begin{aligned}
I_6&\le \tfrac{\alpha}{16}\int V_{xt}^2 dx+C\int (1+t)^{2\lambda}|\bar{v}_x|^2 V_{xx}^2 dx \\
&\le \tfrac{\alpha}{16}\int V_{xt}^2 dx+C\varepsilon^2\delta^2.
\end{aligned} \tag{80}$$

Now, we turn to estimate I_7 as follows:

$$\begin{aligned}
\int (1+t)^{\lambda}F_x V_{xt} tdx=\int\Bigg[&\frac{(1+t)^{2\lambda}}{\alpha}p'(v_+)\bar{v}_{xxt}+\frac{\lambda(1+t)^{2\lambda-1}}{\alpha}p'(v_+)\bar{v}_{xx} \\
&-(1+t)^{\lambda}\left(p(V_x+\bar{v}+\hat{v})-p(\bar{v})-p'(\bar{v})V_x\right)_{xx} \\
&+(1+t)^{\lambda}\left(\left(p'(v_+)-p'(\bar{v})\right)\bar{v}_x\right)_x\Bigg]V_{xt}dx:=\sum_{k=1}^{4}J_k.
\end{aligned} \tag{81}$$

By using Lemma 2.1, Lemma 2.2, Corollary 2.3 and *a priori* assumption (51), we have

$$J_1\le\frac{\alpha}{16}\int V_{xt}^2 dx+C\int (1+t)^{4\lambda}|\bar{v}_{xxt}|^2 dx\le\frac{\alpha}{16}\int V_{xt}^2 dx+C\delta^2, \tag{82}$$

$$J_2\le\frac{\alpha}{16}\int V_{xt}^2 dx+C\int (1+t)^{4\lambda-2}|\bar{v}_{xx}|^2 dx\le\frac{\alpha}{16}\int V_{xt}^2 dx+C\delta^2, \tag{83}$$

$$\begin{aligned} J_3 &= \int (1+t)^{\lambda}\left(p(V_x+\bar{v}+\hat{v})-p(\bar{v})-p'(\bar{v})V_x\right)_x V_{xxt}dx \\ &\le (1+t)^{\lambda}\frac{1}{2}\frac{d}{dt}\int \left(p'(V_x+\bar{v}+\hat{v})-p'(\bar{v})\right)V_{xx}^2dx \\ &\quad -\frac{1}{2}\int (1+t)^{\lambda}\left(p''(V_x+\bar{v}+\hat{v})(V_{xt}+\bar{v}_t+\hat{v}_t)-p''(\bar{v})\bar{v}_t\right)V_{xx}^2dx \\ &\quad +\int (1+t)^{\lambda}\left(p'(V_x+\bar{v}+\hat{v})-p'(\bar{v})-p''(\bar{v})V_x\right)\bar{v}_xV_{xxt}dx \\ &\quad +\int (1+t)^{\lambda}p'(V_x+\bar{v}+\hat{v})\bar{v}_xV_{xxt}dx \\ &\le \frac{1}{2}\frac{d}{dt}\int (1+t)^{\lambda}\left(p'(V_x+\bar{v}+\hat{v})-p'(\bar{v})\right)V_{xx}^2dx \\ &\quad -\frac{\lambda}{2}\int (1+t)^{\lambda-1}\left(p'(V_x+\bar{v}+\hat{v})-p'(\bar{v})\right)V_{xx}^2dx \\ &\quad +C(T)(\varepsilon+\delta)\int (1+t)^{\lambda-1}V_{xx}^2dx \\ &\quad +\int (1+t)^{\lambda}\left(p'(V_x+\bar{v}+\hat{v})-p'(\bar{v})-p''(\bar{v})V_x\right)\bar{v}_xV_{xxt}dx \\ &\quad +\int (1+t)^{\lambda}p'(V_x+\bar{v}+\hat{v})\hat{v}_xV_{xxt}dx \\ &\le \frac{1}{2}\frac{d}{dt}\int (1+t)^{\lambda}\left(p'(V_x+\bar{v}+\hat{v})-p'(\bar{v})\right)V_{xx}^2dx \\ &\quad +\frac{\alpha}{16}\int V_{xt}^2dx+C(T)(\varepsilon+\delta)\int (1+t)^{\lambda-1}V_{xx}^2dx+C(T)\delta, \end{aligned} \tag{84}$$

and

$$\begin{aligned} J_4 &\le \frac{\alpha}{16}\int V_{xt}^2dx + C\int (1+t)^{2\lambda}\left(\left(p'(v_+)-p'(\bar{v})\right)\bar{v}_x\right)_x^2dx \\ &\le \frac{\alpha}{16}\int V_{xt}^2dx + C\delta^2. \end{aligned} \tag{85}$$

Substituting (76)–(85) into (75), and noticing ε,δ sufficiently small, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \left[(1+t)^{\lambda}V_{xt}^2-(1+t)^{\lambda}p'(\bar{v})V_{xx}^2\right]dx+\frac{\alpha}{2}\int V_{xt}^2dx \\ &\quad\le \frac{1}{2}\frac{d}{dt}\int (1+t)^{\lambda}\left(p'(V_x+\bar{v}+\hat{v})-p'(\bar{v})\right)V_{xx}^2dx \\ &\qquad + C\int (1+t)^{\lambda-1}\left(V_{xx}^2+V_{xt}^2\right)dx + C(T)\delta. \end{aligned} \tag{86}$$

Multiplying $(57)_1$ by $-V_{xx}$ and integrating the resulting equality with respect to x over $\mathbb{R}^+$, we have

$$\begin{aligned} &\frac{d}{dt}\int \left[V_xV_{xt}+\frac{\alpha}{2}(1+t)^{-\lambda}V_x^2\right]dx \\ &\qquad +\frac{\alpha\lambda}{2}\int (1+t)^{-\lambda-1}V_x^2dx-\int p'(\bar{v})V_{xx}^2dx \\ &\quad=\int V_{xt}^2dx+\int p''(\bar{v})\bar{v}_xV_xV_{xx}dx-\int FV_{xx}dx. \end{aligned} \tag{87}$$

It is easy to see that

$$\int p''(\bar{v})\bar{v}_xV_xV_{xx}dx \le \frac{C_0}{12}\int V_{xx}^2dx + C\int |\bar{v}_x|^2V_x^2dx \le \frac{C_0}{12}\int V_{xx}^2dx + C\varepsilon^2\delta^2. \tag{88}$$

From (26), we have

$$\begin{aligned} -\int FV_{xx}dx = -\int \Big[&\frac{(1+t)^{\lambda}}{\alpha}p'(v^+)\bar{v}_{xt}+\frac{\lambda(1+t)^{\lambda-1}}{\alpha}p'(v_+)\bar{v}_x \\ &-\left(p(V_x+\bar{v}+\hat{v})-p(\bar{v})-p'(\bar{v})V_x\right)_x+\left(p'(v_+)-p'(\bar{v})\right)\bar{v}_x\Big]V_{xx}dx. \end{aligned} \tag{89}$$

By using Lemma 2.1, Lemma 2.2, Corollary 2.3 and *a priori* assumption (51), we have

$$\begin{aligned} &-\int \frac{(1+t)^{\lambda}}{\alpha}p'(v_+)\bar{v}_{xt}V_{xx}dx \\ &\quad\le \frac{C_0}{12}\int V_{xx}^2dx + C\int (1+t)^{2\lambda}\left|\bar{v}_{xt}\right|^2dx \\ &\quad\le \frac{C_0}{12}\int V_{xx}^2dx + C\delta^2, \end{aligned} \tag{90}$$

$$\begin{aligned} &-\int \frac{\lambda(1+t)^{\lambda-1}}{\alpha}p'(v_+)\bar{v}_xV_{xx}dx \\ &\quad\le \frac{C_0}{12}\int V_{xx}^2dx + C\int (1+t)^{2\lambda-2}\left|\bar{v}_x\right|^2dx \\ &\quad\le \frac{C_0}{12}\int V_{xx}^2dx + C\delta^2, \end{aligned} \tag{91}$$

$$\begin{aligned} &\int \left(p(V_x+\bar{v}+\hat{v})-p(\bar{v})-p'(\bar{v})V_x\right)_xV_{xx}dx \\ &=\int \left(p'(V_x+\bar{v}+\hat{v})-p'(\bar{v})\right)V_{xx}^2dx \\ &\quad+\int \left(p'(V_x+\bar{v}+\hat{v})-p'(\bar{v})-p''(\bar{v})V_x\right)\bar{v}_xV_{xx}dx \\ &\quad+\int p'(V_x+\bar{v}+\hat{v})\hat{v}_xV_{xx}dx \\ &\le C\int \left(|V_x|+|\hat{v}|\right)V_{xx}^2dx+\frac{C_0}{12}\int V_{xx}^2dx \\ &\quad+\int |\bar{v}_x|^2\left(|\hat{v}|^2+|V_x|^4\right)dx+C\int |\hat{v}_x|^2dx \\ &\le C(\varepsilon+\delta)\int V_{xx}^2dx+\frac{C_0}{12}\int V_{xx}^2dx+C\delta^2, \end{aligned} \tag{92}$$

and

$$\begin{aligned} &\int \left(p'(v_+)-p'(\bar{v})\right)\bar{v}_xV_{xx}dx \\ &\le \frac{C_0}{12}\int V_{xx}^2dx + C\int \left(\left(p'(v_+)-p'(\bar{v})\right)\bar{v}_x\right)^2dx \\ &\le \frac{C_0}{12}\int V_{xx}^2dx + C\delta^2. \end{aligned} \tag{93}$$

Substituting (88)–(93) into (87), and using the smallness of δ,ε, we have

$$\begin{aligned} &\frac{d}{dt}\int \left[V_xV_{xt}+\frac{\alpha}{2}(1+t)^{-\lambda}V_x^2\right]dx \\ &\qquad+\frac{\alpha\lambda}{2}\int (1+t)^{-\lambda-1}V_x^2dx+\frac{C_0}{2}\int V_{xx}^2dx \\ &\quad\le \int V_{xt}^2dx + C\delta^2. \end{aligned} \tag{94}$$

Multiplying (86) by h and adding up the resulting inequality and (94), we get

$$\begin{aligned}&\frac{d}{dt}\int\left[V_xV_{xt}+\frac{\alpha}{2}(1+t)^{-\lambda}V_x^2+\frac{h}{2}(1+t)^{\lambda}V_{xt}^2-\frac{h}{2}(1+t)^{\lambda}p'(\overline{v})V_{xx}^2\right]dx\\&\quad+\frac{\alpha\lambda}{2}\int(1+t)^{-\lambda-1}V_x^2dx+\frac{C_0}{2}\int V_{xx}^2dx+\left(\frac{\alpha h}{2}-1\right)\int V_{xt}^2dx\\&\le\frac{h}{2}\frac{d}{dt}\int(1+t)^{\lambda}(p'(V_x+\overline{v}+\widehat{v})-p'(\overline{v}))V_{xx}^2dx\\&\quad+Ch\int(1+t)^{\lambda-1}\left(V_{xx}^2+V_{xt}^2\right)dx+C\delta^2.\end{aligned}\tag{95}$$

Taking $h=6/\alpha$, using Gronwall's inequality on $[0,t]$ and (53) , we have

$$\|V_{xx}\|^2+\|V_{xt}\|^2+\int_0^t\left(\|V_{xx}\|^2+\|V_{xt}\|^2\right)ds\le C(T)\left(\|V_0\|_2^2+\|U_0\|_1^2+\delta\right).\tag{96}$$

This proves (154).

Step 3. Similar calculations to (73) and (96), we can get the high order estimate (3.5). The details are omitted.

From (53), (54), and (55), by using the continuity technique, we can verify the *a priori* assumption (51) is true provided $\|V_0\|_3^2+\|U_0\|_2^2+\delta$ sufficiently small. The proof of Proposition 3.1 is complete.

4. Decay Rates

The main goal of this section is to obtain the decay rates of the solution $(V,U)(x,t)$. We devote ourselves to the estimates of the solution $(V,U)(x,t)$ under the *a priori* assumption

$$\begin{aligned}N_1(T):=\sup_{0<t<T}&\Big\{\|V_x(\cdot,t)\|_{L^\infty}+(1+t)\|V_{xt}(\cdot,t)\|_{L^\infty}\\&+(1+t)^{\lambda+1/2}\|V_{xx}(\cdot,t)\|_{L^\infty}\Big\}\le\varepsilon,\end{aligned}\tag{97}$$

where $0<\varepsilon\ll1, 0<T<\infty$.

Lemma 4.1. *Under the assumptions of Theorem 2.4, if ε is small enough, it holds that*

$$\begin{aligned}&\|V\|^2+(1+t)^{2\lambda}\left(\|V_t\|^2+\|V_x\|^2\right)+\int_0^t(1+s)^{\lambda}\left(\|V_t\|^2+\|V_x\|^2\right)ds\\&\quad\le C\left(\|V_0\|_1^2+\|U_0\|^2+\delta\right),\quad \text{for } 0\le\lambda<\frac{3}{5}.\end{aligned}\tag{98}$$

For $3/5<\lambda<1$, we have

$$\begin{aligned}&(1+t)^{(3/2)-(5\lambda/2)}\|V\|^2+(1+t)^{(3/2)-(\lambda/2)}\left(\|V_t\|^2+\|V_x\|^2\right)\\&\quad\le C\left(\|V_0\|_1^2+\|U_0\|^2+\delta\right),\end{aligned}\tag{99}$$

and

$$\begin{aligned}&\int_0^t\left[(1+s)^{\beta-\lambda-1}\|V\|^2+(1+s)^{\beta}\left(\|V_t\|^2+\|V_x\|^2\right)\right]ds\\&\quad\le C(1+t)^{\beta+(3\lambda/2)-(3/2)}\left(\|V_0\|_1^2+\|U_0\|^2+\delta\right),\end{aligned}\tag{100}$$

for any $3/2-3\lambda/2<\beta<\lambda$.

Proof. Multiplying $(57)_1$ by $(1+t)^{\beta}V$ and integrating the resulting equality with respect to x over $\mathbb{R}^+$, we obtain

$$\begin{aligned}&\frac{d}{dt}\int\left[(1+t)^{\beta}VV_t+\frac{\alpha}{2}(1+t)^{\beta-\lambda}V^2\right]dx\\&\quad+\frac{\alpha(\lambda-\beta)}{2}\int(1+t)^{\beta-\lambda-1}V^2dx-\int(1+t)^{\beta}p'(\overline{v})V_x^2dx\\&=\int(1+t)^{\beta}V_t^2dx+\frac{\beta}{2}\frac{d}{dt}\int(1+t)^{\beta-1}V^2dx\\&\quad+\frac{\beta(1-\beta)}{2}\int(1+t)^{\beta-2}V^2dx+\int(1+t)^{\beta}FVdx.\end{aligned}\tag{101}$$

Now, we estimate the last term in the right hand of (101), from (39), we have

$$\begin{aligned}\int(1+t)^{\beta}FVdx=&\int\Big[\frac{(1+t)^{\beta+\lambda}}{\alpha}p'(v_+)\overline{v}_{xt}+\frac{\lambda(1+t)^{\beta+\lambda-1}}{\alpha}\\&\cdot p'(v_+)\overline{v}_x-(1+t)^{\beta}\left(p(V_x+\overline{v}+\widehat{v})-p(\overline{v})-p'(\overline{v})V_x\right)\\&+(1+t)^{\beta}\left(p'(\overline{v}+)-p'(\overline{v})\right)\overline{v}_x\Big]Vdx.\end{aligned}\tag{102}$$

From Lemma 2.2, the Cauchy–Schwarz's inequality and noticing $p'(v)\le-C_0<0$, we have

$$\begin{aligned}\int\frac{(1+t)^{\beta+\lambda}}{\alpha}p'(v_+)\overline{v}_{xt}Vdx&=-\int\frac{(1+t)^{\beta+\lambda}}{\alpha}p'(v_+)\overline{v}_tV_xdx\\&\le\frac{C_0}{8}\int(1+t)^{\beta}V_x^2dx+C\int(1+t)^{\beta+2\lambda}|\overline{v}_t|^2dx\\&\le\frac{C_0}{8}\int(1+t)^{\beta}V_x^2dx+C\delta^2(1+t)^{\beta+(3\lambda/2)-(5/2)},\end{aligned}\tag{103}$$

and for some constant $\kappa>1, \upsilon>0$, which will be determined below

$$\begin{aligned}&\int\frac{\lambda(1+t)^{\beta+\lambda-1}}{\alpha}p'(v_+)\overline{v}_xVdx\\&\le\upsilon\int(1+t)^{-\kappa}V^2dx+\frac{C}{\upsilon}\int(1+t)^{2\beta+2\lambda+\kappa-2}|\overline{v}_x|^2dx\\&\le\upsilon\int(1+t)^{-\kappa}V^2dx+\frac{C\delta^2}{\upsilon}(1+t)^{2\beta+\kappa+(\lambda/2)-(7/2)}.\end{aligned}\tag{104}$$

By using Lemmas 2.1, Lemma 2.2 and *a priori* assumption (97), we have

$$\begin{aligned}&-\int(1+t)^{\beta}\left(p(V_x+\overline{v}+\widehat{v})-p(\overline{v})-p'(\overline{v})V_x\right)_xVdx\\&=\int(1+t)^{\beta}\left(p(V_x+\overline{v}+\widehat{v})-p(\overline{v})-p'(\overline{v})V_x\right)V_xdx\\&\le\frac{C_0}{8}\int(1+t)^{\beta}V_x^2dx+C\int(1+t)^{\beta}\left(|\widehat{v}|^2+|V_x|^4+|\widehat{v}|^4\right)dx\\&\le\frac{C_0}{8}\int(1+t)^{\beta}V_x^2dx+C\varepsilon^2\int(1+t)^{\beta}V_x^2dx+C\delta^2e^{-ct\vartheta},\end{aligned}\tag{105}$$

and

$$\begin{aligned}&\int(1+t)^{\beta}\left(p'(v_+)-p'(\overline{v})\right)\overline{v}_xVdx\\&\le \upsilon\int(1+t)^{-\kappa}V^2dx\\&\quad+\frac{C}{\upsilon}\int\left(\left(p'(v_+)-p'(\overline{v})\right)\overline{v}_x\right)^2(1+t)^{2\beta+\kappa}dx\\&\le \upsilon\int(1+t)^{-\kappa}V^2dx+\frac{C\delta^2}{\upsilon}(1+t)^{2\beta+\kappa-(5\lambda/2)-(5/2)}.\end{aligned}\tag{106}$$

Substituting (102)–(106) into (101), and noting the smallness of ε, we have

$$\begin{aligned}&\frac{d}{dt}\int\left[(1+t)^{\beta}VV_t+\frac{\alpha}{2}(1+t)^{\beta-\lambda}V^2\right]dx+\frac{\alpha(\lambda-\beta)}{2}\int(1+t)^{\beta-\lambda-1}V^2dx\\&\quad+\frac{C_0}{2}\int(1+t)^{\beta}V_x^2dx\le\int(1+t)^{\beta}V_t^2dx+\frac{\beta}{2}\frac{d}{dt}\int(1+t)^{\beta-1}V^2dx\\&\quad+\frac{\beta(1-\beta)}{2}\int(1+t)^{\beta-2}V^2dx+2\upsilon\int(1+t)^{-\kappa}V^2dx\\&\quad+C\delta^2(1+t)^{\beta+(3\lambda/2)-(5/2)}+\frac{C\delta^2}{\upsilon}(1+t)^{2\beta+\kappa+(\lambda/2)-(7/2)}\\&\quad+\frac{C\delta^2}{\upsilon}(1+t)^{2\beta+\kappa-(5\lambda/2)-(7/2)}.\end{aligned}\tag{107}$$

Next, multiplying $(57)_1$ by $(1+t)^{\beta+\lambda}V_t$ and integrating the resulting equation with respect to x over $\mathbb{R}^+$, we can get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left[(1+t)^{\beta+\lambda}V_t^2-(1+t)^{\beta+\lambda}p'(\overline{v})V_x^2\right]dx\\&\quad+\int\alpha(1+t)^{\beta}V_t^2dx=-\frac{1}{2}\int(1+t)^{\beta+\lambda}p''(\overline{v})\overline{v}_tV_x^2dx\\&\quad-\frac{\beta+\lambda}{2}\int(1+t)^{\beta+\lambda-1}p'(\overline{v})V_x^2dx+\frac{\beta+\lambda}{2}\int(1+t)^{\beta+\lambda-1}V_t^2dx\\&\quad+\int(1+t)^{\beta+\lambda}FV_tdx\le C\int(1+t)^{\beta+\lambda-1}\left(V_x^2+V_t^2\right)dx\\&\quad+\int(1+t)^{\beta+\lambda}FV_tdx.\end{aligned}\tag{108}$$

Now, we estimate the last term in the right hand of (108) as follows:

$$\begin{aligned}\int(1+t)^{\beta+\lambda}FV_tdx=\int\Bigg[&\frac{(1+t)^{\beta+2\lambda}}{\alpha}p'(v_+)\overline{v}_{xt}+\frac{\lambda(1+t)^{\beta+2\lambda-1}}{\alpha}p'(v_+)\overline{v}_x\\&-(1+t)^{\beta+\lambda}\left(p(V_x+\overline{v}+\hat{v})-p(\overline{v})-p'(\overline{v})V_x\right)_x\\&+(1+t)^{\beta+\lambda}\left(p'(v_+)-p'(\overline{v})\right)\overline{v}_x\Bigg]V_tdx.\end{aligned}\tag{109}$$

By using Lemma 2.1, Lemma 2.2 and *a priori* assumption (97), we have

$$\begin{aligned}&\int\frac{(1+t)^{\beta+2\lambda}}{\alpha}p'(v_+)\overline{v}_{xt}V_tdx\\&\le\frac{\alpha}{4}\int(1+t)^{\beta}V_t^2dx+C\int(1+t)^{\beta+4\lambda}\left|\overline{v}_{xt}\right|^2dx\\&\le\frac{\alpha}{4}\int(1+t)^{\beta}V_t^2dx+C\delta^2(1+t)^{\beta+(5\lambda/2)-(7/2)},\end{aligned}\tag{110}$$

$$\begin{aligned}&\int\frac{\lambda(1+t)^{\beta+2\lambda-1}}{\alpha}p'(v_+)\overline{v}_xV_tdx\\&\le\frac{\alpha}{4}\int(1+t)^{\beta}V_t^2dx+C\int(1+t)^{\beta+4\lambda-2}\left|\overline{v}_x\right|^2dx\\&\le\frac{\alpha}{4}\int(1+t)^{\beta}V_t^2dx+C\delta^2(1+t)^{\beta+(5\lambda/2)-(7/2)},\end{aligned}\tag{111}$$

$$\begin{aligned}&-\int(1+t)^{\beta+\lambda}\left(p(V_x+\overline{v}+\hat{v})-p(\overline{v})-p'(\overline{v})V_x\right)_xV_tdx\\&=\int(1+t)^{\beta+\lambda}\left(p(V_x+\overline{v}+\hat{v})-p(\overline{v})-p'(\overline{v})V_x\right)V_{xt}dx\\&=(1+t)^{\beta+\lambda}\frac{d}{dt}\int\left(\int_{\overline{v}}^{V_x+\overline{v}+\hat{v}}p(s)ds-p(\overline{v})V_x-\frac{1}{2}p'(\overline{v})V_x^2\right)dx\\&\quad-\int(1+t)^{\beta+\lambda}\left(p(V_x+\overline{v}+\hat{v})-p(\overline{v})-p'(\overline{v})V_x-\frac{1}{2}p''(\overline{v})V_x^2\right)\overline{v}_tdx\\&\quad-\int(1+t)^{\beta+\lambda}p(V_x+\overline{v}+\hat{v})\hat{v}_tdx\\&\le\frac{d}{dt}\int(1+t)^{\beta+\lambda}\left(\int_{\overline{v}}^{V_x+\overline{v}+\hat{v}}p(s)ds-p(\overline{v})V_x-\frac{1}{2}p'(\overline{v})V_x^2\right)dx\\&\quad-(\beta+\lambda)\int(1+t)^{\beta+\lambda-1}\left(\int_{\overline{v}}^{V_x+\overline{v}+\hat{v}}p(s)ds-p(\overline{v})V_x-\frac{1}{2}p'(\overline{v})V_x^2\right)dx\\&\quad+C\delta\int(1+t)^{\beta+\lambda-1}\left(|\overline{v}|+|V_x|^3\right)dx+C\int(1+t)^{\beta+\lambda}|\hat{v}_t|dx\\&\le\frac{d}{dt}\int(1+t)^{\beta+\lambda}\left(\int_{\overline{v}}^{V_x+\overline{v}+\hat{v}}p(s)ds-p(\overline{v})V_x-\frac{1}{2}p'(\overline{v})V_x^2\right)dx\\&\quad+C(\varepsilon+\delta)\int(1+t)^{\beta+\lambda-1}V_x^2dx+C\delta^2e^{-ct^{\theta}}(1+t)^{\beta+\lambda},\end{aligned}\tag{112}$$

and

$$\begin{aligned}&\int(1+t)^{\beta+\lambda}\left(p'(v_+)-p'(\overline{v})\right)\overline{v}_xV_tdx\\&\le\frac{\alpha}{4}\int(1+t)^{\beta}V_t^2dx+C\int(1+t)^{\beta+2\lambda}\left(\left(p'(v_+)-p'(\overline{v})\right)\overline{v}_x\right)^2dx\\&\le\frac{\alpha}{4}\int(1+t)^{\beta}V_t^2dx+C\delta^2(1+t)^{\beta-(\lambda/2)-(5/2)}.\end{aligned}\tag{113}$$

Substituting (110)–(113) into (108), and noting the smallness of ε, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left[(1+t)^{\beta+\lambda}V_t^2-(1+t)^{\beta+\lambda}p'(\overline{v})V_x^2\right]dx+\frac{\alpha}{4}\int(1+t)^{\beta}V_t^2dx\\&\le\frac{d}{dt}\int(1+t)^{\beta+\lambda}\left(\int_{\overline{v}}^{V_x+\overline{v}+\hat{v}}p(s)ds-p(\overline{v})V_x-\frac{1}{2}p'(\overline{v})V_x^2\right)dx\\&\quad+C\int(1+t)^{\beta+\lambda-1}\left(V_x^2+V_t^2\right)dx+C\delta^2(1+t)^{\beta+(5\lambda/2)-(7/2)}\\&\quad+C\delta^2(1+t)^{\beta-(\lambda/2)-(5/2)}.\end{aligned}\tag{114}$$

Thus we have (114) $\times\, h$ + (107), and we have

$$\begin{aligned}&\frac{d}{dt}\int\Big[(1+t)^{\beta}VV_t+\frac{\alpha}{2}(1+t)^{\beta-\lambda}V^2\\&\quad+\frac{h}{2}(1+t)^{\beta+\lambda}V_t^2-\frac{h}{2}(1+t)^{\beta+\lambda}p'(\overline{v})V_x^2\Big]dx\\&\quad+\frac{\alpha(\lambda-\beta)}{2}\int(1+t)^{\beta-\lambda-1}V^2dx+\frac{C_0}{2}\int(1+t)^{\beta}V_x^2dx\\&\quad+\left(\frac{\alpha h}{4}-1\right)\int(1+t)^{\beta}V_t^2dx\\&\le h\frac{d}{dt}\int(1+t)^{\beta+\lambda}\left(\int_{\overline{v}}^{V_x+\overline{v}+\hat{v}}p(s)ds-p(\overline{v})V_x-\frac{1}{2}p'(\overline{v})V_x^2\right)dx\\&\quad+\frac{\beta}{2}\frac{d}{dt}\int(1+t)^{\beta-1}V^2dx+\frac{\beta(1-\beta)}{2}\int(1+t)^{\beta-2}V^2dx\\&\quad+2\upsilon\int(1+t)^{-\kappa}V^2dx+Ch\int(1+t)^{\beta+\lambda-1}\left(V_x^2+V_t^2\right)dx\\&\quad+C\delta^2(1+t)^{\beta+(3\lambda/2)-(5/2)}+\frac{C\delta^2}{\upsilon}(1+t)^{2\beta+\kappa+(\lambda/2)-(7/2)}\\&\quad+\frac{C\delta^2}{\upsilon}(1+t)^{2\beta+\kappa-(5\lambda/2)-(5/2)}.\end{aligned}\tag{115}$$

Case 1 ($0 \le \lambda < 3/5$). It is easy to know that $1 < 5/2 - 5\lambda/2$. Therefore, we can take $\beta = \lambda$, $\upsilon = 1/2$, and there exist constant κ satisfying $1 < \kappa < \min\{5/2 - 5\lambda/2, \lambda/2 + 3/2, 2 - \lambda\}$. Then, we have

$$\begin{aligned}
&\frac{d}{dt}\int\left[(1+t)^{\lambda}VV_t + \frac{\alpha}{2}V^2 + \frac{h}{2}(1+t)^{2\lambda}V_t^2 - \frac{h}{2}(1+t)^{2\lambda}p'(\bar{v})V_x^2\right]dx \\
&\quad + \frac{C_0}{2}\int(1+t)^{\lambda}V_x^2dx + \left(\frac{\alpha h}{4} - 1\right)\int(1+t)^{\lambda}V_t^2dx \\
&\le h\frac{d}{dt}\int(1+t)^{2\lambda}\left(\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}}p(s)ds - p(\bar{v})V_x - \frac{1}{2}p'(\bar{v})V_x^2\right)dx \\
&\quad + \frac{\lambda}{2}\frac{d}{dt}\int(1+t)^{\lambda-1}V^2dx + \frac{\lambda(1-\lambda)}{2}\int(1+t)^{\lambda-2}V^2dx \\
&\quad + \int(1+t)^{-\kappa}V^2dx + Ch\int(1+t)^{2\lambda-1}\left(V_x^2 + V_t^2\right)dx \\
&\quad + C\delta^2(1+t)^{\kappa+(5\lambda/2)-(7/2)} + C\delta^2(1+t)^{\kappa-(\lambda/2)-(5/2)}.
\end{aligned} \tag{116}$$

Let T_0 be sufficiently large such that if $t \ge T_0$, it holds

$$\begin{cases}
Ch(1+t)^{\lambda-1} \le \frac{C_0}{4}, \\
Ch(1+t)^{\lambda-1} \le \frac{1}{2}\left(\frac{\alpha h}{4} - 1\right), \\
\frac{\lambda}{2}(1+t)^{\lambda-1} \le \frac{1}{4}.
\end{cases} \tag{117}$$

Fixing $h = 12/\alpha$, from (116) and (117) we have

$$\begin{aligned}
&\frac{d}{dt}H(t) + \frac{C_0}{2}\int(1+t)^{\lambda}V_x^2dx + \left(\frac{\alpha h}{4} - 1\right)\int(1+t)^{\lambda}V_t^2dx \\
&\le C\frac{d}{dt}\int(1+t)^{2\lambda}\left(\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}}p(s)ds - p(\bar{v})V_x - \frac{1}{2}p'(\bar{v})V_x^2\right)dx \\
&\quad + \frac{\lambda}{2}\frac{d}{dt}\int(1+t)^{\lambda-1}V^2dx \quad + C\int(1+t)^{-\kappa}V^2dx + \frac{C_0}{4}\int(1+t)^{\lambda}V_x^2dx \\
&\quad + \frac{1}{2}\left(\frac{\alpha h}{4} - 1\right)\int(1+t)^{\lambda}V_t^2dx + C\delta^2(1+t)^{\kappa+(5\lambda/2)-(7/2)} \\
&\quad + C\delta^2(1+t)^{\kappa-(\lambda/2)-(5/2)},
\end{aligned} \tag{118}$$

where

$$H(t) \sim \|V\|^2 + (1+t)^{2\lambda}\left\|V_x\right\|^2 + (1+t)^{2\lambda}\left\|V_t\right\|^2. \tag{119}$$

For any $t \in [T_0, +\infty)$, we have

$$\begin{aligned}
&\frac{d}{dt}H(t) + \frac{C_0}{4}\int(1+t)^{\lambda}V_x^2dx + \frac{1}{2}\left(\frac{\alpha h}{4} - 1\right)\int(1+t)^{\lambda}V_t^2dx \\
&\le C\frac{d}{dt}\int(1+t)^{2\lambda}\left(\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}}p(s)ds - p(\bar{v})V_x - \frac{1}{2}p'(\bar{v})V_x^2\right)dx \\
&\quad + \frac{\lambda}{2}\frac{d}{dt}\int(1+t)^{\lambda-1}V^2dx + C(1+t)^{-\kappa}H(t) \\
&\quad + C\delta^2(1+t)^{\kappa+(5\lambda/2)-(7/2)} + C\delta^2(1+t)^{\kappa-(\lambda/2)-(5/2)}.
\end{aligned} \tag{120}$$

Using (117) and Gronwall's inequality on $[T_0, t]$, noting $1 < \kappa < \min\{5/2 - 5\lambda/2, \lambda/2 + 3/2\}$, $0 \le \lambda < (3/5)$, we have

$$\begin{aligned}
&H(t) + C\int_{T_0}^{t}(1+s)^{\lambda}\left(\left\|V_x\right\|^2 + \left\|V_t\right\|^2\right)ds \\
&\quad \le C\int(1+t)^{2\lambda}\left(\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}}p(s)ds - p(\bar{v})V_x - \frac{1}{2}p'(\bar{v})V_x^2\right)dx \\
&\qquad + \frac{\lambda}{2}\int(1+t)^{\lambda-1}V^2dx + C\left(H(T_0) + \delta\right) \\
&\quad \le C\varepsilon\int(1+t)^{2\lambda}\left|V_x\right|^2dx + \frac{1}{4}\int V^2dx + C\left(H(T_0) + \delta\right),
\end{aligned} \tag{121}$$

which together with (73) and Proposition 3.1 deduce (98) in view of the smallness of ε.

Case 2 ($3/5 < \lambda < 1$). In this case, it is easy to know that $\lambda > (3/2) - (3\lambda/2)$. Therefore, we can take $3/2 - 3\lambda/2 < \beta < \lambda$, $\kappa = \lambda - \beta + 1$, and $\nu = \alpha(\lambda - \beta)/8 > 0$. Then, from (115), we have

$$\begin{aligned}
&\frac{d}{dt}\int\left[(1+t)^{\beta}VV_t + \frac{\alpha}{2}(1+t)^{\beta-\lambda}V^2\right. \\
&\quad \left. + \frac{h}{2}(1+t)^{\beta+\lambda}V_t^2 - \frac{h}{2}(1+t)^{\beta+\lambda}p'(\bar{v})V_x^2\right]dx \\
&\quad + \frac{\alpha(\lambda-\beta)}{4}\int(1+t)^{\beta-\lambda-1}V^2dx + \frac{C_0}{2}\int(1+t)^{\beta}V_x^2dx \\
&\quad + \left(\frac{\alpha h}{4} - 1\right)\int(1+t)^{\beta}V_t^2dx \\
&\le h\frac{d}{dt}\int(1+t)^{\beta+\lambda}\left(\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}}p(s)ds - p(\bar{v})V_x - \frac{1}{2}p'(\bar{v})V_x^2\right)dx \\
&\quad + \frac{\beta}{2}\frac{d}{dt}\int(1+t)^{\beta-1}V^2dx + \frac{\beta(1-\beta)}{2}\int(1+t)^{\beta-2}V^2dx \\
&\quad + Ch\int(1+t)^{\beta+\lambda-1}\left(V_x^2 + V_t^2\right)dx + C\delta^2(1+t)^{\beta+(3\lambda/2)-(5/2)}.
\end{aligned} \tag{122}$$

Let T_1 be sufficiently large such that if $t \ge T_1$, it holds

$$\begin{cases}
Ch(1+t)^{\lambda-1} \le \frac{C_0}{4}, \\
Ch(1+t)^{\lambda-1} \le \frac{1}{2}\left(\frac{\alpha h}{4} - 1\right), \\
\frac{\beta}{2}(1+t)^{\lambda-1} \le \frac{1}{4}.
\end{cases} \tag{123}$$

Fixing $h = 12/\alpha$, from (112), we have

$$\begin{aligned}
&\frac{d}{dt}H_1(t) + \frac{C_0}{4}\int(1+t)^{\beta}V_x^2dx + \frac{1}{2}\left(\frac{\alpha h}{4} - 1\right)\int(1+t)^{\beta}V_t^2dx \\
&\quad + \frac{\alpha(\lambda-\beta)}{4}\int(1+t)^{\beta-\lambda-1}V^2dx \le C\frac{d}{dt}\int(1+t)^{\beta+\lambda} \\
&\quad \cdot\left(\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}}p(s)ds - p(\bar{v})V_x - \frac{1}{2}p'(\bar{v})V_x^2\right)dx + \frac{\beta}{2}\frac{d}{dt}\int(1+t)^{\beta-1}V^2dx \\
&\quad + \frac{\beta(1-\beta)}{2}\int(1+t)^{\beta-2}V^2dx + C\delta^2(1+t)^{\beta+(3\lambda/2)-(5/2)},
\end{aligned} \tag{124}$$

where

$$H_1(t) \sim (1+t)^{\beta-\lambda}\|V\|^2 + (1+t)^{\beta+\lambda}\left\|V_x\right\|^2 + (1+t)^{\beta+\lambda}\left\|V_t\right\|^2. \tag{125}$$

Then, from (124), we know

$$\begin{aligned}
&\frac{d}{dt}H_1(t) + \frac{C_0}{4}\int(1+t)^{\beta}V_x^2dx + C\int(1+t)^{\beta}V_t^2dx \\
&\qquad + C\int(1+t)^{\beta-\lambda-1}V^2dx \le C\frac{d}{dt}\int(1+t)^{\beta+\lambda} \\
&\qquad \cdot\left(\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}}p(s)ds - p(\bar{v})V_x - \frac{1}{2}p'(\bar{v})V_x^2\right)dx \\
&\qquad + \frac{\beta}{2}\frac{d}{dt}\int(1+t)^{\beta-1}V^2dx \\
&\qquad + C(1+t)^{\lambda-2}H_1(t) + C\delta^2(1+t)^{\beta+(3\lambda/2)-(5/2)},
\end{aligned} \tag{126}$$

for any $t \in [T_1, \infty)$.

Using Gronwall's inequality on $[T_1, t]$, (123) and noting $\beta + 3\lambda/2 - 5/2 > -1, 3/5 < \lambda < 1$, we have

$$\begin{aligned} H_1(t) &+ C\int_{T_1}^{t}\left[(1+s)^{\beta}\left(\|V_x\|^2 + \|V_t\|^2\right) + (1+s)^{\beta-\lambda-1}\|V\|^2\right]ds \\ &\le C\int (1+t)^{\beta+\lambda}\left\{\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}} p(s)ds - p(\bar{v})V_x - \frac{1}{2}p'(\bar{v})V_x^2\right\}dx \\ &+ \frac{\beta}{2}\int (1+t)^{\beta-1}V^2dx + H_1(T_1) + C\delta^2(1+t)^{\beta+(3\lambda/2)-(3/2)} \\ &\le C\varepsilon\int (1+t)^{\beta+\lambda}\left|V_x\right|^2dx + \frac{1}{4}\int (1+t)^{\beta-\lambda}V^2dx \\ &+ H_1(T_1) + C\delta^2(1+t)^{\beta+(3\lambda/2)-(3/2)}, \end{aligned} \tag{127}$$

which together with (73) and Proposition 3.1 deduce (99) and (100), in view of the smallness of ε and $\beta + 3\lambda/2 - 3/2 > 0$.

Hence, we complete the proof of Lemma 4.1.

Furthermore, we can get the better decay rate of the functions V_x and V_t as follows. □

Lemma 4.2. *Under the assumptions of Theorem 2.4, if ε is small enough, it holds that*

$$\begin{aligned} &(1+t)^{\lambda+1}\left(\|V_t\|^2 + \|V_x\|^2\right) + \int_0^t (1+s)\|V_t\|^2ds \\ &\le C\left(\|V_0\|_1^2 + \|U_0\|^2 + \delta\right), \quad \text{for} \quad 0 \le \lambda < \frac{3}{5}. \end{aligned} \tag{128}$$

For $3/5 < \lambda < 1$, *we have*

$$(1+t)^{\frac{5}{2}-\frac{3\lambda}{2}}\left(\|V_t\|^2 + \|V_x\|^2\right) \le C\left(\|V_0\|_1^2 + \|U_0\|^2 + \delta\right), \tag{129}$$

and

$$\int_0^t (1+s)^{\beta-\lambda+1}\|V_t\|^2ds \le C(1+t)^{\beta+\frac{3\lambda}{2}-\frac{3}{2}}\left(\|V_0\|_1^2 + \|U_0\|^2 + \delta\right), \tag{130}$$

for any $3/2 - (3\lambda)/2 < \beta < \lambda$.

Proof. Multiplying (117) by $(1+t)^{1-\lambda}$, for the case of $0 \le \lambda < 3/5$, and noting $\beta = \lambda$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\left[(1+t)^{1+\lambda}V_t^2 - (1+t)^{1+\lambda}p'(\bar{v})V_x^2\right]dx + C\int (1+t)V_t^2dx \\ &\le \frac{d}{dt}\int (1+t)^{1+\lambda}\left(\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}} p(s)ds - p(\bar{v})V_x - \frac{1}{2}p'(\bar{v})V_x^2\right)dx \\ &- (1-\lambda)\int (1+t)^{\lambda}\left(\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}} p(s)ds - p(\bar{v})V_x - \frac{1}{2}p'(\bar{v})V_x^2\right)dx \\ &+ C\int\left((1+t)^{\lambda}V_t^2 - (1+t)^{\lambda}p'(\bar{v})V_x^2\right)dx \\ &+ C\int (1+t)^{\lambda}V_x^2dx + C\delta^2(1+t)^{(5\lambda/2)-(5/2)} + C\delta^2(1+t)^{-(\lambda/2)-(3/2)}. \end{aligned} \tag{131}$$

Integrating the above inequality in t over $(0, t)$, using (98), we get

$$(1+t)^{\lambda+1}\left(\|V_t\|^2 + \|V_x\|^2\right) + \int_0^t (1+s)\|V_t\|^2ds \le C\left(\|V_0\|_1^2 + \|U_0\|^2 + \delta\right), \tag{132}$$

for $0 \le \lambda < 3/5$.

This completes the proof of (128). For the case of $3/5 < \lambda < 1$, we can use the similar method to obtain (129) and (131).

Next, we will derive decay rates on the higher derivatives of the global solution $V(x, t)$. □

Lemma 4.3. *Under the assumptions of Theorem 2.4, if ε is small enough, it holds that*

$$\begin{aligned} &(1+t)^{2\lambda+2}\left(\|V_{xt}\|^2 + \|V_{xx}\|^2\right) \\ &+ \int_0^t\left((1+s)^{\lambda+2}\|V_{xt}\|^2 + (1+s)^{2\lambda+1}\|V_{xx}\|^2\right)ds \\ &\le C\left(\|V_0\|_2^2 + \|U_0\|_1^2 + \delta\right), \quad \text{for} \quad 0 \le \lambda < \frac{3}{5}. \end{aligned} \tag{133}$$

For $3/5 < \lambda < 1$, *we have*

$$(1+t)^{\frac{7}{2}-\frac{\lambda}{2}}\left(\|V_{xt}\|^2 + \|V_{xx}\|^2\right) \le C\left(\|V_0\|_2^2 + \|U_0\|_1^2 + \delta\right), \tag{134}$$

and

$$\begin{aligned} &\int_0^t\left((1+s)^{\beta+\lambda+1}\|V_{xx}\|^2 + (1+s)^{\beta+2}\|V_{xt}\|^2\right)ds \\ &\le C(1+t)^{\beta+(3\lambda/2)-(3/2)}\left(\|V_0\|_2^2 + \|U_0\|^1 + \delta\right), \end{aligned} \tag{135}$$

for any $3/2 - (3\lambda)/2 < \beta < \lambda$.

Proof. Multiplying (74) by $(1+t)^{\beta+\lambda}V_{xt}$ and integrating the resulting equation with respect to x over $\mathbb{R}^+$, by using integrations by parts, we can get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\left[(1+t)^{\beta+\lambda}V_{xt}^2 - (1+t)^{\beta+\lambda}p'(\bar{v})V_{xx}^2\right]dx \\ &+ \int \alpha(1+t)^{\beta}V_{xt}^2dx = \frac{\beta+\lambda}{2}\int (1+t)^{\beta+\lambda-1}V_{xt}^2dx \\ &- \frac{\beta+\lambda}{2}\int (1+t)^{\beta+\lambda-1}p'(\bar{v})V_{xx}^2dx - \frac{1}{2}\int (1+t)^{\beta+\lambda}p''(\bar{v})\bar{v}_tV_{xx}^2dx \\ &- \int (1+t)^{\beta+\lambda}p'''(\bar{v})\left|\bar{v}_x\right|^2V_xV_{xt}dx - \int (1+t)^{\beta+\lambda}p''(\bar{v})\bar{v}_{xx}V_xV_{xt}dx \\ &- \int (1+t)^{\beta+\lambda}p''(\bar{v})\bar{v}_xV_{xx}V_{xt}dx + \int (1+t)^{\beta+\lambda}F_xV_{xt}dx := \sum_{k=1}^{7} L_k. \end{aligned} \tag{136}$$

By using the Cauchy–Schwarz's inequality, Lemma 2.2 and Corollary 2.3 to address the following estimates

$$L_1 + L_2 \le C\int (1+t)^{\beta+\lambda-1}\left(V_{xt}^2 + V_{xx}^2\right)dx, \tag{137}$$

$$L_3 \le C\delta\int (1+t)^{\beta+(\lambda/2)-(3/2)}V_{xx}^2dx, \tag{138}$$

$$\begin{aligned} L_4 &\le \frac{\alpha}{16}\int (1+t)^{\beta}V_{xt}^2dx + C\int (1+t)^{\beta+2\lambda}\left|\bar{v}_x\right|^4V_x^2dx \\ &\le \frac{\alpha}{16}\int (1+t)^{\beta}V_{xt}^2dx + C\delta^4\int (1+t)^{\beta-2\lambda-4}V_x^2dx, \end{aligned} \tag{139}$$

$$\begin{aligned} L_5 &\le \frac{\alpha}{16}\int (1+t)^{\beta}V_{xt}^2dx + C\int (1+t)^{\beta+2\lambda}\left|\bar{v}_{xx}\right|^2V_x^2dx \\ &\le \frac{\alpha}{16}\int (1+t)^{\beta}V_{xt}^2dx + C\delta^2\int (1+t)^{\beta-\lambda-3}V_x^2dx, \end{aligned} \tag{140}$$

and

$$
\begin{aligned}
L_6 &\leq \frac{\alpha}{16}\int (1+t)^{\beta}V_{xt}^2dx + C\int (1+t)^{\beta+2\lambda}\left|\overline{v}_x\right|^2 V_{xx}^2dx \\
&\leq \frac{\alpha}{16}\int (1+t)^{\beta}V_{xt}^2dx + C\delta^2\int (1+t)^{\beta-2}V_{xx}^2dx.
\end{aligned} \tag{141}
$$

Now, we turn to estimate L_7 as follows:

$$
\begin{aligned}
L_7 = \int (1+t)^{\beta+\lambda}F_xV_{xt}dx = \int \Big[&\frac{(1+t)^{\beta+2\lambda}}{\alpha}p'(v_+)\overline{v}_{xxt} + \frac{\lambda(1+t)^{\beta+2\lambda-1}}{\alpha} \\
&\cdot p'(v_+)\overline{v}_{xx} - (1+t)^{\beta+\lambda}\left(p\left(V_x+\overline{v}+\widehat{v}\right) - p(\overline{v}) - p'(\overline{v})V_x\right)_{xx} \\
&+ (1+t)^{\beta+\lambda}\left(\left(p'(v_+) - p'(\overline{v})\right)\overline{v}_x\right)_x \Big] V_{xt}dx := \sum_{k=1}^{4} M_k.
\end{aligned} \tag{142}
$$

By using Lemma 2.1, Lemma 2.2, Corollary 2.3 and *a priori* assumption (97), we have

$$
\begin{aligned}
M_1 &\leq \frac{\alpha}{16}\int (1+t)^{\beta}V_{xt}^2dx + C\int (1+t)^{\beta+4\lambda}\left|\overline{v}_{xxt}\right|^2dx \\
&\leq \frac{\alpha}{16}\int (1+t)^{\beta}V_{xt}^2dx + C\delta^2(1+t)^{\beta+(3\lambda/2)-(9/2)},
\end{aligned} \tag{143}
$$

$$
\begin{aligned}
M_2 &\leq \frac{\alpha}{16}\int (1+t)^{\beta}V_{xt}^2dx + C\int (1+t)^{\beta+4\lambda-2}\left|\overline{v}_{xx}\right|^2dx \\
&\leq \frac{\alpha}{16}\int (1+t)^{\beta}V_{xt}^2dx + C\delta^2(1+t)^{\beta+(3\lambda/2)-(9/2)},
\end{aligned} \tag{144}
$$

$$
\begin{aligned}
M_3 &= \int (1+t)^{\beta+\lambda}\left(p\left(V_x+\overline{v}+\widehat{v}\right) - p(\overline{v}) - p'(\overline{v})V_x\right)_x V_{xxt}dx \\
&\leq (1+t)^{\beta+\lambda}\frac{1}{2}\frac{d}{dt}\int \left(p'\left(V_x+\overline{v}+\widehat{v}\right) - p'(\overline{v})\right)V_{xx}^2dx - \frac{1}{2}\int (1+t)^{\beta+\lambda}\left(p''\left(V_x+\overline{v}+\widehat{v}\right)\left(V_{xt}+\overline{v}_t+\widehat{v}_t\right) - p''(\overline{v})\overline{v}_t\right)V_{xx}^2dx \\
&\quad + \int (1+t)^{\beta+\lambda}\left(p'\left(V_x+\overline{v}+\widehat{v}\right) - p'(\overline{v}) - p''(\overline{v})V_x\right)\overline{v}_xV_{xxt}dx + \int (1+t)^{\beta+\lambda}p'\left(V_x+\overline{v}+\widehat{v}\right)\widehat{v}_xV_{xxt}dx \\
&\leq \frac{1}{2}\frac{d}{dt}\int (1+t)^{\beta+\lambda}\left(p'\left(V_x+\overline{v}+\widehat{v}\right) - p'(\overline{v})\right)V_{xx}^2dx - \frac{\beta+\lambda}{2}\int (1+t)^{\beta+\lambda-1}\left(p'\left(V_x+\overline{v}+\widehat{v}\right) - p'(\overline{v})\right)V_{xx}^2dx \\
&\quad + C(\varepsilon+\delta)\int (1+t)^{\beta+\lambda-1}V_{xx}^2dx - \int (1+t)^{\beta+\lambda}\overline{v}_{xx}V_{xt}\left(p'\left(V_x+\overline{v}+\widehat{v}\right) - p'(\overline{v}) - p''(\overline{v})V_x\right)dx \\
&\quad - \int (1+t)^{\beta+\lambda}\overline{v}_xV_{xt}\left(p''\left(V_x+\overline{v}+\widehat{v}\right) - p''(\overline{v})\right)V_{xx}dx - \int (1+t)^{\beta+\lambda}\overline{v}_x^2V_{xt}\left(p''\left(V_x+\overline{v}+\widehat{v}\right) - p''(\overline{v}) - p'''(\overline{v})V_x\right)dx \\
&\quad - \int (1+t)^{\beta+\lambda}V_{xt}\overline{v}_x\widehat{v}_xp''\left(V_x+\overline{v}+\widehat{v}\right)dx - \int (1+t)^{\beta+\lambda}V_{xt}\widehat{v}_xp''\left(V_x+\overline{v}+\widehat{v}\right)\left(V_{xx}+\overline{v}+\widehat{v}\right)dx \\
&\quad - \int (1+t)^{\beta+\lambda}V_{xt}\widehat{v}_{xx}p'\left(V_x+\overline{v}+\widehat{v}\right)dx \\
&\leq \frac{1}{2}\frac{d}{dt}\int (1+t)^{\beta+\lambda}\left(p'\left(V_x+\overline{v}+\widehat{v}\right) - p'(\overline{v})\right)V_{xx}^2dx + \frac{\alpha}{16}\int (1+t)^{\beta}V_{xt}^2dx \\
&\quad + C(\varepsilon+\delta)\int (1+t)^{\beta+\lambda-1}V_{xx}^2dx + C\delta^2\int (1+t)^{\beta-2}V_x^2dx + C\delta^2(1+t)^{\beta+(3\lambda/2)-(9/2)},
\end{aligned} \tag{145}
$$

and

$$
\begin{aligned}
M_4 &\leq \frac{\alpha}{16}\int (1+t)^{\beta}V_{xt}^2dx \\
&\quad + C\int (1+t)^{\beta+2\lambda}\left(\left(p'(v_+) - p'(\overline{v})\right)\overline{v}_x\right)_x^2dx \\
&\leq \frac{\alpha}{16}\int (1+t)^{\beta}V_{xt}^2dx + C\delta^2(1+t)^{\beta-(3\lambda/2)-(7/2)}.
\end{aligned} \tag{146}
$$

Substituting (137)–(146) into (136), and noticing ε, δ sufficiently small, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \left[(1+t)^{\beta+\lambda}V_{xt}^2 - (1+t)^{\beta+\lambda}p'(\overline{v})V_{xx}^2\right]dx \\
&\quad + \frac{\alpha}{2}\int (1+t)^{\beta}V_{xt}^2dx \\
&\leq \frac{1}{2}\frac{d}{dt}\int (1+t)^{\beta+\lambda}\left(p'\left(V_x+\overline{v}+\widehat{v}\right) - p'(\overline{v})\right)V_{xx}^2dx \\
&\quad + C\int (1+t)^{\beta+\lambda-1}\left(V_{xx}^2 + V_{xt}^2\right)dx \\
&\quad + C\delta^2\int (1+t)^{\beta-2}V_x^2dx \\
&\quad + C\delta^2(1+t)^{\beta+(3\lambda/2)-(9/2)} + C\delta^2(1+t)^{\beta-(3\lambda/2)-(7/2)}.
\end{aligned} \tag{147}
$$

Multiplying $(57)_1$ by $-(1+t)^{\beta}V_{xx}$, and integrating the resulting equality with respect to x over $\mathbb{R}^+$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int \left[(1+t)^{\beta}V_xV_{xt} + \frac{\alpha}{2}(1+t)^{\beta-\lambda}V_x^2\right]dx \\
&\quad + \frac{\alpha(\lambda-\beta)}{2}\int (1+t)^{\beta-\lambda-1}V_x^2dx - \int (1+t)^{\beta}p'(\overline{v})V_{xx}^2dx \\
&= \int (1+t)^{\beta}V_{xt}^2dx + \frac{\beta}{2}\frac{d}{dt}\int (1+t)^{\beta-1}V_x^2dx \\
&\quad + \frac{\beta(1-\beta)}{2}\int (1+t)^{\beta-2}V_x^2dx + \int (1+t)^{\beta}p''(\overline{v})\overline{v}_xV_xV_{xx}dx \\
&\quad - \int (1+t)^{\beta}FV_{xx}dx.
\end{aligned} \tag{148}
$$

It is easy to see that

$$
\begin{aligned}
&\int (1+t)^{\beta}p''(\overline{v})\overline{v}_xV_xV_{xx}dx \leq \frac{C_0}{10}\int (1+t)^{\beta}V_{xx}^2dx \\
&\quad + C\int (1+t)^{\beta}\left|\overline{v}_x\right|^2V_x^2dx \leq \frac{C_0}{10}\int (1+t)^{\beta}V_{xx}^2dx \\
&\quad + C\delta^2\int (1+t)^{\beta-2\lambda-2}V_x^2dx.
\end{aligned} \tag{149}
$$

From (39), we have

$$-\int (1+t)^{\beta} F V_{xx} dx = -\int \Big[\frac{(1+t)^{\beta+\lambda}}{\alpha} p'(v_+)\overline{v}_{xt} + \frac{\lambda(1+t)^{\beta+\lambda-1}}{\alpha} p'(v_+)\overline{v}_x - (1+t)^{\beta}\left(p(V_x+\overline{v}+\widehat{v}) - p(\overline{v}) - p'(\overline{v})V_x\right)_x + (1+t)^{\beta}\left(p'(v_+) - p'(\overline{v})\right)\overline{v}_x \Big] V_{xx} dx. \tag{150}$$

By using Lemma 2.1, Lemma 2.2, Corollary 2.3 and *a priori* assumption (97), we have

$$-\int \frac{(1+t)^{\beta+\lambda}}{\alpha} p'(v_+)\overline{v}_{xt} V_{xx} dx \le \frac{C_0}{10}\int (1+t)^{\beta} V_{xx}^2 dx + C\int (1+t)^{\beta+2\lambda}\left|\overline{v}_{xt}\right|^2 dx \le \frac{C_0}{10}\int (1+t)^{\beta} V_{xx}^2 dx + C\delta^2 (1+t)^{\beta+(\lambda/2)-(7/2)}, \tag{151}$$

$$-\int \frac{\lambda(1+t)^{\beta+\lambda-1}}{\alpha} p'(v_+)\overline{v}_x V_{xx} dx \le \frac{C_0}{10}\int (1+t)^{\beta} V_{xx}^2 dx + C\int (1+t)^{\beta+2\lambda-2}\left|\overline{v}_x\right|^2 dx \le \frac{C_0}{10}\int (1+t)^{\beta} V_{xx}^2 dx + C\delta^2 (1+t)^{\beta+(\lambda/2)-(7/2)}, \tag{152}$$

$$\begin{aligned}
&\int (1+t)^{\beta}\left(p(V_x+\overline{v}+\widehat{v}) - p(\overline{v}) - p'(\overline{v})V_x\right)_x V_{xx} dx \\
&= (1+t)^{\beta}\int \left(p'(V_x+\overline{v}+\widehat{v}) - p'(\overline{v})\right)V_{xx}^2 dx \\
&+ (1+t)^{\beta}\int \left(p'(V_x+\overline{v}+\widehat{v}) - p'(\overline{v}) - p''(\overline{v})V_x\right)\overline{v}_x V_{xx} dx \\
&+ (1+t)^{\beta}\int p'(V_x+\overline{v}+\widehat{v})\widehat{v}_x V_{xx} dx \le C\int (1+t)^{\beta}\left(|V_x| + |\widehat{v}|\right)V_{xx}^2 dx \\
&+ \frac{C_0}{10}\int (1+t)^{\beta} V_{xx}^2 dx + \int (1+t)^{\beta}\left|\overline{v}_x\right|^2\left(|\widehat{v}|^2 + |V_x|^4\right)dx \\
&+ C\int (1+t)^{\beta}\left|\widehat{v}_x\right|^2 dx \le C(\varepsilon+\delta)\int (1+t)^{\beta} V_{xx}^2 dx \frac{C_0}{10}\int (1+t)^{\beta} V_{xx}^2 dx \\
&+ C\varepsilon^2\delta^2\int (1+t)^{\beta-\lambda-1} V_x^2 dx + C\delta^2 (1+t)^{\beta+(\lambda/2)-(7/2)},
\end{aligned} \tag{153}$$

and

$$\int (1+t)^{\beta}\left(p'(v_+) - p'(\overline{v})\right)\overline{v}_x V_{xx} dx \le \frac{C_0}{10}\int (1+t)^{\beta} V_{xx}^2 dx + C\int (1+t)^{\beta}\left(\left(p'(v_+) - p'(\overline{v})\right)\overline{v}_x\right)^2 dx \le \frac{C_0}{10}\int (1+t)^{\beta} V_{xx}^2 dx + C\delta^2 (1+t)^{\beta-(5\lambda/2)-(5/2)}. \tag{154}$$

Substituting (149)–(154) into (158), and using the smallness of δ, ε, we have

$$\begin{aligned}
&\frac{d}{dt}\int \left[(1+t)^{\beta} V_x V_{xt} + \frac{\alpha}{2}(1+t)^{\beta-\lambda} V_x^2\right] dx \\
&+ \frac{\alpha(\lambda-\beta)}{2}\int (1+t)^{\beta-\lambda-1} V_x^2 dx + \frac{C_0}{2}\int (1+t)^{\beta} V_{xx}^2 dx \\
&\le \int (1+t)^{\beta} V_{xt}^2 dx + \frac{\beta}{2}\frac{d}{dt}\int (1+t)^{\beta-1} V_x^2 dx \\
&+ \frac{\beta(1-\beta)}{2}\int (1+t)^{\beta-2} V_x^2 dx + C\delta^2\int (1+t)^{\beta-\lambda-1} V_x^2 dx \\
&+ C\delta^2 (1+t)^{\beta+(\lambda/2)-(7/2)} + C\delta^2 (1+t)^{\beta-(5\lambda/2)-(5/2)}.
\end{aligned} \tag{155}$$

Multiplying (147) by h and adding up the resulting inequality and (155), we get

$$\begin{aligned}
&\frac{d}{dt}\int \Big[(1+t)^{\beta} V_x V_{xt} + \frac{\alpha}{2}(1+t)^{\beta-\lambda} V_x^2 \\
&+ \frac{h}{2}(1+t)^{\beta+\lambda} V_{xt}^2 - \frac{h}{2}(1+t)^{\beta+\lambda} p'(\overline{v}) V_{xx}^2\Big] dx \\
&+ \frac{\alpha(\lambda-\beta)}{2}\int (1+t)^{\beta-\lambda-1} V_x^2 dx + \frac{C_0}{2}\int (1+t)^{\beta} V_{xx}^2 dx \\
&+ \left(\frac{\alpha h}{2} - 1\right)\int (1+t)^{\beta} V_{xt}^2 dx \\
&\le \frac{h}{2}\frac{d}{dt}\int (1+t)^{\beta+\lambda}\left(p'(V_x+\overline{v}+\widehat{v}) - p'(\overline{v})\right)V_{xx}^2 dx \\
&+ \frac{\beta}{2}\frac{d}{dt}\int (1+t)^{\beta-1} V_x^2 dx + C\int (1+t)^{\beta-\lambda-1} V_x^2 dx \\
&+ Ch\int (1+t)^{\beta+\lambda-1}\left(V_{xx}^2 + V_{xt}^2\right)dx \\
&+ C\delta^2 (1+t)^{\beta+(\lambda/2)-(7/2)} + C\delta^2 (1+t)^{\beta-(5\lambda/2)-(5/2)}.
\end{aligned} \tag{156}$$

Multiplying (157) by $(1+t)^{\lambda+1}$, we have

$$\begin{aligned}
&\frac{d}{dt}\int \Big[(1+t)^{\beta+\lambda+1} V_x V_{xt} + \frac{\alpha}{2}(1+t)^{\beta+1} V_x^2 \\
&+ \frac{h}{2}(1+t)^{\beta+2\lambda+1} V_{xt}^2 - \frac{h}{2}(1+t)^{\beta+2\lambda+1} p'(\overline{v}) V_{xx}^2\Big] dx \\
&+ \frac{\alpha(\lambda-\beta)}{2}\int (1+t)^{\beta} V_x^2 dx + \frac{C_0}{2}\int (1+t)^{\beta+\lambda+1} V_{xx}^2 dx \\
&+ \left(\frac{\alpha h}{2} - 1\right)\int (1+t)^{\beta+\lambda+1} V_{xt}^2 dx \\
&\le \frac{h}{2}\frac{d}{dt}\int (1+t)^{\beta+2\lambda+1}\left(p'(V_x+\overline{v}+\widehat{v}) - p'(\overline{v})\right)V_{xx}^2 dx \\
&+ \frac{\beta}{2}\frac{d}{dt}\int (1+t)^{\beta+\lambda} V_x^2 dx + C\int (1+t)^{\beta} V_x^2 dx \\
&+ Ch\int (1+t)^{\beta+2\lambda}\left(V_{xx}^2 + V_{xt}^2\right)dx + C\delta^2 (1+t)^{\beta+(3\lambda/2)-(5/2)} \\
&+ C\delta^2 (1+t)^{\beta-(3\lambda/2)-(3/2)} + C\int (1+t)^{\beta+2\lambda}\left(p'(V_x+\overline{v}+\widehat{v}) - p'(\overline{v})\right)V_{xx}^2 dx.
\end{aligned} \tag{157}$$

Case 1 $(0 \le \lambda < 3/5)$. $\beta = \lambda, h = 6/\alpha$. There exists constant $T_2 > 0$, such that for any $t \ge T_2$, it holds $Ch(1+t)^{\lambda-1} \le \min\{C_0/4, 1/2(\alpha h/2 - 1)\}$. Therefore, if $t \ge T_2$, from (157), we have

$$\begin{aligned}
&\frac{d}{dt}H_2(t) + C\int (1+t)^{2\lambda+1}\left(V_{xx}^2 + V_{xt}^2\right)dx \\
&\le \frac{3}{\alpha}\frac{d}{dt}\int (1+t)^{3\lambda+1}\left(p'(V_x+\overline{v}+\widehat{v}) - p'(\overline{v})\right)V_{xx}^2 dx \\
&+ \frac{\lambda}{2}\frac{d}{dt}\int (1+t)^{2\lambda} V_x^2 dx + C\int (1+t)^{\lambda} V_x^2 dx \\
&+ C\delta^2 (1+t)^{(5\lambda/2)-(5/2)} + C\delta^2 (1+t)^{-(\lambda/2)-(3/2)},
\end{aligned} \tag{158}$$

where

$$H_2(t) \sim (1+t)^{\lambda+1}\left\|V_x\right\|^2 + (1+t)^{3\lambda+1}\left\|V_{xx}\right\|^2 + (1+t)^{3\lambda+1}\left\|V_{xt}\right\|^2. \tag{159}$$

Integrating (158) on $[T_2, t]$ with respect to the time t and using (98), (126), $0 \le \lambda < 3/5$, we have

$$(1+t)^{3\lambda+1}\left(\|V_{xx}\|^2+\|V_{xt}\|^2\right)+\int_{T_2}^t(1+s)^{2\lambda+1}\left(\|V_{xx}\|^2+\|V_{xt}\|^2\right)ds \le C\left(H_2(T_2)+\|V_0\|_1^2+\|U_0\|^2+\delta\right). \quad (160)$$

From (160), (53) and (54), we have for any $t>0$

$$(1+t)^{3\lambda+1}\left(\|V_{xx}\|^2+\|V_{xt}\|^2\right)+\int_0^t(1+s)^{2\lambda+1}\left(\|V_{xx}\|^2+\|V_{xt}\|^2\right)ds \le C\left(\|V_0\|_2^2+\|U_0\|_1^2+\delta\right). \quad (161)$$

Multiply (147) by $(1+t)^2$ with $\beta=\lambda$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left[(1+t)^{2+2\lambda}V_{xt}^2-(1+t)^{2+2\lambda}p'(\overline{v})V_{xx}^2\right]dx\\&+\frac{\alpha}{2}\int(1+t)^{\lambda+2}V_{xt}^2dx\le\frac{1}{2}\frac{d}{dt}\int(1+t)^{2+2\lambda}\left(p'(V_x+\overline{v}+\widehat{v})-p'(\overline{v})\right)V_{xx}^2dx\\&+C\int(1+t)^{2\lambda+1}\left(V_{xx}^2+V_{xt}^2\right)dx+C\delta^2\int(1+t)^{\lambda}V_x^2dx\\&+C\delta^2(1+t)^{(5\lambda/2)-(5/2)}+C\delta(1+t)^{-(\lambda/2)-(3/2)}.\end{aligned} \quad (162)$$

Integrating (162) on $(0,t)$, using (98), (161) and $0\le\lambda<3/5$, we have (133).

Case 2 ($3/5<\lambda<1$). In this case, we take $3/2-3\lambda/2<\beta<\lambda$. There exists constant $T_3>0$, such that for any $t\ge T_3$, it holds $Ch(1+t)^{\lambda-1}\le\min\{C_0/4,1/2((\alpha h)/2-1)\}$, where we choose $h=6/\alpha$. Therefore if $t\ge T_3$, from (157), we have

$$\begin{aligned}&\frac{d}{dt}H_3(t)+C\int(1+t)^{\beta+\lambda+1}\left(V_{xx}^2+V_{xt}^2\right)dx+C\int(1+t)^{\beta}V_x^2dx\\&\le\frac{h}{2}\frac{d}{dt}\int(1+t)^{\beta+2\lambda+1}\left(p'(V_x+\overline{v}+\widehat{v})-p'(\overline{v})\right)V_{xx}^2dx\\&+\frac{\beta}{2}\frac{d}{dt}\int(1+t)^{\beta+\lambda}V_x^2dx+C\int(1+t)^{\beta}V_x^2dx\\&+C\delta^2(1+t)^{\beta+(3\lambda/2)-(5/2)},\end{aligned} \quad (163)$$

where

$$H_3(t)\sim(1+t)^{\beta+1}\|V_x\|^2+(1+t)^{\beta+2\lambda+1}\left(\|V_{xx}\|^2+\|V_{xt}\|^2\right). \quad (164)$$

Integrating (163) on $[T_3,t]$ with respect to the time t over $[T_3,t]$ and using (98), (128), $3/5<\lambda<1$, we have

$$H_3(t)+\int_{T_3}^t(1+t)^{\beta+\lambda+1}\left(\|V_{xx}\|^2+\|V_{xt}\|^2\right)ds \le H_3(T_3)+C(1+t)^{\beta+(3\lambda/2)-(3/2)}\left(\|V_0\|_1^2+\|U_0\|^2+\delta\right), \quad (165)$$

which, together with (51), (52), and $\beta+3\lambda/2-3/2>0$, yields

$$(1+t)^{\beta+2\lambda+1}\left(\|V_{xx}\|^2+\|V_{xt}\|^2\right)+\int_0^t(1+s)^{\beta+\lambda+1}\left(\|V_{xx}\|^2+\|V_{xt}\|^2\right)ds \le C(1+t)^{\beta+(3\lambda/2)-(3/2)}\left(\|V_0\|_2^2+\|U_0\|_1^2+\delta\right). \quad (166)$$

Finally, multiplying (147) by $(1+t)^2$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left[(1+t)^{\beta+\lambda+2}V_{xt}^2-(1+t)^{\beta+\lambda+2}p'(\overline{v})V_{xx}^2\right]dx\\&+\frac{\alpha}{2}\int(1+t)^{\beta+2}V_{xt}^2dx\le\frac{1}{2}\frac{d}{dt}\int(1+t)^{\beta+\lambda+2}\left(p'(V_x+\overline{v}+\widehat{v})-p'(\overline{v})\right)V_{xx}^2dx\\&+C\int(1+t)^{\beta+\lambda+1}\left(V_{xx}^2+V_{xt}^2\right)dx+C\int(1+t)^{\beta}V_x^2dx\\&+C\delta^2(1+t)^{\beta+(3\lambda/2)-(5/2)}+C\delta(1+t)^{\beta-(3\lambda/2)-(3/2)}.\end{aligned} \quad (167)$$

Integrating the above inequality over $(0,t)$ and using (99), (100), and (166), we get

$$(1+t)^{\beta+\lambda+2}\left(\|V_{xx}\|^2+\|V_{xt}\|^2\right)+\int_0^t(1+s)^{\beta+2}\|V_{xt}\|^2ds \le C(1+t)^{\beta+\frac{3\lambda}{2}-\frac{3}{2}}\left(\|V_0\|_2^2+\|U_0\|_1^2+\delta\right), \quad (168)$$

which combining with (166) gives (134) and (135). This completes the proof of Lemma 4.3.

Similar calculations to Lemma 4.3 yield the following lemma, the proof of which is omitted. □

Lemma 4.4. *Under the assumptions of Theorem 2.4, if ε is small enough, it holds that*

$$(1+t)^{3\lambda+3}\left(\|V_{xxt}\|^2+\|V_{xxx}\|^2\right)+\int_0^t\left((1+s)^{3+2\lambda}\|V_{xxt}\|^2+(1+s)^{3\lambda+2}\|V_{xxx}\|^2\right)ds \le C\left(\|V_0\|_3^2+\|U_0\|_2^2+\delta\right),\quad for\ 0\le\lambda<\frac{3}{5}. \quad (169)$$

For $3/5<\lambda<1$, we have

$$(1+t)^{(9/2)+(\lambda/2)}\left(\|V_{xxt}\|^2+\|V_{xxx}\|^2\right)\le C\left(\|V_0\|_3^2+\|U_0\|_2^2+\delta\right) \quad (170)$$

and

$$\int_0^t\left((1+s)^{\beta+2\lambda+2}\|V_{xxx}\|^2+(1+s)^{\beta+\lambda+3}\|V_{xxt}\|^2\right)ds \le C(1+t)^{\beta+(3\lambda/2)-(3/2)}\left(\|V_0\|_3^2+\|U_0\|_2^2+\delta\right), \quad (171)$$

for any $3/2-3\lambda/2<\beta<\lambda$.

With Lemmas 4.1–4.4 in hand, by the Sobolev inequality, it is easy to know that if $0\le\lambda<3/5$,

$$\|V_x(\cdot,t)\|_{L^\infty}\le C(1+t)^{-(3/4)(\lambda+1)}\left(\|V_0\|_3+\|U_0\|_2+\delta^{1/2}\right)\le\frac{\varepsilon}{2}, \quad (172)$$

$$\|V_{xt}(\cdot,t)\|_{L^\infty}\le C(1+t)^{-(5/4)(\lambda+1)}\left(\|V_0\|_3+\|U_0\|_2+\delta^{1/2}\right)\le\frac{\varepsilon}{2}(1+t)^{-1}, \quad (173)$$

and

$$\|V_{xx}(\cdot,t)\|_{L^\infty}\le C(1+t)^{-(5/4)(\lambda+1)}\left(\|V_0\|_3+\|U_0\|_2+\delta^{1/2}\right) \le\frac{\varepsilon}{2}(1+t)^{-(\lambda+1/2)}. \quad (174)$$

And if $3/5<\lambda<1$, we get

$$\|V_x(\cdot,t)\|_{L^\infty}\le C(1+t)^{-(3-\lambda/2)}\left(\|V_0\|_3+\|U_0\|_2+\delta^{1/2}\right)\le\frac{\varepsilon}{2}, \quad (175)$$

$$\|V_{xt}(\cdot,t)\|_{L^\infty} \le C(1+t)^{-2}\left(\|V_0\|_3+\|U_0\|_2+\delta^{1/2}\right) \le \frac{\varepsilon}{2}(1+t)^{-1}, \tag{176}$$

and

$$\|V_{xx}(\cdot,t)\|_{L^\infty} \le C(1+t)^{-2}\left(\|V_0\|_3+\|U_0\|_2+\delta^{1/2}\right) \le \frac{\varepsilon}{2}(1+t)^{-(\lambda+1/2)}. \tag{177}$$

provided $\|V_0\|_3+\|U_0\|_2+\delta \le 1$. Up to now, we thus close *a priori* assumption (97) about (V_x, V_{xt}, V_{xx}) from Lemmas 4.1–4.4.

For the decay rates of $U = V_t$, we give the following two lemmas.

Lemma 4.5. *Under the assumptions of Theorem 2.4, if* ε *is small enough, it holds that*

$$\begin{aligned}&(1+t)^2\|U\|^2+(1+t)^{\lambda+3}\left(\|U_t\|^2+\|U_x\|^2\right)\\&\quad+\int_0^t\left((1+s)^{\lambda+2}\|U_x\|^2+(1+s)^3\|U_t\|^2\right)ds\\&\quad\le C\left(\|V_0\|_2^2+\|U_0\|_1^2+\delta\right), \quad \text{for } 0\le\lambda<\frac{3}{5}.\end{aligned} \tag{178}$$

For $3/5<\lambda<1$, *we have*

$$\begin{aligned}&(1+t)^{(7/2)-(5\lambda/2)}\|U\|^2+(1+t)^{(9/2)-(3\lambda/2)}\left(\|U_t\|^2+\|U_x\|^2\right)\\&\quad\le C\left(\|V_0\|_2^2+\|U_0\|_1^2+\delta\right),\end{aligned} \tag{179}$$

and

$$\begin{aligned}&\int_0^t\left[(1+s)^{\beta+2}\|U_x\|^2+(1+s)^{\beta-\lambda+3}\|U_t\|^2\right]ds\\&\quad\le C(1+t)^{\beta+(3\lambda/2)-(3/2)}\left(\|V_0\|_2^2+\|U_0\|_1^2+\delta\right),\end{aligned} \tag{180}$$

for any $(3/2)-(3\lambda/2)<\beta<\lambda$.

Proof. Differentiating $(57)_1$ with respect to t, we get

$$U_{tt}+\left(p'(\overline{v})V_x\right)_{xt}+\frac{\alpha}{(1+t)^\lambda}U_t=\frac{\alpha\lambda}{(1+t)^{\lambda+1}}U+F_t. \tag{181}$$

Multiplying (181) by $(1+t)^{\beta+\lambda}U_t$, and integrating the resulting equality in x over $\mathbb{R}^+$, and after tedious calculations, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left[(1+t)^{\beta+\lambda}U_t^2-(1+t)^{\beta+\lambda}p'(\overline{v})U_x^2\right]dx+\frac{\alpha}{2}\int(1+t)^\beta U_t^2dx\\&\le\frac{1}{2}\frac{d}{dt}\int(1+t)^{\beta+\lambda}\left(p'(V_x+\overline{v}+\widehat{v})-p'(\overline{v})\right)U_x^2dx+C\int(1+t)^{\beta-2}U^2dx\\&\quad+C\int(1+t)^{\beta+\lambda-1}U_x^2dx+\frac{\beta+\lambda}{2}\int(1+t)^{\beta+\lambda-1}U_t^2dx\\&\quad+C\delta^2\int(1+t)^{\beta+\lambda-3}V_x^2dx\\&\quad+C\delta^2\int(1+t)^{\beta+2\lambda-2}V_{xx}^2dx+C\delta(1+t)^{\beta+(5\lambda/2)-(11/2)}.\end{aligned} \tag{182}$$

Similarly, multiplying (181) by $(1+t)^\beta U$, from Lemma 4.2 and after complicated calculations, we have

$$\begin{aligned}&\frac{d}{dt}\int\left[(1+t)^\beta UU_t+\frac{\alpha}{2}(1+t)^{\beta-\lambda}U^2\right]dx\\&\quad+\frac{\alpha(\lambda-\beta)}{2}\int(1+t)^{\beta-\lambda-1}U^2dx+\frac{C_0}{2}\int(1+t)^\beta U_x^2dx\\&\le\int(1+t)^\beta U_t^2dx+\frac{\beta}{2}\frac{d}{dt}\int(1+t)^{\beta-1}U^2dx\\&\quad+\frac{\beta(1-\beta)}{2}\int(1+t)^{\beta-\lambda-1}U^2dx\\&\quad+C\delta^2\int(1+t)^{\beta-2}V_x^2dx+C\delta^2(1+t)^{\beta+(3\lambda/2)-(9/2)}.\end{aligned} \tag{183}$$

Multiplying (172) by h and adding up the resulting inequality and (173), we get

$$\begin{aligned}&\frac{d}{dt}\int\Big[(1+t)^\beta UU_t+\frac{\alpha}{2}(1+t)^{\beta-\lambda}U^2+\frac{h}{2}(1+t)^{\beta+\lambda}U_t^2\\&\quad-\frac{h}{2}(1+t)^{\beta+\lambda}p'(\overline{v})U_x^2\Big]dx\\&\quad+\frac{\alpha(\lambda-\beta)}{2}\int(1+t)^{\beta-\lambda-1}U^2dx+\frac{C_0}{2}\int(1+t)^\beta U_x^2dx\\&\quad+\left(\frac{\alpha h}{2}-1\right)\int(1+t)^\beta U_t^2dx\le\frac{h}{2}\frac{d}{dt}\int(1+t)^{\beta+\lambda}\\&\quad\cdot\left(p'(V_x+\overline{v}+\widehat{v})-p'(\overline{v})\right)U_x^2dx+\frac{\beta}{2}\frac{d}{dt}\int(1+t)^{\beta-1}U^2dx\\&\quad+C\int(1+t)^{\beta-\lambda-1}U^2dx+C\int(1+t)^{\beta+\lambda-1}\left(U_x^2+U_t^2\right)dx\\&\quad+C\delta^2\int(1+t)^{\beta-2}V_x^2dx+C\delta^2\int(1+t)^{\beta+2\lambda-2}V_{xx}^2dx\\&\quad+C\delta^2(1+t)^{\beta+(3\lambda/2)-(9/2)}.\end{aligned} \tag{184}$$

Case 1 $(0\le\lambda<3/5)$. In this case, we take $\beta=\lambda, h=6/\alpha$. Multiplying (184) by $(1+t)^{1-\lambda}, (1+t)^{2-2\lambda}, (1+t)^2$ respectively, then integrating the resulting equations and (4.88) with respect to t over $[0,t]$ respectively, we have

$$\begin{aligned}&(1+t)^2\|U\|^2+(1+t)^{2\lambda+2}\left(\|U_t\|^2+\|U_x\|^2\right)\\&\quad+\int_0^t(1+s)^{\lambda+2}\left(\|U_x\|^2+\|U_t\|^2\right)ds\\&\quad\le C\left(\|V_0\|_2^2+\|U_0\|_1^2+\delta\right).\end{aligned} \tag{185}$$

Multiplying (182) by $(1+t)^{3-\lambda}$, and integrating the resulting equation with respect to t over $[0,t]$, we have

$$\begin{aligned}&(1+t)^{\lambda+3}\left(\|U_t\|^2+\|U_x\|^2\right)+\int_0^t(1+s)^3\|U_t\|^2ds\\&\quad\le C\left(\|V_0\|_2^2+\|U_0\|_1^2+\delta\right).\end{aligned} \tag{186}$$

Combining (183) and (184), we get (178).

Case 2 $(3/5<\lambda<1)$. In this case, we take $3/2-3\lambda/2<\beta<\lambda$, $h=6/\alpha$. There exists constant $T_5>0$, such that for any $t\ge T_5$, it holds $C(1+t)^{\lambda-1}\le\min\{C_0/4, 1/2((\alpha h/2)-1)\}$. Multiplying (184) by $(1+t)^2$, we have

$$\frac{d}{dt}\int\Big[(1+t)^{\beta+2}UU_t+\frac{\alpha}{2}(1+t)^{\beta-\lambda+2}U^2+\frac{h}{2}(1+t)^{\beta+\lambda+2}U_t^2 - \frac{h}{2}(1+t)^{\beta+\lambda+2}p'(\overline{v})U_x^2\Big]dx + \frac{C_0}{2}\int(1+t)^{\beta+2}U_x^2dx+\left(\frac{\alpha h}{2}-1\right)\int(1+t)^{\beta+2}U_t^2dx \le \frac{h}{2}\frac{d}{dt}\int(1+t)^{\beta+\lambda+2}\left(p'(V_x+\overline{v}+\widehat{v})-p'(\overline{v})\right)U_x^2dx + \frac{\beta}{2}\frac{d}{dt}\int(1+t)^{\beta+1}U^2dx+C\int(1+t)^{\beta-\lambda+1}U^2dx + C\int(1+t)^{\beta+\lambda+1}\left(U_x^2+U_t^2\right)dx+C\delta^2\int(1+t)^{\beta}V_x^2dx + C\delta^2\int(1+t)^{\beta+2\lambda}V_{xx}^2dx+C\delta^2(1+t)^{\beta+(3\lambda/2)-(5/2)}. \tag{187}$$

If $t\ge T_5$, integrating (187) over (T_5,t), using (99), (100), (127), (128), (132) and (135), we get

$$H_5(t)+\int_{T_5}^{t}(1+s)^{\beta+2}\left(\|U_x\|^2+\|U_t\|^2\right)ds \le H_5(T_5)+C(1+t)^{\beta+(3\lambda/2)-(3/2)}\left(\|V_0\|_2^2+\|U_0\|_1^2+\delta\right), \tag{188}$$

where

$$H_5(t)\sim(1+t)^{\beta-\lambda+2}\|U\|^2+(1+t)^{\beta+\lambda+2}\left(\|U_x\|^2+\|U_t\|^2\right). \tag{189}$$

Next, if $0<t\le T_5$, from (187), we know

$$\frac{d}{dt}H_5(t)+C\int(1+t)^{\beta+2}U_x^2dx+C\int(1+t)^{\beta+2}U_t^2dx \le \frac{d}{dt}\int(1+t)^{\beta+\lambda+2}\left(p'(V_x+\overline{v}+\widehat{v})-p'(\overline{v})\right)U_x^2dx + C(1+t)^{-1}H_5(t)+C\delta^2\int(1+t)^{\beta}V_x^2dx + C\delta^2\int(1+t)^{\beta+2\lambda}V_{xx}^2dx+C\delta^2(1+t)^{\beta+(3\lambda/2)-(5/2)}. \tag{190}$$

Then using Gronwall's inequality, (99), (100), (134), and (135), and noting that $\beta+3\lambda/2>3/2$, we get

$$H_5(t)+\int_0^t(1+s)^{\beta+2}\left(\|U_x\|^2+\|U_t\|^2\right)ds \le C(1+t)^{\beta+(3\lambda/2)-(3/2)}\left(\|V_0\|_2^2+\|U_0\|_1^2+\delta\right),\ \text{for any}\ t\in(0,T_5]. \tag{191}$$

From (191) and (188), we have

$$H_5(t)+\int_0^t(1+s)^{\beta+2}\left(\|U_x\|^2+\|U_t\|^2\right)ds \le C(1+t)^{\beta+(3\lambda/2)-(3/2)}\left(\|V_0\|_2^2+\|U_0\|_1^2+\delta\right),\quad \text{for any}\ t\in(0,\infty). \tag{192}$$

Multiplying (182) by $(1+t)^{3-\lambda}$, and integrating the resulting equation with respect to t over $[0,t]$, we have

$$(1+t)^{\beta+3}\left(\|U_x\|^2+\|U_t\|^2\right)+\int_0^t(1+s)^{\beta-\lambda+3}\|U_t\|^2ds \le C(1+t)^{\beta+(3\lambda/2)-(3/2)}\left(\|V_0\|_2^2+\|U_0\|_1^2+\delta\right). \tag{193}$$

Combining (192) and (193), we get (179) and (180). Therefore we completed the proof of Lemma 4.5.

Similar to the proof of Lemma 4.5, we can get the following estimates. □

Lemma 4.6. *Under the assumptions of Theorem 2.4, if ε,δ are small enough, it holds that*

$$(1+t)^{2\lambda+4}\left(\|U_{xt}\|^2+\|U_{xx}\|^2\right)+\int_0^t\left((1+s)^{2\lambda+3}\|U_{xx}\|^2+(1+s)^{\lambda+4}\|U_{xt}\|^2\right)ds \le C\left(\|V_0\|_3^2+\|U_0\|_2^2+\delta\right),\quad \text{for}\ 0\le\lambda<\frac{3}{5}. \tag{194}$$

For $3/5<\lambda<1$, *we have*

$$(1+t)^{(11/2)-(\lambda/2)}\left(\|U_{xt}\|^2+\|U_{xx}\|^2\right)\le C\left(\|V_0\|_3^2+\|U_0\|_2^2+\delta\right), \tag{195}$$

and

$$\int_0^t\left[(1+s)^{\beta+\lambda+3}\|U_{xx}\|^2+(1+s)^{\beta+4}\|U_{xt}\|^2\right]ds \le C(1+t)^{\beta-(3/2)+(3\lambda/2)}\left(\|V_0\|_3^2+\|U_0\|_2^2+\delta\right), \tag{196}$$

for any $3/2-3\lambda/2<\beta<\lambda$.

Recalling Lemmas 4.1–4.6, we completed the desired decay estimates (42), (43), and (44).

By taking $\beta=\lambda=1/7$, $\nu=1$ and $\kappa=1$ in Lemmas 4.1–4.6, Theorem 2.6 can be shown by slightly modifying the proof of Theorem 2.4, and therefore, we omit its proof here.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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