Hindawi
Advances in Mathematical Physics
Volume 2020, Article ID 2676919, 7 pages
https://doi.org/10.1155/2020/2676919

Research Article

Representation of Manifolds for the Stochastic Swift-Hohenberg Equation with Multiplicative Noise

Yanfeng Guo [1,2] and Donglong Li [1]

[1] *School of Science, Guangxi University of Science and Technology, Liuzhou, Guangxi, China 545006*
[2] *School of Mathematics and Physics, China University of Geosciences Wuhan, Wuhan, Hubei, China 430074*

Correspondence should be addressed to Yanfeng Guo; guoyan_feng@163.com

Received 27 August 2019; Revised 11 November 2019; Accepted 21 November 2019; Published 8 January 2020

Guest Editor: David Carfì

Representation of approximation for manifolds of the stochastic Swift-Hohenberg equation with multiplicative noise has been investigated via non-Markovian reduced system. The approximate parameterizations of the small scales for the large scales are given in the process of seeking for stochastic parameterizing manifolds, which are obtained as pullback limits of some backward-forward systems depending on the time-history of the dynamics of the low modes in a mean square sense through the nonlinear terms. When the corresponding pullback limits of some backward-forward systems are efficiently determined, the corresponding non-Markovian reduced systems can be obtained for researching good modeling performances in practice.

1. Introduction

Recently, more and more authors have paid attention to considering the approximation problems of manifolds for the stochastic partial differential equations (SPDEs). For decades, various approximating methods have been given to solve these problems, such as amplitude equations approach [1–4] and the manifolds-based approaches [5–11].

In this paper, approximation of manifolds for the stochastic Swift-Hohenberg equation with multiplicative noise will be investigated in Stratonovich sense [12]. It is well known that there have been some authors to consider the approximation of manifolds in large probability sense [1, 2, 13] and they have obtained some results until now. In addition, approximation in parameterizing manifold for a stochastic Swift-Hohenberg equation with additive noise have been investigated by us in [14]. Furthermore, it is needed to consider the problems in Stratonovich sense. Until now, there have been few consideration from the point of view of approximation in parameterizing manifold under the pathwise sense for the stochastic Swift-Hohenberg equation with multiplicative noise. The ideas in [14] can be used to consider the approximation of manifold for some stochastic equations with multiplicative noise. Because the different difficulties come from different noise terms, there are some different methods and techniques in studying stochastic equations with multiplicative noise. Here, we investigate the corresponding problems for the stochastic Swift-Hohenberg equation with multiplicative noise with pathwise and obtain some new results for it. The results obtained in this paper are different from those in [14], although there are some similar sentences in some manuscripts. It is well known that various noises cause various stochastic processes for stochastic equations with different noises. The main differences from results in [14] are given by some formulas with various mathematics meanings, in which some different stochastic functions are used. Because the different difficulties mainly come from various noise terms, there are some new difficulties coming from the multiplicative noise solved in our manuscript. So, some different techniques are used in studying stochastic equations with multiplicative noise.

We will extend the strategy introduced in [5, 6] to the stochastic Swift-Hohenberg equation [12] with multiplicative noise and obtain the approximation of parameterizing manifolds and corresponding non-Markovian reduced system. The key idea is mainly based on the approximate parameterizations of the small scales for the large scales via the stochastic parameterizing manifolds. Random manifolds will improve the partial knowledge of the solutions of SPDEs in mean square error, when it is compared with its projection onto the resolved modes. Approximation of parameterizing manifolds can be obtained by representing the modes with high wave numbers as the pullback limit depend on the time-history of the modes with low wave numbers for the corresponding backward-forward systems. Some conditions with nonresonance conditions below are given and weaker than those in the classical stochastic invariant manifold theory (see [7, 15, 16] and references therein). On the base of these approximations of parameterizing manifolds, when the corresponding pullback limits of some backward-forward systems are efficiently determined, the corresponding non-Markovian stochastic reduced systems are given to reach good modeling performances in practice and take the form of stochastic differential equations with random coefficients, which convey memory effects via the history of the Wiener process and arise from the nonlinear interactions between the low modes embedded in the noise bath. These random coefficients show an exponential decay of correlations, whose rate depends explicitly on the gaps of the nonresonance conditions. In fact, it is possible to achieve very good parameterizing quality for the stochastic Swift-Hohenberg equation with multiplicative noise from our results. And the performances from the reduced system can be numerically assessed for a corresponding optimal or suboptimal control problem.

The paper is organized as follows. In Section 2, we give our functional framework, some definitions about parameterizing manifolds and some properties of some stochastic processes being used. We have devoted Section 3 to studying the representation of approximation of parameterizing manifolds as pullback limits of the corresponding backward-forward systems for a stochastic Swift-Hohenberg equation with multiplicative noise. In Section 4, on the basis of the approximation of parameterizing manifolds, the non-Markovian stochastic reduced systems involving random coefficients are obtained for the stochastic Swift-Hohenberg with multiplicative noise.

2. Preliminaries

The functional framework spaces are a pair of Hilbert spaces (H_1, H) such that H_1 is compactly and densely embedded in H. Let $A : H_1 \longrightarrow H$ be a sectorial operator [16] such that $-A$ is stable in the sense that its spectrum satisfies $\mathrm{Re}\,(\sigma(-A)) < 0$. And we consider interpolated spaces H_α between H_1 and H with $\alpha \in [0, 1)$ along with the perturbations of the linear operator $-A$ given by a one parameter family B_λ of bounded linear operators from H_α to H, depending continuously on λ. Define $L_\lambda = -A + B_\lambda$, which maps H_1 into H.

A local stochastic Swift-Hohenberg equation with multiplicative noise in Stratonovich sense [1] is written as follows:

$$\begin{aligned} du &= \left(\lambda u - (1+\Delta)^2 u - u^3\right)dt + \sigma u \circ dW_t \\ &:= (L_\lambda u + F(u))dt + \sigma u \circ dW_t, \end{aligned} \tag{1}$$

with Dirichlet boundary conditions $u(0, t; \omega) = u(l, t; \omega) = 0$, $t > 0$ and initial condition $u(x, 0; \omega) = u_0(x)$, $x \in (0, l)$, where λ is a parameterizing variable, σ is positive, and u_0 is some appropriate initial datum with $H = L^2(0, l)$ and $H_1 = H^4(0, l) \cap H_0^2(0, l)$; $W(t)$ is a standard real valued one-dimensional Brownian motion [17] with paths in $C_0(\mathbb{R}, \mathbb{R})$; and Ω being endowed with its corresponding Borel σ-algebra $\mathscr{F}$, its filtration $\mathscr{F}_t$, the Wiener measure $\mathbb{P}$.

Let $F(u) = -u^3$, which is a continuous triple nonlinear mapping from H^α into H, where $\alpha > (1/3)$. Obviously, function $F(u)$ is a mapping from H_1 into H. Assume $L_\lambda = -A + B_\lambda$, where $B_\lambda = \lambda$ and $A = (1+\Delta)^2$ is closed self-adjoint linear operator with dense domain $D(A)$ in $H = L^2(D)$. The operator L_λ is self-adjoint with an orthonormal basis of eigenfunctions $\{e_k = \sqrt{2/l}\sin(k\pi x/l)\}_{k\in\mathbb{N}}$ in H with corresponding eigenvalues $\{\beta_k(\lambda) = \lambda - (1 - (k^2\pi^2/l^2))^2\}_{k\in\mathbb{N}}$.

Then, problem (1) can be rewritten as

$$du = (L_\lambda u + F(u))dt + \sigma u \circ dW_t, \tag{2}$$

with initial condition $u(x, 0; \omega) = u_0(x)$, $x \in (0, l)$ and Dirichlet boundary conditions $u(0, t; \omega) = u(l, t; \omega) = 0$, $t > 0$. Now, we investigate the random dynamical systems of system (2) in the sense of parameterizing manifolds in [6, 17]. The stochastic parameterizing manifolds are mainly considered for local stochastic Swift-Hohenberg equation with multiplicative noise (2). Firstly, a stochastic parameterizing manifold $\mathfrak{M}$ is seen as the graph of a random function h^{pm}, which is a mapping from H^c to $H^{\alpha,s}$ and provides approximation parameterizations of the high part $u_s(t, \omega) = P_s u(t, \omega)$ by using of the low part $u_c(t, \omega) = P_c u(t, \omega)$. The scalar Langevin equation,

$$dz + z\,dt = \sigma dW, \tag{3}$$

is given. A unique stationary solution $z(\theta_t\omega)$ of this equation is called the stationary Ornstein-Uhlenbeck (OU) process. By simply integrating on the both sides of (3), the identity

$$\int_0^t z_\sigma(\theta_s\omega)ds + z_\sigma(\theta_t\omega) = z_\sigma(\omega) + \sigma W_t(\omega), \quad \forall t \in \mathbb{R}, \tag{4}$$

holds, which is important for representation of approximation.

3. Representation of Manifolds with Multiplicative Noise

Making use of the method in [6], we investigate the local stochastic Swift-Hohenberg (equation (2)) with multiplicative noise in Stratonovich sense. One considers the following backward-forward system associated with SPDE (2).

$$d\widehat{u}_c^{(1)} = L_\lambda^c \widehat{u}_c^{(1)} ds + \sigma \widehat{u}_c^{(1)} \circ dW_s, \quad s \in [-T, 0], \tag{5}$$

$$\widehat{u}_c^{(1)}(s, \omega)|_{s=0} = \xi \in H^c, \tag{6}$$

$$d\widehat{u}_s^{(1)} = \left(L_\lambda^s \widehat{u}_s^{(1)} + P_s F\left(\widehat{u}_c^{(1)}(s-T,\omega)\right)\right) ds + \sigma \widehat{u}_s^{(1)} \circ dW_{s-T}, \quad s \in [0,T], \tag{7}$$

$$\widehat{u}_s^{(1)}(s,\theta_{-T}\omega)|_{s=0} = 0, \tag{8}$$

where $L_\lambda^c := P_c L_\lambda$ and $L_\lambda^s := P_s L_\lambda$. From system (5), (6), (7), and (8), we know that the initial value of $\widehat{u}_c^{(1)}$ is represented in fiber ω and the initial value of $\widehat{u}_s^{(1)}$ is prescribed in fiber $\theta_{-T}\omega$.

It is possible to obtain the solution of system (5) and (6) by using a backward-forward integration procedure due to the partial coupling between the equations constituting this system, where $\widehat{u}_c^{(1)}$ forces the evolution equation of $\widehat{u}_s^{(1)}$ but not reciprocally. In addition, since $u_c^{(1)}$ is emanated backward from ξ in H^c and forces the equation ruling the evolution of $\widehat{u}_s^{(1)}$, thus $\widehat{u}_s^{(1)}$ depends naturally on ξ. One emphasizes this dependence as $\widehat{u}_s^{(1)}[\xi]$ in the whole paper.

The nonresonance conditions should be given in following theorem, under which the pullback limit of $\widehat{u}_s^{(1)}[\xi]$ exists. Now, representation of an analytical description of such parameterizing manifolds will be provided. In particularly, it emphasizes the dependence on the part of the noise path of the manifolds.

Theorem 1. *Consider the SPDE (2) in the functional setting of Section 2, with F assumed to be a trilinear function. Let $\mathcal{J} := \{1,\cdots,m\}$ with $m = \dim(H^c)$.*

Suppose also $\beta_n(\lambda) < 0$ for all $n > m$. Furthermore, assume that the following nonresonance conditions for all $(i_1, i_2, i_3) \in \mathcal{J}^3$, $n > m$,

$$\begin{aligned} &\text{if} < F(e_{i_1}, e_{i_2}, e_{i_3}), e_n > \neq 0, \\ &\text{then}\ \beta_{i_1} + \beta_{i_2} + \beta_{i_3} - \beta_n > 0 \end{aligned} \tag{9}$$

hold. Then, the pullback limit of the solution $\widehat{u}_s^{(1)}[\xi](T, \theta_{-T}\omega; 0)$ of (7) and (8) exists and is given by

$$\begin{aligned} \widehat{h}_\lambda^{(1)}(\xi,\omega) &= \lim_{T\to+\infty} \widehat{u}_s^{(1)}[\xi](T,\theta_{-T\omega};0)| \\ &= \int_{-\infty}^0 e^{-L_\lambda^s \tau + 2\sigma W_\tau(\omega)} P_s F\left(e^{L_\lambda^c \tau}\xi\right) d\tau, \forall \xi \in H^c, \quad \omega \in \Omega, \end{aligned} \tag{10}$$

where $\widehat{u}_c^{(1)}(s,\omega;\xi)$ is the solution of (5) and (6)

$$\widehat{u}_c^{(1)}(s,\omega;\xi) = e^{L_\lambda^c s + \sigma W_s(\omega)}\xi. \tag{11}$$

Moreover, $\widehat{h}_\lambda^{(1)}$ has the following analytic expression:

$$\begin{aligned} \widehat{h}_\lambda^{(1)}(\xi,\omega) = &\sum_{n=m+1}^{\infty} \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \sum_{i_3=1}^{m} \xi_{i_1}\xi_{i_2}\xi_{i_3} M_{n,\lambda}^{i_1 i_2 i_3}(\omega) \\ &< F(e_{i_1}, e_{i_2}, e_{i_3}), \quad e_n > e_n, \end{aligned} \tag{12}$$

where $\xi_i = <\xi, e_i>, i = 1, \cdots, m$, and

$$\begin{aligned} M_{n,\lambda}^{i_1 i_2 i_3}(\omega) = &\int_{-\infty}^0 e^\tau \left(\beta_{i_1}(\lambda) + \beta_{i_2}(\lambda) + \beta_{i_3}(\lambda) - \beta_n(\lambda)\right) \\ &+ 2\sigma W_\tau(\omega)_{d\tau}. \end{aligned} \tag{13}$$

Proof. Firstly, from (5), (6), (7), and (8), one introduces two processes $u_c^{(1)}$ and $u_s^{(1)}$ for $\widehat{u}_c^{(1)}$ and $\widehat{u}_s^{(1)}$ as follows:

$$\begin{aligned} u_c^{(1)}(s,\omega;\xi) &= e^{-z_\sigma(\theta_s\omega)} \widehat{u}_c^{(1)}\left(s,\omega; e^{z_\sigma(\theta_s\omega)}\xi\right), \\ &\qquad s \in [-T,0], \quad \xi \in H^c, \\ u_s^{(1)}[\xi](s,\theta_{-T}\omega;0) &= e^{-z_\sigma(\theta_{s-T}\omega)} \widehat{u}_s^{(1)}\left[e^{z_\sigma(\omega)}\xi\right](s,\theta_{-T}\omega;0), \\ &\qquad s \in [0,T]. \end{aligned} \tag{14}$$

Here, via the above transformation processes, the backward-forward system (5), (6), (7), and (8) is transformed into the following system of random differential equations:

$$\frac{du_c^{(1)}}{ds} = L_\lambda^c u_c^{(1)} + z_\sigma(\theta_s\omega) u_c^{(1)}, \quad s \in [-T,0], \tag{15}$$

$$u_c^{(1)}(s,\omega)|_{s=0} = \xi \in H^c, \tag{16}$$

$$\begin{aligned} \frac{du_s^{(1)}}{ds} = &L_\lambda^s u_s^{(1)} + z_\sigma(\theta_{s-T}\omega) u_s^{(1)} \\ &+ e^{2z_\sigma(\theta_{s-T}\omega)} P_s F\left(u_c^{(1)}(s-T,\omega)\right), \quad s \in [0,T], \end{aligned} \tag{17}$$

$$u_s^{(1)}(s,\theta_{-T}\omega)|_{s=0} = 0. \tag{18}$$

Using the variation of constants method, we can formally obtain the solution of (15) and (16), which is followed by making use of an integration by parts performed to the resulting stochastic convolution terms

$$u_c^{(1)}(s,\omega;\xi) = e^{L_\lambda^c s + \int_0^s z_\sigma(\theta_r\omega)dr}\xi. \tag{19}$$

Similarly, the solution of (17) and (18) can be also obtained at T, which is formed

$$u_s^{(1)}[\xi](T,\theta_{-T}\omega;0) = \int_{-T}^0 e^{-\tau L_\lambda^s + 2z_\sigma(\omega)dr + 2\sigma W_\tau(\omega)} P_s F\left(e^{\tau L_\lambda^c}\xi\right) d\tau, \tag{20}$$

where $u_c^{(1)}(\cdot,\omega;\xi)$ is taken as a form of (19). When $T \longrightarrow +\infty$, since condition (9), the limit of (20) exists, which is formed

$$\begin{aligned} h_\lambda^{(1)}(\xi,\omega) &= \lim_{T\to+\infty} u_s^{(1)}[\xi](T,\theta_{-T}\omega;0) \\ &= \int_{-\infty}^0 e^{-\tau L_\lambda^s + 2z_\sigma(\omega)dr + 2\sigma W_\tau(\omega)} P_s F\left(e^{\tau L_\lambda^c}\xi\right) d\tau. \end{aligned} \tag{21}$$

Secondly, one investigates the analysis presentation of this limit. Propose that

$$h_\lambda^{(1)}(\xi,\omega)=\sum_{n=m+1}^{\infty} h_\lambda^{(1),n}(\xi,\omega)e_n, \tag{22}$$

where

$$\begin{aligned} h_\lambda^{(1),n}(\xi,\omega)=&\sum_{i_1,i_2,i_3=1}^{m} e^{2z_\sigma(\omega)}\xi_{i_1}\xi_{i_2}\xi_{i_3}M_{n,\lambda}^{i_1i_2i_3}(\omega)\\ &< F\left(e_{i_1},e_{i_2},e_{i_3}\right),\quad e_n > . \end{aligned} \tag{23}$$

Here, $i_j=1,\cdots,m$ and $\xi_{i_j}=<\xi,e_{i_j}>, j=1,2,3,$ and

$$M_{n,\lambda}^{i_1i_2i_3}(\omega)=\int_{-\infty}^{0} e^{\left(\sum_{j=1}^{3}\beta_{1_j}(\lambda)-\beta_n(\lambda)\right)s+2\sigma W_s(\omega)} ds. \tag{24}$$

According to the same assumptions and the inverse transformation, (11) can be immediately obtained from (19), and the analytic expression of $\widehat{h}_\lambda^{(1)}$ has the form of (12).

Furthermore, the approximation $\widehat{h}_\lambda^{(1)}$ can be provided by the above theorem, which constitutes a parameterizing manifold function of SPDE (2). Moreover, the random coefficients $M_{n,\lambda}$ have decaying property of correlations when it is checked by similar calculations performing for the proof of Lemma 5.1 in [6], which are solutions of auxiliary SDEs

$$dM=\left(1-\left(\sum_{j=1}^{3}\beta_{i_j}(\lambda)-\beta_n(\lambda)\right)M\right)dt-\sigma M\circ dW_t. \tag{25}$$

Remark 2. Here, the random coefficients $M_{n,\lambda}^{i_1i_2i_3}(\omega)$ satisfied the stochastic equation (25) and are different from M_i in [14], since the stochastic processes' transformations are various for stochastic equations with multiplicative noises. So, $M_{n,\lambda}^{i_1i_2i_3}(\omega)$ in this paper and M_i in [14] have different representations by formulas. In this paper, the transformations of stochastic processes are more difficult than those in [14]. So, the random coefficients $M_{n,\lambda}^{i_1i_2i_3}(\omega)$ are much more complex than those in [14]. Then, these differences hold in the whole paper.

4. PM-Based Non-Markovian Reduced System with Multiplicative Noise

In this section, the PM-based non-Markovian reduced system of problem (2) is investigated in two cases that are in two subspaces, $H^c=\text{span}\{e_1\}$ or $H^c=\text{span}\{e_1,e_2\}$, respectively.

When one projects (2) into the subspace H_c, it yields that

$$du_c=(L_\lambda^c u_c+P_cF(u_c+u_s))dt+\sigma u\circ dW_t, \tag{26}$$

where $u_c=P_cu$ with P_c being the canonical projector on subspace H^c. By replacing $u_s(t,\omega)=P_su(t,\omega)$ with (12), the pullback limit $\widehat{h}_\lambda^{(1)}(\xi,\theta_t\omega)$, one yields the following reduced system

$$d\xi=\left(L_\lambda^c\xi+P_cF\left(\xi+\widehat{h}_\lambda^{(1)}(\xi,\theta_t\omega)\right)\right)dt+\sigma\xi\circ dW_t, \tag{27}$$

which provides an approximation of the SPDE dynamics projected onto the low modes.

From (12), the random coefficients of $e_i(x) (i=1,2,\cdots)$ contained in the expansion of $\widehat{h}_\lambda^{(1)}$ exhibit the decaying property of correlations. Therefore, extrinsic memory effects in the Stratonovich sense are conveyed by the drift part of (27), making such reduced systems be non-Markovian (see [18, 19]).

The analytic form of $\widehat{h}_\lambda^{(1)}$ from (12) can be used. The nonlinear interactions $F_n^{i_1i_2i_3}=<F(e_{i_1},e_{i_2},e_{i_3}),\ e_n>$, have the following form. When $m=1$,

$$\begin{aligned} &F_3^{i_1i_2i_3}=\frac{1}{2l},\quad n=3,\\ &F_n^{i_1i_2i_3}=0,\quad n=2\quad\text{or}\quad n\geq 4. \end{aligned} \tag{28}$$

When $m=2$,

$$\begin{aligned} &F_n^{i_1i_2i_3}=\frac{1}{2l},\quad \textbf{\textit{when}}\quad i_1+i_2+i_3=n\quad\text{or}\quad i_1-i_2-i_3=n,\\ &F_n^{i_1i_2i_3}=-\frac{1}{2l},\quad \textbf{\textit{when}}\quad i_1+i_2-i_3=n\quad\text{or}\quad i_1-i_2+i_3=n,\\ &F_n^{i_1i_2i_3}=0,\quad n\geq 7. \end{aligned} \tag{29}$$

Firstly, we investigate the system in case $m=1$. Since approximation of parameterizing manifolds have been obtained in Section 3, then one yields that

$$\widehat{h}_\lambda^{(1)}(\xi,\omega)=\frac{1}{2l}\xi_1^3M_{3,\lambda}^{111}(\omega)e_3, \tag{30}$$

where $\xi_1=<\xi,e_1>$, and

$$M_{3,\lambda}^{111}(\omega)=\int_{-\infty}^{0} e^{(3\beta_1(\lambda)-\beta_3(\lambda))s+2\sigma W_s(\omega)}ds. \tag{31}$$

In this case, the approximation is simple. However, it is not enough to present the performances of the corresponding dynamics. Furthermore, parameterizing manifolds in two-dimensional case for low mode are considered, which perform more dynamics than in the above case.

Secondly, when $m=2$, then one can obtain the more complex results than in the case of $m=1$.

Here,

$$\begin{aligned}\widehat{h}_\lambda^{(1)}(\xi,\omega) &= \sum_{n-m+1}^{\infty}\sum_{i_1,i_2,i_3=1}^{m}\xi_{i_1}\xi_{i_2}\xi_{i_3}M_{n,\lambda}^{i_1i_2i_3}(\omega) < F\left(e_{i_1},e_{i_2},e_{i_3}\right),e_n > e_n \\ &= \frac{1}{2l}\left(\xi_1^3M_{3,\lambda}^{111} - 3\xi_1\xi_2^2M_{3,\lambda}^{122}\right)e_3 + \frac{3}{2l}\xi_1^2\xi_2M_{4,\lambda}^{112}e_4 \\ &\quad + \frac{3}{2l}\xi_1\xi_2^2M_{5,\lambda}^{122}e_5 + \frac{1}{2l}\xi_2^3M_{6,\lambda}^{222}e_6,\end{aligned} \tag{32}$$

where $\xi_1 = \langle\xi, e_1\rangle$, $\xi_2 = \langle\xi, e_2\rangle$, and

$$\begin{aligned}M_{3,\lambda}^{111}(\omega) &= \int_{-\infty}^{0}e^{(3\beta_1(\lambda)-\beta_3(\lambda))s+2\sigma W_s(\omega)}ds,\ M_{3,\lambda}^{122}(\omega) \\ &= \int_{-\infty}^{0}e^{(\beta_1(\lambda)+2\beta_2(\lambda)-\beta_3(\lambda))s+2\sigma W_s(\omega)}ds, \\ M_{4,\lambda}^{112}(\omega) &= \int_{-\infty}^{0}e^{(2\beta_1(\lambda)+\beta_2(\lambda)-\beta_4(\lambda))s+2\sigma W_s(\omega)}ds, \\ M_{5,\lambda}^{122}(\omega) &= \int_{-\infty}^{0}e^{(\beta_1(\lambda)+2\beta_2(\lambda)-\beta_5(\lambda))s+2\sigma W_s(\omega)}ds,\ M_{6,\lambda}^{222}(\omega) \\ &= \int_{-\infty}^{0}e^{(3\beta_2(\lambda)-\beta_6(\lambda))s+2\sigma W_s(\omega)}ds.\end{aligned} \tag{33}$$

However, it is complex to use directly the analytic formula of $\widehat{h}_\lambda^{(1)}$ to obtain the vector $P_cF(\xi + \widehat{h}_\lambda^{(1)}(\xi,\theta_t\omega))$ as ξ is various in H^c in spite of any case in fact. So, we can use $\widehat{u}_s^{(1)}[\xi(t,\omega)](t+T,\theta_T\omega;0)$ to take the place of $\widehat{h}_\lambda^{(1)}(\xi,\theta_t\omega)$ on the fly along a trajectory $\xi(t,\omega)$ of interest, where $\widehat{u}_s^{(1)}$ is given by integrating on both sides of the backward-forward system (5), (6), (7), and (8), when T is chosen sufficiently large [6]. Then, it is natural to study the reduced system as follows:

$$\begin{aligned}d\xi_t &= \left(L_\lambda^c\xi_t + P_cF\left(\xi_t + \widehat{u}_s^{(1)}[\xi(t,\omega)](t+T,\theta_T\omega;0)\right)\right)dt \\ &\quad + \sigma\xi_t\circ dW_t,\ \xi(0,\omega) = \phi, \quad t>0,\end{aligned} \tag{34}$$

where ϕ is appropriately chosen according to the SPDE initial datum and $\widehat{u}_s^{(1)}[\xi(t,\omega)]$ is given from the following system:

$$\begin{aligned}d\widehat{u}_c^{(1)} &= L_\lambda^c\widehat{u}_c^{(1)}ds + \sigma\widehat{u}_c^{(1)}\circ dW_s,\ \widehat{u}_c^{(1)}(s,\omega)|_{s=t} = \xi(t,\omega), \\ &\qquad s\in[t-T,t], \\ d\widehat{u}_s^{(1)} &= \left(L_\lambda^s\widehat{u}_s^{(1)} + P_sF\left(\widehat{u}_c^{(1)}(s-T,\omega)\right)\right)ds + \sigma\widehat{u}_s^{(1)}\circ dW_{s-T}, \\ &\qquad \widehat{u}_s^{(1)}(s,\theta_{-T}\omega)|_{s=t} = 0,\ s\in[t,t+T].\end{aligned} \tag{35}$$

Now, we give the corresponding non-Markovian systems from the above system. Investigating them in two cases $m=1$ and $m=2$.

Firstly, when $m=1$, one denotes $\xi_1(t,\omega) = \xi_1(t,\omega)e_1$, with $\xi_1(t,\omega) = <\xi(t,\omega), e_1>$. Then, the system can be written as in coordinate form

$$d\xi_1 = \beta_1(\lambda)\xi_1 - \frac{3}{2l}\left(\xi_1^3 - \xi_1^2y_3^{(1)} + \xi_1\left[y_3^{(1)}\right]^2\right)dt + \sigma\xi_1\circ dW_t, \quad t>0, \tag{36}$$

with $\xi_1(0,\omega) = \langle\phi, e_1\rangle$, where $\xi_t = \xi_1(t,\omega)e_1$ and $y_j^{(1)}$, $j=2,\cdots$, are given from the following system:

$$\begin{aligned}dy_1^{(1)} &= \beta_1(\lambda)_{y_1}^{(1)}ds + \sigma y_1^{(1)}\circ dW_s, \quad s\in[t-T,t], \\ dy_2^{(1)} &= \beta_2(\lambda)y_2^{(1)}ds + \sigma y_2^{(1)}\circ dW_{s-T}, \quad s\in[t,t+T], \\ dy_3^{(1)} &= \left(\beta_3(\lambda)y_3^{(1)} + \frac{1}{2l}\left[y_1^{(1)}(s-T,\omega)\right]^3\right)ds + \sigma y_3^{(1)}\circ dW_{s-T}, \\ &\qquad s\in[t,t+T], \\ dy_j^{(1)} &= \beta_j(\lambda)y_j^{(1)}ds + \sigma y_j^{(1)}\circ dW_{s-T}, \quad s\in[t,t+T], \quad j=4,\cdots,\end{aligned} \tag{37}$$

with $y_1^{(1)}(s,\omega)|_{s=t} = \xi_1(t,\omega)$, $y_j^{(1)}(s,\theta_{-T}\omega)|_{s=t} = 0$, $j=2\cdots$.

Secondly, when $m=2$, one denotes $\xi(t,\omega) = \xi_1(t,\omega)e_1 + \xi_2(t,\omega)e_2$, with $\xi_i(t,\omega) = <\xi(t,\omega), e_i>$, $i=1,2$. Then, for $t>0$, the corresponding system can be written as in coordinate form

$$\begin{aligned}d\xi_1 &= \left\{\beta_1(\lambda)\xi_1 - \frac{3}{2l}\xi_1^3 - \frac{3}{l}\xi_1\xi_2^2 + \frac{3}{2l}\left(\left(\xi_1^2-\xi_2^2\right)y_3^{(1)}\right.\right. \\ &\quad \left. + 2\xi_1\xi_2y_4^{(1)} + \xi_2^2y_5^{(1)}\right) + \frac{3\xi_1}{l}\left(y_3^{(1)}y_5^{(1)} + y_4^{(1)}y_6^{(1)}\right. \\ &\quad \left. - \sum_{i=3}^{6}\left[y_i^{(1)}\right]^2\right) + \frac{3\xi_2}{l}\left(y_3^{(1)}y_6^{(1)} - \sum_{i=4}^{6}y_{i-1}^{(1)}y_i^{(1)}\right) \\ &\quad - \frac{1}{l}\left(\left[y_3^{(1)}\right]^2y_5^{(1)} + 2y_3^{(1)}y_4^{(1)}y_6^{(1)}\right) \\ &\quad \left. - \frac{1}{2l}y_3^{(1)}\left(y_3^{(1)}y_5^{(1)} + y_4^{(1)}y_6^{(1)}\right)\right\}dt + \sigma\xi_1\circ dW_t, \\ d\xi_2 &= \left\{\beta_2(\lambda)\xi_1 - \frac{3}{l}\xi_1^2\xi_2 - \frac{3}{2l}\xi_2^3 + \frac{3}{2l}\left(\xi_1^2y_4^{(1)}\right.\right. \\ &\quad \left. + 2\xi_1\xi_2\left(-y_3^{(1)} + y_5^{(1)}\right) + \xi_2^2y_6^{(1)}\right) \\ &\quad + \frac{3\xi_1}{l}\left(y_3^{(1)}y_6^{(1)} - \sum_{i=4}^{6}y_{i-1}^{(1)}y_i^{(1)}\right) - \frac{3\xi_2}{l}\sum_{i=3}^{6}\left[y_i^{(1)}\right]^2 \\ &\quad - \frac{3}{2l}\left(\left[y_3^{(1)}\right]^2y_4^{(1)} + 2y_3^{(1)}y_4^1y_5^{(1)} + y_3^{(1)}y_5^{(1)}y_6^{(1)}\right. \\ &\quad \left.\left. + \left[y_4^{(1)}\right]^2y_6^{(1)}\right)\right\}dt + \sigma\xi_2\circ dW_t,\end{aligned} \tag{38}$$

with $\xi_1(0,\omega)=\langle\phi,e_1\rangle$, $\xi_2(0,\omega)=\langle\phi,e_2\rangle$, where $\xi_t=\xi_1(t,\omega)e_1+\xi_2(t,\omega)e_2$ and $y_j^{(1)}, j=3,\cdots$, are given from following system

$$
\begin{aligned}
dy_1^{(1)} &= \beta_1(\lambda)y_1^{(1)}ds+\sigma y_1^{(1)}\circ dW_s, \quad s\in[t-T,t],\\
dy_2^{(1)} &= \beta_2(\lambda)y_2^{(1)}ds+\sigma y_2^{(1)}\circ dW_s, \quad s\in[t-T,t],\\
dy_3^{(1)} &= \left(\beta_3(\lambda)y_3^{(1)}+\frac{1}{2l}\left(\left[y_1^{(1)}(s-T,\omega)\right]^3\right.\right.\\
&\quad \left.\left.-3y_1^{(1)}(s-T,\omega)\left[y_2^{(1)}(s-T,\omega)\right]^2\right)\right)ds\\
&\quad +\sigma_3 y_3^{(1)}\circ dW_{s-T}, \quad s\in[t,t+T],\\
dy_4^{(1)} &= \left(\beta_4(\lambda)y_4^{(1)}+\frac{3}{2l}\left[y_1^{(1)}(s-T,\omega)\right]^2 y_2^{(1)}(s-T,\omega)\right)ds\\
&\quad +\sigma y_4^{(1)}\circ dW_{s-T}, \quad s\in[t,t+T],\\
dy_5^{(1)} &= \left(\beta_5(\lambda)y_4^{(1)}+\frac{3}{2l}y_1^{(1)}(s-T,\omega)\left[y_2^{(1)}(s-T,\omega)\right]2\right)ds\\
&\quad +\sigma y_5^{(1)}\circ dW_{s-T}, \quad s\in[t,t+T],\\
dy_6^{(1)} &= \left(\beta_6(\lambda)y_6^{(1)}+\frac{1}{2l}\left[y_2^{(1)}(s-T,\omega)\right]^3\right)ds\\
&\quad +\sigma y_6^{(1)}\circ dW_{s-T}, \quad s\in[t,t+T],\\
dy_j^{(1)} &= \beta_j(\lambda)y_j^{(1)}ds+\sigma y_j^{(1)}\circ dW_{s-T}, \quad s\in[t,t+T],\ j=7,\cdots,
\end{aligned}
\tag{39}
$$

with $y_1^{(1)}(s,\omega)|_{s=t}=\xi_1(t,\omega), y_2^{(1)}(s,\omega)|_{s=t}=\xi_2(t,\omega), y_j^{(1)}(s,\theta_{-T}\omega)|_{s=t}=0,\ j=3,\cdots$..

From the above equations, the representations of approximation for manifold and the corresponding reduced non-Markovian systems for stochastic Swift-Hohenberg equation with multiplicative noise are obtained. And the performances given by the above non-Markovian reduced system should have approximate dynamics on the H^c modes in modeling of the pathwise SPDE (2). It is more important that one should give partial dynamics in approximation sense on the H^c modes in modeling of the pathwise SPDEs in practice. The performances from the reduced system may be numerically assessed for a corresponding optimal or suboptimal control problems in the deduced processes. The numerical results will be further shown in the future. The processes deduced in this manuscript offered an idea in order to further investigate the approximation of stochastic manifold for some quantum stochastic equations with multiplicative noise.

Data Availability

Some data and ideas in our previous work in reference [14] were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (Nos. 11861013 and 11771444), Guangxi Natural Science Foundation (No. 2017GXNSFAA198221), Promotion of the Basic Capacity of Middle and Young Teachers in Guangxi Universities (No. 2017KY0340), the Fundamental Research Funds for the Central Universities, and the China University of Geosciences (Wuhan) (No. 2018061).

References

[1] D. Blömker, M. Hairer, and G. A. Pavliotis, *Stochastic Swift-Hohenberg Equation near a Change of Stability*, 2007.

[2] D. Blömker and W. Wang, "Qualitative properties of local random invariant manifolds for SPDEs with quadratic nonlinearity," *Journal of Dynamics and Differential Equations*, vol. 22, no. 4, pp. 677–695, 2010.

[3] D. Blomker, *Amplitude Equations for Stochastic Partial Differential Equations. Vol.3 of Interdisciplinary Mathematical Sciences*, World Scientific Publishing, Singapore, 2007.

[4] G. Chen, J. Duan, and J. Zhang, "Geometric shape of invariant manifolds for a class of stochastic partial differential equations," *Journal of Mathematical Physics*, vol. 52, no. 7, p. 072702, 2011.

[5] M. D. Chekroun, H. H. Liu, and S. H. Wang, "Non-Markovian reduced systems for stochastic partial differential equations: the additive noise case," https://arxiv.org/abs/1311.3069v1.

[6] M. D. Chekroun, H. H. Liu, and S. H. Wang, *Stochastic Parameterizing Manifolds and Non-Markovian Reduced Equations–Stochastic Manifolds for Nonlinear SPDEs II*, Springer Briefs in Mathematics, Springer-Verlag, New York, NY, USA, 2015.

[7] J. Duan, K. Lu, and B. Schmalfuss, "Invariant manifolds for stochastic differential equations," *Annals of Probability*, vol. 31, pp. 2109–2135, 2003.

[8] J. Duan, K. Lu, and B. Schmalfuss, "Smooth stable and unstable manifolds for stochastic evolutionary equations," *Journal of Dynamics and Differential Equations*, vol. 16, no. 4, pp. 949–972, 2004.

[9] J.-P. Eckmann and C. E. Wayne, "Propagating fronts and the center manifold theorem," *Communications in Mathematical Physics*, vol. 136, no. 2, pp. 285–307, 1991.

[10] H. Fu, X. Liu, and J. Duan, "Slow manifolds for multi-time-scale stochastic evolutionary systems," *Communications in Mathematical Sciences*, vol. 11, no. 1, pp. 141–162, 2013.

[11] X. Sun, J. Duan, and X. Li, "An impact of noise on invariant manifolds in nonlinear dynamical systems," *Journal of Mathematical Physics*, vol. 51, no. 4, p. 042702, 2010.

[12] J. Swift and P. C. Hohenberg, "Hydrodynamic fluctuations at the convective instability," *Physical Review A*, vol. 15, no. 1, pp. 319–328, 1977.

[13] G. Lin, H. Gao, J. Duan, and V. J. Ervin, "Asymptotic dynamical difference between the nonlocal and local Swift-Hohenberg models," *Journal of Mathematical Physics*, vol. 41, no. 4, pp. 2077–2089, 2000.

[14] Y. F. Guo and J. Duan, "Approximation representation of parameterizing manifold and non-Markovian reduced systems for a stochastic Swift-Hohenberg equation," *Applied Mathematics Letters*, vol. 52, pp. 112–117, 2016.

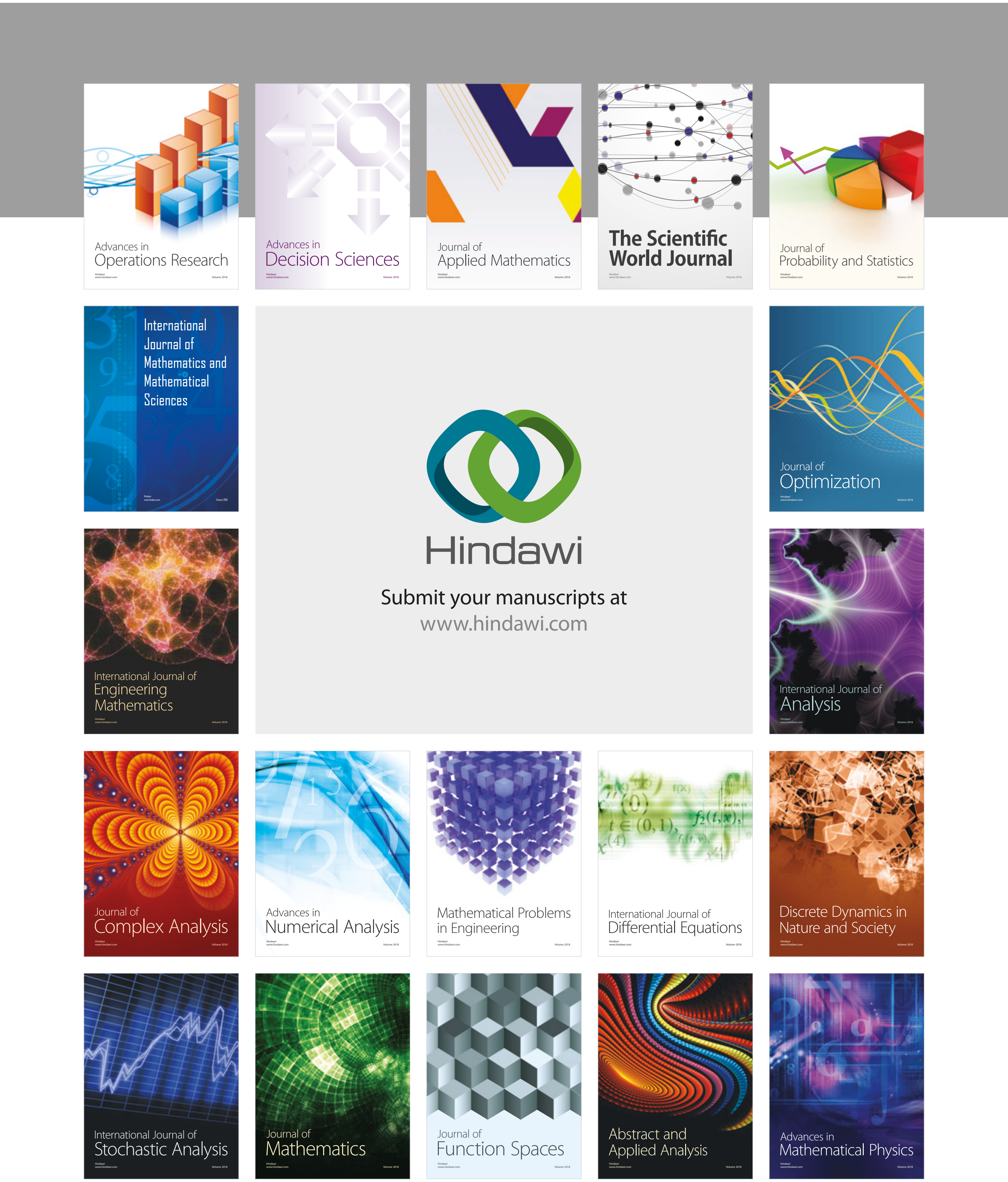
[15] J. Duan and W. Wang, *Effective Dynamics of Stochastic Partial Differential Equations*, Elsevier, London, UK, 2014.

[16] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Spinger-Varlag, Berlin, Germany, 1981.

[17] L. Arnold, *Random Dynamical Systems*, Spinger-Verlag, New York, NY, USA, 1998.

[18] M. Hairer, "Ergodic properties of a class of non-Markovian processes," in *Trends in Stochastic Analysis*, vol. 353 of London Mathematical Society Lecture Note Series, , pp. 65–98, Cambridge University Press, 2009.

[19] M. Hairer and A. Ohashi, "Ergodic theory for SDEs with extrinsic memory," *Annals of Probability*, vol. 35, no. 5, pp. 1950–1977, 2007.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
The Scientific
World Journal
Journal of
Probability and Statistics
International
Journal of
Mathematics and
Mathematical
Sciences
Hindawi
Submit your manuscripts at
www.hindawi.com
Journal of
Optimization
International Journal of
Engineering
Mathematics
International Journal of
Analysis
Journal of
Complex Analysis
Advances in
Numerical Analysis
Mathematical Problems
in Engineering
International Journal of
Differential Equations
Discrete Dynamics in
Nature and Society
International Journal of
Stochastic Analysis
Journal of
Mathematics
Journal of
Function Spaces
Abstract and
Applied Analysis
Advances in
Mathematical Physics