Research Article

Global Existence for Stochastic Strongly Dissipative Zakharov Equations

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The stochastic strongly dissipative Zakharov equations with white noise are studied. On the basis of the time uniform a priori estimates, we prove the existence and uniqueness of solutions in energy spaces $\mathscr{E}_1$ and $\mathscr{E}_2$, by using the standard Galerkin approximation method of stochastic partial differential equations.

1. Introduction

The Zakharov system, derived by Zakharov in 1972 [1], describes the interaction between Langmuir (dispersive) and ion acoustic (approximately nondispersive) waves in an unmagnetized plasma. The usual Zakharov system defined in the space $\mathbb{R}^{d+1}$ is given by

$$
\begin{aligned}
iE_t + \Delta E &= nE, \\
n_{tt} - \Delta n &= \Delta |E|^2,
\end{aligned}
\tag{1}
$$

where the wave fields $E(x,t)$ and $n(x,t)$ are complex and real, respectively. It has become commonly accepted that the Zakharov system is a general model to govern interaction of dispersive nondispersive waves.

In the past decades, the Zakharov system has been studied by many authors [2–10]. In [4], the authors established globally in time existence and uniqueness of smooth solution for a generalized Zakharov equation in a two-dimensional case for small initial date and proved global existence of smooth solution in one spatial dimension without any small assumption for initial data. In [7, 8], the uniqueness and existence of the global classical solution of the periodic initial value problem for the system of Zakharov equations and the system of generalized Zakharov equations have been proved.

In order to better qualitative agreement, it is necessary to include damping effects or effects of the loss of energy. In a realistic physical system, dissipation must be included into each equation. In [11], the author studied the following dissipative Zakharov equations:

$$
\frac{1}{\lambda^2} n_{tt} - n_{xx} - |E|^2_{xx} - \alpha n_t = f,
\tag{2}
$$

$$
iE_t + E_{xx} - nE + i\gamma E = g,
\tag{3}
$$

where positive constants α and γ are damping coefficients and f and g are external forces. The author obtained the long time behavior of (2) and (3) on a bounded interval with initial conditions and homogeneous Dirichlet boundary conditions. The asymptotic behaviors of the solution for (2) and (3) in 1D-2D have been investigated (see [11–13]).

In [14], the authors considered the following strongly dissipative Zakharov equations:

$$
\frac{1}{\lambda^2} n_{tt} - n_{xx} - |E|^2_{xx} - \alpha n_{txx} = f,
\tag{4}
$$

$$
iE_t + E_{xx} - nE + i\gamma E = g.
\tag{5}
$$

They studied the Cauchy problem of (4) and (5) and proved the existence of the maximal attractor.

In recent years, the importance of taking random effects into account in modeling, analyzing, simulating, and predicting complex phenomena has been widely recognized in geophysical and climate dynamics, materials science, chemistry, biology, and other areas [15–18]. Stochastic partial differential equations are appropriate mathematical models for complex systems under random influences or noise. Usually, the noise can be regarded as a simple approximation of turbulence in fluids.

In [19], stochastic dissipative Zakharov equations have been studied on a regular domain D in the Itô sense. The authors proved the existence and uniqueness of solutions. Then, a global random attractor was constructed. Further, the existence of a stationary measure was proved. In [20], the existence and uniqueness of solutions are obtained. Moreover, the asymptotic behaviors of the solutions for the stochastic dissipative quantum Zakharov equations with white noise were also investigated.

In the present paper, we consider the following random forced strongly dissipative Zakharov equations on a regular domain D in the Itô sense:

$$\frac{1}{\lambda^2} n_{tt} - n_{xx} - |E|^2_{xx} - \alpha n_{txx} = f + \dot{W}_1,$$

$$iE_t + E_{xx} - nE + i\gamma E = g + \dot{W}_2,$$
(6)

where the parameters $\gamma > 0$ and $\alpha > 0$, and W_1 and W_2 are independent $L^2(D)$-valued Wiener processes, which will be detailed in the next section. Here, $\dot{W}$ denotes the general derivative of the Wiener processes W with respect to time.

The rest of this paper is organized as follows. In Section 2, some functional setting and some conditions are given. In Section 3, a series of time uniform a priori estimates are given in different energy spaces. Some of the technique of estimates and Itô's formula are used frequently. In Section 4, we obtain the existence and uniqueness of solutions for the stochastic strongly dissipative Zakharov equations by using the standard Galerkin approximation method in different investigated spaces. Various positive constants are denoted by C throughout this paper. Now, we state the main results of the paper.

2. Preliminary

In this section, we give a detailed description of the stochastic strongly dissipative Zakharov equations. Let $D = [0, l]$ with $0 < l < \infty$. Consider the following strongly dissipative stochastic Zakharov equations on a regular domain D:

$$\frac{1}{\lambda^2} n_{tt} - n_{xx} - |E|^2_{xx} - \alpha n_{txx} = f + \dot{W}_1,$$
(7)

$$iE_t + E_{xx} - nE + i\gamma E = g + \dot{W}_2, \quad t \in R^+, x \in D,$$
(8)

with initial conditions

$$E|_{t=0} = E_0(x), n|_{t=0} = n_0(x), n_t|_{t=0} = n_1(x), \quad x \in D,$$
(9)

and Dirichlet boundary conditions

$$E = 0, n = 0, t \in R^+, x \in \partial D;$$
(10)

here, f, g, α, γ, and λ are given, with $\alpha > 0$ and $\gamma > 0$.

As in [14], to study the solution of strongly dissipative stochastic Zakharov equations, we set $m = n_t + \varepsilon n$, where $\varepsilon > 0$ is small enough and will be chosen later. So we have the following equations:

$$m = n_t + \varepsilon n,$$
(11)

$$\frac{1}{\lambda^2} m_t - \frac{\varepsilon}{\lambda^2} m - \alpha m_{xx} + \frac{\varepsilon^2}{\lambda^2} n - (1 - \varepsilon\alpha) n_{xx}$$
$$- |E|^2_{xx} = f + \dot{W}_1,$$
(12)

$$iE_t + E_{xx} - nE + i\gamma E$$
$$= g + \dot{W}_2, \quad t \in R^+, x \in D,$$
(13)

$$E|_{t=0} = E_0(x), n|_{t=0} = n_0(x), m|_{t=0}$$
$$= n_1(x) + \varepsilon n_0(x) = m_0(x), \quad x \in D,$$
(14)

$$E = 0, n = 0, m = 0, t \in R^+, x \in \partial D.$$
(15)

Here, we give a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. The expectation operator with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$. The stochastic terms $W_1(t)$ and $W_2(t)$ on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ are defined by

$$W_1(t) = q_1(x)\omega_1(t), W_2(t) = q_2(x)\omega_2(t),$$
(16)

where $\omega_1(t)$ is a standard real-valued Wiener process and $\omega_2(t)$ is a standard complex-valued Wiener process independent of $\omega_1(t)$. In addition, $q_1(x)$ and $q_2(x)$ are sufficiently smooth functions.

We will work on the usual functional spaces $L^2(D)$, $H^m(D)$, and $H_0^1(D)$. The inner product on $L^2(D)$ will be denoted by $(u, v) = \text{Re} \int_D u(x)\bar{v}(x)dx, u, v \in L^2(D)$ and the norm by $\|u\|_0 = \|u\|_{L^2} = (\int_D |u|^2 dx)^{1/2}$. The general p-norm of $L^p(D)$ $(p \geq 1)$ is denoted by $\|u\|_{L^p} = (\int_D |u|^p dx)^{1/p}$. And the norm on $H^m(D)$ is denoted by $\|u\|_m = (\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|^2_{L^2(D)})^{1/2}$. The real and imaginary parts of a complex number A are denoted, respectively, by $\text{Re } A$ and $\text{Im } A$.

Now, we define spaces $\mathscr{E}_i, i = 0, 1, 2$, as $\mathscr{E}_0 = H^{-1}(D) \times L^2(D) \times H_0^1(D)$, $\mathscr{E}_1 = L^2(D) \times H_0^1(D) \times (H^2(D) \cap H_0^1(D))$, and $\mathscr{E}_2 = H_0^1(D) \times (H^2(D) \cap H_0^1(D)) \times \{u \in H^3(D) \cap H_0^1(D), \triangle u \in H_0^1(D)\}$. Endow $\mathscr{E}_i$ with the usual product norm $\|\cdot\|_{\mathscr{E}_i}$. Then, $\mathscr{E}_2 \subset \mathscr{E}_1 \subset \mathscr{E}_0$ with compact embeddings.

In our approach, we need the following lemmas. Let $(\mathscr{X}, \|\cdot\|_{\mathscr{X}}) \subset (\mathscr{Y}, \|\cdot\|_{\mathscr{Y}}) \subset (\mathscr{Z}, \|\cdot\|_{\mathscr{Z}})$ be three reflective Banach spaces and $\mathscr{X} \subset \mathscr{Y}$ with compact and dense embedding. Define the Banach space

$$\mathscr{G} = \left\{ v : v \in L^2(0, T; \mathscr{X}), \frac{dv}{dt} \in L^2(0, T; \mathscr{Z}) \right\},$$
(17)

with

$$\|v\|_{\mathscr{G}}^2 = \int_0^T \|v(s)\|_{\mathscr{X}}^2 ds + \int_0^T \left\|\frac{dv}{dt}(s)\right\|_{\mathscr{X}}^2 ds, \quad v \in \mathscr{G}. \quad (18)$$

Then, we have the following lemma about compactness result (see [21]).

Lemma 1. *If K is bounded in $\mathscr{G}$, then K is precompact in $L^2 (0, T; \mathscr{Y})$.*

Another lemma is needed for some maximal estimates on the stochastic integral. Assume U and H are separable Hilbert spaces, W is a Q-Wiener process on U_0 with $U_0 = Q^{1/2}U$. And let $L_2^0 = L_2^0(U_0, H)$ be the space of Hilbert-Schmidt operators from U_0 to H. We have the following results (see [22]).

Lemma 2. *For any $r \geq 1$ and any L_2^0-valued predictable process $\Phi(t), t \in [0, T]$, we have*

$$\mathbb{E}\int_0^t \Phi(s)dW(s) = 0,$$

$$\mathbb{E}\left(\sup_{s\in[0,t]}\left|\int_0^s \Phi(\sigma)dW(\sigma)\right|^{2r}\right)$$

$$\leq c_r \sup_{s\in[0,t]}\mathbb{E}\left(\left|\int_0^s \Phi(\sigma)dW(\sigma)\right|^{2r}\right) \quad (19)$$

$$\leq C_r\mathbb{E}\left(\left|\int_0^t \|\Phi(s)\|_{L_2^0}^2 ds\right|^r\right),$$

where c_r and C_r are some positive constants dependent on r.

3. Time Uniform A Priori Estimates

In this section, in the sense of expectation, we give a priori estimates for (n_t, n, E) in different spaces $\mathscr{C}_0$, $\mathscr{C}_1$, and $\mathscr{C}_2$.

Lemma 3. *Assume $g, E_0, q_2 \in L^2(D)$. Then, for any $T > 0$ and $p \geq 1$, $E \in L^{2p}(\Omega; L^\infty(0, T; L^2(D))) \cap L^\infty(0,\infty;L^{2p}(\Omega, L^2(D)))$.*

Proof. Applying Itô's formula to $\|E\|_0^2$, by (13), we have

$$\frac{d}{dt}\|E\|_0^2 = 2 \operatorname{Re}\int_D E_t \bar{E}dx + \|q_2\|_0^2$$

$$= -2\gamma\|E\|_0^2 + 2 \operatorname{Im}\int_D g\bar{E}dx \quad (20)$$

$$+ 2 \operatorname{Im}\int_D \dot{W}_2 \bar{E}dx + \|q_2\|_0^2.$$

Integrating (20) from 0 to t, by Young inequality, we obtain

$$\|E\|_0^2 = \|E_0\|_0^2 - 2\gamma\int_0^t \|E(s)\|_0^2 ds + 2\int_0^t \operatorname{Im}\int_D g\bar{E}dxds$$

$$+ 2\int_0^t \operatorname{Im}\int_D \dot{W}_2\bar{E}dxds + \|q_2\|_0^2 t$$

$$\leq \|E_0\|_0^2 - 2\gamma\int_0^t \|E(s)\|_0^2 ds + 2\int_0^t \|g\|_0\|E\|_0 ds$$

$$+ 2\int_0^t \operatorname{Im}\int_D \dot{W}_2\bar{E}dxds + \|q_2\|_0^2 t \quad (21)$$

$$\leq \|E_0\|_0^2 - \gamma\int_0^t \|E(s)\|_0^2 ds + \left(\frac{1}{\gamma}\|g\|_0^2 + \|q_2\|_0^2\right)t$$

$$+ 2\int_0^t \operatorname{Im}\int_D \dot{W}_2\bar{E}dxds.$$

Taking expectation on both sides of (21), by Lemma 2, we get

$$\mathbb{E}\|E\|_0^2 \leq \mathbb{E}\|E_0\|_0^2 - \gamma\mathbb{E}\int_0^t \|E(s)\|_0^2 ds + \left(\frac{1}{\gamma}\|g\|_0^2 + \|q_2\|_0^2\right)t. \quad (22)$$

Then, by Gronwall inequality, we get

$$\mathbb{E}\|E\|_0^2 \leq e^{-\gamma t}\mathbb{E}\|E_0\|_0^2 + \frac{1}{\gamma^2}\|g\|_0^2 + \frac{1}{\gamma}\|q_2\|_0^2 \leq C, \quad (23)$$

where C is independent of T.

By (23), we obtain $E \in L^\infty(0, \infty; L^2(\Omega, L^2(D)))$.

Taking the supremum and expectation on both sides of (21), by Lemma 2 and (23), for any $T > 0$, there exists a positive constant C_T; we have

$$\mathbb{E}\sup_{0\leq t\leq T}\|E\|_0^2 \leq \mathbb{E}\|E_0\|_0^2 + \mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t \operatorname{Im}\int_D \dot{W}_2\bar{E}dxds\right|^2$$

$$+ \left(\frac{1}{\gamma}\|g\|_0^2 + \|q_2\|_0^2\right)T + 1$$

$$\leq \mathbb{E}\|E_0\|_0^2 + C\|q_2\|_0^2\mathbb{E}\int_0^T \|E\|_0^2 ds \quad (24)$$

$$+ \left(\frac{1}{\gamma}\|g\|_0^2 + \|q_2\|_0^2\right)T + 1$$

$$\leq C_T\left(\mathbb{E}\|E_0\|_0^4 + \|g\|_0^4 + \|q_2\|_0^4 + 1\right)$$

$$\leq C_T(E_0, g, q_2),$$

where $C_T(E_0, g, q_2)$ is a constant depending on $\|E_0\|_0$, $\|g\|_0$, $\|q_2\|_0$ and T.

By (24), we obtain $E \in L^2(\Omega; L^\infty(0, T; L^2(D)))$.

By the above estimates, we can further give an estimate of $\|E\|_0^{2p}$ for any $p \geq 1$. Now applying Itô's formula, Young inequality, and Hölder inequality, we have

$$\frac{d}{dt}\|E\|_0^{2p} = p\|E\|_0^{2(p-1)}\frac{d}{dt}\|E\|_0^2 + 2p(p-1)\|E\|_0^{2(p-1)}\|q_2\|_0^2$$

$$= p\|E\|_0^{2(p-1)}\left(2\,\mathrm{Re}\int_D E_t\bar{E}dx + \|q_2\|_0^2\right)$$

$$\quad + 2p(p-1)\|E\|_0^{2(p-1)}\|q_2\|_0^2$$

$$= p\|E\|_0^{2(p-1)}\left(-2\gamma\|E(t)\|_0^2\right)$$

$$\quad + 2p\|E\|_0^{2(p-1)}\,\mathrm{Im}\int_D g\bar{E}dx$$

$$\quad + 2p\|E\|_0^{2(p-1)}\,\mathrm{Im}\int_D \dot{W}_2\bar{E}dx$$

$$\quad + p(2p-1)\|E\|_0^{2(p-1)}\|q_2\|_0^2$$

$$\leq -2\gamma p\|E\|_0^{2p} + 2p\|E\|_0^{2(p-1)}\|g\|_0\|E\|_0$$

$$\quad + 2p\|E\|_0^{2(p-1)}\,\mathrm{Im}\int_D \dot{W}_2\bar{E}dx$$

$$\quad + p(2p-1)\|E\|_0^{2(p-1)}\|q_2\|_0^2$$

$$\leq -\gamma p\|E\|_0^{2p} + \frac{p}{\gamma}\|E\|_0^{2(p-1)}\|g\|_0^2$$

$$\quad + 2p\|E\|_0^{2(p-1)}\,\mathrm{Im}\int_D \dot{W}_2\bar{E}dx$$

$$\quad + p(2p-1)\|E\|_0^{2(p-1)}\|q_2\|_0^2$$

$$\leq -\gamma p\|E\|_0^{2p} + \frac{\gamma p}{2}\|E\|_0^{2p} + 2p\|E\|_0^{2(p-1)}\,\mathrm{Im}\int_D \dot{W}_2\bar{E}dx$$

$$\quad + C\big(\|g\|_0, \|q_2\|_0\big) = -\frac{\gamma p}{2}\|E\|_0^{2p}$$

$$\quad + 2p\|E\|_0^{2(p-1)}\,\mathrm{Im}\int_D \dot{W}_2\bar{E}dx + C\big(\|g\|_0, \|q_2\|_0\big).$$

$$(25)$$

Integrating (25) from 0 to t, we obtain

$$\|E\|_0^{2p} \leq \|E_0\|_0^{2p} - \frac{\gamma p}{2}\int_0^t \|E\|_0^{2p}ds + C\big(\|g\|_0, \|q_2\|_0\big)t$$

$$\quad + 2p\int_0^t \|E\|_0^{2(p-1)}\,\mathrm{Im}\int_D \dot{W}_2\bar{E}dxds. \tag{26}$$

Taking expectation on both sides of (26), we get

$$\mathbb{E}\|E\|_0^{2p} \leq \mathbb{E}\|E_0\|_0^{2p} - \frac{\gamma p}{2}\mathbb{E}\int_0^t \|E\|_0^{2p}ds$$

$$\quad + C\big(\|g\|_0, \|q_2\|_0\big)t. \tag{27}$$

Then, by Gronwall inequality, we get

$$\mathbb{E}\|E\|_0^{2p} \leq e^{-\gamma pt/2}\mathbb{E}\|E_0\|_0^{2p} + C\big(\|g\|_0, \|q_2\|_0\big) \leq C(g, q_2), \tag{28}$$

where $C(g, q_2)$ is independent of T.

By (28), we obtain $E \in L^\infty(0,\infty;L^{2p}(\Omega, L^2(D)))$.

Taking the supremum and expectation on both sides of (26), by Lemma 2 and (28), for any $T > 0$, there exists a positive constant C_T; we have

$$\mathbb{E}\sup_{0\leq t\leq T}\|E\|_0^{2p} \leq \mathbb{E}\|E_0\|_0^{2p} + C\big(\|g\|_0, \|q_2\|_0\big)T + 1$$

$$\quad + p^2\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t \|E\|_0^{2(p-1)}\,\mathrm{Im}\int_D \dot{W}_2\bar{E}dxds\right|^2$$

$$\leq \mathbb{E}\|E_0\|_0^{2p} + C\big(\|g\|_0, \|q_2\|_0\big)T + 1$$

$$\quad + Cp^2\|q_2\|_0^2\mathbb{E}\int_0^T \|E\|_0^{4p-2}ds$$

$$\leq C_T\big(\|E_0\|_0, \|g\|_0, \|q_2\|_0\big) \leq C_T(E_0, g, q_2),$$

$$(29)$$

where $C_T(E_0, g, q_2)$ is a constant depending on $\|E_0\|_0$, $\|g\|_0$, $\|q_2\|_0$ and T.

By (29), we obtain $E \in L^{2p}(\Omega; L^\infty(0, T; L^2(D)))$.

Lemma 3 is proved completely.

Lemma 4. *Assume* $(n_1, n_0, E_0) \in \mathscr{E}_0, f, q_1 \in H^{-1}(D)$ *and* $g, q_2 \in H_0^1(D)$. *Then, for any* $T > 0$ *and* $p \geq 1$, $(n_t, n, E) \in L^{2p}(\Omega; L^\infty(0, T; \mathscr{E}_0)) \cap L^\infty(0, \infty; L^{2p}(\Omega, \mathscr{E}_0))$.

Proof. Applying Itô's formula to $\|m\|_{-1}^2$, by (12), we have

$$\frac{1}{2\lambda^2}\frac{d}{dt}\|m\|_{-1}^2 = \int_D \frac{1}{\lambda^2}m_t(-\Delta)^{-1}mdx + \frac{\lambda^2}{2}\|q_1\|_{-1}^2 = \frac{\varepsilon}{\lambda^2}\|m\|_{-1}^2$$

$$\quad - \alpha\|m\|_0^2 - \frac{\varepsilon^2}{\lambda^2}\int_D (-\Delta)^{-1/2}n(-\Delta)^{-1/2}mdx$$

$$\quad - (1-\varepsilon\alpha)\int_D nmdx - \int_D |E|^2 mdx$$

$$\quad + \int_D f(-\Delta)^{-1}mdx + \int_D \dot{W}_1(-\Delta)^{-1}mdx$$

$$\quad + \frac{\lambda^2}{2}\|q_1\|_{-1}^2.$$

$$(30)$$

Since $m = n_t + \varepsilon n$, from (30), we obtain

$$\frac{d}{dt}\left[\frac{1}{2\lambda^2}\|m\|_{-1}^2 + \frac{1}{2}(1-\varepsilon\alpha)\|n\|_0^2 + \frac{\varepsilon^2}{2\lambda^2}\|n\|_{-1}^2\right]$$

$$= \frac{\varepsilon}{\lambda^2}\|m\|_{-1}^2 - \alpha\|m\|_0^2 - \frac{\varepsilon^3}{\lambda^2}\|n\|_{-1}^2 - \varepsilon(1-\varepsilon\alpha)\|n\|_0^2$$

$$\quad - \int_D |E|^2 mdx + \int_D f(-\Delta)^{-1}mdx$$

$$\quad + \int_D \dot{W}_1(-\Delta)^{-1}mdx + \frac{\lambda^2}{2}\|q_1\|_{-1}^2 \tag{31}$$

$$\leq \left(\frac{\varepsilon}{\lambda^2} - \alpha\lambda_1\right)\|m\|_{-1}^2 - \frac{\varepsilon^3}{\lambda^2}\|n\|_{-1}^2 - \varepsilon(1-\varepsilon\alpha)\|n\|_0^2$$

$$\quad - \int_D |E|^2 mdx + \int_D f(-\Delta)^{-1}mdx$$

$$\quad + \int_D \dot{W}_1(-\Delta)^{-1}mdx + \frac{\lambda^2}{2}\|q_1\|_{-1}^2.$$

where λ_1 is the first eigenvalue of $-\Delta$.

By (31), we get

$$
\begin{aligned}
\frac{d}{dt} &\left[\frac{1}{2\lambda^2} \|m\|_{-1}^2 + \frac{1}{2}(1-\varepsilon\alpha)\|n\|_0^2 + \frac{\varepsilon^2}{2\lambda^2}\|n\|_{-1}^2 \right] \\
&+ \left(\alpha\lambda_1 - \frac{\varepsilon}{\lambda^2} \right) \|m\|_{-1}^2 + \varepsilon(1-\varepsilon\alpha)\|n\|_0^2 + \frac{\varepsilon^3}{\lambda^2}\|n\|_{-1}^2 \\
&\leq -\int_D |E|^2 m\,dx + \int_D f(-\Delta)^{-1} m\,dx + \frac{\lambda^2}{2}\|q_1\|_{-1}^2 \\
&+ \int_D \dot{W}_1 (-\Delta)^{-1} m\,dx.
\end{aligned}
\tag{32}
$$

Applying Itô's formula to $\|E_x\|_0^2$, by (13), we have

$$
\frac{d}{dt}\|E_x\|_0^2 = 2\,\mathrm{Re}\int_D E_x \bar{E}_{tx}\,dx + \|q_2\|_1^2.
\tag{33}
$$

Substituting $E_{xx} = -iE_t + nE - i\gamma E + g + \dot{W}_2$ into (33), we have

$$
\begin{aligned}
\frac{d}{dt}\|E_x\|_0^2 &= -2\,\mathrm{Re}\int_D E_{xx}\bar{E}_t\,dx + \|q_2\|_1^2 \\
&= -2\,\mathrm{Re}\int_D \left(nE - i\gamma E + g + \dot{W}_2 \right)\bar{E}_t\,dx + \|q_2\|_1^2 \\
&= -2\,\mathrm{Re}\int_D nE\bar{E}_t\,dx + 2\,\mathrm{Re}\int_D \gamma\bar{E}(-iE_t)\,dx \\
&\quad - 2\,\mathrm{Re}\int_D g\bar{E}_t\,dx - 2\,\mathrm{Re}\int_D \dot{W}_2\bar{E}_t\,dx + \|q_2\|_1^2 \\
&= -2\,\mathrm{Re}\int_D nE\bar{E}_t\,dx - 2\gamma\|E_x\|_0^2 - 2\gamma\int_D n|E|^2\,dx \\
&\quad - 2\,\mathrm{Re}\int_D \gamma\bar{E}g\,dx - 2\,\mathrm{Re}\int_D \gamma\bar{E}\dot{W}_2\,dx \\
&\quad - 2\,\mathrm{Re}\int_D g\bar{E}_t\,dx - 2\,\mathrm{Re}\int_D \dot{W}_2\bar{E}_t\,dx + \|q_2\|_1^2 \\
&= -2\,\mathrm{Re}\int_D nE\bar{E}_t\,dx - 2\gamma\|E_x\|_0^2 - 2\gamma\int_D n|E|^2\,dx \\
&\quad + \|q_2\|_1^2 - 2\,\mathrm{Re}\int_D \left(\gamma\bar{E} + \bar{E}_t\right)g\,dx \\
&\quad - 2\,\mathrm{Re}\int_D \left(\gamma\bar{E} + \bar{E}_t\right)\dot{W}_2\,dx.
\end{aligned}
\tag{34}
$$

In addition, applying Itô's formula to $\int_D n|E|^2 dx$, we have

$$
\frac{d}{dt}\int_D n|E|^2\,dx = \int_D n_t|E|^2\,dx + 2\,\mathrm{Re}\int_D nE\bar{E}_t\,dx + \int_D n|q_2|^2\,dx.
\tag{35}
$$

Substituting (35) into (34), we obtain

$$
\begin{aligned}
\frac{d}{dt} &\left(\|E_x\|_0^2 + \int_D n|E|^2\,dx \right) \\
&= -2\gamma\|E_x\|_0^2 - 2\gamma\int_D n|E|^2\,dx + \|q_2\|_1^2 \\
&\quad - 2\,\mathrm{Re}\int_D \left(\gamma\bar{E} + \bar{E}_t\right)g\,dx - 2\,\mathrm{Re}\int_D \left(\gamma\bar{E} + \bar{E}_t\right)\dot{W}_2\,dx \\
&\quad + \int_D n_t|E|^2\,dx + \int_D n|q_2|^2\,dx \\
&= -2\gamma\|E_x\|_0^2 - (2\gamma + \varepsilon)\int_D n|E|^2\,dx + \|q_2\|_1^2 \\
&\quad + \int_D n|q_2|^2\,dx - 2\,\mathrm{Re}\int_D \left(\gamma\bar{E} + \bar{E}_t\right)g\,dx \\
&\quad - 2\,\mathrm{Re}\int_D \left(\gamma\bar{E} + \bar{E}_t\right)\dot{W}_2\,dx + \int_D m|E|^2\,dx.
\end{aligned}
\tag{36}
$$

Taking $0 < \varepsilon \leq \min\{1/\alpha, \alpha\lambda_1\lambda^2/2\}$, and by (32) and (36), we have

$$
\begin{aligned}
\frac{d}{dt} &\left[\frac{1}{2\lambda^2}\|m\|_{-1}^2 + \frac{1}{2}(1-\varepsilon\alpha)\|n\|_0^2 + \frac{\varepsilon^2}{2\lambda^2}\|n\|_{-1}^2 \right] \\
&+ \frac{d}{dt}\left(\|E_x\|_0^2 + \int_D n|E|^2\,dx \right) + \frac{\varepsilon}{\lambda^2}\|m\|_{-1}^2 \\
&+ \varepsilon(1-\varepsilon\alpha)\|n\|_0^2 + \frac{\varepsilon^3}{\lambda^2}\|n\|_{-1}^2 + 2\gamma\|E_x\|_0^2 \\
&+ (2\gamma + \varepsilon)\int_D n|E|^2\,dx \\
&\leq \frac{1}{2}\frac{d}{dt}\left[\frac{1}{\lambda^2}\|m\|_{-1}^2 + (1-\varepsilon\alpha)\|n\|_0^2 + \frac{\varepsilon^2}{\lambda^2}\|n\|_{-1}^2 \right] \\
&+ \frac{d}{dt}\left(\|E_x\|_0^2 + \int_D n|E|^2\,dx \right) + \left(\alpha\lambda_1 - \frac{\varepsilon}{\lambda^2} \right)\|m\|_{-1}^2 \\
&+ \varepsilon(1-\varepsilon\alpha)\|n\|_0^2 + \frac{\varepsilon^3}{\lambda^2}\|n\|_{-1}^2 + 2\gamma\|E_x\|_0^2 \\
&+ (2\gamma + \varepsilon)\int_D n|E|^2\,dx \leq \int_D f(-\Delta)^{-1}m\,dx \\
&+ \int_D \dot{W}_1(-\Delta)^{-1}m\,dx + \frac{\lambda^2}{2}\|q_1\|_{-1}^2 + \|q_2\|_1^2 \\
&- 2\,\mathrm{Re}\int_D \left(\gamma\bar{E} + \bar{E}_t\right)g\,dx - 2\,\mathrm{Re}\int_D \left(\gamma\bar{E} + \bar{E}_t\right)\dot{W}_2\,dx \\
&+ \int_D n|q_2|^2\,\mathrm{d}x.
\end{aligned}
\tag{37}
$$

By Hölder inequality and Young inequality, we have the following estimates:

$$-2\,\mathrm{Re}\int_D\left(\gamma\bar{E}+\bar{E}_t\right)gdx \le 2\|E_x\|_0\|g_x\|_0 + 2\|g\|_{L^\infty}\|n\|_0\|E\|_0$$

$$-2\,\mathrm{Re}\int_D i\dot{W}_2\bar{g}dx \le \frac{\gamma}{2}\|E_x\|_0^2$$

$$+\frac{2}{\gamma}\|g_x\|_0^2 + C_1\|g_x\|_0^4$$

$$+\frac{\varepsilon(1-\varepsilon\alpha)}{6}\|n\|_0^2 + C_2\|E\|_0^4$$

$$+2\,\mathrm{Im}\int_D \dot{W}_2\bar{g}dx,$$

$$(\gamma+\varepsilon)\left|\int_D n|E|^2dx\right| \le C\|n\|_0\|E\|_{L^4}^2 \le C\|n\|_0\left(\|E\|_0^{3/2}\|E_x\|_0^{1/2}\right)$$

$$\le \frac{\varepsilon(1-\varepsilon\alpha)}{6}\|n\|_0^2 + \frac{\gamma}{2}\|E_x\|_0^2 + C_3\|E\|_0^6,$$

$$\int_D f(-\Delta)^{-1}mdx \le \|f\|_{-1}\|m\|_{-1}$$

$$\le \frac{\varepsilon}{2\lambda^2}\|m\|_{-1}^2 + \frac{\lambda^2}{2\varepsilon}\|f\|_{-1}^2,$$

$$\int_D n|q_2|^2dx \le \|n\|_0\|q_2\|_{L^4}^2 \le \|n\|_0\left(C\|q_2\|_0^{3/2}\|q_{2x}\|_0^{1/2}\right)$$

$$\le \frac{\varepsilon(1-\varepsilon\alpha)}{6}\|n\|_0^2 + C\left(\|q_{2x}\|_0^2 + \|q_2\|_0^6\right).$$

$$(38)$$

By (38), we obtain

$$\frac{d}{dt}\left[\frac{1}{2\lambda^2}\|m\|_{-1}^2 + \frac{1}{2}(1-\varepsilon\alpha)\|n\|_0^2 + \frac{\varepsilon^2}{2\lambda^2}\|n\|_{-1}^2\right]$$

$$+\frac{d}{dt}\left(\|E_x\|_0^2 + \int_D n|E|^2dx\right) + \frac{\varepsilon}{\lambda^2}\|m\|_{-1}^2$$

$$+\varepsilon(1-\varepsilon\alpha)\|n\|_0^2 + \frac{\varepsilon^3}{\lambda^2}\|n\|_{-1}^2 + 2\gamma\|E_x\|_0^2$$

$$+(2\gamma+\varepsilon)\int_D n|E|^2dx$$

$$\le \int_D f(-\Delta)^{-1}mdx + \int_D \dot{W}_1(-\Delta)^{-1}mdx$$

$$+\frac{\lambda^2}{2}\|q_1\|_{-1}^2 + \|q_2\|_1^2 - 2\,\mathrm{Re}\int_D\left(\gamma\bar{E}+\bar{E}_t\right)gdx \qquad (39)$$

$$-2\,\mathrm{Re}\int_D\left(\gamma\bar{E}+\bar{E}_t\right)\dot{W}_2dx + \int_D n|q_2|^2dx$$

$$\le \frac{\varepsilon}{2\lambda^2}\|m\|_{-1}^2 + \frac{\lambda^2}{2\varepsilon}\|f\|_{-1}^2 + \int_D \dot{W}_1(-\Delta)^{-1}mdx$$

$$+\frac{\lambda^2}{2}\|q_1\|_{-1}^2 + \|q_2\|_1^2 + \frac{\gamma}{2}\|E_x\|_0^2 + \frac{2}{\gamma}\|g_x\|_0^2$$

$$+C_1\|g_x\|_0^4 + \frac{\varepsilon(1-\varepsilon\alpha)}{6}\|n\|_0^2 + C_2\|E\|_0^4$$

$$+2\,\mathrm{Im}\int_D \dot{W}_2\bar{g}dx - 2\,\mathrm{Re}\int_D\left(\gamma\bar{E}+\bar{E}_t\right)\dot{W}_2dx$$

$$+\frac{\varepsilon(1-\varepsilon\alpha)}{6}\|n\|_0^2 + C\left(\|q_2\|_1^2 + \|q_2\|_0^6\right).$$

By (39), we get

$$\frac{d}{dt}\left[\frac{1}{2\lambda^2}\|m\|_{-1}^2 + \frac{1}{2}(1-\varepsilon\alpha)\|n\|_0^2 + \frac{\varepsilon^2}{2\lambda^2}\|n\|_{-1}^2\right]$$

$$+\frac{d}{dt}\left(\|E_x\|_0^2 + \int_D n|E|^2dx\right) + \frac{\varepsilon}{\lambda^2}\|m\|_{-1}^2$$

$$+\varepsilon(1-\varepsilon\alpha)\|n\|_0^2 + \frac{\varepsilon^3}{\lambda^2}\|n\|_{-1}^2 + 2\gamma\|E_x\|_0^2 + \gamma\int_D n|E|^2dx$$

$$\le \frac{\varepsilon}{2\lambda^2}\|m\|_{-1}^2 + \frac{\lambda^2}{2\varepsilon}\|f\|_{-1}^2 + \int_D \dot{W}_1(-\Delta)^{-1}mdx$$

$$+\frac{\lambda^2}{2}\|q_1\|_{-1}^2 + \|q_2\|_1^2 + \frac{\gamma}{2}\|E_x\|_0^2 + \frac{2}{\gamma}\|g_x\|_0^2$$

$$+\frac{\varepsilon(1-\varepsilon\alpha)}{6}\|n\|_0^2 + C_2\|E\|_0^4 + C_1\|g_x\|_0^4$$

$$+2\,\mathrm{Im}\int_D \dot{W}_2\bar{g}dx - 2\,\mathrm{Re}\int_D\left(\gamma\bar{E}+\bar{E}_t\right)\dot{W}_2dx$$

$$+\frac{\varepsilon(1-\varepsilon\alpha)}{6}\|n\|_0^2 + C\left(\|q_2\|_1^2 + \|q_2\|_0^6\right)$$

$$+\frac{\varepsilon(1-\varepsilon\alpha)}{6}\|n\|_0^2 + \frac{\gamma}{2}\|E_x\|_0^2 + C_3\|E\|_0^6.$$

$$(40)$$

Now, taking $\beta = \min\{\varepsilon, \gamma\}$ and letting $H_0(t) = (1/2\lambda^2)\|m\|_{-1}^2 + ((1-\varepsilon\alpha)/2)\|n\|_0^2 + (\varepsilon^2/2\lambda^2)\|n\|_{-1}^2 + \|E_x\|_0^2 + \int_D n|E|^2dx$, the above inequality is changed into

$$\frac{d}{dt}H_0(t) + \beta H_0(t) \le \frac{d}{dt}\left[\frac{1}{2\lambda^2}\|m\|_{-1}^2 + \frac{1}{2}(1-\varepsilon\alpha)\|n\|_0^2\right.$$

$$\left.+\frac{\varepsilon^2}{2\lambda^2}\|n\|_{-1}^2\right] + \frac{d}{dt}\left(\|E_x\|_0^2\right.$$

$$\left.+\int_D n|E|^2dx\right) + \frac{\varepsilon}{2\lambda^2}\|m\|_{-1}^2$$

$$+\frac{\varepsilon(1-\varepsilon\alpha)}{2}\|n\|_0^2 + \frac{\varepsilon^3}{\lambda^2}\|n\|_{-1}^2$$

$$+\gamma\|E_x\|_0^2 + \gamma\int_D n|E|^2dx$$

$$\le C\left(\|f\|_{-1}, \|g\|_1, \|q_1\|_{-1}, \|q_2\|_1\right)$$

$$+\int_D \dot{W}_1(-\Delta)^{-1}mdx + C\|E\|_0^6$$

$$+2\,\mathrm{Im}\int_D \dot{W}_2\bar{g}dx$$

$$-2\,\mathrm{Re}\int_D\left(\gamma\bar{E}+\bar{E}_t\right)\dot{W}_2dx$$

$$\le C(f, g, q_1, q_2) + \int_D \dot{W}_1(-\Delta)^{-1}mdx$$

$$+C\|E\|_0^6 + 2\,\mathrm{Im}\int_D \dot{W}_2\bar{g}dx$$

$$-2\,\mathrm{Re}\int_D\left(\gamma\bar{E}+\bar{E}_t\right)\dot{W}_2dx.$$

$$(41)$$

Integrating (41) from 0 to t, we obtain

$$H_0(t) - H_0(0) + \beta \int_0^t H_0(s)ds$$

$$\leq C \int_0^t \|E(s)\|_0^6 ds + C(f, g, q_1, q_2)t$$

$$+ \int_0^t \int_D \dot{W}_1(-\Delta)^{-1}mdxds + 2\int_0^t \mathrm{Im} \int_D \dot{W}_2\bar{g}dxds \qquad (42)$$

$$- 2\int_0^t \mathrm{Re} \int_D \left(\gamma\bar{E} + \bar{E}_t\right)\dot{W}_2 dxds.$$

Taking expectation on both sides of the above inequality, by (28), we get

$$\mathbb{E}H_0(t) - \mathbb{E}H_0(0) + \beta\mathbb{E}\int_0^t H_0(s)ds$$

$$\leq C\mathbb{E}\int_0^t \|E(s)\|_0^6 ds + C(f, g, q_1, q_2)t \qquad (43)$$

$$\leq C(E_0, f, g, q_1, q_2)t.$$

By (23) and Gronwall inequality, we have

$$\mathbb{E}H_0(t) \leq e^{-\beta t}\mathbb{E}H_0(0) + C(E_0, f, g, q_1, q_2) \qquad (44)$$

$$\leq C(E_0, f, g, q_1, q_2),$$

where $C(E_0, f, g, q_1, q_2)$ is independent of T.

Since

$$\left|\int_D n|E|^2 dx\right| \leq C\|n\|_0 \|E\|_{L^4}^2 \leq C\|n\|_0 \left(\|E\|_0^{3/2}\|E_x\|_0^{1/2}\right)$$

$$\leq \frac{1-\varepsilon\alpha}{4}\|n\|_0^2 + \frac{1}{2}\|E_x\|_0^2 + C\|E\|_0^6, \qquad (45)$$

we have

$$C\left(\|m\|_{-1}^2 + \|n\|_0^2 + \|n\|_{-1}^2 + \|E_x\|_0^2\right) - C\|E\|_0^6$$

$$\leq H_0(t) \leq C\left(\|m\|_{-1}^2 + \|n\|_0^2 + \|n\|_{-1}^2 + \|E_x\|_0^2\right) + C\|E\|_0^6. \qquad (46)$$

By (41) and (46), we have

$$\frac{d}{dt}H_0(t) \leq C(f, g, q_1, q_2) + \int_D \dot{W}_1(-\Delta)^{-1}mdx$$

$$+ C\|E\|_0^6 + 2\,\mathrm{Im}\int_D \dot{W}_2\bar{g}dx \qquad (47)$$

$$- 2\,\mathrm{Re}\int_D \left(\gamma\bar{E} + \bar{E}_t\right)\dot{W}_2 dx.$$

Therefore, by (28) and (44), we have

$$\mathbb{E}\left(\|m\|_{-1}^2 + \|n\|_0^2 + \|n\|_{-1}^2 + \|E_x\|_0^2\right)$$

$$\leq C\left(\mathbb{E}H_0(t) + C\mathbb{E}\|E\|_0^6\right)$$

$$\leq C\left[e^{-\beta t}\mathbb{E}H_0(0) + C(E_0, f, g, q_1, q_2)\right] \qquad (48)$$

$$+ C\left[e^{-3\gamma t/2}\mathbb{E}\|E_0\|_0^6 + C\left(\|g\|_0, \|q_2\|_0\right)\right]$$

$$\leq C(E_0, f, g, q_1, q_2),$$

where $C(E_0, f, g, q_1, q_2)$ is independent of T.

By (48), we obtain $(n_t, n, E) \in L^\infty(0, \infty; L^2(\Omega, \mathscr{E}_0))$.

Further, we give an estimate of $H_0^p(t)$ for any $p \geq 1$. First as

$$\|nE\|_{-1}^2 = \int_D (-\Delta)^{-1}(nE)n\bar{E}dx \leq \left\|(-\Delta)^{-1}nE\right\|_{L^\infty}\|n\|_0\|E\|_0$$

$$\leq C\left\|(-\Delta)^{-1}nE\right\|_1\|n\|_0\|E\|_0 \leq C\|nE\|_{-1}\|n\|_0\|E\|_0, \qquad (49)$$

by (47), we have

$$\|nE\|_{-1}^2 \leq C\|n\|_0^2\|E\|_0^2 \leq C\left(H_0(t) + \|E\|_0^6\right)\|E\|_0^2, \qquad (50)$$

$$\|m\|_{-1}^2 + \|n\|_0^2 + \|n\|_{-1}^2 + \|E_x\|_0^2 \leq C\left(H_0(t) + \|E\|_0^6\right). \qquad (51)$$

Applying Itô's formula to $H_0^p(t)$, by (41), (50), (51), and Young inequality, we obtain

$$\frac{d}{dt}H_0^p(t) = pH_0^{p-1}(t)\frac{d}{dt}H_0(t) + \frac{1}{2}p(p-1)CH_0^{p-2}(t)$$

$$\cdot \left(\|E\|_1^2\|q_2\|_1^2 + \|g\|_0^2\|q_2\|_0^2\right) + \frac{1}{2}p(p-1)CH_0^{p-2}(t)$$

$$\cdot \left(\|nE\|_{-1}^2\|q_2\|_1^2 + \|m\|_{-1}^2\|q_2\|_{-1}^2\right)$$

$$\leq pH_0^{p-1}(t)\left[-\beta H_0(t) + C(f, g, q_1, q_2) + C\|E\|_0^6\right]$$

$$+ pH_0^{p-1}(t)\left[\int_D \dot{W}_1(-\Delta)^{-1}mdx + 2\,\mathrm{Im}\int_D \dot{W}_2\bar{g}dx\right]$$

$$+ pH_0^{p-1}(t)\left[-2\,\mathrm{Re}\int_D \left(\gamma\bar{E} + \bar{E}_t\right)\dot{W}_2 dx\right]$$

$$+ \frac{1}{2}p(p-1)CH_0^{p-2}(t)\left(\|E\|_1^2\|q_2\|_1^2 + \|g\|_0^2\|q_2\|_0^2\right)$$

$$+ \frac{1}{2}p(p-1)CH_0^{p-2}(t)\left(\|nE\|_{-1}^2\|q_2\|_1^2\right)$$

$$+ \frac{1}{2}p(p-1)CH_0^{p-2}(t)\left(\|m\|_{-1}^2\|q_2\|_{-1}^2\right)$$

$$
\begin{aligned}
&\leq -\beta p H_0^p(t) + p H_0^{p-1}(t) \left[C(f, g, q_1, q_2) + C\|E\|_0^6 \right] \\
&\quad + p H_0^{p-1}(t) \left[\int_D \dot{W}_1 (-\Delta)^{-1} m dx + 2 \operatorname{Im} \int_D \dot{W}_2 \bar{g} dx \right] \\
&\quad + p H_0^{p-1}(t) \left[-2 \operatorname{Re} \int_D \left(\gamma \bar{E} + \bar{E}_t \right) \dot{W}_2 dx \right] \\
&\quad + \frac{1}{2} p(p-1) C H_0^{p-2}(t) \left[2 \left(H_0(t) + C\|E\|_0^6 \right) \right] \|q_2\|_1^2 \\
&\quad + \frac{1}{2} p(p-1) C H_0^{p-2}(t) \|g\|_0^2 \|q_2\|_0^2 \\
&\quad + \frac{1}{2} p(p-1) C H_0^{p-2}(t) \left[\left(H_0(t) + \|E\|_0^6 \right) C\|E\|_0^2 \right] \|q_2\|_1^2 \\
&\quad + \frac{1}{2} p(p-1) C H_0^{p-2}(t) \left[2\lambda^2 \left(H_0(t) + C\|E\|_0^6 \right) \right] \|q_2\|_{-1}^2 \\
&\leq -\frac{\beta p}{2} H_0^p(t) + C(E_0, f, g, q_1, q_2) + C \left(\|E\|_0^{6p} + \|E\|_0^{3p} \right) \\
&\quad + p H_0^{p-1}(t) \left[\int_D \dot{W}_1 (-\Delta)^{-1} m dx + 2 \operatorname{Im} \int_D \dot{W}_2 \bar{g} dx \right] \\
&\quad + p H_0^{p-1}(t) \left[-2 \operatorname{Re} \int_D \left(\gamma \bar{E} + \bar{E}_t \right) \dot{W}_2 dx \right].
\end{aligned}
\tag{52}
$$

Integrating from 0 to t and taking expectation on both sides of the above inequality, by Hölder inequality and (28), we derive

$$
\begin{aligned}
\mathbb{E} H_0^p(t) &\leq \mathbb{E} H_0^p(0) - \frac{\beta p}{2} \mathbb{E} \int_0^t H_0^p(s) ds \\
&\quad + C(E_0, f, g, q_1, q_2) t.
\end{aligned}
\tag{53}
$$

By Gronwall inequality, we have

$$
\begin{aligned}
\mathbb{E} H_0^p(t) &\leq e^{-\beta p t/2} \mathbb{E} H_0^p(0) + C(E_0, f, g, q_1, q_2) \\
&\leq C(E_0, f, g, q_1, q_2, n_0, n_1),
\end{aligned}
\tag{54}
$$

where $C(E_0, f, g, q_1, q_2, n_0, n_1)$ is independent of T.

By (28), (51), and (54), we obtain $(n_t, n, E) \in L^\infty(0, \infty; L^{2p}(\Omega, \mathscr{E}_0))$.

On the other hand, integrating from 0 to t on both sides of (47), deduce

$$
\begin{aligned}
H_0(t) &\leq H_0(0) + C(f, g, q_1, q_2) t \\
&\quad + \left| \int_0^t \int_D \dot{W}_1 (-\Delta)^{-1} m dx ds \right|^2 \\
&\quad + C \int_0^t \|E\|_0^6 ds + \left| \int_0^t \operatorname{Im} \int_D \dot{W}_2 g dx ds \right|^2 \\
&\quad + \left| \int_0^t \operatorname{Re} \int_D \left(\gamma \bar{E} + \bar{E}_t \right) \dot{W}_2 dx ds \right|^2 + C.
\end{aligned}
\tag{55}
$$

Now, taking the supremum and expectation on both sides of the above inequality, we have

$$
\begin{aligned}
\mathbb{E} \sup_{0 \leq t \leq T} H_0(t) &\leq \mathbb{E} H_0(0) + C(f, g, q_1, q_2) T + C \\
&\quad + \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \int_D \dot{W}_1 (-\Delta)^{-1} m dx ds \right|^2 \\
&\quad + C \mathbb{E} \sup_{0 \leq t \leq T} \int_0^t \|E\|_0^6 ds \\
&\quad + \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \operatorname{Im} \int_D \dot{W}_2 g dx ds \right|^2 \\
&\quad + \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \operatorname{Re} \int_D \left(\gamma \bar{E} + \bar{E}_t \right) \dot{W}_2 dx ds \right|^2 \\
&\leq \mathbb{E} H_0(0) + C(f, g, q_1, q_2) T \\
&\quad + C\|q_1\|_{-1}^2 \mathbb{E} \int_0^T \|m\|_{-1}^2 dt + C \mathbb{E} \int_0^T \|E\|_0^6 ds \\
&\quad + C\|q_2\|_0^2 \mathbb{E} \int_0^T \|g\|_0^2 dt + C \\
&\quad + \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \operatorname{Re} \int_D \left(-i\bar{E}_{xx} + in\bar{E} \right. \right. \\
&\quad \left. \left. + i\bar{g} + i\bar{W}_2 \right) \dot{W}_2 dx ds \right|^2 \\
&\leq \mathbb{E} H_0(0) + C(f, g, q_1, q_2) T \\
&\quad + C\|q_1\|_{-1}^2 \mathbb{E} \int_0^T \|m\|_{-1}^2 dt + C \mathbb{E} \int_0^T \|E\|_0^6 ds \\
&\quad + C\|q_2\|_0^2 \mathbb{E} \int_0^T \|g\|_0^2 dt + C \\
&\quad + C\|q_2\|_1^2 \mathbb{E} \int_0^T \|E_x\|_0^2 dt + C\|q_2\|_1^2 \mathbb{E} \int_0^T \|nE\|_{-1}^2 dt \\
&\leq \mathbb{E} H_0(0) + C(f, g, q_1, q_2) T + C \\
&\quad + C \mathbb{E} \int_0^T \|E\|_0^6 ds + C\|q_2\|_0^2 \mathbb{E} \int_0^T \|g\|_0^2 dt \\
&\quad + C\|q_1\|_{-1}^2 \mathbb{E} \int_0^T 2\lambda^2 \left[H_0(t) + C\|E\|_0^6 \right] dt \\
&\quad + C\|q_2\|_1^2 \mathbb{E} \int_0^T 2 \left[H_0(t) + C\|E\|_0^6 \right] dt \\
&\quad + C\|q_2\|_1^2 \mathbb{E} \int_0^T C \left[H_0(t) + \|E\|_0^6 \right] \|E\|_0^2 dt.
\end{aligned}
\tag{56}
$$

Then, by (28), (54), and (56), we obtain

$$
\mathbb{E} \sup_{0 \leq t \leq T} H_0(t) \leq C_T(E_0, f, g, q_1, q_2, n_1, n_0),
\tag{57}
$$

where C_T depends on the initial data.

Therefore, by (48), we have

$$
\mathbb{E} \sup_{0 \leq t \leq T} \|(n_t, n, E)\|_{\mathscr{E}_0}^2 \leq C_T(E_0, f, g, q_1, q_2, n_1, n_0).
\tag{58}
$$

By (58), we obtain $E \in L^2(\Omega; L^\infty(0, T; \mathscr{E}_0))$.

In addition, from (52), we have

$$\frac{d}{dt}H_0^p(t) \le C(E_0, f, g, q_1, q_2) + C\left(\|E\|_0^{6p} + \|E\|_0^{3p}\right)$$
$$+ pH_0^{p-1}(t)\left[\int_D \dot{W}_1(-\Delta)^{-1}mdx\right.$$
$$\left. + 2\,\mathrm{Im}\int_D \dot{W}_2\bar{g}dx\right] + pH_0^{p-1}(t)$$
$$\cdot \left[-2\,\mathrm{Re}\int_D (\gamma\bar{E} + \bar{E}_t)\dot{W}_2 dx\right]. \tag{59}$$

Integrating from 0 to t and taking the supremum and the expectation on both sides of (59), we have

$$\mathbb{E}\sup_{0\le t\le T}H_0^p(t) \le C(E_0, f, g, q_1, q_2)T$$
$$+ C\mathbb{E}\sup_{0\le t\le T}\int_0^t \left(\|E\|_0^{6p} + \|E\|_0^{3p}\right)ds$$
$$+ pH_0^{p-1}(t)\int_D \dot{W}_1(-\Delta)^{-1}mdx$$
$$+ 2pH_0^{p-1}(t)\,\mathrm{Im}\int_D \dot{W}_2\bar{g}dx$$
$$+ pH_0^{p-1}(t)\left[-2\,\mathrm{Re}\int_D \left(-i\bar{E}_{xx}\right.\right.$$
$$\left.\left. + in\bar{E} + i\bar{g} + i\overline{\dot{W}_2}\right)\dot{W}_2 dx\right]$$
$$\le C(E_0, f, g, q_1, q_2)T$$
$$+ \mathbb{E}\sup_{0\le t\le T}\int_0^t \left(\|E\|_0^{6p} + \|E\|_0^{3p}\right)ds$$
$$+ \mathbb{E}\sup_{0\le t\le T}\left|\int_0^t H_0^{p-1}(t)\int_D \dot{W}_1(-\Delta)^{-1}mdxds\right|^2$$
$$+ \mathbb{E}\sup_{0\le t\le T}\left|\int_0^t H_0^{p-1}(t)\,\mathrm{Im}\int_D \dot{W}_2\bar{g}dxds\right|^2$$
$$+ C + \mathbb{E}\sup_{0\le t\le T}\left|\int_0^t H_0^{p-1}(t)\,\mathrm{Im}\int_D \bar{E}_{xx}\dot{W}_2 dxds\right|^2$$
$$+ \mathbb{E}\sup_{0\le t\le T}\left|\int_0^t H_0^{p-1}(t)\,\mathrm{Im}\int_D n\bar{E}\dot{W}_2 dxds\right|^2. \tag{60}$$

By (50), (51), Hölder inequality, and Young inequality, we have the following estimates:

$$C\mathbb{E}\sup_{0\le t\le T}\int_0^t \left(\|E\|_0^{6p} + \|E\|_0^{3p}\right)ds \le C\mathbb{E}\int_0^T \left(\|E\|_0^{6p} + \|E\|_0^{3p}\right)ds,$$

$$\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t H_0^{p-1}(t)\int_D \dot{W}_1(-\Delta)^{-1}mdxds\right|^2$$
$$\le C\|q_1\|_{-1}^2 \mathbb{E}\int_0^T H_0^{2p-2}(t)\|m\|_{-1}^2 ds$$

$$\le C\|q_1\|_{-1}^2 \mathbb{E}\int_0^T H_0^{2p-2}(t)\left[C\left(H_0(t) + \|E\|_0^6\right)\right]ds$$
$$\le C\|q_1\|_{-1}^2 \mathbb{E}\int_0^T \left[H_0^{2p}(t) + \|E\|_0^{6p} + 1\right]ds,$$

$$\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t H_0^{p-1}(t)\,\mathrm{Im}\int_D \dot{W}_2\bar{g}dxds\right|^2$$
$$\le C\|q_2\|_0^2 \mathbb{E}\int_0^T H_0^{2p-2}(t)\|g\|_0^2 ds$$
$$\le C\|q_2\|_0^2 \mathbb{E}\int_0^T \left[H_0^{2p}(t) + \|g\|_0^{2p}\right]ds$$
$$\le C\|q_2\|_0^2 \mathbb{E}\int_0^T \left[H_0^{2p}(t) + 1\right]ds,$$

$$\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t H_0^{p-1}(t)\,\mathrm{Im}\int_D \bar{E}_{xx}\dot{W}_2 dxds\right|^2$$
$$\le C\|q_{2x}\|_0^2 \mathbb{E}\int_0^T H_0^{2p-2}(t)\|E_x\|_0^2 ds$$
$$\le C\|q_{2x}\|_0^2 \mathbb{E}\int_0^T H_0^{2p-2}(t)\left[C\left(H_0(t) + \|E\|_0^6\right)\right]ds$$
$$\le C\|q_{2x}\|_0^2 \mathbb{E}\int_0^T \left[H_0^{2p}(t) + \|E\|_0^{6p} + 1\right]ds,$$

$$\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t H_0^{p-1}(t)\,\mathrm{Im}\int_D n\bar{E}\dot{W}_2 dxds\right|^2$$
$$\le C\|q_2\|_1^2 \mathbb{E}\int_0^T H_0^{2p-2}(t)\|nE\|_{-1}^2 ds$$
$$\le C\|q_2\|_1^2 \mathbb{E}\int_0^T H_0^{2p-2}(t)\left[C\left(H_0(t) + \|E\|_0^6\right)\|E\|_0^2\right]ds$$
$$\le C\|q_2\|_1^2 \mathbb{E}\int_0^T \left[H_0^{2p}(t) + \|E\|_0^{4p} + \|E\|_0^{8p}\right]ds. \tag{61}$$

By (28), (54), (60), and (61), we have

$$\mathbb{E}\sup_{0\le t\le T}H_0^p(t) \le C\mathbb{E}H_0^p(0) + C_T(E_0, f, g, q_1, q_2, n_1, n_0)$$
$$\le C_T(E_0, f, g, q_1, q_2, n_1, n_0). \tag{62}$$

Moreover, by (29), (51), and (62), we have

$$C\mathbb{E}\sup_{0\le t\le T}\left(\|m\|_{-1}^{2p} + \|n\|_0^{2p} + \|n\|_{-1}^{2p} + \|E_x\|_0^{2p}\right)$$
$$\le C\mathbb{E}\sup_{0\le t\le T}\left[C\left(H_0(t) + \|E\|_0^6\right)\right]^{2p}$$
$$\le C\mathbb{E}\sup_{0\le t\le T}\left[H_0^{2p}(t) + \|E\|_0^{12p}\right]$$
$$\le C_T(E_0, f, g, q_1, q_2, n_1, n_0), \tag{63}$$

where C_T depends on the initial data.

Therefore, by (63), we have

$$\mathbb{E} \sup_{0 \le t \le T} \|(n_t, n, E)\|_{\mathscr{E}_0}^{2p} \le C_T(E_0, f, g, q_1, q_2, n_1, n_0). \quad (64)$$

By (64), we obtain $E \in L^{2p}(\Omega; L^\infty(0, T; \mathscr{E}_0))$.
Lemma 4 is proved completely.

Lemma 5. *Assume* $(n_1, n_0, E_0) \in \mathscr{E}_1$, $f \in L^2(D)$, $q_1 \in H_0^1(D)$, *and* $g, q_2 \in H^2(D) \cap H_0^1(D)$. *Then, for any* $T > 0$ *and* $p \ge 1$, $(n_t, n, E) \in L^{2p}(\Omega; L^\infty(0, T; \mathscr{E}_1)) \cap L^\infty(0, \infty; L^{2p}(\Omega, \mathscr{E}_1))$.

Proof. Applying Itô's formula to $\|m\|_0^2$, by (12), we have

$$\frac{1}{2\lambda^2} \frac{d}{dt} \|m\|_0^2 = \int_D \frac{1}{\lambda^2} m_t m dx + \frac{\lambda^2}{2} \|q_1\|_0^2$$

$$= \frac{\varepsilon}{\lambda^2} \|m\|_0^2 - \alpha \|m\|_1^2 - \frac{\varepsilon^2}{\lambda^2} \int_D n m dx$$

$$+ (1 - \varepsilon\alpha) \int_D n_{xx} m dx + \int_D |E|_{xx}^2 m dx \quad (65)$$

$$+ \int_D f m dx + \int_D \dot{W}_1 m dx + \frac{\lambda^2}{2} \|q_1\|_0^2.$$

Since $m = n_t + \varepsilon n$, from (65), we obtain

$$\frac{d}{dt} \left(\frac{1}{2\lambda^2} \|m\|_0^2 + \frac{\varepsilon^2}{2\lambda^2} \|n\|_0^2 + \frac{1 - \varepsilon\alpha}{2} \|n_x\|_0^2 \right)$$

$$= \frac{\varepsilon}{\lambda^2} \|m\|_0^2 - \alpha \|m\|_1^2 - \frac{\varepsilon^3}{\lambda^2} \|n\|_0^2 - \varepsilon(1 - \varepsilon\alpha) \|n_x\|_0^2$$

$$+ \int_D |E|_{xx}^2 m dx + \int_D f m dx + \int_D \dot{W}_1 m dx + \frac{\lambda^2}{2} \|q_1\|_0^2$$

$$\le \left(\frac{\varepsilon}{\lambda^2} - \alpha\lambda_1 \right) \|m\|_0^2 - \frac{\varepsilon^3}{\lambda^2} \|n\|_0^2 - \varepsilon(1 - \varepsilon\alpha) \|n_x\|_0^2$$

$$+ \int_D |E|_{xx}^2 m dx + \int_D f m dx + \int_D \dot{W}_1 m dx + \frac{\lambda^2}{2} \|q_1\|_0^2, \quad (66)$$

where λ_1 is the first eigenvalue of $-\Delta$. By (66), we obtain

$$\frac{d}{dt} \left(\frac{1}{2\lambda^2} \|m\|_0^2 + \frac{\varepsilon^2}{2\lambda^2} \|n\|_0^2 + \frac{1 - \varepsilon\alpha}{2} \|n_x\|_0^2 \right)$$

$$- \left(\frac{\varepsilon}{\lambda^2} - \alpha\lambda_1 \right) \|m\|_0^2 + \frac{\varepsilon^3}{\lambda^2} \|n\|_0^2 + \varepsilon(1 - \varepsilon\alpha) \|n_x\|_0^2$$

$$\le \int_D |E|_{xx}^2 m dx + \int_D f m dx + \int_D \dot{W}_1 m dx + \frac{\lambda^2}{2} \|q_1\|_0^2. \quad (67)$$

Applying Itô's formula to $\|E_{xx}\|_0^2$, by (13), we have

$$\frac{d}{dt} \|E_{xx}\|_0^2 = 2 \operatorname{Re} \int_D E_{xx} \bar{E}_{txx} dx + \|q_2\|_2^2. \quad (68)$$

Substituting $E_{xx} = -iE_t + nE - i\gamma E + g + \dot{W}_2$ into (68), we have

$$\frac{d}{dt} \|E_{xx}\|_0^2 = 2 \operatorname{Re} \int_D \left(-iE_t + nE - i\gamma E + g + \dot{W}_2 \right) \bar{E}_{txx} dx$$

$$+ \|q_2\|_2^2 = 2 \operatorname{Re} \int_D nE \bar{E}_{txx} dx$$

$$+ 2 \operatorname{Re} \int_D g \bar{E}_{txx} dx + 2 \operatorname{Re} \int_D \dot{W}_2 \bar{E}_{txx} dx$$

$$+ \|q_2\|_2^2 + 2 \operatorname{Re} \int_D \left(\gamma \bar{E}_{xx} \right)$$

$$\cdot \left(-E_{xx} + nE - i\gamma E + g + \dot{W}_2 \right) dx$$

$$= 2 \operatorname{Re} \int_D nE \left(\bar{E}_{txx} + \gamma \bar{E}_{xx} \right) dx$$

$$+ 2 \operatorname{Re} \int_D g \left(\bar{E}_{txx} + \gamma \bar{E}_{xx} \right) dx$$

$$+ 2 \operatorname{Re} \int_D \dot{W}_2 \left(\bar{E}_{txx} + \gamma \bar{E}_{xx} \right) dx$$

$$- 2\gamma \|E_{xx}\|_0^2 + \|q_2\|_2^2. \quad (69)$$

From (66) and (69), we obtain

$$\frac{d}{dt} \left(\frac{1}{2\lambda^2} \|m\|_0^2 + \frac{\varepsilon^2}{2\lambda^2} \|n\|_0^2 + \frac{1 - \varepsilon\alpha}{2} \|n_x\|_0^2 + \|E_{xx}\|_0^2 \right)$$

$$- \left(\frac{\varepsilon}{\lambda^2} - \alpha\lambda_1 \right) \|m\|_0^2 + \frac{\varepsilon^3}{\lambda^2} \|n\|_0^2 + \varepsilon(1 - \varepsilon\alpha) \|n_x\|_0^2$$

$$\le \int_D |E|_{xx}^2 m dx + \int_D f m dx + \int_D \dot{W}_1 m dx + \frac{\lambda^2}{2} \|q_1\|_0^2$$

$$+ 2 \operatorname{Re} \int_D nE \left(\bar{E}_{txx} + \gamma \bar{E}_{xx} \right) dx$$

$$+ 2 \operatorname{Re} \int_D g \left(\bar{E}_{txx} + \gamma \bar{E}_{xx} \right) dx$$

$$+ 2 \operatorname{Re} \int_D \dot{W}_2 \left(\bar{E}_{txx} + \gamma \bar{E}_{xx} \right) dx - 2\gamma \|E_{xx}\|_0^2 + \|q_2\|_2^2, \quad (70)$$

since

$$\frac{d}{dt} \left(-2 \operatorname{Re} \int_D nE \bar{E}_{xx} dx \right)$$

$$= - \operatorname{Re} \int_D n_t |E|_{xx}^2 dx + 2 \operatorname{Re} \int_D n_t |E_x|^2 dx$$

$$- 2 \operatorname{Re} \int_D nE \bar{E}_{txx} dx - 2 \operatorname{Re} \int_D nE_t \bar{E}_{xx} dx$$

$$- 2 \operatorname{Re} \int_D n q_2 \bar{q}_{2xx} dx. \quad (71)$$

Substituting (71) into (70), we have

$$\frac{d}{dt}\left(\frac{1}{2\lambda^2}\|m\|_0^2 + \frac{\varepsilon^2}{2\lambda^2}\|n\|_0^2 + \frac{1-\varepsilon\alpha}{2}\|n_x\|_0^2 + \|E_{xx}\|_0^2\right)$$

$$- \left(\frac{\varepsilon}{\lambda^2} - \alpha\lambda_1\right)\|m\|_0^2 + \frac{\varepsilon^3}{\lambda^2}\|n\|_0^2 + \varepsilon(1-\varepsilon\alpha)\|n_x\|_0^2$$

$$\leq \int_D |E|_{xx}^2 m dx + \int_D f m dx + \int_D \dot{W}_1 m dx$$

$$+ \frac{\lambda^2}{2}\|q_1\|_0^2 + 2\,\mathrm{Re}\int_D nE\bar{E}_{txx}dx$$

$$+ 2\gamma\,\mathrm{Re}\int_D nE\bar{E}_{xx}dx + 2\,\mathrm{Re}\int_D g\left(\bar{E}_{txx} + \gamma\bar{E}_{xx}\right)dx$$

$$+ 2\,\mathrm{Re}\int_D \dot{W}_2\left(\bar{E}_{txx} + \gamma\bar{E}_{xx}\right)dx - 2\gamma\|E_{xx}\|_0^2 + \|q_2\|_2^2$$

$$\leq \int_D |E|_{xx}^2 m dx + \int_D f m dx + \int_D \dot{W}_1 m dx$$

$$+ \frac{\lambda^2}{2}\|q_1\|_0^2 - \mathrm{Re}\int_D n_t|E|_{xx}^2 dx + 2\,\mathrm{Re}\int_D n_t|E_x|^2 dx$$

$$+ \frac{d}{dt}\left(2\,\mathrm{Re}\int_D nE\bar{E}_{xx}dx\right) - 2\,\mathrm{Re}\int_D nE_t\bar{E}_{xx}dx$$

$$- 2\,\mathrm{Re}\int_D nq_2\bar{q}_{2xx}dx + 2\gamma\,\mathrm{Re}\int_D nE\bar{E}_{xx}dx$$

$$+ 2\,\mathrm{Re}\int_D g\left(\bar{E}_{txx} + \gamma\bar{E}_{xx}\right)dx$$

$$+ 2\,\mathrm{Re}\int_D \dot{W}_x\left(\bar{E}_{txx} + \gamma\bar{E}_{xx}\right)dx - 2\gamma\|E_{xx}\|_0^2 + \|q_2\|_2^2.$$

$$(72)$$

Taking $0 < \varepsilon \leq \min\{1/\alpha, \alpha\lambda_1\lambda^2/2\}$, and by (72), we have

$$\frac{d}{dt}\left(\frac{1}{2\lambda^2}\|m\|_0^2 + \frac{\varepsilon^2}{2\lambda^2}\|n\|_0^2 + \frac{1-\varepsilon\alpha}{2}\|n_x\|_0^2\right)$$

$$+ \frac{d}{dt}\left(\|E_{xx}\|_0^2 - 2\,\mathrm{Re}\int_D nE\bar{E}_{xx}dx\right)$$

$$+ \frac{\varepsilon}{\lambda^2}\|m\|_0^2 + \frac{\varepsilon^3}{\lambda^2}\|n\|_0^2 + \varepsilon(1-\varepsilon\alpha)\|n_x\|_0^2$$

$$+ 2\gamma\|E_{xx}\|_0^2 - 2\gamma\,\mathrm{Re}\int_D nE\bar{E}_{xx}dx$$

$$\leq \int_D |E|_{xx}^2 m dx + \int_D f m dx + \int_D \dot{W}_1 m dx$$

$$+ \frac{\lambda^2}{2}\|q_1\|_0^2 - \mathrm{Re}\int_D n_t|E|_{xx}^2 dx$$

$$+ 2\,\mathrm{Re}\int_D n_t|E_x|^2 dx + \|q_2\|_2^2 - 2\,\mathrm{Re}\int_D nE_t\bar{E}_{xx}dx$$

$$- 2\,\mathrm{Re}\int_D nq_2\bar{q}_{2xx}dx + 2\,\mathrm{Re}\int_D g\left(\bar{E}_{txx} + \gamma\bar{E}_{xx}\right)dx$$

$$+ 2\,\mathrm{Re}\int_D \dot{W}_2\left(\bar{E}_{txx} + \gamma\bar{E}_{xx}\right)dx$$

$$\leq \int_D f m dx + \int_D \dot{W}_1 m dx + \frac{\lambda^2}{2}\|q_1\|_0^2$$

$$- \varepsilon\,\mathrm{Re}\int_D n_x|E|_x^2 dx + 2\,\mathrm{Re}\int_D m|E_x|^2 dx$$

$$- 2\varepsilon\,\mathrm{Re}\int_D n|E_x|^2 dx + \|q_2\|_2^2$$

$$- 2\,\mathrm{Re}\int_D nE_t\bar{E}_{xx}dx - 2\,\mathrm{Re}\int_D nq_2\bar{q}_{2xx}dx$$

$$+ 2\,\mathrm{Re}\int_D g\left(\bar{E}_{txx} + \gamma\bar{E}_{xx}\right)dx$$

$$+ 2\,\mathrm{Re}\int_D \dot{W}_2\left(\bar{E}_{txx} + \gamma\bar{E}_{xx}\right)dx.$$

$$(73)$$

By Hölder inequality and Young inequality, we have the following estimates:

$$\int_D f m dx \leq \|f\|_0\|m\|_0 \leq C\|f\|_0^2 + \frac{\varepsilon}{4\lambda^2}\|m\|_0^2,$$

$$-\varepsilon\,\mathrm{Re}\int_D n_x|E|_x^2 dx + 2\,\mathrm{Re}\int_D m|E_x|^2 dx - 2\varepsilon\,\mathrm{Re}\int_D n|E_x|^2 dx$$

$$\leq \varepsilon\|n_x\|_0\|E_x\|_{L^4}^2 + 2\|m\|_0\|E_x\|_{L^4}^2 + 2\varepsilon\|n\|_0\|E_x\|_{L^4}^2$$

$$\leq \left(\varepsilon\|n_x\|_0 + 2\|m\|_0 + 2\varepsilon\|n\|_0\right)\|E_x\|_0^{3/2}\|E_{xx}\|_0^{1/2}$$

$$\leq \frac{\varepsilon(1-\varepsilon\alpha)}{6}\|n_x\|_0^2 + \frac{\varepsilon}{4\lambda^2}\|m\|_0^2 + \frac{\varepsilon^3}{3\lambda^2}\|n\|_0^2$$

$$+ \frac{\gamma}{5}\|E_{xx}\|_0^2 + C\|E_x\|_0^6,$$

$$-2\,\mathrm{Re}\int_D nE_t\bar{E}_{xx}dx = 2\,\mathrm{Re}\int_D \left(in\bar{E}_{xx}\right)\left(iE_t\right)dx$$

$$= 2\,\mathrm{Re}\int_D in^2\bar{E}_{xx}Edx + 2\gamma\,\mathrm{Re}\int_D n\bar{E}_{xx}Edx$$

$$+ 2\,\mathrm{Re}\int_D in\bar{E}_{xx}gdx + 2\,\mathrm{Re}\int_D in\bar{E}_{xx}\dot{W}_2dx$$

$$\leq \|n\|_{L^4}^2\|E_{xx}\|_0\|E\|_{L^\infty} + \|n\|_0\|E_{xx}\|_0\|E\|_{L^\infty}$$

$$+ \|n\|_0\|E_{xx}\|_0\|g\|_{L^\infty} - 2\,\mathrm{Im}\int_D n\bar{E}_{xx}\dot{W}_2dx$$

$$\leq \|n\|_0^{3/2}\|n_x\|_0^{1/2}\|E_{xx}\|_0\|E_x\|_0 + \|n\|_0\|E_{xx}\|_0\|E_x\|_0$$

$$+ \|n\|_0\|E_{xx}\|_0\|g_x\|_0 - 2\,\mathrm{Im}\int_D n\bar{E}_{xx}\dot{W}_2dx$$

$$\leq C\|n\|_0^{12} + \frac{\varepsilon(1-\varepsilon\alpha)}{6}\|n_x\|_0^2 + \frac{\gamma}{5}\|E_{xx}\|_0^2$$

$$+ C\|E_x\|_0^8 + C\|n\|_0^4 + \frac{2\gamma}{5}\|E_{xx}\|_0^2 + C\|E_x\|_0^4$$

$$+ C\|g_x\|_0^4 - 2\,\mathrm{Im}\int_D n\bar{E}_{xx}\dot{W}_2dx$$

$$\leq \frac{\varepsilon(1-\varepsilon\alpha)}{6}\|n_x\|_0^2 + \frac{3\gamma}{5}\|E_{xx}\|_0^2 + C\left(\|n\|_0^{12} + \|E_x\|_0^8\right)$$

$$+ C\left(\|n\|_0^4 + \|E_x\|_0^4 + \|g_x\|_0^4\right) - 2\,\mathrm{Im}\int_D n\bar{E}_{xx}\dot{W}_2dx,$$

$$-2 \operatorname{Re} \int_D n q_2 \bar{q}_{2xx} dx \leq \frac{\varepsilon^3}{3\lambda^2} \|n\|_0^2 + C\|q_{2xx}\|_0^4,$$

$$
\begin{aligned}
&2 \operatorname{Re} \int_D \left(\gamma \bar{E}_{xx} + \bar{E}_{txx}\right) g \, dx \\
&= 2 \operatorname{Re} \int_D \left(-i\bar{E}_{xx} + in\bar{E} + i\bar{g} + i\bar{W}_2\right)_{xx} g \, dx \\
&\leq 2\|E_{xx}\|_0 \|g_{xx}\|_0 + 2\|g_x\|_{L^\infty} \left(\|n_x\|_0 \|E\|_0 + \|n\|_0 \|E_x\|_0\right) \\
&\quad - 2 \operatorname{Re} \int_D i\dot{W}_2 \bar{g}_{xx} dx \leq \frac{\gamma}{5} \|E_{xx}\|_0^2 + C\|g_{xx}\|_0^2 + C\|g_{xx}\|_0^4 \\
&\quad + \frac{\varepsilon(1-\varepsilon\alpha)}{6} \|n_x\|_0^2 + C\|E\|_0^4 + C\|g_{xx}\|_0^4 \\
&\quad + \frac{\varepsilon^3}{4\lambda^2} \|n\|_0^2 + C\|E_x\|_0^4 + 2 \operatorname{Im} \int_D \dot{W}_2 \bar{g}_{xx} dx \\
&\leq \frac{\gamma}{5} \|E_{xx}\|_0^2 + \frac{\varepsilon(1-\varepsilon\alpha)}{6} \|n_x\|_0^2 + \frac{\varepsilon^3}{3\lambda^2} \|n\|_0^2 \\
&\quad + 2 \operatorname{Im} \int_D \dot{W}_2 \bar{g}_{xx} dx \\
&\quad + C\left(\|g_{xx}\|_0^2 + \|g_{xx}\|_0^4 + \|E\|_0^4 + \|E_x\|_0^4\right).
\end{aligned}
\tag{74}
$$

By (73), we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2\lambda^2}\|m\|_0^2 + \frac{\varepsilon^2}{2\lambda^2}\|n\|_0^2 + \frac{1-\varepsilon\alpha}{2}\|n_x\|_0^2\right) \\
&\quad + \frac{d}{dt}\left(\|E_{xx}\|_0^2 - 2\operatorname{Re}\int_D nE\bar{E}_{xx}dx\right) \\
&\quad + \frac{\varepsilon}{\lambda^2}\|m\|_0^2 + \frac{\varepsilon^3}{\lambda^2}\|n\|_0^2 + \varepsilon(1-\varepsilon\alpha)\|n_x\|_0^2 \\
&\quad + 2\gamma\|E_{xx}\|_0^2 - 2\gamma\operatorname{Re}\int_D nE\bar{E}_{xx}dx \\
&\leq C\|f\|_0^2 + \frac{\varepsilon}{4\lambda^2}\|m\|_0^2 + \int_D \dot{W}_1 m\,dx + \frac{\lambda^2}{2}\|q_1\|_0^2 \\
&\quad + \frac{\varepsilon(1-\varepsilon\alpha)}{6}\|n_x\|_0^2 + \frac{\varepsilon}{4\lambda^2}\|m\|_0^2 + \frac{\varepsilon^3}{6\lambda^2}\|n\|_0^2 \\
&\quad + \frac{\gamma}{5}\|E_{xx}\|_0^2 + C\|E_x\|_0^6 + \|q_2\|_2^2 + \frac{\varepsilon(1-\varepsilon\alpha)}{6}\|n_x\|_0^2 \\
&\quad + \frac{3\gamma}{5}\|E_{xx}\|_0^2 + C\left(\|n\|_0^{12} + \|E_x\|_0^8\right) \\
&\quad + C\left(\|n\|_0^4 + \|E_x\|_0^4 + \|g_x\|_0^4\right) - 2\operatorname{Im}\int_D n\bar{E}_{xx}\dot{W}_2 dx \\
&\quad + \frac{\varepsilon^3}{6\lambda^2}\|n\|_0^2 + C\|q_{2xx}\|_0^4 + \frac{\gamma}{5}\|E_{xx}\|_0^2 \\
&\quad + \frac{\varepsilon(1-\varepsilon\alpha)}{6}\|n_x\|_0^2 + \frac{\varepsilon^3}{6\lambda^2}\|n\|_0^2 + 2\operatorname{Im}\int_D \dot{W}_2\bar{g}_{xx}dx \\
&\quad + C\left(\|g_{xx}\|_0^2 + \|g_{xx}\|_0^4 + \|E\|_0^4 + \|E_x\|_0^4\right) \\
&\quad + 2\operatorname{Re}\int_D \dot{W}_2\left(\bar{E}_{txx} + \gamma\bar{E}_{xx}\right)dx.
\end{aligned}
\tag{75}
$$

By (75), we get

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2\lambda^2}\|m\|_0^2 + \frac{\varepsilon^2}{2\lambda^2}\|n\|_0^2 + \frac{1-\varepsilon\alpha}{2}\|n_x\|_0^2\right) \\
&\quad + \frac{d}{dt}\left(\|E_{xx}\|_0^2 - 2\operatorname{Re}\int_D nE\bar{E}_{xx}dx\right) \\
&\quad + \frac{\varepsilon}{2\lambda^2}\|m\|_0^2 + \frac{\varepsilon^3}{2\lambda^2}\|n\|_0^2 + \frac{\varepsilon(1-\varepsilon\alpha)}{2}\|n_x\|_0^2 \\
&\quad + \gamma\|E_{xx}\|_0^2 - 2\gamma\operatorname{Re}\int_D nE\bar{E}_{xx}dx \\
&\leq C\|f\|_0^2 + \int_D \dot{W}_1 m\,dx + \frac{\lambda^2}{2}\|q_1\|_0^2 + C\|E_x\|_0^6 \\
&\quad + \|q_2\|_2^2 + C\left(\|n\|_0^{12} + \|E_x\|_0^8\right) + C\left(\|n\|_0^4 + \|E_x\|_0^4 + \|g_x\|_0^4\right) \\
&\quad - 2\operatorname{Im}\int_D n\bar{E}_{xx}\dot{W}_2 dx + C\|q_{2xx}\|_0^4 + 2\operatorname{Im}\int_D \dot{W}_2\bar{g}_{xx}dx \\
&\quad + C\left(\|g_{xx}\|_0^2 + \|g_{xx}\|_0^4 + \|E\|_0^4 + \|E_x\|_0^4\right) \\
&\quad + 2\operatorname{Re}\int_D \dot{W}_2\left(\bar{E}_{txx} + \gamma\bar{E}_{xx}\right)dx \\
&\leq C\left(\|f\|_0, \|q_1\|_0, \|q_2\|_2, \|g_x\|_0, \|g_{xx}\|_0,\right) \\
&\quad + C\left(\|E_x\|_0, \|n\|_0, \|E\|_0\right) + \int_D \dot{W}_1 m\,dx \\
&\quad - 2\operatorname{Im}\int_D n\bar{E}_{xx}\dot{W}_2 dx + 2\operatorname{Re}\int_D \dot{W}_2\left(\bar{E}_{txx} + \gamma\bar{E}_{xx}\right)dx.
\end{aligned}
\tag{76}
$$

Now, taking $\beta = \min\{\varepsilon, \gamma\}$ and letting $H_1(t) = (1/2\lambda^2)$ $\|m\|_0^2 + (\varepsilon^2/2\lambda^2)\|n\|_0^2 + ((1-\varepsilon\alpha)/2)\|n_x\|_0^2 + \|E_{xx}\|_0^2 - 2\operatorname{Re}$ $\int_D nE\bar{E}_{xx}dx$, by (76), the above inequality is changed into

$$
\begin{aligned}
&\frac{d}{dt}H_1(t) + \beta H_1(t) \\
&\quad \leq C\left(\|f\|_0, \|q_1\|_0, \|q_2\|_2, \|g_x\|_0, \|g_{xx}\|_0\right) \\
&\quad\quad + C\left(\|E_x\|_0, \|n\|_0, \|E\|_0\right) + \int_D \dot{W}_1 m\,dx \\
&\quad\quad - 2\operatorname{Im}\int_D n\bar{E}_{xx}\dot{W}_2 dx + 2\operatorname{Re}\int_D \dot{W}_2\left(\bar{E}_{txx} + \gamma\bar{E}_{xx}\right)dx \\
&\quad \leq C(f, g, q_1, q_2, E_0, n_0, n_1) + \int_D \dot{W}_1 m\,dx \\
&\quad\quad - 2\operatorname{Im}\int_D n\bar{E}_{xx}\dot{W}_2 dx + 2\operatorname{Re}\int_D \dot{W}_2\left(\bar{E}_{txx} + \gamma\bar{E}_{xx}\right)dx.
\end{aligned}
\tag{77}
$$

Integrating (77) from 0 to t and taking expectation on both sides of the above inequality, we get

$$
\begin{aligned}
&\mathbb{E}H_1(t) - \mathbb{E}H_1(0) + \beta\mathbb{E}\int_0^t H_1(s)ds \\
&\quad \leq C(f, g, q_1, q_2, E_0, n_0, n_1)t.
\end{aligned}
\tag{78}
$$

By Gronwall inequality, we have

$$\mathbb{E}H_1(t) \le e^{-\beta t}\mathbb{E}H_1(0) + C(f, g, q_1, q_2, E_0, n_0, n_1)$$
$$\le C(f, g, q_1, q_2, E_0, n_0, n_1), \tag{79}$$

where $C(f, g, q_1, q_2, E_0, n_0, n_1)$ is independent of T.

Since

$$\left| 2 \operatorname{Re} \int_D n E \bar{E}_{xx} dx \right| \le 2\|E\|_{L_\infty}\|n\|_0\|E_{xx}\|_0$$
$$\le C\|E_x\|_0\|n\|_0\|E_{xx}\|_0 \tag{80}$$
$$\le \frac{1}{2}\|E_{xx}\|_0^2 + C\left(\|E_x\|_0^4 + \|n\|_0^4\right),$$

we have

$$C\left(\|m\|_0^2 + \|n\|_0^2 + \|n_x\|_0^2 + \|E_{xx}\|_0^2\right) - C\left(\|E_x\|_0^4 + \|n\|_0^4\right)$$
$$\le H_1(t) \le C\left(\|m\|_0^2 + \|n\|_0^2 + \|n_x\|_0^2 + \|E_{xx}\|_0^2\right)$$
$$+ C\left(\|E_x\|_0^4 + \|n\|_0^4\right). \tag{81}$$

By (77) and (81), we get

$$\frac{d}{dt}H_1(t) \le C(f, g, q_1, q_2, E_0, n_0, n_1)$$
$$+ \int_D \dot{W}_1 m dx - 2 \operatorname{Im} \int_D n\bar{E}_{xx}\dot{W}_2 dx \tag{82}$$
$$+ 2 \operatorname{Re} \int_D \dot{W}_2\left(\bar{E}_{txx} + \gamma\bar{E}_{xx}\right)dx.$$

By (28), (54), (79), and (81), we get

$$\mathbb{E}\left(\|m\|_0^2 + \|n\|_0^2 + \|n_x\|_0^2 + \|E_{xx}\|_0^2\right)$$
$$\le \mathbb{E}H_1(t) + C\mathbb{E}\left(\|E_x\|_0^4 + \|n\|_0^4\right) \tag{83}$$
$$\le C(f, g, q_1, q_2, E_0, n_0, n_1),$$

where $C(f, g, q_1, q_2, E_0, n_0, n_1)$ is independent of T.

By (83), we obtain $(n_t, n, E) \in L^\infty(0, \infty; L^2(\Omega, \mathscr{E}_1))$.

Further, we give an estimate of $H_1^p(t)$ for any $p \ge 1$. First as

$$\|nE\|_0^2 = \int_D nEn\bar{E}dx \le \|nE\|_0\|n\|_0\|E\|_{L^\infty}$$
$$\le C\|nE\|_0\|n\|_0\|E_x\|_0,$$
$$\|nE_{xx}\|_{-1}^2 = \int_D (-\Delta)^{-1}(nE_{xx})\left(n\bar{E}_{xx}\right)dx \tag{84}$$
$$\le \left\|(-\Delta)^{-1}nE\right\|_{L^\infty}\|n\|_0\|E_{xx}\|_0$$
$$\le C\left\|(-\Delta)^{-1}nE_{xx}\right\|_1\|n\|_0\|E_{xx}\|_0$$
$$\le C\|nE_{xx}\|_{-1}\|n\|_0\|E_{xx}\|_0.$$

By (81) and (84), we have

$$\|nE\|_0^2 \le C\|n\|_0^2\|E_x\|_0^2$$
$$\le C\left(H_1(t) + \|E_x\|_0^4 + \|n\|_0^4\right)\|E_x\|_0^2, \tag{85}$$

$$\|nE_{xx}\|_{-1}^2 \le C\|n\|_0^2\|E_{xx}\|_0^2$$
$$\le C\left(H_1(t) + \|E_x\|_0^4 + \|n\|_0^4\right)\|E_{xx}\|_0^2, \tag{86}$$

$$\|m\|_0^2 + \|n\|_0^2 + \|n_x\|_0^2 + \|E_{xx}\|_0^2$$
$$\le C\left(H_1(t) + \|E_x\|_0^4 + \|n\|_0^4\right). \tag{87}$$

Applying Itô's formula to $H_1^p(t)$, by (77), (85), (86), (87), and Young inequality, we obtain

$$\frac{d}{dt}H_1^p(t) = pH_0^{p-1}(t)\frac{d}{dt}H_1(t) + \frac{1}{2}p(p-1)CH_0^{p-2}(t)$$
$$\cdot \left(\|E\|_2^2\|q_2\|_2^2 + \|g\|_1^2\|q_2\|_1^2 + \|nE\|_0^2\|q_2\|_2^2\right)$$
$$+ \frac{1}{2}p(p-1)CH_0^{p-2}(t)$$
$$\cdot \left(\|nE_{xx}\|_{-1}^2\|q_2\|_1^2 + \|m\|_0^2\|q_1\|_0^2\right)$$
$$= pH_0^{p-1}(t)\left[-\beta H_1(t) + C(f, g, q_1, q_2, E_0, n_0, n_1)\right.$$
$$+ \int_D \dot{W}_1 m dx\right] + pH_0^{p-1}(t)\left[-2 \operatorname{Im} \int_D n\bar{E}_{xx}\dot{W}_2 dx\right.$$
$$+ 2 \operatorname{Re} \int_D \dot{W}_2\left(\bar{E}_{txx} + \gamma\bar{E}_{xx}\right)dx\right] + \frac{1}{2}p(p-1)CH_0^{p-2}$$
$$\cdot (t)\left(\|E\|_2^2\|q_2\|_2^2 + \|g\|_1^2\|q_2\|_1^2 + \|nE\|_0^2\|q_2\|_2^2\right)$$
$$+ \frac{1}{2}p(p-1)CH_0^{p-2}(t)$$
$$\cdot \left(\|nE_{xx}\|_{-1}^2\|q_2\|_1^2 + \|m\|_0^2\|q_1\|_0^2\right)$$
$$\le -\beta pH_0^p(t) + C(f, g, q_1, q_2, E_0, n_0, n_1)$$
$$+ pH_0^{p-1}(t)\int_D \dot{W}_1 m dx + pH_0^{p-1}(t)$$
$$\cdot \left[-2 \operatorname{Im} \int_D n\bar{E}_{xx}\dot{W}_2 dx + 2 \operatorname{Re} \int_D \dot{W}_2\left(\bar{E}_{txx}\right.\right.$$
$$\left.\left.+ \gamma\bar{E}_{xx}\right)dx\right] + \frac{1}{2}p(p-1)CH_0^{p-2}(t)$$
$$\cdot \left(\|E\|_2^2\|q_2\|_2^2 + \|g\|_1^2\|q_2\|_1^2 + \|nE\|_0^2\|q_2\|_2^2\right)$$
$$+ \frac{1}{2}p(p-1)CH_0^{p-2}(t)$$
$$\cdot \left(\|nE_{xx}\|_{-1}^2\|q_2\|_1^2 + \|m\|_0^2\|q_1\|_0^2\right)$$
$$\le -\beta pH_0^p(t) + C(f, g, q_1, q_2, E_0, n_0, n_1)$$
$$+ pH_0^{p-1}(t)\int_D \dot{W}_1 m dx + pH_0^{p-1}(t)$$

$$
\cdot \left[-2 \operatorname{Im} \int_D n\bar{E}_{xx}\dot{W}_2 dx + 2 \operatorname{Re} \int_D \dot{W}_2 \left(\bar{E}_{txx} \right. \right.
$$

$$
\left. \left. + \gamma \bar{E}_{xx} \right) dx \right] + \frac{1}{2} p(p-1) C H_0^{p-2}(t) \|E\|_2^2 \|q_2\|_2^2
$$

$$
+ \frac{1}{2} p(p-1) C H_0^{p-2}(t) \|g\|_1^2 \|q_2\|_1^2
$$

$$
+ \frac{1}{2} p(p-1) C H_0^{p-2}(t) \|nE\|_0^2 \|q_2\|_2^2
$$

$$
+ \frac{1}{2} p(p-1) C H_0^{p-2}(t) \|nE_{xx}\|_{-1}^2 \|q_2\|_1^2
$$

$$
+ \frac{1}{2} p(p-1) C H_0^{p-2}(t) \|m\|_0^2 \|q_1\|_0^2
$$

$$
\leq -\beta p H_0^p(t) + C(f, g, q_1, q_2, E_0, n_0, n_1)
$$

$$
+ p H_0^{p-1}(t) \int_D \dot{W}_1 m dx + p H_0^{p-1}(t)
$$

$$
\cdot \left[-2 \operatorname{Im} \int_D n\bar{E}_{xx}\dot{W}_2 dx + 2 \operatorname{Re} \int_D \dot{W}_2 \left(\bar{E}_{txx} \right. \right.
$$

$$
\left. \left. + \gamma \bar{E}_{xx} \right) dx \right] + \frac{1}{2} p(p-1) C H_0^{p-2}(t) \left(H_1(t) \right.
$$

$$
\left. + \|E_x\|_0^4 + \|n\|_0^4 \right) \|q_2\|_2^2 + \frac{1}{2} p(p-1) C H_0^{p-2}
$$

$$
\cdot (t) \|g\|_1^2 \|q_2\|_1^2 + \frac{1}{2} p(p-1) C H_0^{p-2}(t) \left(H_1(t) \right.
$$

$$
\left. + \|E_x\|_0^4 + \|n\|_0^4 \right) \|E_x\|_0^2 \|q_2\|_2^2 + \frac{1}{2} p(p-1)
$$

$$
\cdot C H_0^{p-2}(t) \left(H_1(t) + \|E_x\|_0^4 + \|n\|_0^4 \right) H_1(t) \|q_2\|_1^2
$$

$$
+ \frac{1}{2} p(p-1) C H_0^{p-2}(t) \left(H_1(t) + \|E_x\|_0^4 \right.
$$

$$
\left. + \|n\|_0^4 \right) \|E_x\|_0^4 \|q_2\|_1^2 + \frac{1}{2} p(p-1) C H_0^{p-2}(t)
$$

$$
\cdot \left(H_1(t) + \|E_x\|_0^4 + \|n\|_0^4 \right) \|n\|_0^4 \|q_2\|_1^2 + \frac{1}{2} p
$$

$$
\cdot (p-1) C H_0^{p-2}(t) \left(H_1(t) + \|E_x\|_0^4 + \|n\|_0^4 \right) \|q_1\|_0^2
$$

$$
\leq -\frac{\beta p}{2} H_0^p(t) + C(f, g, q_1, q_2, E_0, n_0, n_1)
$$

$$
+ p H_0^{p-1}(t) \int_D \dot{W}_1 m dx + p H_0^{p-1}(t)
$$

$$
\cdot \left[-2 \operatorname{Im} \int_D n\bar{E}_{xx}\dot{W}_2 dx + 2 \operatorname{Re} \int_D \dot{W}_2 \left(\bar{E}_{txx} \right. \right.
$$

$$
\left. \left. + \gamma \bar{E}_{xx} \right) dx \right] + C \left(\|E_x\|_0^{4p} + \|E_x\|_0^{6p} + \|n\|_0^{4p} \right).
$$

$$
\tag{88}
$$

Taking expectation on both sides of the above inequality, by Hölder inequality and (88), we derive

$$
\mathbb{E} H_1^p(t) \leq \mathbb{E} H_1^p(0) - \frac{\beta p}{2} \mathbb{E} \int_0^t H_1^p(s) ds
$$

$$
+ C(E_0, f, g, q_1, q_2, n_0, n_1) t.
$$

$$
\tag{89}
$$

By Gronwall inequality, we have

$$
\mathbb{E} H_1^p(t) \leq e^{-\beta p t/2} \mathbb{E} H_1^p(0) + C(E_0, f, g, q_1, q_2, n_0, n_1)
$$

$$
\leq C(E_0, f, g, q_1, q_2, n_0, n_1),
$$

$$
\tag{90}
$$

where $C(E_0, f, g, q_1, q_2, n_0, n_1)$ is independent of T.

By (87) and (90), we obtain $(n_t, n, E) \in L^\infty(0, \infty; L^{2p}(\Omega, \mathscr{E}_1))$.

By the same approach of Lemma 4, we obtain

$$
\mathbb{E} \sup_{0 \leq t \leq T} H_1(t) \leq C_T(E_0, f, g, q_1, q_2, n_1, n_0),
$$

$$
\tag{91}
$$

where C_T depends on the initial data. Therefore, by (47), we have

$$
\mathbb{E} \sup_{0 \leq t \leq T} \|(n_t, n, E)\|_{\mathscr{E}_1}^2 \leq C_T(E_0, f, g, q_1, q_2, n_1, n_0).
$$

$$
\tag{92}
$$

By (92), we obtain $E \in L^2(\Omega; L^\infty(0, T; \mathscr{E}_1))$.
Moreover, we get

$$
\mathbb{E} \sup_{0 \leq t \leq T} H_1^p(t) \leq C \mathbb{E} H_1^p(0) + C_T(E_0, f, g, q_1, q_2, n_1, n_0)
$$

$$
\leq C_T(E_0, f, g, q_1, q_2, n_1, n_0).
$$

$$
\tag{93}
$$

By (87), we have

$$
C \mathbb{E} \sup_{0 \leq t \leq T} \left(\|m\|_0^{2p} + \|n\|_0^{2p} + \|n_x\|_{-1}^{2p} + \|E_{xx}\|_0^{2p} \right)
$$

$$
\leq C \mathbb{E} \sup_{0 \leq t \leq T} \left[C \left(H_1(t) + \|E_x\|_0^4 + \|n\|_0^4 \right) \right]^{2p}
$$

$$
\leq C \mathbb{E} \sup_{0 \leq t \leq T} \left(H_1^{2p}(t) + \|E_x\|_0^{8p} + \|n\|_0^{8p} \right)
$$

$$
\leq C_T(E_0, f, g, q_1, q_2, n_1, n_0),
$$

$$
\tag{94}
$$

where C_T depends on the initial data. Therefore, by (94), we have

$$
\mathbb{E} \sup_{0 \leq t \leq T} \|(n_t, n, E)\|_{\mathscr{E}_1}^{2p} \leq C_T(E_0, f, g, q_1, q_2, n_1, n_0).
$$

$$
\tag{95}
$$

By (95), we obtain $E \in L^{2p}(\Omega; L^\infty(0, T; \mathscr{E}_1))$.
Lemma 5 is proved completely.

Lemma 6. *Assume* $(n_1, n_0, E_0) \in \mathscr{E}_2$, $f \in H_0^1(D)$, $g, q_1 \in H^2(D) \cap H_0^1(D)$, *and* $q_2 \in H^3(D) \cap H_0^1(D)$. *Then, for any* $T > 0$ *and* $p \geq 1$, $(n_t, n, E) \in L^{2p}(\Omega; L^\infty(0, T; \mathscr{E}_2)) \cap L^\infty(0, \infty; L^{2p}(\Omega, \mathscr{E}_2))$.

Proof. Applying Itô's formula to $\|m_x\|_0^2$, by (12), we have

$$
\begin{aligned}
\frac{1}{2\lambda^2}\frac{d}{dt}\|m_x\|_0^2 &= -\int_D \frac{1}{\lambda^2} m_t m_{xx} dx + \frac{\lambda^2}{2}\|q_1\|_1^2 \\
&= \int_D \left[\frac{\varepsilon}{\lambda^2} m + \alpha m_{xx} - \frac{\varepsilon^2}{\lambda^2} n\right](-m_{xx})dx \\
&\quad + \int_D \left[(1-\varepsilon\alpha)n_{xx} + |E|_{xx}^2 + f + \dot{W}_1\right] \\
&\quad \cdot (-m_{xx})dx + \frac{\lambda^2}{2}\|q_1\|_1^2 \\
&= \frac{\varepsilon}{\lambda^2}\|m_x\|_0^2 - \alpha\|m\|_2^2 - \frac{\varepsilon^2}{\lambda^2}\int_D n_x m_x dx \\
&\quad - (1-\varepsilon\alpha)\int_D n_{xx} m_{xx} dx \\
&\quad - \int_D |E|_{xx}^2 m_{xx} dx - \int_D f m_{xx} dx \\
&\quad - \int_D \dot{W}_1 m_{xx} dx + \frac{\lambda^2}{2}\|q_1\|_1^2.
\end{aligned}
\tag{96}
$$

Since $m = n_t + \varepsilon n$, from (96), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2\lambda^2}\|m_x\|_0^2 + \frac{\varepsilon^2}{2\lambda^2}\|n_x\|_0^2 + \frac{1-\varepsilon\alpha}{2}\|n_{xx}\|_0^2\right] \\
&= \frac{\varepsilon}{\lambda^2}\|m_x\|_0^2 - \alpha\|m\|_2^2 - \frac{\varepsilon^3}{\lambda^2}\|n_x\|_0^2 \\
&\quad - \varepsilon(1-\varepsilon\alpha)\|n_{xx}\|_0^2 - \int_D |E|_{xx}^2 m_{xx} dx \\
&\quad + \int_D f_x m_x dx - \int_D \dot{W}_1 m_{xx} dx + \frac{\lambda^2}{2}\|q_1\|_1^2,
\end{aligned}
\tag{97}
$$

where λ_1 is the first eigenvalue of $-\Delta$.

By (97), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2\lambda^2}\|m_x\|_0^2 + \frac{\varepsilon^2}{2\lambda^2}\|n_x\|_0^2 + \frac{1-\varepsilon\alpha}{2}\|n_{xx}\|_0^2\right] \\
&\quad - \left(\frac{\varepsilon}{\lambda^2} - \alpha\lambda_1\right)\|m_x\|_0^2 + \frac{\varepsilon^3}{\lambda^2}\|n_x\|_0^2 + \varepsilon(1-\varepsilon\alpha)\|n_{xx}\|_0^2 \\
&\leq -\int_D |E|_{xx}^2 m_{xx} dx + \int_D f_x m_x dx - \int_D \dot{W}_1 m_{xx} dx \\
&\quad + \frac{\lambda^2}{2}\|q_1\|_1^2.
\end{aligned}
\tag{98}
$$

Applying Itô's formula to $\|E_{xxx}\|_0^2$, by (13), we have

$$
\begin{aligned}
\frac{d}{dt}\|E_{xxx}\|_0^2 &= 2\,\mathrm{Re}\int_D E_{xxx}\bar{E}_{txxx}dx + \|q_2\|_3^2 \\
&= -2\,\mathrm{Re}\int_D E_{xx}\bar{E}_{txxxx}dx + \|q_2\|_3^2.
\end{aligned}
\tag{99}
$$

Substituting $E_{xx} = -iE_t + nE - i\gamma E + g + \dot{W}_2$ into (99), we have

$$
\begin{aligned}
\frac{d}{dt}\|E_{xxx}\|_0^2 &= -2\,\mathrm{Re}\int_D \left(-iE_t + nE - i\gamma E + g + \dot{W}_2\right)\bar{E}_{txxxx}dx \\
&\quad + \|q_2\|_3^2 = -2\,\mathrm{Re}\int_D nE\bar{E}_{txxxx}dx \\
&\quad - 2\,\mathrm{Re}\int_D \left(\gamma\bar{E}_{xxxx}\right)\left(E_{xx} + nE - i\gamma E + g + \dot{W}_2\right)dx \\
&\quad - 2\,\mathrm{Re}\int_D g\bar{E}_{txxxx}dx - 2\,\mathrm{Re}\int_D \dot{W}_2\bar{E}_{txxxx}dx + \|q_2\|_3^2 \\
&= -2\gamma\|E_{xxx}\|_0^2 - 2\,\mathrm{Re}\int_D nE\left(\bar{E}_{txxxx} + \gamma\bar{E}_{xxxx}\right)dx \\
&\quad - 2\,\mathrm{Re}\int_D \dot{W}_2\left(\bar{E}_{txxxx} + \gamma\bar{E}_{xxxx}\right)dx \\
&\quad - 2\,\mathrm{Re}\int_D g\left(\bar{E}_{txxxx} + \gamma\bar{E}_{xxxx}\right)dx + \|q_2\|_3^2.
\end{aligned}
\tag{100}
$$

From (98) and (100), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2\lambda^2}\|m_x\|_0^2 + \frac{\varepsilon^2}{2\lambda^2}\|n_x\|_0^2 + \frac{1-\varepsilon\alpha}{2}\|n_{xx}\|_0^2 + \|E_{xxx}\|_0^2\right) \\
&\quad - \left(\frac{\varepsilon}{\lambda^2} - \alpha\lambda_1\right)\|m_x\|_0^2 + \frac{\varepsilon^3}{\lambda^2}\|n_x\|_0^2 + \varepsilon(1-\varepsilon\alpha)\|n_{xx}\|_0^2 \\
&\leq -\int_D |E|_{xx}^2 m_{xx} dx + \int_D f_x m_x dx - \int_D \dot{W}_1 m_{xx} dx \\
&\quad + \frac{\lambda^2}{2}\|q_1\|_1^2 - 2\gamma\|E_{xxx}\|_0^2 - 2\,\mathrm{Re}\int_D nE\left(\bar{E}_{txxxx} + \gamma\bar{E}_{xxxx}\right)dx \\
&\quad - 2\,\mathrm{Re}\int_D \dot{W}_2\left(\bar{E}_{txxxx} + \gamma\bar{E}_{xxxx}\right)dx \\
&\quad - 2\,\mathrm{Re}\int_D g\left(\bar{E}_{txxxx} + \gamma\bar{E}_{xxxx}\right)dx + \|q_2\|_3^2.
\end{aligned}
\tag{101}
$$

Taking $0 < \varepsilon \leq \min\{1/\alpha, \alpha\lambda_1\lambda^2/2\}$, and by (101), we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2\lambda^2}\|m_x\|_0^2 + \frac{\varepsilon^2}{2\lambda^2}\|n_x\|_0^2 + \frac{1-\varepsilon\alpha}{2}\|n_{xx}\|_0^2 + \|E_{xxx}\|_0^2\right) \\
&\quad + \frac{\varepsilon}{\lambda^2}\|m_x\|_0^2 + \frac{\varepsilon^3}{\lambda^2}\|n_x\|_0^2 + \varepsilon(1-\varepsilon\alpha)\|n_{xx}\|_0^2 + 2\gamma\|E_{xxx}\|_0^2 \\
&\leq -\int_D |E|_{xx}^2 m_{xx} dx + \int_D f_x m_x dx - \int_D \dot{W}_1 m_{xx} dx \\
&\quad + \frac{\lambda^2}{2}\|q_1\|_1^2 - 2\,\mathrm{Re}\int_D nE\left(\bar{E}_{txxxx} + \gamma\bar{E}_{xxxx}\right)dx \\
&\quad - 2\,\mathrm{Re}\int_D \dot{W}_2\left(\bar{E}_{txxxx} + \gamma\bar{E}_{xxxx}\right)dx \\
&\quad - 2\,\mathrm{Re}\int_D g\left(\bar{E}_{txxxx} + \gamma\bar{E}_{xxxx}\right)dx + \|q_2\|_3^2.
\end{aligned}
\tag{102}
$$

Since

$$
\begin{aligned}
2\,\mathrm{Re} & \int_D nE\bigl(\bar{E}_{txxxx} + \gamma\bar{E}_{xxxx}\bigr)dx \\
&= 2\,\mathrm{Re}\int_D n_{xx}\mathrm{E}\bigl(\bar{E}_{txx} + \gamma\bar{E}_{xx}\bigr)dx \\
&\quad + 2\,\mathrm{Re}\int_D nE_{xx}\bigl(\bar{E}_{txx} + \gamma\bar{E}_{xx}\bigr)dx \\
&\quad + 4\,\mathrm{Re}\int_D n_x E_x\bigl(\bar{E}_{txx} + \gamma\bar{E}_{xx}\bigr)dx,
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt} & \int_D n_{xx}|E|_{xx}^2 dx \\
&= \int_D n_{txx}|E|_{xx}^2 dx + 2\,\mathrm{Re}\int_D n_{xx}\bar{E}_t E_{xx}dx \\
&\quad + 2\,\mathrm{Re}\int_D n_{xx}\bar{E}E_{txx}dx - 4\frac{d}{dt}\,\mathrm{Re}\int_D n_x E_x\bar{E}_{xx}dx \\
&\quad + 4\,\mathrm{Re}\int_D n_{tx}E_x\bar{E}_{xx}dx + 4\,\mathrm{Re}\int_D n_x q_{2x}\bar{q}_{2xx}dx \\
&\quad + 2\,\mathrm{Re}\int_D n_{xx}\bar{q}_2 q_{2xx}dx + 2\,\mathrm{Re}\int_D \mathrm{n}_{xx}\bar{q}_{2x}q_{2x}dx,
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\int_D n|E_{xx}|^2 dx &= \int_D n_t|E_{xx}|^2 dx + 2\,\mathrm{Re}\int_D nE_{xx}\bar{E}_{txx}dx \\
&\quad + \int_D n|q_{2xx}|^2 dx,
\end{aligned}
$$

$$
\frac{d}{dt}\left(2\,\mathrm{Re}\int_D n_x E_x\bar{E}_{xx}dx\right)
$$

$$(103)$$

then we deduce

$$
\begin{aligned}
2\,\mathrm{Re} & \int_D nE\bigl(\bar{E}_{txxxx} + \gamma\bar{E}_{xxxx}\bigr)dx \\
&= \frac{d}{dt}\int_D n_{xx}|E|_{xx}^2 dx + \frac{d}{dt}\int_D n|E_{xx}|^2 dx \\
&\quad + \frac{d}{dt}\left(8\,\mathrm{Re}\int_D n_x E_x\bar{E}_{xx}dx\right) - \int_D n_{txx}|E|_{xx}^2 dx \\
&\quad - 2\,\mathrm{Re}\int_D n_{xx}\bar{E}_t E_{xx}dx - 4\,\mathrm{Re}\int_D n_{tx}E_x\bar{E}_{xx}dx \\
&\quad - \int_D n_t|E_{xx}|^2 dx + 2\gamma\,\mathrm{Re}\int_D n_{xx}E\bar{E}_{xx}dx \\
&\quad - 4\,\mathrm{Re}\int_D n_{tx}E_x\bar{E}_{xx}dx - 4\,\mathrm{Re}\int_D n_x E_{tx}\bar{E}_{xx}dx \\
&\quad + 2\gamma\,\mathrm{Re}\int_D nE_{xx}\bar{E}_{xx}dx + 4\gamma\,\mathrm{Re}\int_D n_x E_x\bar{E}_{xx}dx \\
&\quad - 4\,\mathrm{Re}\int_D n_x q_{2x}\bar{q}_{2xx}dx - \int_D n|q_{2xx}|^2 dx \\
&\quad - 4\,\mathrm{Re}\int_D n_x q_{2x}\bar{q}_{2xx}dx - 2\,\mathrm{Re}\int_D n_{xx}\bar{q}_2 q_{2xx}dx \\
&\quad - 2\,\mathrm{Re}\int_D n_{xx}\bar{q}_{2x}q_{2x}dx.
\end{aligned}
$$

$$(104)$$

Substituting (104) into (102), we have

$$
\begin{aligned}
\frac{d}{dt} & \left(\frac{1}{2\lambda^2}\|m_x\|_0^2 + \frac{\varepsilon^2}{2\lambda^2}\|n_x\|_0^2 + \frac{1-\varepsilon\alpha}{2}\|n_{xx}\|_0^2 + \|E_{xxx}\|_0^2\right) \\
&\cdot \frac{d}{dt}\left(\int_D n_{xx}|E|_{xx}^2 dx + \int_D n|E_{xx}|^2 dx + 8\,\mathrm{Re}\int_D n_x E_x\bar{E}_{xx}dx\right) \\
&\quad + \frac{\varepsilon}{\lambda^2}\|m_x\|_0^2 + \frac{\varepsilon^3}{\lambda^2}\|n_x\|_0^2 + \varepsilon(1-\varepsilon\alpha)\|n_{xx}\|_0^2 + 2\gamma\|E_{xxx}\|_0^2 \\
&\leq -\int_D |E|_{xx}^2 m_{xx}dx + \int_D f_x m_{xx}dx - \int_D \dot{W}_1 m_{xx}dx \\
&\quad + \frac{\lambda^2}{2}\|q_1\|_1^2 + \int_D n_{txx}|E|_{xx}^2 dx + 2\,\mathrm{Re}\int_D n_{xx}\bar{E}_t E_{xx}dx \\
&\quad + 4\,\mathrm{Re}\int_D n_{tx}E_x\bar{E}_{xx}dx + \int_D n_t|E_{xx}|^2 dx \\
&\quad - 2\gamma\,\mathrm{Re}\int_D n_{xx}E\bar{E}_{xx}dx + 4\,\mathrm{Re}\int_D n_{tx}E_x\bar{E}_{xx}dx \\
&\quad + 4\,\mathrm{Re}\int_D n_x E_{tx}\bar{E}_{xx}dx - 2\gamma\,\mathrm{Re}\int_D nE_{xx}\bar{E}_{xx}dx \\
&\quad - 4\gamma\,\mathrm{Re}\int_D n_x E_x\bar{E}_{xx}dx + 4\,\mathrm{Re}\int_D n_x q_{2x}\bar{q}_{2xx}dx \\
&\quad + \int_D n|q_{2xx}|^2 dx + 4\,\mathrm{Re}\int_D n_x q_{2x}\bar{q}_{2xx}dx \\
&\quad + 2\,\mathrm{Re}\int_D n_{xx}\bar{q}_2 q_{2xx}dx + 2\,\mathrm{Re}\int_D n_{xx}\bar{q}_{2x}q_{2x}dx \\
&\quad - 2\,\mathrm{Re}\int_D \dot{W}_2\bigl(\bar{E}_{txxxx} + \gamma\bar{E}_{xxxx}\bigr)dx \\
&\quad - 2\,\mathrm{Re}\int_D g\bigl(\bar{E}_{txxxx} + \gamma\bar{E}_{xxxx}\bigr)dx + \|q_2\|_3^2.
\end{aligned}
$$

$$(105)$$

4. Existence and Uniqueness of Solutions

By the above a priori estimates, we prove the existence and uniqueness of solutions for stochastic Equations (7) and (8) with initial boundary conditions (9) and (10) in spaces $\mathscr{E}_1$.

Theorem 7. *Assume* $(n_1, n_0, E_0) \in \mathscr{E}_1$, $f \in L^2(D)$, $q_1 \in H_0^1(D)$, *and* $g, q_2 \in H^2(D) \cap H_0^1(D)$. *Then, there is a unique solution* $(n_t, n, E) \in L^\infty(\mathbb{R}^+, \mathscr{E}_1)$ *almost surely for Equations (7) and (8). And* (n_t, n, E) *is continuous from* $\mathbb{R}^+$ *to* $\mathscr{E}_1$.

Proof. First, we assume $(n_1, n_0, E_0) \in \mathscr{E}_1$. Let $\{e_i\}_{i=1}^\infty$ be the eigenvectors of Laplace operator $-\Delta$ on D with Dirichlet boundary condition, which is also an orthonormal basis of $L^2(D)$. Let P^k be the projection from $L^2(D)$ onto the space spanned by $\{e_i : i = 1, 2, \cdots, k\}$. We define (n_t^k, n^k, E^k) as the Galerkin approximate solution of the following equations:

$$
\frac{1}{\lambda^2}n_{tt}^k - n_{xx}^k - P^k\Bigl|E^k\Bigr|_{xx}^2 - \alpha n_{txx}^k = f^k + \dot{W}_1^k,
$$

$$(106)$$

$$iE_t^k + E_{xx}^k - P^k\left(n^k E^k\right) + i\gamma E^k$$
$$= g^k + \dot{W}_2^k, \quad t \in R^+, x \in D, \tag{107}$$

$$n^k(0) = P^k n_0, n_t^k(0) = P^k n_1, E^k(0) = P^k E_0. \tag{108}$$

Here, $f^k = P^k f$, $g^k = P^k g$, $\dot{W}_1^k = P^k \dot{W}_1$, and $\dot{W}_2^k = P^k \dot{W}_2$, and P^k commutes with the operator Δ. Now, we will treat the above equations pathwise by introducing the random processes solving

$$\frac{1}{\lambda^2} \eta_{tt} - \eta_{xx} - \alpha \eta_{txx} = \dot{W}_1,$$
$$i\xi_t + \xi_{xx} + i\gamma\xi = \dot{W}_2, \tag{109}$$

with Dirichlet boundary conditions

$$\eta|_{\partial D} = 0, \xi|_{\partial D} = 0, \tag{110}$$

and initial conditions

$$\eta(x, 0) = 0, \eta_t(x, 0) = 0, \xi(x, 0) = 0, \quad x \in D. \tag{111}$$

Following the same method as in Section 3, for any $T > 0$ and almost all sample point $\omega \in \Omega$, we have

$$\eta_t \in C\left(0, T; H^1(D)\right), \eta \in C\left(0, T; H^1(D) \cap H_0^1(D)\right), \tag{112}$$

$$\xi \in C\left(0, T; H^3(D) \cap H_0^1(D)\right), \tag{113}$$

and satisfy the following estimates:

$$\mathbb{E}\left(\|\eta_t(t)\|_1^2 + \|\eta(t)\|_2^2 + \|\xi(t)\|_3^2\right) \leq C, \tag{114}$$

for some positive constant C independent of T. And for any $T > 0$,

$$\mathbb{E} \sup_{0 \leq t \leq T}\left(\|\eta_t(t)\|_1^2 + \|\eta(t)\|_2^2 + \|\xi(t)\|_3^2\right) \leq C_T, \tag{115}$$

for some positive constant C_T. Now, let $R^{k,M} = (N_t^{k,M}, N^{k,M}, F^{k,M})$ be the solution of the following equations:

$$\frac{1}{\lambda^2} N_{tt}^{k,M} - N_{xx}^{k,M} - \chi_M\left(\left|R^{k,M}\right|_{\mathscr{E}_1}\right) P^k \left|E^k\right|_{xx}^2 - \alpha N_{txx}^{k,M} = f^k, \tag{116}$$

$$iF_t^{k,M} + F_{xx}^{k,M} - \chi_M\left(\left|R^{k,M}\right|_{\mathscr{E}_1}\right) P^k\left(n^{k,M} E^{k,M}\right) + i\gamma F^{k,M} = g^k, \tag{117}$$

with initial conditions

$$N_t^{k,M}(x, 0) = P^k n_1, N^{k,M}(x, 0) = P^k n_0, F^{k,M}(x, 0) = P^k E_0. \tag{118}$$

Here, $n^{k,M} = N^{k,M} + P^k \eta$ and $E^{k,M} = F^{k,M} + P^k\xi$. And $\chi_M \in C_0^\infty(\mathbb{R})$ such that $\chi_M = 1$ for $|r| \leq M$ and $\chi_M = 0$ for $|r| \geq M$. Note that (116) and (117) are random differential equations with Lipschitz nonlinearity in finite dimension. Then, for almost all sample point $\omega \in \Omega$, we have a unique solution $(N_t^{k,M}, N^{k,M}, F^{k,M})$ for (116) and (117). Define the stopping time by $\tau_M = \inf\left\{t > 0 : |R|_{\mathscr{E}_1}^{k,M} \geq M\right\}$, if the set $\left\{|R|_{\mathscr{E}_1}^{k,M} \geq M\right\}$ is nonempty; otherwise, $\tau_M = \infty$.

Since τ_M is increasing in M, let $\tau_\infty = \lim_{M\to\infty}\tau_M$ almost surely. And for $t < \tau_M$, we have

$$\left(N_t^{k,M}, N^{k,M}, F^{k,M}\right) + \left(P^k \eta_t, P^k \eta, P^k \xi\right), \tag{119}$$

satisfying (106) and (107). By the estimates given in Section 3 and (114) and (115), for any $t \geq 0$, we have

$$\mathbb{E}\left|\left(N_t^{k,M}, N^{k,M}, F^{k,M}\right)\right|_{\mathscr{E}_1}^2 \leq C(n_0, n_1, E_0, f, g, q_1, q_2), \tag{120}$$

with $C(n_0, n_1, E_0, f, g, q_1, q_2)$ independent of T and M, and for any $T > 0$,

$$\mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_M} \left|\left(N_t^{k,M}, N^{k,M}, F^{k,M}\right)\right|_{\mathscr{E}_1}^2 \leq C_T(n_0, n_1, E_0, f, g, q_1, q_2), \tag{121}$$

with $C(n_0, n_1, E_0, f, g, q_1, q_2)$ independent of M; here, $T \wedge \tau_M = \min\{T, \tau_M\}$. On the other hand, we have

$$\mathbb{E}\left|\left(N_t^{k,M}(T \wedge \tau_M), N^{k,M}(T \wedge \tau_M), F^{k,M}(T \wedge \tau_M)\right)\right|_{\mathscr{E}_1}^2$$
$$\geq \mathbb{E}\left[I(\tau_M \leq T)\left|\left(N_t^{k,M}(T \wedge \tau_M), N^{k,M}(T \wedge \tau_M),\right.\right.\right.$$
$$\left.\left.\left. F^{k,M}(T \wedge \tau_M)\right)\right|_{\mathscr{E}_1}^2\right] \geq M^2 \mathbb{P}(\tau_M \leq T), \tag{122}$$

where $I(\tau_M \leq T) = 1$ for $\tau_M \leq T$ and $I(\tau_M \leq T) = 0$ for $\tau_M > T$. Then, according to (120), we have

$$\mathbb{P}(\tau_M \leq T) \leq \frac{1}{M^2} C(n_0, n_1, E_0, f, g, q_1, q_2). \tag{123}$$

According to the above estimate and Borel-Cantelli lemma, for any $T > 0$, we have $\mathbb{P}(\tau_M > T) = 1$. So we know

$$\left(N_t^k, N^k, F^k\right) = \lim_{M\to\infty}\left(N_t^{k,M}, N^{k,M}, F^{k,M}\right), \tag{124}$$

satisfying the following random differential equations:

$$\frac{1}{\lambda^2} N_{tt}^k - N_{xx}^k - P^k\left|E^k\right|_{xx}^2 - \alpha N_{txx}^k = f^k, \tag{125}$$

$$iF_t^k + F_{xx}^k - P^k\left(n^k E^k\right) + i\gamma F^k = g^k, \tag{126}$$

with initial conditions

$$N^k(0) = P^k n_0, N_t^k(0) = P^k n_1, F^k(0) = P^k E_0. \quad (127)$$

Then, (N_t^k, N^k, F^k) satisfies the estimates (120) and (121), and for any $t \geq 0$,

$$\left(n_t^k(t), n^k(t), E^k(t)\right) = \left(N_t^k(t), N^k(t), F^k(t)\right)$$
$$+ \left(P^k \eta_t(t), P^k \eta(t), P^k \xi(t)\right), \quad (128)$$

is the unique global solution of (106), (107), and (108). Now, we will consider (125) and (126) for fixed ω. First, by (121), for any $T > 0$,

$$\mathbb{P}\left(\bigcap_{L=1}^{\infty} \bigcup_{l=1}^{\infty} \bigcap_{k=l}^{\infty} \left\{\sup_{0 \leq t \leq T} \left\| \left(N_t^k(t), N^k(t), F^k(t)\right) \right\|_{\mathscr{E}_1}^2 \geq L \right\}\right) = 0. \quad (129)$$

Let

$$\tilde{\Omega} = \bigcap_{L=1}^{\infty} \bigcup_{l=1}^{\infty} \bigcap_{k=l}^{\infty} \left\{\sup_{0 \leq t \leq T} \left\| \left(N_t^k(t), N^k(t), F^k(t)\right) \right\|_{\mathscr{E}_1}^2 \leq L \right\}. \quad (130)$$

Then, $\mathbb{P}(\Omega \tilde{\Omega}) = 0$. Now, for any fixed $\omega \in \tilde{\Omega}$, there is an $r(\omega)$ with $0 < r(\omega) < \infty$ such that

$$\sup_{0 \leq t \leq T} \left\| \left(N_t^k(t), N^k(t), F^k(t)\right) \right\|_{\mathscr{E}_1}^2 \leq r(\omega). \quad (131)$$

Then, we can extract a subsequence still denoted by (N_t^k, N^k, F^k) such that for any $T > 0$, N_t^k converges to N_t weakly star in $L^\infty(0, T; L^2(D))$, N^k converges to N weakly star in $L^\infty(0, T; H_0^1(D))$, and F^k converges to F weakly star in $L^\infty(0, T; H^2 \cap H_0^1(D))$.

These convergences are sufficient to pass the limit $k \to \infty$ in linear terms, but we need a strong convergence of F^k for nonlinear terms. In fact from (126) and estimate (131), we know F^k is bounded in $L^\infty(0, T; L^2(D))$. Then, by Lemma 1, we can further extract a subsequence still denoted by F^k such that F^k converges to F strongly in $L^2(0, T; H_0^1(D))$. Then, by a standard procedure, we can pass the limit $k \to \infty$ to show that $(N_t, N, F) \in L^\infty(0, T; \mathscr{E}_1)$ is a weak solution of

$$\frac{1}{\lambda^2} N_{tt} - N_{xx} - |E|_{xx}^2 - \alpha N_{txx} = f,$$
$$iF_t + F_{xx} - nE + i\gamma F = g, \quad (132)$$

with initial conditions

$$N_t(x, 0) = n_1(x), N(x, 0) = n_0(x), F(x, 0) = E_0(x), \quad x \in D. \quad (133)$$

Then, $(n_t, n, F) = (N_t, N, F) + (\eta_t, \eta, \xi)$ is a solution of (7) and (8) and satisfies the estimates in Section 3.

Now, we prove the continuity of the solution. In fact, for $\omega \in \tilde{\Omega}, f \in L^2(D)$, and $|E|_{xx}^2 \in L^\infty(0, T; L_0^2(D))$, we have

$$\frac{1}{\lambda^2} N_{tt} - N_{xx} \in L^\infty(0, T; L_0^2(D)). \quad (134)$$

Then, by Lemma 4.1 and Chapter II of [23], we have for almost all $\omega \in \Omega$

$$N \in C(0, T; H_0^1(D)), N_t \in C(0, T; L^2(D)). \quad (135)$$

By a similar method, noticing $g \in H_0^1(D)$ and $F_t \in L^\infty(0, T; L_0^2(D))$ almost surely, by [21], we have

$$F \in C\left(0, T; H_0^2(D) \bigcap H_0^1(D)\right). \quad (136)$$

Then, by (113) and the definition of N and F,

$$(n_t, n, F) \in C(0, T; \mathscr{E}_1), \quad (137)$$

almost surely. Now, as the noise is additive, we can follow the same approach as in [11]. The solution (n_t, n, E) is unique in $L^\infty(0, T; \mathscr{E}_1)$ almost surely and is continuous from $[0, T]$ to $\mathscr{E}_1$ almost surely. Then, for T is arbitrary, by the estimates in Section 3, we derive Theorem 7.

Using the same method as above, we obtain the following results.

Theorem 8. *Assume $(n_1, n_0, E_0) \in \mathscr{E}_2$, $f \in H_0^1(D)$, $g, q_1 \in H^2(D) \cap H_0^1(D)$, and $q_2 \in H^3(D) \cap H_0^1(D)$. Then, there is a unique solution $(n_t, n, E) \in L^\infty(\mathbb{R}^+, \mathscr{E}_2)$ almost surely for Equations (7) and (8). And (n_t, n, E) is continuous from $\mathbb{R}^+$ to $\mathscr{E}_1$.*

5. Conclusion

In [10], the authors studied the initial boundary value problem for a generalized Zakharov system. The authors proved the global existence and uniqueness of the generalized solution to the problem by a priori estimates and Galerkin method. In this paper, we discuss the random forced strongly dissipative Zakharov equations, which are the generalized Zakharov system under random influences in [10]. We proved the existence and uniqueness of solutions in energy spaces $\mathscr{E}_1$ and $\mathscr{E}_2$. The results of this paper are a good supplement to the results in [10].

Conflicts of Interest

Acknowledgments

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