Hindawi
Advances in Mathematical Physics
Volume 2020, Article ID 2076074, 7 pages
https://doi.org/10.1155/2020/2076074

Research Article

Three Types Generalized Z_n-Heisenberg Ferromagnet Models

Yinfei Zhou, Shuchao Wan, Yang Bai, and Zhaowen Yan

School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China

Correspondence should be addressed to Zhaowen Yan; yanzw@imu.edu.cn

Received 13 May 2019; Revised 9 July 2019; Accepted 31 July 2019; Published 31 January 2020

Academic Editor: Matteo Beccaria

By taking values in a commutative subalgebra $gl(n, \mathbb{C})$, we construct a new generalized Z_n-Heisenberg ferromagnet model in (1+1)-dimensions. The corresponding geometrical equivalence between the generalized Z_n-Heisenberg ferromagnet model and Z_n-mixed derivative nonlinear Schrödinger equation has been investigated. The Lax pairs associated with the generalized systems have been derived. In addition, we construct the generalized Z_n-inhomogeneous Heisenberg ferromagnet model and Z_n-Ishimori equation in (2+1)-dimensions. We also discuss the integrable properties of the multi-component systems. Meanwhile, the generalized Z_n-nonlinear Schrödinger equation, Z_n-Davey–Stewartson equation and their Lax representation have been well studied.

1. Introduction

The Heisenberg ferromagnet (HF) model is one of the most investigated integrable systems which plays an important role in the two-dimensional (2D) gravity theory [1] and anti-de Sitter/conformal field theories [2, 3]. It is proved that the HF model is gauge and geometric equivalent to the nonlinear Schrödinger (NLS) equation [4, 5]. (1+1)-dimensional generalized HF models involving inhomogeneous and higher order deformed HF models have been analyzed [6, 7]. The deformed HF models in (2+1)-dimensions also have been investigated, such as the higher order HF models [8, 9], the HF models with self-consistent potentials [10], the Ishimori equation [11], and inhomogeneous HF models [12, 13].

Multi-component version of the integrable systems has deserved much attention due to its wide application in multiple orthogonal polynomials, representation theory, random matrix model, the related Riemann-Hilbert problems, and Brownian motions [14–18]. Many important integrable systems have been extended to their multi-component counterparts, such as multi-component KP [19, 20], multi-component Toda systems [14], and multi-component BKP [21]. After considering commutative subalgebra of diagonal matrices, Bogdanov et al. [22] constructed the generalized multicomponent KP hierarchy which involves N independent generalized scalar KP hierarchies. Starting from the maximal commutative subalgebra of $gl(m, \mathbb{C})$, one [23, 24] constructed a new Z_m-Kadomtsev–Petviashvili (KP) hierarchy and investigated the existence of τ-functions. Meanwhile, the relation between dispersionless reduced Z_m-KP hierarchy and Frobenius manifold has been discussed. Recently, Li et al. [25] constructed the extended multi-component Toda hierarchy and extended multi-component bigraded Toda hierarchy. By virtue of taking values in a matrix-valued differential algebra set, they also establish a class of Hirota quadratic equation, which may be useful in Gromov–Witten theory and noncommutative symplectic geometry. In [25], one has defined the new multi-component sinh–Gordon systems by considering commutative subalgebra of $gl(n, \mathbb{C})$ and established their Bäcklund transformations. A natural problem then arises as to how to construct the corresponding extended HF models. With this motivation, this paper will be devoted to constructing three types commutative multi-component generalized HF models by taking values in commutative subalgebra. Furthermore their corresponding geometrical and gauge equivalent counterparts shall be discussed.

This paper is organized as follows. In Section 2, we present a brief review of some elementary facts about the Z_n-HF model and Z_n-NLS equation. Section 3 is devoted to constructing the generalized Z_n-HF models and establishing the geometrical equivalence with the Z_n-mixed derivative NLSE. In Section 4, we investigate the generalized Z_n-inhomogeneous HF models and their structure and integrability. In addition, we deduce the multi-component Ishimori equation and discuss its corresponding gauge equivalent counterpart. The last section will be devoted to a summary and discussion.

2. Z_n-Heisenberg Ferromagnet Model

The Heisenberg ferromagnet (HF) model in (1+1)-dimensions [4] is an important integrable equation which reads as

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}, \tag{1}$$

where $\mathbf{S}$ denotes the spin vector, $\mathbf{S} = (S_1, S_2, S_3)$ and satisfies the constraint $\mathbf{S}^2 = 1$.

The matrix form of the HF model can be expressed as

$$iS_t = \frac{1}{2}[S, S_{xx}], \tag{2}$$

Where $S = \sum_{i=1}^{3} S_i \sigma_i$, $S^2 = I$, $trS = 0$ and $\sigma_i (i = 1, 2, 3)$ are Pauli matrices.

Let S take values in a commutative subalgebra $Z_n = \mathbb{C}[\Gamma]/(\Gamma^n)$ and $\Gamma = \left(\delta_{ij,j+1}\right)_{ij} \in gl(n, \mathbb{C})$. From the equation (2), we obtain

$$i\tilde{S}_t = \frac{1}{2}\left[\tilde{S}, \tilde{S}_{xx}\right], \tag{3}$$

Where $\tilde{S}^2 = I$, I is an identity matrix. Suppose $\tilde{S}$ can be expressed as

$$\tilde{S}_i = S_{i0}E + S_{i1}\Gamma + S_{i2}\Gamma^2 + \ldots + S_{i(n-1)}\Gamma^{n-1}. \tag{4}$$

Then $\tilde{S}(x, t)$ can be divided into n parts

$$\tilde{S} = S_0 + S_1 + S_2 + \ldots + S_{n-1}, \tag{5}$$

where

$$S_k = \mathbf{S}_k \cdot \mathbf{X}_k,\ \mathbf{S}_k = (S_{k1}, S_{k2}, S_{k3}),\ \boldsymbol{X}_k = (X_{k1}, X_{k2}, X_{k3}),$$
$$X_{k1} = \begin{pmatrix} 0 & \Gamma^k \\ \Gamma^k & 0 \end{pmatrix}, \quad X_{k2} = \begin{pmatrix} 0 & i\Gamma^k \\ -i\Gamma^k & 0 \end{pmatrix}, \quad X_{k3} = \begin{pmatrix} \Gamma^k & 0 \\ 0 & -\Gamma^k \end{pmatrix} \tag{6}$$

and when $k = 0$, $\Gamma^0 = E$, E is a identity matrix. Then we may derive the following theorems.

Theorem 1. *The following equation holds*

$$iS_{kt} = \frac{1}{2}\sum_{i+j=k}\left[S_i, S_{jxx}\right], \quad 0 \le k \le n-1. \tag{7}$$

Proof. By choosing the coefficient of Γ^k for two sides of the identity (3), (3) leads to (7), which will be referred to as the Z_n-HF model.

The integrability condition of (7) is as the following linear systems

$$\begin{aligned}\Phi_x\left(x, t, \lambda_j\right) &= U\left(x, t, \lambda_j\right)\Phi\left(x, t, \lambda_j\right),\\ \Phi_t\left(x, t, \lambda_j\right) &= V\left(x, t, \lambda_j\right)\Phi\left(x, t, \lambda_j\right),\end{aligned} \tag{8}$$

where

$$\begin{aligned}U &= i\sum_{l=0}^{n-1}\sum_{j+k=l}\lambda_j \mathbf{S}_k \cdot \mathbf{X}_l,\\ V &= 2i\sum_{l=0}^{n-1}\sum_{t+j+k=l}\lambda_t\lambda_j \mathbf{S}_k \cdot \mathbf{X}_l + \frac{1}{2}\sum_{l=0}^{n-1}\sum_{t+j+k=l}\lambda_t \mathbf{S}_j \cdot \mathbf{S}_{kx} \cdot \mathbf{X}_l.\end{aligned} \tag{9}$$

Substituting (5) and (6) into (3), we obtain the following corollary:

Corollary 2. *The vector form of the Z_n-HF model:*

$$\mathbf{S}_{kt} = \sum_{i+j=k}\mathbf{S}_i \times \mathbf{S}_{jxx}, \quad 0 \le k \le n-1, \tag{10}$$

here we use the property

$$\begin{aligned}\left[\mathbf{S}_i \cdot \mathbf{X}_i, \mathbf{S}_{jxx} \cdot \mathbf{X}_j\right] &= 2i\,\mathbf{S}_i \times \mathbf{S}_{jxx} \cdot \mathbf{X}_{i+j}, \quad i + j \le n-1,\\ &\qquad when\, i + j < n-1, \mathrm{X}_{i+j} = 0.\end{aligned} \tag{11}$$

This proves that the Z_n-HF is geometrical equivalent to the following Z_n-NLSE.

Theorem 3. *The following identity holds*

$$i\varphi_{kt} + \varphi_{kxx} + 2\sum_{i+j+l=k}\varphi_i\overline{\varphi}_j\varphi_l = 0,\ 0 \le k \le n-1. \tag{12}$$

Proof. From NLS equation, we obtain

$$i\tilde{\varphi}_t + \tilde{\varphi}_{xx} + 2\left|\tilde{\varphi}\right|^2\tilde{\varphi} = 0, \tag{13}$$

where

$$\tilde{\varphi} = \varphi_0 E + \varphi_1\Gamma + \varphi_2\Gamma^2 + \ldots + \varphi_{n-1}\Gamma^{n-1}. \tag{14}$$

By choosing the coefficients of Γ^k for the identity (13), (13) leads to the Z_n-NLS equation (12). □

The Lax pair of (12) can be represented as

$$\begin{aligned}\Phi'_x\left(x, t, \lambda_j\right) &= U'\left(x, t, \lambda_j\right)\Phi'\left(x, t, \lambda_j\right),\\ \Phi'_t\left(x, t, \lambda_j\right) &= V'\left(x, t, \lambda_j\right)\Phi'\left(x, t, \lambda_j\right),\end{aligned} \tag{15}$$

where

$$\begin{aligned}U' &= \sum_{j=0}^{n-1}\left(-i\lambda_j\sum + M_j(x, t)\right)\Gamma^j,\\ V' &= \sum_{j=0}^{n-1}\left(-2i\sum_{j+k=l}\lambda_j\lambda_k\sum\cdot\Gamma^l + 2\sum_{j+k=l}\lambda_\alpha M_\beta(x, t)\cdot\Gamma^l\right.\\ &\quad \left. -iM_{lx}(x, t)\cdot\Gamma^l - i\sum_{k+t=l}M_k(x, t)M_t(x, t)\cdot\Gamma^l\right)\cdot\sum,\end{aligned} \tag{16}$$

and

$$\sum = diag\{E, -E\}, \quad M_j(x, t) = \begin{pmatrix} 0 & \varphi_j \\ -\overline{\varphi}_j & 0 \end{pmatrix}, \tag{17}$$

where E is a identity matrix.

3. Generalized Z_n-Heisenberg Ferromagnet Model in (1+1)-Dimensions

Let us consider the integrable deformed HF model [28]

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \frac{\epsilon}{2}(\mathbf{S}_x \cdot \mathbf{S}_x)\mathbf{S}_x, \tag{18}$$

where ϵ is a deformation parameter.

By expanding $\mathbf{S} = \mathbf{S}_0 E + \mathbf{S}_1 \Gamma + \ldots + \mathbf{S}_{n-1}\Gamma^{n-1}$, we obtain the generalized Z_n-Heisenberg ferromagnet model in (1+1)-dimensions

$$\mathbf{S}_{jt} = \sum_{\alpha+\beta=j}\left(\mathbf{S}_\alpha \times \mathbf{S}_{\beta xx}\right) + \frac{\epsilon}{2}\sum_{a+b+c=j}\left[\left(\mathbf{S}_{ax}\cdot\mathbf{S}_{bx}\right)\cdot\mathbf{S}_{cx}\right], \quad 0 \le j \le n-1, \tag{19}$$

where ϵ is a deformation parameter. When $\epsilon = 0$, Eq. (19) reduces to the Z_n-HF model (10). The Lax representation of the generalized Z_n-HF equation (19) is given by

$$\begin{aligned}\tilde{U} &= \epsilon\sum_{k=0}^{n-1}\sum_{m+n+p+q=k}\lambda_m\lambda_n\mathbf{S}_p\cdot\mathbf{X}_q + \epsilon\sum_{k=0}^{n-1}\sum_{m+n+p=k}\lambda_m\mathbf{S}_{nx}\cdot\mathbf{X}_p,\\
\tilde{V} &= \epsilon\sum_{k=0}^{n-1}\sum_{m+n+p+q+l=k}\lambda_m\lambda_n\mathbf{S}_p\times\mathbf{S}_{qx}\cdot\mathbf{X}_l\\
&\quad - \epsilon\sum_{k=0}^{n-1}\sum_{m+n+p+q+l+w=k}\lambda_m\lambda_n\lambda_p\lambda_q\mathbf{S}_l\cdot\mathbf{X}_w\\
&\quad + \epsilon\sum_{k=0}^{n-1}\sum_{m+n+p+q=k}\lambda_m\left(\mathbf{S}_n\times\mathbf{S}_{pxx}\right)\cdot\mathbf{X}_q\\
&\quad + 2\epsilon^2\sum_{k=0}^{n-1}\sum_{m+n+p+q+l+w=k}\lambda_m\lambda_n\left(\mathbf{S}_{px}\cdot\mathbf{S}_{qx}\right)\mathbf{S}_l\cdot\mathbf{X}_w\\
&\quad + 2\epsilon^2\sum_{k=0}^{n-1}\sum_{m+n+p+q+l=k}\lambda_m\left(\mathbf{S}_{nx}\cdot\mathbf{S}_{px}\right)\mathbf{S}_{qx}\cdot\mathbf{X}_l\\
&\quad - \epsilon^2\sum_{k=0}^{n-1}\sum_{m+n+p+q+l=k}\lambda_m\lambda_n\lambda_p\mathbf{S}_{qx}\cdot\mathbf{X}_l\\
&\quad - \frac{3}{2}\epsilon^2\sum_{k=0}^{n-1}\sum_{m+n+p+q+l=k}\lambda_m\lambda_n\left(\mathbf{S}_{px}\cdot\mathbf{S}_{qx}\right)\cdot\mathbf{X}_l,\end{aligned}\tag{20}$$

where $\lambda_j(0 \le j \le n-1)$ are spectral parameters.

In order to derive the geometrical equivalent counterpart of (19), we introduce the multi-component Serret-Frenet equation

$$\mathbf{t}_{jx} = \sum_{m+l=j}k_m\mathbf{n}_l, \quad \mathbf{b}_{jx} = -\sum_{m+l=j}\tau_m\mathbf{n}_l, \quad \mathbf{n}_{jx} = \sum_{m+l=j}\left(\tau_m\mathbf{b}_l - k_m\mathbf{t}_l\right). \tag{21}$$

By introducing the multi-component Hasimoto function

$$\varphi_{jx} = \sum_{\alpha+\beta=j}k_{\alpha x}b_\beta + i\sum_{p+q+r=j}\left(k_p\tau_q b_r\right), \quad 0 \le j \le n-1, \tag{22}$$

where

$$b_\beta = \sum_{m_1\cdot 1+m_2\cdot 2+\ldots+m_{n-1}\cdot(n-1)=\beta}\frac{1}{m_1!m_2!\ldots m_{n-1}!}A_1^{m_1}A_2^{m_2}\ldots A_{n-1}^{m_{n-1}}\cdot\exp\left(i\int_{-\infty}^{x}\tau_0\,dx'\right), \tag{23}$$

here

$$A_{i_t} = \exp\left(i\int_{-\infty}^{x}\tau_{i_t}\,dx'\right). \tag{24}$$

Identifying $\mathbf{S}_j(0 \le j \le n-1)$ in (19) with the tangent vector of a curve $\mathbf{t}_j$, we obtain

$$\begin{aligned}\mathbf{t}_{jt} &= \sum_{\alpha+\beta=j}\left(\mathbf{t}_\alpha\times\mathbf{t}_{\beta xx}\right) + \frac{\epsilon}{2}\sum_{a+b+c=j}\left[\left(\mathbf{t}_{ax}\cdot\mathbf{t}_{bx}\right)\cdot\mathbf{t}_{cx}\right],\\
&= \sum_{p+q=j}\left(\eta_p\mathbf{n}_q + \zeta_p\mathbf{b}_q\right) + \frac{\epsilon}{2}\sum_{m+n+p+q=j}k_m k_n k_p\mathbf{n}_q, \quad 0 \le j \le n-1.\end{aligned}\tag{25}$$

Then we have

$$\gamma^j = -\sum_{p+q=j}\left(\eta^p + i\zeta^q\right)b_q = -i\varphi_{jx} - \frac{\epsilon}{2}\sum_{a+b+c=j}\overline{\varphi}_a\varphi_b\varphi_c. \tag{26}$$

By the equation

$$R_{jx} = \frac{i}{2}\sum_{\alpha+\beta=j}\left(\gamma^\alpha\overline{\varphi}^\beta - \overline{\gamma}^\alpha\varphi^\beta\right) = \frac{1}{2}\left(\sum_{a+b=j}\overline{\varphi}_a\varphi_b\right)_x, \tag{27}$$

one finds that the time evolution equation satisfies the following equation

$$\varphi_{jt} + \gamma_{jx} - i\sum_{\alpha+\beta=j}R_\alpha\varphi_\beta = 0. \tag{28}$$

Substituting (26) and (27) into (28) and taking $\varphi_j \to 2\varphi_j$, we derive the Z_n-mixed derivative NLSE equation

$$i\varphi_{jt} + \varphi_{jxx} + 2\sum_{\alpha+\beta+\gamma=j}\varphi_\alpha\overline{\varphi}_\beta\varphi_\gamma - 2i\epsilon\left(\sum_{\alpha+\beta+\gamma=j}\varphi_\alpha\overline{\varphi}_\beta\varphi_\gamma\right)_x = 0, \quad 0 \le j \le n-1. \tag{29}$$

Taking $\epsilon = 0$, the Z_n-mixed derivative NLSE equation (29) degenerates into Z_n-NLSE equation (12). Then we obtain the Lax representation of the Z_n-mixed derivative NLSE equation

$$\begin{aligned}U &= i\sum_{k=0}^{n-1}\sum_{a+b=k}\left(2\epsilon\sum_{m+n=a}\lambda_m\lambda_n + 2\lambda_a\right)\sigma_3^b - \sum_{k=0}^{n-1}\sum_{m+n=k}\left(2\epsilon\lambda_m + 1\right)A_n,\\
V &= \sum_{k=0}^{n-1}\sum_{f+g=k}\Bigg[-8i\epsilon^2\sum_{m+n+p+q=f}\lambda_m\lambda_n\lambda_p\lambda_q - 16i\epsilon\sum_{m+n+p=f}\lambda_m\lambda_n\lambda_p\\
&\quad + \sum_{m+n+p+q=f}\left(-8i + 4i\epsilon^2\varphi_m\overline{\varphi}_n\right)\lambda_p\lambda_q + 4i\epsilon\sum_{m+n+p=f}\lambda_m\varphi_n\overline{\varphi}_p + i\sum_{m+n=f}\varphi_m\overline{\varphi}_n\Bigg]\sigma_3^g\\
&\quad + \sum_{k=0}^{n-1}\sum_{m+n=k}\left(8\epsilon^2\sum_{a+b+c=m}\lambda_a\lambda_b\lambda_c + 12\epsilon\sum_{a+b=m}\lambda_a\lambda_b + 4\lambda_m\right)A_n\\
&\quad - \sum_{k=0}^{n-1}\sum_{m+n=k}\left(2\epsilon\lambda_m + 1\right)B_n,\end{aligned}\tag{30}$$

where

$$A_j = \begin{pmatrix}0 & \varphi_j\\ -\overline{\varphi}_j & 0\end{pmatrix}, \quad \sigma_3^j = \begin{pmatrix}\Gamma^j & 0\\ 0 & -\Gamma^j\end{pmatrix}. \tag{31}$$

and

$$B_j = \begin{pmatrix} 0 & i\varphi_{jx} + 2\epsilon \sum_{a+b+c=j} \overline{\varphi}_a \varphi_b \varphi_c \\ i\varphi_{jx} - 2\epsilon \sum_{a+b+c=j} \overline{\varphi}_a \varphi_b \overline{\varphi}_c & 0 \end{pmatrix}. \tag{32}$$

4. Generalized Z_n-Heisenberg Ferromagnet Model in (2+1)-Dimensions

Many (2+1)-dimensional integrable inhomogeneous Heisenberg ferromagnet equations have been of interest, for instance, Inhomogeneous M-I equation [13] and the Ishimori equation [11]. The Ishimori equation [11] is a well-known (2+1)-dimensional integrable extension of the HF model, which involves an infinite dimensional symmetry algebra with a loop algebra structure and is solved by the inverse scattering transform approach. There is geometrical and gauge equivalence between the Ishimori equation and Davey-Stewartson equation [29, 30]. In this section, we shall derive the multi-component counterparts of two types deformed HF models in (2+1)-dimensions.

4.1. Z_n-Inhomogeneous M-I Equation. Let $\mathbf{S}$ take values in a commutative algebra, we have $\mathbf{S} = \mathbf{S}_0 E + \mathbf{S}_1 \Gamma + \ldots + \mathbf{S}_{n-1}\Gamma^{n-1}$. By means of multi-component generalization, we obtain the generalized Z_n-inhomogeneous Heisenberg ferromagnet model in (2+1)-dimensions

$$\begin{aligned} \mathbf{S}_{jt} = &\sum_{m+n=j} \left(\mathbf{S}_{mx} \times \mathbf{S}_{ny} + \mathbf{S}_m \times \mathbf{S}_{nxy} + u_{mx}\mathbf{S}_n + u_m \mathbf{S}_{nx} + \rho_m \mathbf{S}_{nx}\right), \\ &0 \le j \le n-1, \end{aligned} \tag{33}$$

where

$$u_{jx} = -\sum_{a+b+c=j} S_a \cdot \left(\mathbf{S}_{bx} \times \mathbf{S}_{cy}\right) \tag{34}$$

and the parameters ρ_m satisfy

$$\rho_m = \sum_{a+b=m} \mu_{3a} x_b + \nu_{3m}. \tag{35}$$

When $j = 0$, Eq. (33) reduced to the integrable inhomogeneous Myrzakulov-I equation [13].

The linear problem of the multi-component HF models (33) in (2+1)-dimensions can be expressed as

$$\begin{aligned} \xi_x^n &= \frac{i}{2} \sum_{l=0}^{n} \sum_{\alpha+\beta+\gamma=l} \lambda_\alpha \xi_\gamma \mathbf{S}_\beta \sigma^l, \\ \xi_t^n &= -\sum_{f+g=n} \lambda_f \xi_y^g + \frac{i}{2} \sum_{l=0}^{n} \Bigg[\sum_{\alpha+\beta+p+q=l} \lambda_\alpha \left(\rho_\beta + u_\beta\right) \mathbf{S}_p \xi_q \\ &\quad + \sum_{m+i+j+\gamma=l} \lambda_m \left(\mathbf{S}_i \times \mathbf{S}_{jy}\right) \xi_\gamma \Bigg] \sigma^l, \end{aligned} \tag{36}$$

where

$$\mathbf{S}_\beta = \left(S_{1\beta}, S_{2\beta}, S_{3\beta}\right), \quad \mathbf{X}_l = \sigma^l = \left(\sigma_1^l, \sigma_2^l, \sigma_3^l\right). \tag{37}$$

and

$$\sigma_1^l = \begin{pmatrix} 0 & \Gamma^l \\ \Gamma^l & 0 \end{pmatrix}, \quad \sigma_2^l = \begin{pmatrix} 0 & i\Gamma^l \\ -i\Gamma^l & 0 \end{pmatrix}, \quad \sigma_3^l = \begin{pmatrix} \Gamma^l & 0 \\ 0 & -\Gamma^l \end{pmatrix}. \tag{38}$$

Then the Lax representation of Eq. (33) is given by

$$\begin{aligned} \tilde{F} &= -\frac{i}{2} \sum_{l=0}^{n} \sum_{\alpha+\beta=l} \lambda_\alpha \mathbf{S}_\beta \sigma^l, \\ \tilde{G} &= -\frac{i}{2} \sum_{l=0}^{n} \left[\sum_{p+q+r=l} \lambda_p \left(\rho_q + u_q\right) \mathbf{S}_r + \sum_{i+j+m=l} \lambda_m \left(\mathbf{S}_i \times \mathbf{S}_{jy}\right) \right] \sigma^l. \end{aligned} \tag{39}$$

Now one considers the the geometrical equivalent counterpart of the multi-component Eq. (33). Let us introduce the multi-component Serret-Frenet equation

$$\begin{aligned} \mathbf{t}_{jy} &= -\sum_{p+q+l=j} \frac{u_p x}{k_q} \mathbf{b}_l + \partial_x^{-1} \sum_{p+q=j} k_{py} \mathbf{n}_q - \partial_x^{-1} \sum_{p+q+l+m=j} \frac{\tau_p u_{qx}}{k_l} \mathbf{n}_m, \\ \mathbf{b}_{jy} &= -\sum_{p+q=j} \left(u_p + \partial_x^{-1} \tau_{py}\right) \mathbf{n}_q + \sum_{p+q+l=j} \frac{u_{px}}{k_q} \mathbf{t}_l, \\ \mathbf{n}_{jy} &= \sum_{p+q=j} \left(u_p \mathbf{b}_q + \partial_x^{-1} \tau_{py} \mathbf{b}_q - \partial_x^{-1} k_{py} \mathbf{t}_q\right) + \sum_{p+q+l+m=j} \frac{\tau_p u_{qx}}{k_l} \mathbf{t}_m. \end{aligned} \tag{40}$$

Then we derive the multi-component Hasimoto function

$$\varphi_{jy} = \sum_{\alpha+\beta=j} k_{ay} b_\beta + i \sum_{p+q+r=j} \left(\partial_x^{-1} k_p \tau_{qy} b_r\right), \quad 0 \le j \le n-1. \tag{41}$$

In order to derive the geometrical equivalent counterpart of (33), we identify $S_j (0 \le j \le n-1)$ in the vector form of the Z_n-generalized inhomogeneous HF model in (2+1)-dimensions (33) with the tangent vector of a curve $\mathbf{t}_j$. Then we have

$$\begin{aligned} \mathbf{t}_{jt} &= \sum_{m+l=j} \mathbf{t}_m \times \mathbf{t}_{lxy} + \sum_{m+l=j} \mathbf{t}_{mx} \times \mathbf{t}_{ly} + \sum_{m+l=j} u_{mx} \mathbf{t}_l \\ &\quad + \sum_{m+l=j} u_m \mathbf{S}_{lx} + \sum_{m+l=j} \rho_m \mathbf{S}_{lx}, \\ &= \sum_{p+q=j} k_{py} \mathbf{b}_q + \sum_{p+q+m=j} \left(-k_p \partial_x^{-1} \tau_{qy} \mathbf{n}_m + \rho_p k_q \mathbf{n}_m\right), \\ &\quad 0 \le j \le n-1. \end{aligned} \tag{42}$$

Thus we obtain

$$\gamma^j = -\sum_{p+q=j} (\eta^p + i\zeta^q) b_q = -i\varphi_{jy} - \sum_{a+b=j} \rho_a \varphi_b. \tag{43}$$

By the equation

$$R_{jx} = \frac{i}{2} \sum_{\alpha+\beta=j} \left(\gamma^\alpha \overline{\varphi}^\beta - \overline{\gamma}^\alpha \varphi^\beta\right) = -\frac{1}{2} \partial_y \left(\sum_{a+b=j} \overline{\varphi}_a \varphi_b \right). \tag{44}$$

Substituting (43) and (44) into (28) and taking $R_{jx} \to -R_{jx}$, we derive the Z_n-NLS equation

$$i\varphi_{jt} - \varphi_{jxy} - i \left(\sum_{\alpha+\beta=j} \rho_\alpha \varphi_\beta \right)_x - \sum_{a+b=j} R_a \varphi_b = 0, \quad 0 \le j \le n-1, \tag{45}$$

where

$$R_{jx} = \frac{1}{2}\partial_y\left(\sum_{a+b=j}\overline{\varphi}_a\varphi_b\right), \quad 0 \le j \le n-1. \tag{46}$$

When $j = 0$, the Z_n-NLS equation (45) degrades into the (2+1)-dimensional focusing nonlinear Schrödinger equation equation [13]. The Lax representation of the Z_n-HLS equation can be expressed as

$$\tilde{U} = \frac{1}{2}\begin{pmatrix} i\lambda & \varphi \\ -\overline{\varphi} & -i\lambda \end{pmatrix}, \quad \tilde{V} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ -\overline{\tilde{B}} & -\tilde{A} \end{pmatrix}, \tag{47}$$

where

$$\begin{aligned} \tilde{A} &= -\frac{1}{2}\overset{n-1}{\underset{k=0}{-}} R_k + \frac{i}{2}\sum_{k=0}^{n-1}\sum_{m+n=k}\lambda_m\rho_n, \\ \tilde{B} &= -\frac{1}{2}\sum_{k=0}^{n-1}\varphi_{ky} + \frac{i}{2}\sum_{k=0}^{n-1}\sum_{m+n=k}\rho_m\varphi_n, \end{aligned} \tag{48}$$

where $\lambda_j(0 \le j \le n-1)$ are spectral parameters.

4.2. Z_n-Ishimori Equation. Based on the multi-component generalization, we construct the multi-component Ishimori equation in (2+1)-dimensions

$$\begin{aligned} &i\mathbf{S}_{jt} + \sum_{m+n=j} iu_{my}\mathbf{S}_{nx} + iu_{mx}S_{ny} + \tfrac{1}{2}\left[\mathbf{S}_m, \left(\tfrac{1}{4}\mathbf{S}_{nxx} + \alpha^2\mathbf{S}_{nyy}\right)\right] = 0, \\ &\alpha^2 u_{jyy} - \tfrac{1}{4}u_{jxx} = \sum_{m+n=j}\tfrac{\alpha^2}{4i}tr\left(S_m\left[\mathbf{S}_{ny}, \mathbf{S}_{nx}\right]\right), \quad 0 \le j \le n-1. \end{aligned} \tag{49}$$

The Lax representation of (49) is given by

$$\phi_x = U\phi_y, \quad \phi_t = W\phi_{xx} + V\phi_x, \tag{50}$$

where

$$\begin{aligned} U_j &= -\alpha\mathbf{S}_j, \\ V_j &= -i\mathbf{S}_{jxx} + u_{jy} + \sum_{m+n=j}\left(-i\alpha\mathbf{S}_{ny}\mathbf{S}_m - \alpha^3 u_{mx}\mathbf{S}_n\right), \\ W_j &= -2i\mathbf{S}_j. \end{aligned} \tag{51}$$

In terms of gauge transformation

$$\tilde{\psi}_j = g_j\phi_j. \tag{52}$$

The functions g_j and $\mathbf{S}_j$ can be written as

$$\begin{aligned} g_j &= \left(f_{1j} + \sum_{m+n=j} f_{1m}S_{3n}, \sum_{m+n=j} f_{1m}S_n^-, \sum_{m+n=j} f_{2m}S_n^+,\right. \\ &\qquad \left. -f_{2j} - \sum_{m+n=j} f_{2m}S_{3n}\right)\cdot\left(\omega_{1j}, \omega_{2j}, \omega_{3j}, \omega_{4j}\right), \\ \mathbf{S}_j &= \sum_{a+b+c=j} g_a^{-1}\sigma_{3b}g_c, \end{aligned} \tag{53}$$

where $f_{ij}, \omega_{ij}, S_j^\pm$ satisfy the following equations:

$$\begin{aligned} &S_j^\pm = S_{1j} \pm iS_{2j}, \\ &\sum_{m+n=j} 2\left(1 + S_{3m}\left[\alpha\left(\ln f_{1n\,y} - \frac{1}{2}\left(lnf_{1n}\right)_x\right]\right)\right. \\ &\quad = \frac{1}{2}\left(S_{3jx} + \sum_{m+n=j} S_{3m}S_{3nx} + S_{mx}^-S_n^+\right) \\ &\quad - \alpha\left(S_{3jy} + \sum_{m+n=j} S_{3m}S_{3ny} + S_{my}^-S_n^+\right), \end{aligned} \tag{54}$$

and

$$\begin{aligned} &\sum_{m+n=j} 2\left(1 + S_{3m}\left[\alpha\left(\ln f_{2n}\right)_y + \frac{1}{2}\left(\ln f_{2n}\right)_x\right]\right) \\ &\quad = -\frac{1}{2}\left(S_{3jx} + \sum_{m+n=j} S_{3m}S_{3nx} + S_{mx}^+S_n^-\right) \\ &\quad - \alpha\left(S_{3jy} + \sum_{m+n=j} S_{3m}S_{3ny} + S_{my}^+S_n^-\right), \end{aligned} \tag{55}$$

here

$$\begin{aligned} &\omega_{1j} = \begin{pmatrix} \Gamma^j & 0 \\ 0 & 0 \end{pmatrix}, \omega_{2j} = \begin{pmatrix} 0 & \Gamma^j \\ 0 & 0 \end{pmatrix}, \omega_{3j} = \begin{pmatrix} 0 & 0 \\ \Gamma^j & 0 \end{pmatrix}, \\ &\omega_{4j} = \begin{pmatrix} 0 & 0 \\ 0 & \Gamma^j \end{pmatrix}. \end{aligned} \tag{56}$$

Then it follows that

$$\alpha g_{jy} - \sum_{m+n=j} B_{1m}g_{nx} = \sum_{m+n=j} B_{0m}g_n, \tag{57}$$

where

$$B_{1j} = \begin{pmatrix} \frac{1}{2}\Gamma^j & 0 \\ 0 & -\frac{1}{2}\Gamma^j \end{pmatrix}, \quad B_{0j} = \begin{pmatrix} 0 & q_j \\ p_j & 0 \end{pmatrix}. \tag{58}$$

Thus we obtain the gauge equivalent counterpart of Eq. (49) which can be considered as the Z_n-Davey-Stewartson equation

$$\begin{aligned} &iq_{jt} + \frac{1}{4}q_{jxx} + \alpha^2 q_{jyy} + \sum_{m+n=j} v_m q_n = 0, \\ &\alpha^2 v_{jyy} - \frac{1}{4}v_{jxx} = \sum_{m+n=j} -2\alpha^2\left(p_m q_n\right)_{yy} + \frac{1}{4}\left(p_m q_n\right)_{xx}. \end{aligned} \tag{59}$$

Its Lax reprensentation is given by

$$\begin{aligned} \alpha\tilde{\psi}_y &= \tilde{B}_1\tilde{\psi}_x + \tilde{B}_0\tilde{\psi}, \\ \tilde{\psi}_t &= i\tilde{C}_0\tilde{\psi}_{xx} + \tilde{C}_1\tilde{\psi}_x + \tilde{C}_0\tilde{\psi}. \end{aligned} \tag{60}$$

with

$$\begin{aligned} \tilde{B}_0 &= B_{00} + B_{01} + B_{02} + \ldots + B_{0(n-1)}, \\ \tilde{B}_1 &= B_{10} + B_{11} + B_{12} + \ldots + B_{1(n-1)}, \end{aligned} \tag{61}$$

where

$$
\begin{aligned}
&B_{0j} = \mathbf{B}_{0j} \cdot \mathbf{a}_j, \ \ \mathbf{B}_{0j} = \left(q_j, p_j\right), \quad \mathbf{a}_j = \left(a_{1j}, a_{2j}\right),\\
&a_{1j} = \begin{pmatrix} 0 & \Gamma^j \\ 0 & 0 \end{pmatrix}, \quad a_{2j} = \begin{pmatrix} 0 & 0 \\ \Gamma^j & 0 \end{pmatrix},\\
&B_{1j} = \begin{pmatrix} \frac{1}{2}\Gamma^j & 0 \\ 0 & -\frac{1}{2}\Gamma^j \end{pmatrix}.
\end{aligned} \tag{62}
$$

Similarly,

$$
\begin{aligned}
\tilde{C_0} &= C_{00} + C_{01} + C_{02} + \ldots + C_{0(n-1)},\\
\tilde{C_1} &= C_{10} + C_{11} + C_{12} + \ldots + C_{1(n-1)},\\
\tilde{C_2} &= C_{20} + C_{21} + C_{22} + \ldots + C_{2(n-1)},
\end{aligned} \tag{63}
$$

where

$$
\begin{aligned}
&C_{0j} = \mathbf{C}_{0j} \cdot \mathbf{b}_j, \mathbf{C}_{0j} = \left(c_{11j}, c_{12j}, c_{21j}, c_{22j}\right), \mathbf{b}_j = \left(b_{1j}, b_{2j}, b_{3j}, b_{4j}\right).\\
&b_{1j} = \begin{pmatrix} \Gamma^j & 0 \\ 0 & 0 \end{pmatrix}, b_{2j} = \begin{pmatrix} 0 & \Gamma^j \\ 0 & 0 \end{pmatrix}, b_{3j} = \begin{pmatrix} 0 & 0 \\ \Gamma^j & 0 \end{pmatrix}, b_{4j} = \begin{pmatrix} 0 & 0 \\ 0 & \Gamma^j \end{pmatrix},
\end{aligned} \tag{64}
$$

with

$$
c_{12j} = \frac{i}{2} q_{jx} + i\alpha q_{jy}, \quad c_{21j} = \frac{i}{2} p_{jx} - i\alpha p_{jy}. \tag{65}
$$

Here c_{11j} and c_{22j} are the solution of the following equations

$$
\begin{aligned}
\frac{1}{2} c_{11jx} - \alpha c_{11jy} &= i\left(\frac{1}{2} \sum_{m+n=j} \left(p_m q_n\right)_x + \alpha \sum_{m+n=j} \left(p_m q_n\right)_y\right),\\
-\frac{1}{2} c_{22jx} - \alpha c_{22jy} &= i\left(\frac{1}{2} \sum_{m+n=j} \left(p_m q_n\right)_x - \alpha \sum_{m+n=j} \left(p_m q_n\right)_y\right).
\end{aligned} \tag{66}
$$

C_{1j} and C_{2j} are given by

$$
\begin{aligned}
&C_{1j} = \mathbf{C}_{1j} \cdot \mathbf{c}_j, \quad \mathbf{C}_{1j} = \left(q_j, p_j\right), \mathbf{c}_j = \left(c_{1j}, c_{2j}\right),\\
&c_{1j} = \begin{pmatrix} 0 & i\Gamma^j \\ 0 & 0 \end{pmatrix}, \quad c_{2j} = \begin{pmatrix} 0 & 0 \\ i\Gamma^j & 0 \end{pmatrix},\\
&C_{2j} = \begin{pmatrix} \frac{1}{2}\Gamma^j & 0 \\ 0 & -\frac{1}{2}\Gamma^j \end{pmatrix}.
\end{aligned} \tag{67}
$$

5. Summary and Discussion

Considering the commutative subalgebra $gl(n, \mathbb{C})$, we have constructed three types generalized Z_n-HF models in (1+1) and (2+1)-dimensions. From the geometrical and gauge equivalence point of view, we also establish the corresponding equivalent counterparts of three types generalized Z_n -Heisenberg ferromagnet models. The introduction of new degrees of freedom may emerge from multiscale procedures or regularizations of gradient catastrophes. Their physical meaning and application should be of interest. The methods in the paper may clearly be applied to the other generalized Heisenberg supermagnetic models. Therefore, other types of generalized Z_n-HF models still deserve further study.

Data Availability

All data included in this study are available upon request by contact with the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work is partially supported by National Natural Science Foundation of China (Grant Nos. 11965014, 11605096, and 11601247), National Undergraduate Training Program for Innovation (Grant No. 201810126023). We thank Professor Ke Wu and Professor Wei-Zhong Zhao (CNU, China) for long-term encouragements and supports. The authors thank the valuable suggestions of the referees.

References

[1] L. Martina, O. K. Pashaev, and G. Soliani, "Bright solitons as black holes," *Physical Review D*, vol. 58, no. 8, Article ID 084025, 1998.

[2] G. Arutyunov, S. Frolov, J. Russo, and A. A. Tseytlin, "Spinning strings in AdS5×S5 and integrable systems," *Nuclear Physics B*, vol. 671, pp. 3–50, 2003.

[3] V. A. Kazakov, A. Marshakov, J. A. Minahan, and K. Zarembo, "Classical/quantum integrability in AdS/CFT," *Journal of High Energy Physics*, vol. 2004, no. 05, pp. 024–024, 2004.

[4] M. Lakshmanan, "Continuum spin system as an exactly solvable dynamical system," *Physics Letters A*, vol. 61, no. 1, pp. 53–54, 1977.

[5] V. E. Zakharov and L. A. Takhtajan, "Equivalence of the nonlinear Schrödinger equation and the equation of a Heisenberg ferromagnet," *Theoretical and Mathematical Physics*, vol. 38, pp. 17–23, 1979.

[6] X. J. Duan, M. Deng, W. Z. Zhao, and K. Wu, "The prolongation structure of the inhomogeneous equation of the reaction–diffusion type," *Journal of Physics A: Mathematical and Theoretical*, vol. 40, pp. 3831–3837, 2007.

[7] W. Z. Zhao, Y. Q. Bai, and K. Wu, "Generalized inhomogeneous Heisenberg ferromagnet model and generalized nonlinear Schrödinger equation," *Physics Letters A*, vol. 352, no. 1-2, pp. 64–68, 2006.

[8] M. Lakshmanan, K. Porsezian, and M. Daniel, "Effect of discreteness on the continuum limit of the Heisenberg spin chain," *Physics Letters A*, vol. 133, no. 9, pp. 483–488, 1988.

[9] K. Porsezian, "Nonlinear Schrödinger family on moving space curves: lax pairs, soliton solution and equivalent spin chain fn1," *Chaos Soliton & Fractals*, vol. 9, no. 10, pp. 1709–1722, 1998.

[10] R. Myrzakulov, G. Mamyrbekova, G. Nugmanova, and M. Lakshmanan, "Integrable (2+1)-dimensional spin models with self-consistent potentials," *Symmetry*, vol. 7, no. 3, pp. 1352–1375, 2015.

[11] Y. Ishimori, "Multi-vortex solutions of a two-dimensional nonlinear wave equation," *Progress of Theoretical Physics*, vol. 72, no. 1, pp. 33–37, 1984.

[12] Y. Zhai, S. Albeverio, W. Z. Zhao, and K. Wu, "Prolongation structure of the (2+1)-dimensional integrable Heisenberg ferromagnet model," *Journal of Physics A: Mathematical and General*, vol. 39, no. 9, pp. 2117–2126, 2006.

[13] Z.-H. Zhang, M. Deng, W. Z. Zhao, and K. Wu, "On the (2+1)-dimensional integrable inhomogeneous Heisenberg ferromagnet equation," *Journal of Physical Society of Japan*, vol. 75, Article ID 104002, 2006.

[14] C. Álvarez-Fernández, U. Fidalgo, and M. Mañas, "The multicomponent 2D Toda hierarchy: generalized matrix orthogonal polynomials, multiple orthogonal polynomials and Riemann-Hilbert problems," *Inverse Problems*, vol. 26, no. 5, Article ID 055009, 2010.

[15] M. Mañas and L. M. Alonso, "The multicomponent 2D Toda hierarchy: dispersionless limit," *Inverse Problems*, vol. 25, no. 11, Article ID 115020, 2009.

[16] M. Mañas, L. M. Alonso, and C. Álvarez-Fernández, "The multicomponent 2D Toda hierarchy: discrete flows and string equations," *Inverse Problems*, vol. 25, no. 6, Article ID 065007, 2009.

[17] C. Álvarez-Fernández, U. F. Prieto, and M. Mañas, "Multiple orthogonal polynomials of mixed type: Gauss¨CBorel factorization and the multi-component 2D Toda hierarchy," *Advances in Mathematics*, vol. 227, no. 4, pp. 1451–1525, 2011.

[18] E. Daems and A. B. J. Kuijlaars, "Multiple orthogonal polynomials of mixed type and non-intersecting Brownian motions," *Journal of Approximation Theory*, vol. 146, no. 1, pp. 91–114, 2007.

[19] V. G. Kac and J. W. van de Leur, "The n-component KP hierarchy and representation theory," *Journal of Mathematical Physics*, vol. 44, no. 8, pp. 3245–3293, 2003.

[20] M. Adler, P. van Moerbeke, and P. Vanhaecke, "Moment matrices and multi-component KP, with applications to random matrix theory," *Communications in Mathematical Physics*, vol. 286, no. 1, pp. 1–38, 2009.

[21] C. Z. Li, "Gauge transformation and symmetries of the commutative multi-component BKP hierarchy," *Journal of Physics A: Mathematical and Theoretical*, vol. 49, no. 1, Article ID 015203, 2015.

[22] L. V. Bogdanov and B. G. Konopelchenko, "Analytic-bilinear approach to integrable hierarchies. II. Multicomponent KP and 2D Toda lattice equations," *Journal of Mathematical Physics*, vol. 39, no. 9, pp. 4701–4728, 1998.

[23] I. A. B. Strachan and D. F. Zuo, "Integrability of the Frobenius algebra-valued Kadomtsev-Petviashvili hierarchy," *Journal of Mathematical Physics*, vol. 56, no. 11, Article ID 113509, 2015.

[24] D. F. Zuo, "Local matrix generalizations of W-algebras," 2014, http://arXiv.org/abs/1401.2216.

[25] C. Li and J. He, "On the extended multi-component toda hierarchy," *Mathematical Physics, Analysis and Geometry*, vol. 17, no. 3-4, pp. 377–407, 2014.

[26] X. Yang and C. Li, "Bäcklund transformations of Zn-sine-Gordon systems," *Modern Physics Letters B*, vol. 31, no. 17, Article ID 1750189, 2017.

[27] Z. W. Yan, S. C. Wan, M. Zhu, and C. Z. Li, "On the Zn-Heisenberg ferromagnet model," *Submmited*, 2019.

[28] A. Kundu, "Landau-Lifshitz and higherorder nonlinear systems gauge generated from nonlinear Schrödingertype equations," *Journal of Mathematical Physics*, vol. 25, no. 12, pp. 3433–3438, 1984.

[29] M. Lakshmanan, R. Myrzakulov, S. Vijayalakshmi, and A. K. Danlybaeva, "Motion of curves and surfaces and nonlinear evolution equations in (2+1) dimensions," *Journal of Mathematical Physics*, vol. 39, no. 7, pp. 3765–3771, 1998.

[30] R. A. Leo, L. Martina, and G. Soliani, "Gauge equivalence theory of the noncompact Ishimori model and the Davey¨CStewartson equation," *Journal of Mathematical Physics*, vol. 33, no. 4, pp. 1515–1522, 1992.