Hindawi
Advances in Mathematical Physics
Volume 2020, Article ID 1545043, 6 pages
https://doi.org/10.1155/2020/1545043

Research Article

Existence of Solution of Space–Time Fractional Diffusion-Wave Equation in Weighted Sobolev Space

Kangqun Zhang

Department of Mathematics and Physics, Nanjing Institute of Technology, Nanjing 211167, China

Correspondence should be addressed to Kangqun Zhang; chkqnju@hotmail.com

Received 2 July 2019; Accepted 20 August 2019; Published 7 February 2020

Academic Editor: Zine El Abiddine Fellah

In this paper, we consider Cauchy problem of space-time fractional diffusion-wave equation. Applying Laplace transform and Fourier transform, we establish the existence of solution in terms of Mittag-Leffler function and prove its uniqueness in weighted Sobolev space by use of Mikhlin multiplier theorem. The estimate of solution also shows the connections between the loss of regularity and the order of fractional derivatives in space or in time.

1. Introduction

In this paper, we focus space-time fractional diffusion-wave equation

$$ {}_0^C\left(\frac{\partial}{\partial t}\right)^{\alpha_1} u + (-\Delta)^{\alpha_2/2} u = f(t,x), \quad \text{in} \quad (0,+\infty)\times\mathbb{R}^n, \quad (1)$$

where ${}_0^C(\partial/\partial t)^{\alpha_1}$ stands for the Caputo fractional partial derivative operator of order α_1, $\alpha_1 \in (0,1)\cup(1,2)$, $(-\Delta)^{\alpha_2/2}$ is the fractional Laplace differential operator of order α_2, $\alpha_2 \in (1,2)$.

Fractional derivatives describe the property of memory and heredity of many materials, which is the major advantage compared with integer order derivatives. Fractional diffusion-wave equations are obtained from the classic diffusion equation and wave equation by replacing the integral order derivative terms by fractional derivatives of order $\alpha \in (0,1)\cup(1,2)$. It has attracted considerable attention recently for various reasons, which include modeling of anomalous diffusive and subdiffusive systems, description of fractional random walk, wave propagation phenomenon, multiphase fluid flow problems, and electromagnetic theory. Nigmatullin [1, 2] pointed out that many of the universal electromagnetic, acoustic, and mechanical responses can be modeled accurately using the fractional diffusion-wave equations. Schneider and Wyss [3] presented the diffusion and wave equations in terms of integro-differential equations, and obtained the associated Greens functions in closed form in terms of the Foxs functions. Mbodje and Montseny [4] investigated the existence, uniqueness, and asymptotic decay of the wave equation with fractional derivative feedback, and showed that the method developed can easily be adapted to a wide class of problems involving fractional derivative or integral operators of the time variable. In fact, more numerical algorithms present an efficient method in solving the related problem [5–8]. The development of analytical methods is delayed since there are no analytic solutions in many cases [9–12]. Additional background, survey, and more applications of this field in science, engineering, and mathematics can be found in [13–20] and the references therein.

The fractional wave equation has been researched in all probability for the first time in [21] with the same order in space and in time, i.e., $\alpha_1 = \alpha_2$, where an explicit formula for the fundamental solution of this equation was established. Then this feature was shown to be a decisive factor for inheriting some crucial characteristics of the wave equation like a constant propagation velocity of both the maximum of its fundamental solution and its gravity and mass centers in [22]. Moreover, the first, the second, and the Smith centrovelocities of the damped waves described by the fractional wave equation are constant and depend just on the equation order.

While the fractional wave equation contains fractional derivatives of the same order in space and in time, we establish existence of solution of Cauchy problem to fractional wave equation (1) with different order in space and in time in weighted Sobolev spaces. The powers of the weighted show the connections between the loss of the regularity and the orders of the fractional derivatives in space or in time. The main tools are Laplace transform, Fourier transform, Mikhlin multiplier theorem, and Mittag-Leffler functions. Applying the same technique, we also obtain the existence of solution of fractional diffusion equation.

This paper is organized as follows: In Section 2, the related fractional calculus definition and Laplace transform are introduced, the explicit solution of fractional differential equation is given by use of Mittag-Leffler functions. In Section 3, based on the main result given in Section 2, we show the existence and uniqueness of solution of space-time fractional diffusion-wave equation.

2. Laplace Transform and Fractional Calculus

In this section, we recall some necessary definitions and properties of fractional calculus, then use Laplace transform to consider initial value problem of the related fractional differential equation.

Definition 1 ([19]). The Riemann-Liouville fractional integral of order α of a function $f:(0,\infty)\to\mathbb{R}$ is defined by

$$ {}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds, \quad \Re\alpha > 0. \tag{2}$$

Definition 2 ([19]). The Riemann-Liouville fractional derivative of order $\alpha\in(0,1)\cup(1,2)$ of a function $f:(0,\infty)\to\mathbb{R}$ is defined by

$$ {}_0D_t^{\alpha} f(t) = \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\, {}_0I_t^{\lceil\alpha\rceil-\alpha}(t). \tag{3}$$

Definition 3 ([19]). The Caputo fractional derivative of order α of a function $f:(0,\infty)\to\mathbb{R}$ is defined by

$$ {}_0^CD_t^{\alpha} f(t) = \begin{cases} {}_0D_t^{\alpha} f(t)\left(f(t)-f(0^+)\right), & \alpha\in(0,1) \\ {}_0D_t^{\alpha} f(t)\left(f(t)-f(0^+)-tf'(0^+)\right), & \alpha\in(1,2). \end{cases} \tag{4}$$

The Mittag-Leffler function $E_{\alpha,\beta}(z)$ [23] is represented by

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \Re(\alpha)>0, \Re(\beta)>0, \tag{5}$$

where $\Re(\alpha)$ and $\Re(\beta)$ denotes the real part of the complex numbers α and β, respectively.

Lemma 1 ([23]).

$$ \frac{d}{dy}E_{\alpha,\beta}(y) = \frac{E_{\alpha,\beta-1}(y)-(\beta-1)E_{\alpha,\beta}(y)}{\alpha y}, \tag{6}$$

$$ \frac{d^m}{dy^m}\left(y^{\beta-1}E_{\alpha,\beta}(y^{\alpha})\right) = y^{\beta-m-1}E_{\alpha,\beta-m}(y^{\alpha}), \quad \Re(\beta-m)>0, \quad m\in\mathbb{N}. \tag{7}$$

Lemma 2 ([23]). *Let $\alpha<2$, $\beta\in\mathbb{R}$ and $(\pi\alpha/2)<\mu<\min\{\pi,\pi\alpha\}$. Then we have the following estimate*

$$ \left|E_{\alpha,\beta}(y)\right| \le \frac{M}{1+|y|}, \quad \mu\le|\arg y|\le\pi. \tag{8}$$

where M denotes a positive constant.

Lemma 3 ([24]). *For any $\alpha>0$, $\beta>0$ and $\lambda\in\mathbb{C}$, there is*

$$ \mathfrak{L}\left[t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\right] = s^{\alpha-\beta}\left(s^{\alpha}-\lambda\right)^{-1}, \tag{9}$$

with $\Re s > \|\lambda\|^{1/\alpha}$, where $\Re s$ denotes the real part of the complex number s, the Laplace transform of a function $f(t)$ is defined by

$$ \mathfrak{L}[f](s) = \int_0^{\infty} e^{-st}f(t)dt. \tag{10}$$

The initial value problem of fractional differential equation for $\alpha\in(0,1)$,

$$ \begin{cases} {}_0^CD_t^{\alpha}u(t) = \lambda u(t)+f(t), & t>0, \\ u(0)=u_0, \end{cases} \tag{11}$$

where ${}_0^CD_t^{\alpha}$ stands for a Caputo fractional derivative operator, u_0 is a constant number.

Theorem 1 ([24]). *Consider the problem (11), then there is a explicit solution which is given in the integral form*

$$ u(t) = u_0E_{\alpha,1}(\lambda t^{\alpha}) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^{\alpha}\right)f(s)ds. \tag{12}$$

The initial value problem of fractional differential equation for $\alpha\in(1,2)$,

$$ \begin{cases} {}_0^CD_t^{\alpha}u(t) = \lambda u(t)+f(t), & t>0, \\ u(0)=u_0, \\ u'(0)=u_1, \end{cases} \tag{13}$$

where ${}_0^CD_t^{\alpha}$ denotes a Caputo fractional derivative operator, u_i $(i=0,1)$ is a constant number.

Theorem 2. *Consider the problem (13), then there is a explicit solution which is given in the integral form*

$$ \begin{aligned} u(t) =& u_0E_{\alpha,1}(\lambda t^{\alpha}) + u_1tE_{\alpha,2}(\lambda t^{\alpha}) \\ &+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^{\alpha}\right)f(s)ds. \end{aligned} \tag{14}$$

Proof. According to Definition 1–3, taking Laplace transform with respect to t in both sides of Eq. (13), we obtain

$$ \mathfrak{L}[u](s) = s^{\alpha-1}\left(s^{\alpha}-\lambda\right)^{-1}u_0 + s^{\alpha-2}\left(s^{\alpha}-\lambda\right)^{-1}u_1 + \left(s^{\alpha}-\lambda\right)^{-1}\mathfrak{L}[f(t)]. \tag{15}$$

The inverse Laplace transform using Lemma 3 yields

$$ \mathfrak{L}^{-1}\left[s^{\alpha-1}\left(s^{\alpha}-\lambda\right)^{-1}\right] = E_{\alpha,1}(\lambda t^{\alpha}) \tag{16}$$

$$ \mathfrak{L}^{-1}\left[s^{\alpha-2}\left(s^{\alpha}-\lambda\right)^{-1}\right] = tE_{\alpha,2}(\lambda t^{\alpha}), \tag{17}$$

$$\begin{aligned}&\mathfrak{L}^{-1}\left[(s^\alpha-\lambda)^{-1}\mathfrak{L}[f(t)]\right]\\&\quad=\mathfrak{L}^{-1}\left[(s^\alpha-\lambda)^{-1}\right]*f(t)\\&\quad=t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)*f(t)\\&\quad=\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha f(s)ds.\end{aligned}\tag{18}$$

Then substitute (15–18) into (13) which yields Theorem 2. □

3. Fourier Transform and the Main Result

In this section, based on the results of Theorem 2, Mikhlin multiplier theorem, Mattag-Leffler function and Fourier transf orm, we establish the existence and uniqueness of solution of Cauchy problem of space-time fractional diffusion-wave equation in weighted Sobolev space.

Definition 4 ([25]). The fourier transform of fractional laplace operator $(-\Delta)^\alpha$ is defined by

$$\mathfrak{F}[(-\Delta)^\alpha f](\xi)=|\xi|^{2\alpha}\mathfrak{F}[f](\xi),\tag{19}$$

where f satisfies $(-\Delta)^\alpha f\in L^p(\mathbb{R}^n)$, $p\in[1,+\infty)$.

For more details of Fourier transformation, one can refer to [26, 27].

First, we consider the fractional wave equation, i.e., the case $\alpha_1\in(1,2)$. Taking Fourier transformation of space variables x on Eq. (1) with initial data $u(0,x)=\varphi_1(x)$, $(\partial/\partial t)u(0,x)=\varphi_2(x)$ yields

$$\begin{cases}{}_0^C\left(\frac{\partial}{\partial t}\right)^{\alpha_1}\hat{u}(t,\xi)=-|\xi|^{\alpha_2}\hat{u}(t,\xi)+\hat{f}(t,\xi), & \text{in}\quad(0,+\infty)\times\mathbb{R}^n,\\ \hat{u}(0,\xi)=\hat{\varphi}_1(\xi),\\ \frac{\partial}{\partial t}\hat{u}(0,\xi)=\hat{\varphi}_2(\xi),\end{cases}\tag{20}$$

where $\widehat{u}(t,\xi)=\mathfrak{F}[(u(t,\cdot)]$.

Set $\lambda=-|\xi|^{\alpha_2}$ in (13), according to Theorem 2, the solution of (20) is given by

$$\begin{aligned}\hat{u}(t,\xi)&=\hat{\varphi}_1(\xi)E_{\alpha_1,1}\left(-|\xi|^{\alpha_2}t^{\alpha_1}\right)+\hat{\varphi}_2(\xi)tE_{\alpha_1,2}\left(-|\xi|^{\alpha_2}t^{\alpha_1}\right)\\&\quad+\int_0^t(t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}\left(-|\xi|^{\alpha_2}(t-s)^{\alpha_1}\right)\hat{f}(s,\xi)ds.\end{aligned}\tag{21}$$

In terms of (2) in Lemma 1 and Lemma 2, by mathematical induction, we conclude.

Lemma 4. *For each $k\in\mathbb{Z}^+$ and any $\Re(\alpha)>0$, $\beta\in\mathbb{R}$, $0\le\delta\le 1$, there exists a positive constant C_k such that*

$$|y|^k\left|\frac{d^k}{dy^k}\left(y^\delta E_{\alpha,\beta}(y)\right)\right|\le C_k.\tag{22}$$

Proof. For $k=1$, Lemmas 1 (6) and 2.5 imply (22) holding.

For $k=2$, $y^2(d^2/dy^2)=(y(d/dy))^2-y(d/dy)$. Then it is enough to show $(y(d/dy))^2(y^\delta E_{\alpha,\beta}(y))$ is bounded since $k=1$ holds. By a direct computation in terms of (6) that

$$\begin{aligned}\left(y\frac{d}{dy}\right)^2\left(y^\delta E_{\alpha,\beta}(y)\right)&=\frac{1}{\alpha}y\frac{d}{dy}\left(y^\delta\left(E_{\alpha,\beta-1}(y)-(\beta-1)E_{\alpha,\beta}(y)\right)\right)\\&\quad+\delta y\frac{d}{dy}\left(y^\delta E_{\alpha,\beta}(y)\right).\end{aligned}\tag{23}$$

This also reduces to the case for $k=1$. Hence, (22) holds for $k=2$.

In the following we conclude that $(y(d/dy))^k(y^\delta E_{\alpha,\beta}(y))$ is bounded for any $k\in\mathbb{Z}^+$ by mathematical induction. Assume for $k-1$, there exist

$$|y|^{k-1}\left|\frac{d^{k-1}}{dy^{k-1}}\left(y^\delta E_{\alpha,\beta}(y)\right)\right|\le C_{k-1},\tag{24}$$

$$y^{k-1}\frac{d^{k-1}}{dy^{k-1}}=\sum_{i=1}^{k-1}b_i\left(y\frac{d}{dy}\right)^i,\tag{25}$$

where b_i are constants. Then by use of (25), we have

$$\begin{aligned}y^k\left(\frac{d}{dy}\right)^k\left(y^\delta E_{\alpha,\beta}(y)\right)&=y\frac{d}{dy}\left(\sum_{i=1}^{k-1}b_i\left(y\frac{d}{dy}\right)^i\left(y^\delta E_{\alpha,\beta}(y)\right)\right)\\&=\sum_{i=1}^{k}d_i\left(y\frac{d}{dy}\right)^i\left(y^\delta E_{\alpha,\beta}(y)\right).\end{aligned}\tag{26}$$

It follows (24) and (26) that (22) is holding. □

Corollary 1. *For each $\gamma\in Z^+$ and any $\alpha>0$, $\beta\in\mathbb{R}$, $0\le\delta\le 1$, there exists a positive constant C_γ such that*

$$\left||\xi|^\gamma\frac{\partial^\gamma}{\partial\xi^\gamma}\left(y^\delta E_{\alpha,\beta}(y)\right)\right|\le C_\gamma,\tag{27}$$

$$\left||\xi|^\gamma\frac{\partial^\gamma}{\partial\xi^\gamma}\left(\frac{\partial}{\partial t}\left(y^\delta E_{\alpha,\beta}(y)\right)\right)\right|\le C_\gamma,\tag{28}$$

where $y=-|\xi|^{\alpha_2}t^{\alpha_1}$.

Next, we choose the version of Mihlin's multiplier theorem given in [28] as our Lemma.

Lemma 5. *Let $a(\xi)$ be the symbol of a singular integral operator A in $\mathbb{R}^n$. Suppose that $a(\xi)\in C^\infty(\mathbb{R}^n\backslash\{0\})$, and there is some positive constant M for all $\xi\ne 0$ such that*

$$|\xi|^{|\gamma|}\left|\frac{\partial^\gamma a(\xi)}{\partial\xi^\gamma}\right|\le M,\quad 0\le|\gamma|\le 1+\frac{[n]}{2}.\tag{29}$$

Then, A is a bounded linear operator from $L^p(\mathbb{R}^n)$ into itself for $1<p<+\infty$, and its operator norm depends only on M, n, and p.

Theorem 3. *Set $1<p<+\infty$, $\alpha_i\in(1,2)$, $i=1,2$. Suppose $\varphi_i\in L^p(\mathbb{R}^n)$, $i=1,2$, $f\in C_0^\infty((0,+\infty),L^p(\mathbb{R}^n))$, then there is a unique solution u of Cauchy problem of space-time fractional wave equation which is represented by*

$$\begin{aligned}u(t,x)&=\varphi_1(x)*\mathfrak{F}^{-1}\left[E_{\alpha_1,1}\left(-|\xi|^{\alpha_2}t^{\alpha_1}\right)\right]+\varphi_2(x)\\&\quad*\mathfrak{F}^{-1}\left(tE_{\alpha_1,2}\left(-|\xi|^{\alpha_2}t^{\alpha_1}\right)\right)\\&\quad+\int_0^t(t-s)^{\alpha_1-1}\mathfrak{F}^{-1}\left[E_{\alpha_1,\alpha_1}\left(-|\xi|^{\alpha_2}(t-s)^{\alpha_1}\right)\right]\\&\quad*f(s,x)ds,\end{aligned}\tag{30}$$

and satisfies

$$\begin{aligned}
&\|u(t,\cdot)\|_{L^p(\mathbb{R}^n)} + \left\|t^{\frac{\alpha_1}{\alpha_2}} u(t,\cdot)\right\|_{\dot{H}^{1,p}(\mathbb{R}^n)} \\
&\quad + \left\|t^{\alpha_1\left(1-\frac{\varepsilon}{\alpha_2}\right)} u(t,\cdot)\right\|_{\dot{H}^{\alpha_2-\varepsilon,p}(\mathbb{R}^n)} \\
&\quad + \left\|t^{\alpha_1\left(1-\frac{\varepsilon}{\alpha_2}\right)} {}_0^C\left(\frac{\partial}{\partial t}\right)^{\alpha_1} u(t,\cdot)\right\|_{\dot{H}^{-\varepsilon,p}(\mathbb{R}^n)} \\
&\lesssim \|\varphi_1\|_{L^p(\mathbb{R}^n)} + t\|\varphi_2\|_{L^p(\mathbb{R}^n)} \\
&\quad + t^{\alpha_1}\left(\sup_{s\in(0,t)} \|f(s,\cdot)\|_{L^p(\mathbb{R}^n)} + \sup_{s\in(0,t)} \|f(s,\cdot)\|_{H^{-\varepsilon,p}(\mathbb{R}^n)}\right)\Bigg),
\end{aligned} \tag{31}$$

for any positive number $\varepsilon \ll 1$, where $\dot{H}^{k,p}(\mathbb{R}^n)$ denotes the classical homogeneous Sobolev space.

Proof. Taking inverse Fourier transform on (21), it is easy to obtain (30). Then, It follows (30) that

$$\begin{aligned}
&\|u(t,\cdot)\|_{\dot{H}^{\delta,p}(R^n)} \\
&= \left\|\mathfrak{F}^{-1}\left(|\xi|^\delta \hat{u}(t,\xi)\right)\right\|_{L^p(\mathbb{R}^n)} \\
&\le \left\|\mathfrak{F}^{-1}\left(|\xi|^\delta \hat{\varphi}_1(\xi) E_{\alpha_1,1}\left(-|\xi|^{\alpha_2} t^{\alpha_1}\right)\right)\right\|_{L^p(\mathbb{R}^n)} \\
&\quad + \left\|\mathfrak{F}^{-1}\left(|\xi|^\delta \hat{\varphi}_2(\xi) t E_{\alpha_1,2}\left(-|\xi|^{\alpha_2} t^{\alpha_1}\right)\right)\right\|_{L^p(\mathbb{R}^n)} \\
&\quad + \left\|\mathfrak{F}^{-1}\left(|\xi|^\delta \int_0^t (t-s)^{\alpha_1-1}\, E_{\alpha_1,\alpha_1}\left(-|\xi|^{\alpha_2}(t-s)_1^{\alpha}\right)\hat{f}(s,\xi)ds\right)\right\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \left\|\mathfrak{F}^{-1}\left(\hat{\varphi}_1(\xi) t^{-\frac{\alpha_1\delta}{\alpha_2}}\left(-|\xi|^{\alpha_2} t^{\alpha_1}\right)^{\frac{\delta}{\alpha_2}} E_{\alpha_1,1}\left(-|\xi|^{\alpha_2} t^{\alpha_1}\right)\right)\right\|_{L^p(\mathbb{R}^n)} \\
&\quad + \left\|\mathfrak{F}^{-1}\left(\hat{\varphi}_2(\xi) t^{1-\frac{\alpha_1\delta}{\alpha_2}}\left(-|\xi|^{\alpha_2} t^{\alpha_1}\right)^{\frac{\delta}{\alpha_2}} E_{\alpha_1,2}\left(-|\xi|^{\alpha_2} t^{\alpha_1}\right)\right)\right\|_{L^p(\mathbb{R}^n)} \\
&\quad + \int_0^t (t-s)^{\alpha_1-1-\frac{\delta\alpha_1}{\alpha_2}} \times \Bigg\|\mathfrak{F}^{-1}\Bigg(\left(-|\xi|^{\alpha_2}(t-s)^{\alpha_1}\right)^{\frac{\delta}{\alpha_2}} \\
&\quad \cdot E_{\alpha_1,\alpha_1}\left(-|\xi|^{\alpha_2}(t-s)_1^{\alpha}\right)\hat{f}(s,\xi)\Bigg)\Bigg\|_{L^p(\mathbb{R}^n)} ds.
\end{aligned} \tag{32}$$

Let $y = -|\xi|^{\alpha_2}(t-s)^{\alpha_1}$, then (27) yields that

$$|\xi|^\gamma \left|\frac{\partial^\gamma}{\partial \xi^\gamma} y^{(\delta/\alpha_2)} E_{\alpha,\beta}(y)\right| \le C_\gamma. \tag{33}$$

According to Corollary 1 and Lemma 5 for $0 \le \delta \le \alpha_2$, we have

$$\left\|\mathfrak{F}^{-1}\left(\hat{\varphi}_1(\xi) t^{-(\alpha_1\delta/\alpha_2)} y^{\delta/\alpha_2} E_{\alpha_1,1}(y)\right)\right\|_{L^p(\mathbb{R}^n)} \lesssim t^{-(\alpha_1\delta/\alpha_2)} \|\varphi_1\|_{L^p(\mathbb{R}^n)}, \tag{34}$$

$$\left\|\mathfrak{F}^{-1}\left(\hat{\varphi}_2(\xi) t^{1-(\alpha_1\delta/\alpha_2)} y^{(\delta/\alpha_2)} E_{\alpha_1,2}(y)\right)\right\|_{L^p(\mathbb{R}^n)} \lesssim t^{1-(\alpha_1\delta/\alpha_2)} \|\varphi_1\|_{L^p(\mathbb{R}^n)}, \tag{35}$$

$$\left\|\mathfrak{F}^{-1}\left(y^{(\delta/\alpha_2)} E_{\alpha_1,\alpha_1}(y)\hat{f}(t,\xi)\right)\right\|_{L^p(\mathbb{R}^n)} \lesssim \|f(t,\cdot)\|_{L^p(\mathbb{R}^n)}. \tag{36}$$

Substitute (34–36) into (32), we get

$$\begin{aligned}
&\|u(t,\cdot)\|_{\dot{H}^{\delta,p}(\mathbb{R}^n)} \\
&\lesssim t^{-(\alpha_1\delta/\alpha_2)}\Bigg(\|\varphi_1\|_{L^p(\mathbb{R}^n)} + t\|\varphi_2\|_{L^p(\mathbb{R}^n)} + t^{\alpha_1} \\
&\quad \cdot \int_0^1 (1-s)^{\alpha_1-1-(\alpha_1\delta/\alpha_2)} \|f(st,\cdot)\|_{L^p(\mathbb{R}^n)} ds\Bigg).
\end{aligned} \tag{37}$$

It is easy to verify that $\alpha_1 - 1 - (\alpha_1\delta/\alpha_2) > -1$ holds for $0 \le \delta \le \alpha_2 - \varepsilon$ and $0 < \varepsilon \ll 1$, then we have

$$\int_0^1 (1-s)^{\alpha_1-1-(\alpha_1\delta/\alpha_2)} \|f(st,\cdot)\|_{L^p(\mathbb{R}^n)} ds \lesssim \sup_{s\in(0,t)} \|f(s,\cdot)\|_{L^p(\mathbb{R}^n)}. \tag{38}$$

In terms of (37) and (39), sum up with $\delta = 0, 1, \alpha_2 - \varepsilon$, we arrive at the following estimate

$$\begin{aligned}
&\|u(t,\cdot)\|_{L^{k,p}(\mathbb{R}^n)} + \left\|t^{(\alpha_1/\alpha_2)} u(t,\cdot)\right\|_{\dot{H}^{1,p}(\mathbb{R}^n)} \\
&\quad + \left\|t^{\alpha_1(1-(\varepsilon/\alpha_2))} u(t,\cdot)\right\|_{\dot{H}^{\alpha_2-\varepsilon,p}(\mathbb{R}^n)} \\
&\lesssim \|\varphi_1\|_{L^p(\mathbb{R}^n)} \\
&\quad + t\|\varphi_2\|_{L^p(\mathbb{R}^n)} + t^{\alpha_1} \sup_{s\in(0,t)} \|f(s,\cdot)\|_{L^p(\mathbb{R}^n)},
\end{aligned} \tag{39}$$

with $\delta_k = (\alpha k/2)$, $k = 0, 1$.

For the term ${}_0^C(\partial/\partial t)^{\alpha_1} u(t,\cdot)$, we use Eq. (1) to estimate.

$$\begin{aligned}
&\left\|{}_0^C\left(\frac{\partial}{\partial t}\right)^{\alpha_1} u(t,\cdot)\right\|_{\dot{H}^{-\varepsilon,p}(\mathbb{R}^n)} \\
&= \left\|-(-\Delta)^{\alpha_2} u + f(t,x)\right\|_{\dot{H}^{-\varepsilon,p}(\mathbb{R}^n)} \\
&\lesssim \|u(t,\cdot)\|_{\dot{H}^{\alpha_2-\varepsilon,p}(\mathbb{R}^n)} + \|f(t,\cdot)\|_{\dot{H}^{-\varepsilon,p}(\mathbb{R}^n)} \\
&\lesssim t^{-\alpha_1}\Bigg(\|\varphi_1\|_{L^p(\mathbb{R}^n)} + t\|\varphi_2\|_{L^p(\mathbb{R}^n)} \\
&\quad + t^{\alpha_1}\left(\sup_{s\in(0,t)} \|f(s,\cdot)\|_{L^p(\mathbb{R}^n)} + \sup_{s\in(0,t)} \|f(s,\cdot)\|_{H^{-\varepsilon,p}(\mathbb{R}^n)}\right)\Bigg).
\end{aligned} \tag{40}$$

Combing (39) and (40), we arrive at (31).

Then, we complete Theorem 3.

Last, we consider fractional diffusion equation (1), i.e., the case $\alpha_1 \in (0,1)$. Taking Fourier transformation of space variables x on Eq. (1) with the initial datum $u(0,x) = \varphi(x)$ yields

$$\begin{cases} {}_0^C\left(\frac{\partial}{\partial t}\right)^{\alpha_1} \hat{u}(t,\xi) = -|\xi|^{\alpha_2}\hat{u}(t,\xi) + \hat{f}(t,\xi), & \text{in} \quad (0,+\infty)\times\mathbb{R}^n, \\ \hat{u}(0,\xi) = \hat{\varphi}(\xi). \end{cases} \tag{41}$$

In terms of Theorem 1, we solve initial value problem of space-time fractional diffusion equation by taking a similar procedure in proving Theorem 3, then we directly give the conclusion without proof. □

Theorem 4. *Set $1 < p < +\infty$, $\alpha_1 \in (0,1)$, $\alpha_2 \in (1,2)$. Suppose $\varphi \in L^p(\mathbb{R}^n)$, $f \in C_0^\infty((0,+\infty), L^p(\mathbb{R}^n))$, then there is a unique solution u of problem (41) which is represented by*

$$\begin{aligned}
u(t,x) &= \varphi_1(x) * \mathfrak{F}^{-1}\left[E_{\alpha_1,1}\left(-|\xi|^{\alpha_2} t^{\alpha_1}\right)\right] \\
&\quad + \int_0^t (t-s)^{\alpha_1-1} \mathfrak{F}^{-1}\left[E_{\alpha_1,\alpha_1}\left(-|\xi|^{\alpha_2}(t-s)^{\alpha_1}\right)\right] \\
&\quad * f(s,x)ds,
\end{aligned} \tag{42}$$

and satisfies

$$
\begin{aligned}
&\|u(t,\cdot)\|_{L^p(\mathbb{R}^n)} \\
&\quad + \left\|t^{(\alpha_1/\alpha_2)}u(t,\cdot)\right\|_{\dot{H}^{1,p}(\mathbb{R}^n)} \\
&\quad + \left\|t^{\alpha_1(1-(\varepsilon/\alpha_2))}u(t,\cdot)\right\|_{\dot{H}^{\alpha_2-\varepsilon,p}(\mathbb{R}^n)} \\
&\quad + \left\|t^{\alpha_1(1-(\varepsilon/\alpha_2))}{}_0^C\left(\frac{\partial}{\partial t}\right)^{\alpha_1}u(t,\cdot)\right\|_{\dot{H}^{-\varepsilon,p}(\mathbb{R}^n)} \\
&\lesssim \|\varphi\|_{L^p(\mathbb{R}^n)} + t^{\alpha_1}\left(\sup_{s\in(0,t)}\|f(t,\cdot)\|_{L^p(\mathbb{R}^n)} + \sup_{s\in(0,t)}\|f(s,\cdot)\|_{H^{-\varepsilon,p}(\mathbb{R}^n)}\right)\Bigg),
\end{aligned}
\tag{43}
$$

for any positive number $\varepsilon \ll 1$, *where* $\dot{H}^{\delta,p}(\mathbb{R}^n)$ *denotes the classical homogeneous Sobolev space.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflict of interests.

Acknowledgments

This work was supported by NNSF of China (Grant No.11326152), NSF of Jiangsu Province of China (Grant No.BK20130736), NSF of the Jiangsu Higher Education Institutions of China (18KJB110013) and NSF of Nanjing Institute of Technology(CKJB201709).

References

[1] R. R. Nigmatullin, "To the theoretical explanation of the universal response," Phys. Status Solidi (B): Basic Res, vol. 123, no. 2, pp. 739–745, 1984.

[2] R. R.Nigmatullin, "Realization of the generalized transfer equation in a medium with fractal geometry," Phys. Status Solidi (B): Basic Res, vol. 133, no. 1, pp. 425–430, 1986.

[3] W. R. Schneider and W. Wyss, "Fractional diffusion and wave equations," *Journal of Mathematical Physics*, vol. 30, no. 1, pp. 134–144, 1989.

[4] B. Mbodje and G. Montseny, "Boundary fractional derivative control of the wave equation," *IEEE Transactions on Automatic Control*, vol. 40, no. 2, pp. 378–382, 1995.

[5] H. Jafari and V. Daftardar-Gejji, "Solving linear and nonlinear fractional diffusion and wave equations by Adomian decomposition," *Applied Mathematics and Computation*, vol. 180, no. 2, pp. 488–497, 2006.

[6] F. Liu, M. Meerschaert, R. J. McGough, P. Zhuang, and Q. Liu, "Numerical methods for solving the multi-term time-fractional wave-diffusion equation," *Fractional Calculus and Applied Analysis*, vol. 16, no. 1, pp. 9–25, 2013.

[7] Z. M. Odibat and S. Momani, "Approximate solutions for boundary value problems of time-fractional wave equation," *Applied Mathematics and Computation*, vol. 181, no. 1, pp. 767–774, 2006.

[8] N. H. Sweilam, M. M. Khader, and A. M. Nagy, "Numerical solution of two-sided space-fractional wave equation using finite difference method," *Journal of Computational and Applied Mathematics*, vol. 235, no. 8, pp. 2832–2841, 2011.

[9] H. Jafari and S. Seifi, "Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 5, pp. 2006–2012, 2009.

[10] K. Sakamoto and M. Yamamoto, "Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems," *Journal of Mathematical Analysis and Applications*, vol. 382, no. 1, pp. 426–447, 2011.

[11] T. Sandev, Z. Tomovski, J. L. A. Dubbeldam, and A. Chechkin, "Generalized diffusion-wave equation with memory kernel," *Journal of Physics A: Mathematical and Theoretical*, vol. 52, no. 1, p. 015201, 2019.

[12] Y. J. Yang, D. Baleanu, and X. J. Yang, "Analysis of fractal wave equations by local fractional Fourier series method," *Advances in Mathematical Physics*, vol. 2013, pp. 1–6, 2013.

[13] P. L. Butzer and U. Westphal, "An introduction to fractional calculus," in *Applications of Fractional Calculus in Physics*, R. Hilfer, Ed., pp. 1–85, World Scientific, Singapore, 2000.

[14] V. Kiryakova, "Generalized fractional calculus and applications", Pitman Research Notes in Mathematics Longman-Wiley, vol. 301, Harlow and Wiley, New York,1994.

[15] F. Mainardi, "Fractional calculus in some basic problems in continuum and statistical mechanics," *Fractals and Fractional Calculus in Continuum Mechanics*, A. Carpinteri and F. Mainardi, Eds., pp. 291–348, Springer-Verlag, NY, 1997.

[16] R. Metzler and T. F. Nonnenmacher, "Space-and time-fractional diffusion and wave equations, fractional Fokker–Planck equations, and physical motivation," *Chemical Physics*, vol. 284, no. 1-2, pp. 67–90, 2002.

[17] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, NY, 1993.

[18] K. B. Oldham and J. Spanier, *The Fractional Calculus*, vol. 111, Academic Press, NY, 1974.

[19] I. Podlubny, *Fractional Differential Equations*, Academic Press, NY, 1999.

[20] K. Zhang, "Existence results for a generalization of the time-fractional diffusion equation with variable coefficients," *Boundary Value Problems*, vol. 2019, no. 1, p. 10, 2019.

[21] R. Gorenflo, A. Iskenderov, and Y. Luchko, "Mapping between solutions of fractional diffusion-wave equations," *Fractional Calculus and Applied Analysis*, vol. 3, no. 1, pp. 75–86, 2000.

[22] Y. Luchko, "Fractional wave equation and damped waves," *Journal of Mathematical Physics*, vol. 54, no. 3, p. 031505, 2013.

[23] H. J. Haubold, A. M. Mathai, and R. K. Saxena, "Mittag-Leffler functions and their applications," *Journal of Applied Mathematics*, vol. 2011, pp. 1–51, 2011.

[24] K. Li and J. Peng, "Laplace transform and fractional differential equations," *Applied Mathematics Letters*, vol. 24, no. 12, pp. 2019–2023, 2011.

[25] M. Kwaśnicki, "Ten equivalent definitions of the fractional laplace operator," *Fractional Calculus and Applied Analysis*, vol. 20, no. 1, pp. 7–51, 2017.

[26] V. Namias, "The fractional order Fourier transform and its application to quantum mechanics," *IMA Journal of Applied Mathematics*, vol. 25, no. 3, pp. 241–265, 1980.

[27] H. M. Ozaktas and M. A. Kutay, "The fractional fourier transform," in *2001 European Control Conference (ECC)*, pp. 1477–1483, IEEE, Porto, Portugal, 2001.

[28] J. U. Kim, "An L^p a priori estimate for the Tricomi equation in the upper half space," *Transactions of the American Mathematical Society*, vol. 351, no. 11, pp. 4611–4628, 1999.