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Research Article

Breather Wave Solutions and Interaction Solutions for Two Mixed Calogero-Bogoyavlenskii-Schiff and Bogoyavlensky-Konopelchenko Equations

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In this paper, based on a bilinear differential equation, we study the breather wave solutions by employing the extended homoclinic test method. By constructing the different forms, we also consider the interaction solutions. Furthermore, it is natural to analyse dynamic behaviors of three-dimensional plots.

1. Introduction

Recently, great attention has been paid to the study about exact solutions of nonlinear partial differential equations. So, it becomes more important to seek exact solutions of nonlinear partial differential equations (NLPDEs), which occur in many fields, such as chemistry, biology, optics, classical mechanics, acoustics, engineering, and social sciences. At present, many mathematicians have proposed a large number of methods to seek exact solutions, such as Bäcklund transformation [1], Hirota bilinear methods [2], homoclinic breather limit approach [3, 4], and Darboux transformation [5–12]. Among these methods, the Hirota bilinear method is one of the most critical and powerful methods. Recently, some new exact solutions of nonlinear partial differential equations have been constructed [13–22] by means of bilinear operator theories, so it has become an important research direction to study the dynamic properties of these new equations. In this article, the breather wave solutions will be discussed. On the basis of lump solution [23], the interaction solutions will be obtained.

The two mixed Calogero-Bogoyavlenskii-Schiff (CBS) and Bogoyavlensky-Konopelchenko (BK) equations [23] are usually written as

$$u_t + u_{2x,y} + 3u_x u_y + \delta_1 u_y + \delta_2 w_{2y} + \delta_3 u_x + \delta_4\left(3u_x^2 + u_{3x}\right) + \delta_5\left(3w_{2y}^2 + w_{4y}\right) + \delta_6\left(3u_y w_{2y} + u_{3y}\right) = 0, \tag{1}$$

where $u_x = w$ and $\delta_i, i = 1, \cdots, 6$, are arbitrary constants. When the constants satisfy $\delta_3 = \delta_4 = \delta_5 = \delta_6 = 0$, and $\delta_5 = \delta_6 = 0$, the Calogero-Bogoyavlenskii-Schiff (CBS) and Bogoyavlensky-Konopelchenko (CBS-BK) equations will become a generalized Calogero-Bogoyavlenskill-Schiff (gCBS) equation [24] and a generalized Calagero-Bogoyavlenskii Konopelchenko equation [25], respectively. The CBS equation was first constructed by Bogoyavlenskii and Schiff in different ways [26, 27]. Namely, Bogoyavlenskii used the modified Lax formalism, whereas Schiff derived the same equation by reducing the self-dual Yang-Mills equation. In 2019, a class explicit lump solutions of the CBS-BK equation are constructed by using the Hirota bilinear approaches by Ren et al. [23]. The $(2+1)$-dimensional CBS equation also can be derived from the Korteweg-de Vries equation [28, 29]. Moreover, the BK equation is used as the interaction of a Rieman wave propagation [30], so we called the $(2+1)$-dimensional nonlinear partial differential equation (1) as gCBS-BK equation. These two equations have been widely studied in different ways [29, 31–40].

2. The Bilinear Equation for gCBS-BK Equation

If we take

$$u = 2\partial_x \ln f, \tag{2}$$

where $f(x, y, t)$ is an unknown real function, the bilinear equation of Equation (1) can be presented

$$\left[D_t D_x + D_x^3 D_y + \delta_1 D_x D_y + \delta_2 D_y^2 + \delta_3 D_x^2 + \delta_4 D_x^4 + \delta_5 D_y^4 + \delta_6 D_x D_y^3\right] f \cdot f = 0, \tag{3}$$

where D_t,D_x are all bilinear derivative operators and D -operator [2] is defined by

$$D_x^m D_t^n a(x,t) \cdot b(x,t) = (\partial_t - \partial_{t'})^n \left(\partial_x - \partial_{x'}\right)^m a(x,t) b\left(x', t'\right)\Big|_{x'=x, t'=t}, \tag{4}$$

where m and n are the positive integers, $a(x,t)$ is the function of x and t, and $b(x,t)$ is the function of the formal variables x' and t'.

3. Breather Wave Solutions of CBS-BK Equation

In this section, we will use the extended homoclinic text method [41, 42] to get the breather wave solutions of Equation (1). To start with,

$$f(x,y,t) = k_1 \exp(\xi_1) + \exp(-\xi_1) + k_2 \cos(\xi_2) + a_9, \tag{5}$$

where ξ_1 and ξ_2 are defined by

$$\begin{cases} \xi_1 = a_1 x + a_2 y + a_3 t + a_4, \\ \xi_2 = a_5 x + a_6 y + a_7 t + a_8, \end{cases} \tag{6}$$

where a_i, $i = 1, \cdots, 8$, k_1, and k_2 are all real numbers. Substituting Equation (5) into Equation (3), we can get the following.where a_1, a_5, and a_7 are some free real numbers.

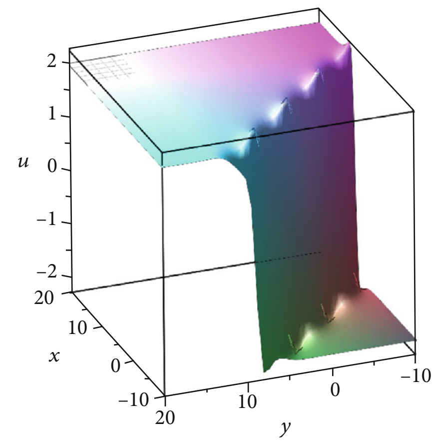

Figure 1: Spatiotemporal structure of solution (8) with the parameter selections $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_6 = 1$, $k_1 = 1$, $\delta_1 = 1$, $\delta_2 = 1$, $\delta_5 = 1$, and $\delta_6 = 1$.

Case 1.

$$\begin{aligned} a_5 &= 0, \\ a_7 &= -\frac{a_6\left(3a_1 a_2^2 \delta_6 - a_1 a_6^2 \delta_6 + 4a_2^3 \delta_5 - 4a_2 a_6^2 \delta_5 + a_1^3 + a_1 \delta_1 + 2a_2 \delta_2\right)}{a_1}, \\ a_9 &= 0. \end{aligned} \tag{7}$$

Substituting Equation (7) into Equation (5), through the transformation (2), we have

$$u(x,y,t) = \frac{2k_1 a_1 \exp(\xi_1) - 2k_1 a_1 \exp(-\xi_1) - 2k_2 a_5 \sin(\xi_2)}{k_1 \exp(\xi_1) + \exp(-\xi_1) + k_2 \cos(\xi_2)}, \tag{8}$$

where ξ_1 and ξ_2 are given by

$$\begin{cases} \xi_1 = a_1 x + a_2 y + a_3 t + a_4, \\ \xi_2 = a_6 y - \dfrac{a_6\left(3a_1 a_2^2 \delta_6 - a_1 a_6^2 \delta_6 + 4a_2^3 \delta_5 - 4a_2 a_6^2 \delta_5 + a_1^3 + a_1 \delta_1 + 2a_2 \delta_2\right)}{a_1} t + a_8, \end{cases} \tag{9}$$

where a_i, $i = 1, \cdots, 8$, δ_1, δ_2, δ_5, and δ_6 are real numbers. Figure 1 described the evolution of solution (8).

Case 2.

$$\begin{aligned} a_2 &= 0, \\ a_3 &= \frac{a_1 a_7}{a_5}, \\ a_6 &= 0, \\ a_9 &= 0, \end{aligned} \tag{10}$$

Substituting Equation (10) into Equation (5), through the transformation (2), we have

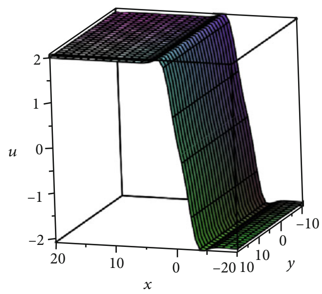

Figure 2: Spatiotemporal structure of solution (11) with the parameter selections $a_1 = 1$, $a_5 = 1$, and $a_7 = 1$.

$$u(x, y, t) = \frac{2k_1 a_1 \exp(\xi_1) - 2k_1 a_1 \exp(-\xi_1) - 2k_2 a_5 \sin(\xi_2)}{k_1 \exp(\xi_1) + \exp(-\xi_1) + k_2 \cos(\xi_2)}, \tag{11}$$

where ξ_1 and ξ_2 are determined by

$$\begin{cases} \xi_1 = a_1 x + \dfrac{a_1 a_7}{a_5} t + a_4, \\ \xi_2 = a_5 x + a_7 t + a_8, \end{cases} \tag{12}$$

where a_1, a_4, a_5, a_7, and a_8 are real numbers. Therefore, the dynamic behavior can be performed in Figure 2.

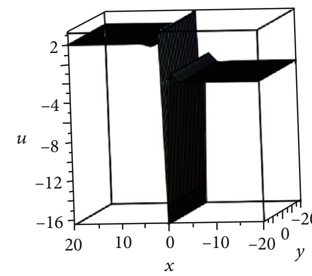

Figure 3: Spatiotemporal structure of solution (14) with the parameter selections $a_1 = 1$, $a_5 = 1$, $a_7 = 1$, $k_1 = 1$, $k_2 = 1$, and $\delta_4 = 1$.

Case 3.

$$\begin{aligned} a_2 &= 0, \\ a_3 &= \frac{a_1\left(2a_1^2 a_5 \delta_4 + 2a_5^3 \delta_4 + a_7\right)}{a_5}, \\ a_6 &= 0, \\ a_9 &= 0, \\ k_1 &= -\frac{k_2^2 a_5^2}{4k_1^2}. \end{aligned} \tag{13}$$

Substituting Equation (13) into Equation (5), through the transformation (2), we have

$$u(x, y, t) = \frac{-\left(2k_2^2 a_5^2/4k_1^2\right) a_1 \exp(\xi_1) + \left(2k_2^2 a_5^2/4k_1^2\right) a_1 \exp(-\xi_1) - 2k_2 a_5 \sin(\xi_2)}{k_1 \exp(\xi_1) + \exp(-\xi_1) + k_2 \cos(\xi_2)}, \tag{14}$$

where ξ_1 and ξ_2 are given by

$$\begin{cases} \xi_1 = a_1 x + \dfrac{a_1\left(2a_1^2 a_5 \delta_4 + 2a_5^3 \delta_4 + a_7\right)}{a_5} t + a_4, \\ \xi_2 = a_5 x + a_7 t + a_8, \end{cases} \tag{15}$$

where a_1, a_4, a_5, a_7, a_8, k_1, k_2, and δ_4 are free real numbers. Figure 3 described the evolution of solution (14).

Case 4. Substituting a_1, k_1, and a_7 into Equation (5), through the transformation (2), we have

$$u(x, y, t) = \frac{2k_1 a_1 \exp(\xi_1) - 2k_1 a_1 \exp(-\xi_1) - 2k_2 a_5 \sin(\xi_2)}{k_1 \exp(\xi_1) + \exp(-\xi_1) + k_2 \cos(\xi_2)}. \tag{16}$$

The evolution of solution (16) is described in Figure 4. ξ_1 and ξ_2 are given by

$$\begin{cases} \xi_1 = -\dfrac{a_2 a_5}{a_6} x + a_2 y + a_3 t + a_4, \\ \xi_2 = a_5 x + a_6 y + \dfrac{8a_2^2 a_6^4 \delta_5 - 8a_6^6 \delta_5 + 4a_2^2 a_5^3 a_6 + 4a_5^3 a_6^3 - 3a_2^2 a_5^2 \delta_3 - 3a_2^3 a_5^3 a_6 \delta_1 - 3a_2^2 a_6^2 \delta_2 - 3a_5^2 a_6^2 \delta_3 - 3a_5 a_6^3 \delta_1 + a_6^4 \delta_2}{3a_5\left(a_2^2 + a_6^2\right)} t + a_8, \end{cases} \tag{17}$$

where a_2, a_3, a_4, a_5, a_6, a_8, δ_1, and δ_2 are real numbers.

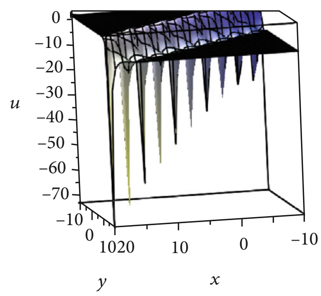

Figure 4: Spatiotemporal structure of solution (16) with the parameter selections $a_2 = 1$, $a_5 = 1$, $a_6 = 1$, $k_2 = 1$, $\delta_1 = 1$, $\delta_2 = 1$, $\delta_3 = 1$, and $\delta_5 = 1$.

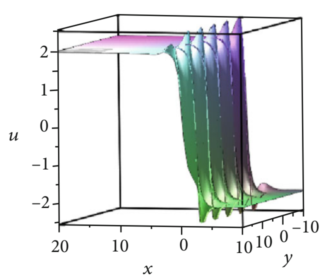

Figure 5: Spatiotemporal structure of solution (19) $a_1 = 1$, $a_5 = 1$, $a_6 = 1$, $k_1 = 1$, $k_2 = 1$, $\delta_3 = 1$, and $\delta_6 = 1$.

Case 5.

$$\begin{aligned} a_2 &= 0, \\ a_3 &= \frac{a_1\left(a_1^2 a_6 - 2a_5\delta_3\right)}{2a_5}, \\ a_7 &= a_6^3\delta_6 + \frac{a_1^2 a_6}{2} + a_5^2 a_6 - a_5\delta_3 - a_6\delta_1. \end{aligned} \tag{18}$$

Substituting Equation (18) into Equation (5), through the transformation (2), we have

$$u(x, y, t) = \frac{2k_1 a_1 \exp(\xi_1) - 2k_1 a_1 \exp(-\xi_1) - 2k_2 a_5 \sin(\xi_2)}{k_1 \exp(\xi_1) + \exp(-\xi_1) + k_2 \cos(\xi_2)}. \tag{19}$$

The evolution of solution (19) is described in Figure 5. ξ_1 and ξ_2 are given by

$$\begin{cases} \xi_1 = a_1 x + \dfrac{a_1\left(a_1^2 a_6 - 2a_5\delta_3\right)}{2a_5} t + a_4, \\ \xi_2 = a_5 x + a_6 y + \left(a_6^3\delta_6 + \dfrac{a_1^2 a_6}{2} + a_5^2 a_6 - a_5\delta_3 - a_6\delta_1\right) t + a_8, \end{cases} \tag{20}$$

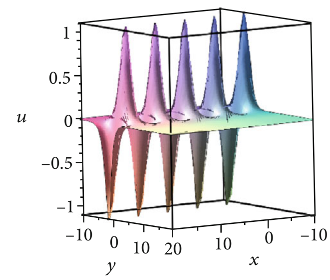

Figure 6: Spatiotemporal structure of solution (22) with the parameter selections $a_2 = 1$, $a_5 = 1$, $a_7 = 1$, $k_1 = 1$, $k_2 = 1$, $\delta_1 = 1$, $\delta_5 = 1$, and $\delta_6 = 1$.

where a_1, a_4, a_5, a_6, a_8, k_1, k_2, δ_3, and δ_6 are free real numbers. The three-dimensional dynamic figure can be drawn as Figure 5.

Case 6.

$$\begin{aligned} a_1 &= 0, \\ a_3 &= -a_2\left(a_2^2\delta_6 - a_5^2 + \delta_1\right), \\ a_6 &= 0. \end{aligned} \tag{21}$$

Substituting Equation (21) into Equation (5), through the transformation (2), we have

$$u(x, y, t) = \frac{2k_1 a_1 \exp(\xi_1) - 2k_1 a_1 \exp(-\xi_1) - 2k_2 a_5 \sin(\xi_2)}{k_1 \exp(\xi_1) + \exp(-\xi_1) + k_2 \cos(\xi_2)}, \tag{22}$$

where ξ_1 and ξ_2 are defined by

$$\begin{cases} \xi_1 = a_2 y - a_2\left(a_2^2\delta_6 - a_5^2 + \delta_1\right) t + a_4, \\ \xi_2 = a_5 x + a_7 t + a_8, \end{cases} \tag{23}$$

where a_2, a_4, a_5, a_7, a_8, k_1, k_2, δ_1, δ_5, and δ_6 are some free real numbers. The figure is given as Figure 6.

Case 7.

$$\begin{aligned} a_2 &= -a_6, \\ a_3 &= \frac{4a_1^3 a_6 - 3a_1^2\delta_3 + 3a_1 a_6\delta_1 - a_6^2\delta_2}{3a_1}, \\ a_7 &= \frac{4a_1^3 a_6 - 3a_1^2\delta_3 - 3a_1 a_6\delta_1 - a_6^2\delta_1}{3a_1}, \\ k_1 &= \frac{k_2^2}{4}, \end{aligned} \tag{24}$$

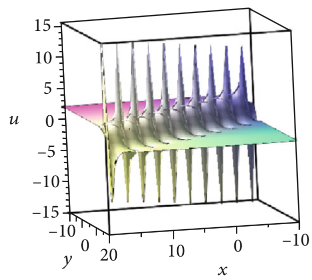

FIGURE 7: Spatiotemporal structure of solution (25) with the parameter selections $a_1 = 1$, $a_6 = 1$, $a_7 = 1$, $k_2 = 1$, $\delta_1 = 1$, $\delta_2 = 1$, $\delta_3 = 1$, and $\delta_5 = 1$.

where a_1, a_6, a_7, k_2, δ_1, δ_2, δ_3, and δ_5 are free real numbers. Substituting Equation (24) into Equation (5), through the transformation (2), we have

$$u(x, y, t) = \frac{\left(2k_2^2/4\right)a_1 \exp(\xi_1) - \left(2k_2^2/4\right)a_1 \exp(-\xi_1) - 2k_2 a_5 \sin(\xi_2)}{\left(k_2^2/4\right)\exp(\xi_1) + \exp(-\xi_1) + k_2 \cos(\xi_2)}. \tag{25}$$

The figure is given as Figure 7.

ξ_1 and ξ_2 are followed by

$$\begin{cases} \xi_1 = a_1 x + a_2 y - \dfrac{4a_1^3 a_6 + 3a_1^2\delta_3 + 3a_1 a_6 \delta_1 + a_6^2\delta_2}{3a_6^2\delta_1} t + a_4, \\ \xi_2 = -a_1 x + a_6 y + \dfrac{4a_1^3 a_6 + 3a_1^2\delta_3 - 3a_1 a_6\delta_1 + a_6^2\delta_1}{3a_1} t + a_8, \end{cases} \tag{26}$$

where a_1, a_4, a_5, a_6, a_7, a_8, k_2, δ_1, δ_2, δ_3, and δ_5 are free real numbers.

Case 8.

$$\begin{aligned} a_3 &= -\frac{4a_1^3 a_6 + 3a_1^2\delta_3 + 3a_1 a_6\delta_1 + a_6^2\delta_2}{3a_6^2\delta_1}, \\ a_5 &= -a_1, \\ a_7 &= \frac{4a_1^3 a_6 + 3a_1^2\delta_3 - 3a_1 a_6\delta_1 + a_6^2\delta_1}{3a_1}, \end{aligned} \tag{27}$$

where a_1, a_6, a_7, k_2, δ_1, δ_2, δ_3, and δ_5 are free real numbers. Substituting Equation (27) into Equation (5), through the transformation (2), we have

$$u(x, y, t) = \frac{2k_1 a_1 \exp(\xi_1) - 2k_1 a_1 \exp(-\xi_1) - 2k_2 a_5 \sin(\xi_2)}{k_1 \exp(\xi_1) + \exp(-\xi_1) + k_2 \cos(\xi_2)}. \tag{28}$$

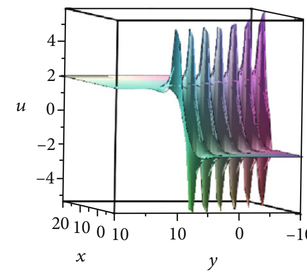

FIGURE 8: Spatiotemporal structure of solution (28) with the parameter selections $a_1 = 1$, $a_6 = 1$, $a_7 = 1$, $k_2 = 1$, $\delta_1 = 1$, $\delta_2 = 1$, $\delta_3 = 1$, and $\delta_5 = 1$.

The figure is drawn as Figure 8.

ξ_1 and ξ_2 are defined by

$$\begin{cases} \xi_1 = a_1 x + a_2 y - \dfrac{4a_1^3 a_6 + 3a_1^2\delta_3 + 3a_1 a_6 \delta_1 + a_6^2\delta_2}{3a_6^2\delta_1} t + a_4, \\ \xi_2 = -a_1 x + a_6 y + \dfrac{4a_1^3 a_6 + 3a_1^2\delta_3 - 3a_1 a_6\delta_1 + a_6^2\delta_1}{3a_1} t + a_8, \end{cases} \tag{29}$$

where a_1, a_2, a_4, a_6, a_7, a_8, k_2, δ_1, δ_2, δ_3, and δ_5 are free real numbers.

4. Interaction Solutions of CBS-BK System

4.1. Interaction between a Lump and One-Kink Soliton. With the help of Maple, we will discuss the interaction between a lump and one-kink soliton by taking $f(x, y, t)$ as a combination of positive quadratic function and one exponential function, that is,

$$f(x, y, t) = \xi_1^2 + \xi_2^2 + \exp(\xi_3) + a_9, \tag{30}$$

where ξ_1, ξ_2, and ξ_3 are defined by

$$\begin{cases} \xi_1 = a_1 x + a_2 y + a_3 t + a_4, \\ \xi_2 = a_5 x + a_6 y + a_7 t + a_8, \\ \xi_3 = k_1 x + p_1 y + q_1 t + r_1, \end{cases} \tag{31}$$

where a_i, $i = 1, \cdots, 9$, p_1, k_1, r_1, and q_1 are all real numbers. In order to get the interaction solutions of Equation (1), substituting Equation (30) into Equation (2),

$$u(x,y,t)=\frac{4a_1\xi_1+4a_2\xi_2+2k_1\exp(\xi_3)}{\xi_1^2+\xi_2^2+\exp(\xi_3)+a_9}, \tag{32}$$

where ξ_1, ξ_2, and ξ_3 are defined by

$$\begin{cases}\xi_1=a_1x+a_2y+a_3t+a_4,\\ \xi_2=a_5x+a_6y+a_7t+a_8,\\ \xi_3=k_1x+p_1y+q_1t+r_1,\end{cases} \tag{33}$$

where a_i, $i=1,\cdots,9$, p_1, k_1, r_1, and q_1 are all real numbers. Substituting Equation (30) into Equation (3), through complex analysis and calculations, we can have the following.

Case 1.

$$\begin{aligned}a_1&=0,\\ a_6&=0,\\ k_1&=0,\\ \delta_2&=0,\\ \delta_4&=0,\\ \delta_5&=0,\\ q_1&=\frac{p_1\left(a_2\delta_6p_1^2-a_3\right)}{a_2},\\ \delta_1&=-\frac{a_7}{a_5},\end{aligned} \tag{34}$$

where a_2, a_3, a_5, a_7, p_1, and δ_6 are free real numbers. Substituting Equation (34) into Equation (32), we have

$$u(x,y,t)=\frac{2a_5(a_5x+a_6y+a_7t+a_8)}{a_9+(a_2y+a_3t+a_4)^2+(a_5x+a_7t+a_8)^2+b_1\exp\left(p_1y+\left(p_1\left(a_2\delta_6p_1^2-a_3\right)/a_2\right)t+r_1\right)}. \tag{35}$$

Case 2.

$$\begin{aligned}a_1&=0,\\ a_5&=0,\\ \delta_2&=0,\\ \delta_5&=0,\\ \delta_6&=0,\\ a_3&=\frac{a_7a_2}{a_6},\\ \delta_1&=\frac{a_6k_1^2+a_7}{a_6},\\ q_1&=-\frac{a_6k_1^3\delta_4+a_6k_1\delta_3-a_7p_1}{a_6},\end{aligned} \tag{36}$$

where a_2, a_6, a_7, k_1, p_1, δ_3, and δ_4 are some free real numbers. Substituting Equation (36) into Equation (32), we have

$$u(x,y,t)=\frac{4a_1\xi_1+4a_2\xi_2+\exp(\xi_3)}{\xi_1^2+\xi_2^2+\exp(\xi_3)+a_9}, \tag{37}$$

where ξ_1, ξ_2, and ξ_3 are defined by

$$\begin{cases}\xi_1=a_2y+\dfrac{a_7a_2}{a_6}t+a_4,\\ \xi_2=a_6y+a_7t+a_8,\\ \xi_3=k_1x+p_1y-\dfrac{a_6k_1^3\delta_4+a_6k_1\delta_3-a_7p_1}{a_6}t+r_1,\end{cases} \tag{38}$$

where a_2, a_4, a_6, a_7, a_8, k_1, p_1, r_1, δ_3, and δ_4 are some free real numbers.

In order to obtain the dynamic feature, we choose Case 2 to analyse. The three-dimensional dynamic graphs are drawn as Figure 9. We can find that the lump waves and the exponential function waves interact with each other and keep moving in the opposite direction.

4.2. Interaction between a Lump and Periodic Waves. In order to get interaction solutions between a lump and periodic waves, we will take f as the combination of positive function and hyperbolic cosine function. Therefore, f can be determined by

$$f(x,y,t)=\xi_1^2+\xi_2^2+b_1\cos(\xi_3)+a_9, \tag{39}$$

where variables are defined by

$$\begin{cases}\xi_1=a_1x+a_2y+a_3t+a_4,\\ \xi_2=a_5x+a_6y+a_7t+a_8,\\ \xi_3=k_1x+p_1y+q_1t+r_1,\end{cases} \tag{40}$$

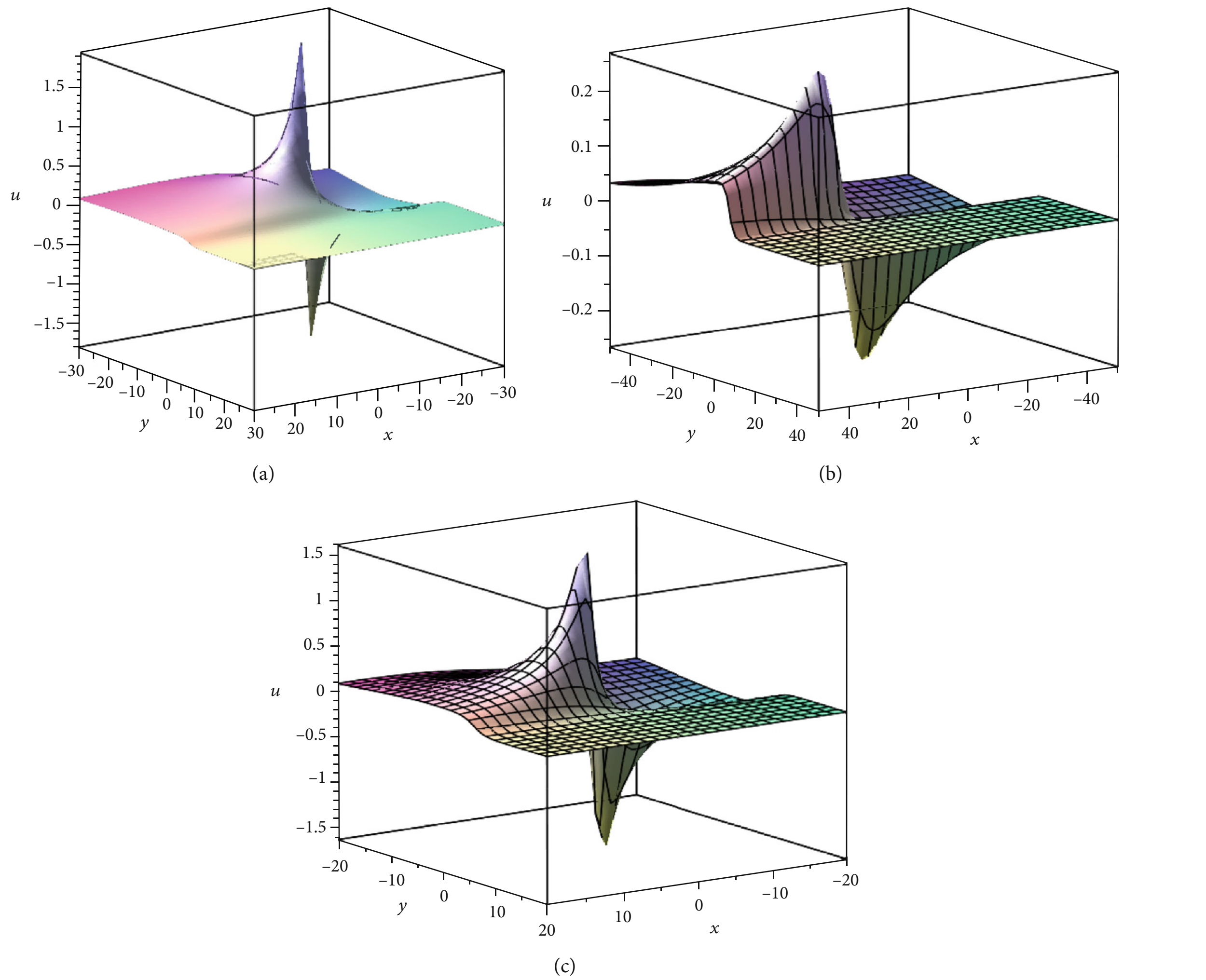

FIGURE 9: Spatiotemporal structure of solution (37) with the parameter selections: (a) $t=-10$, $a_2=1$, $p_1=1$, $a_3=1$, $a_7=1$, $a_5=1$, and $\delta_6=1$; (b) $t=0$, $a_2=1$, $p_1=1$, $a_3=1$, $a_7=1$, $a_5=1$, and $\delta_6=1$; (c) $t=-10$, $a_2=1$, $p_1=1$, $a_3=1$, $a_7=1$, $a_5=1$, and $\delta_6=1$.

where a_i, $i=1,\cdots,9$, p_1, k_1, r_1, and q_1 are all real numbers. Substituting Equation (39) into Equation (2), we can get the interaction solutions of Equation (1):

$$u(x,y,t)=\frac{4a_1\xi_1+4a_2\xi_2-2b_1k_1\sin(\xi_3)}{\xi_1^2+\xi_2^2+b_1\cos(\xi_3)+a_9}, \tag{41}$$

where ξ_1, ξ_2, and ξ_3 are defined by

$$\begin{cases}\xi_1=a_1x+a_2y+a_3t+a_4,\\ \xi_2=a_5x+a_6y+a_7t+a_8,\\ \xi_3=k_1x+p_1y+q_1t+r_1,\end{cases} \tag{42}$$

where a_i, $i=1,\cdots,9$, p_1, k_1, r_1, and q_1 are real numbers. Through long and tedious calculations, we can get the following relations between the parameters.where a_5, a_7, b_1, p_1, δ_1, δ_2, and δ_6 are free real numbers.where a_5, a_6, a_7, p_1, b_1, δ_1, and δ_6 are free real numbers.where a_2, a_3, a_5, a_7, p_1, and δ_6 are some free real numbers.where a_5, p_1, δ_1, and δ_6 are free real numbers.

Case 1.

$$\begin{aligned}&a_1=0,\\&a_2=0,\\&a_3=0,\\&a_6=0,\\&q_1=p_1\left(P_1^2\delta_6-\delta_1\right),\\&\delta_3=-\frac{a_7}{a_5},\\&\delta_4=-\frac{b_1^2p_1^2\delta_2}{a_5^4},\\&\delta_5=\frac{\delta_2}{p_1^2},\end{aligned} \tag{43}$$

Case 2.

$$\begin{aligned} a_1 &= 0, \\ a_2 &= 0, \\ a_3 &= 0, \\ k_1 &= 0, \\ q_1 &= -p_1\delta_1, \\ \delta_2 &= -\frac{a_5 p_1^2 \delta_6}{a_6}, \\ \delta_3 &= \frac{a_6 p_1^2 \delta_6 - 2a_6\delta_1 - 2a_7}{a_5}, \end{aligned} \tag{44}$$

Case 3.

$$\begin{aligned} a_1 &= 0, \\ a_6 &= 0, \\ k_1 &= 0, \\ q_1 &= \frac{p_1\left(a_2 p_1^2 \delta_6 + a_3\right)}{a_2}, \\ \delta_1 &= -\frac{a_3}{a_2}, \\ \delta_2 &= 0, \\ \delta_3 &= -\frac{a_7}{a_5}, \\ \delta_4 &= 0, \\ \delta_5 &= 0, \end{aligned} \tag{45}$$

Case 4.

$$\begin{aligned} a_2 &= 0, \\ a_3 &= -a_1\delta_3, \\ k_1 &= 0, \\ q_1 &= p_1^3\delta_6 - p_1\delta_1, \\ \delta_2 &= 0, \\ \delta_4 &= 0, \\ \delta_5 &= 0, \end{aligned} \tag{46}$$

When we change the coefficients of the equation, the value of Equation (47) will be different accordingly. In order to obtain the dynamic feature, we choose Case 2 to analyse. Taking Equation (44) into Equation (41), we can get

$$u(x, y, t) = \frac{2a_5(a_5 + a_7 t + a_8)}{a_9 + a_4^2 + (a_5 x + a_6 y + a_7 t + a_8)^2 + b_1 \cos\left(p_1 y - p_1\delta_1 t + r_1\right)}. \tag{47}$$

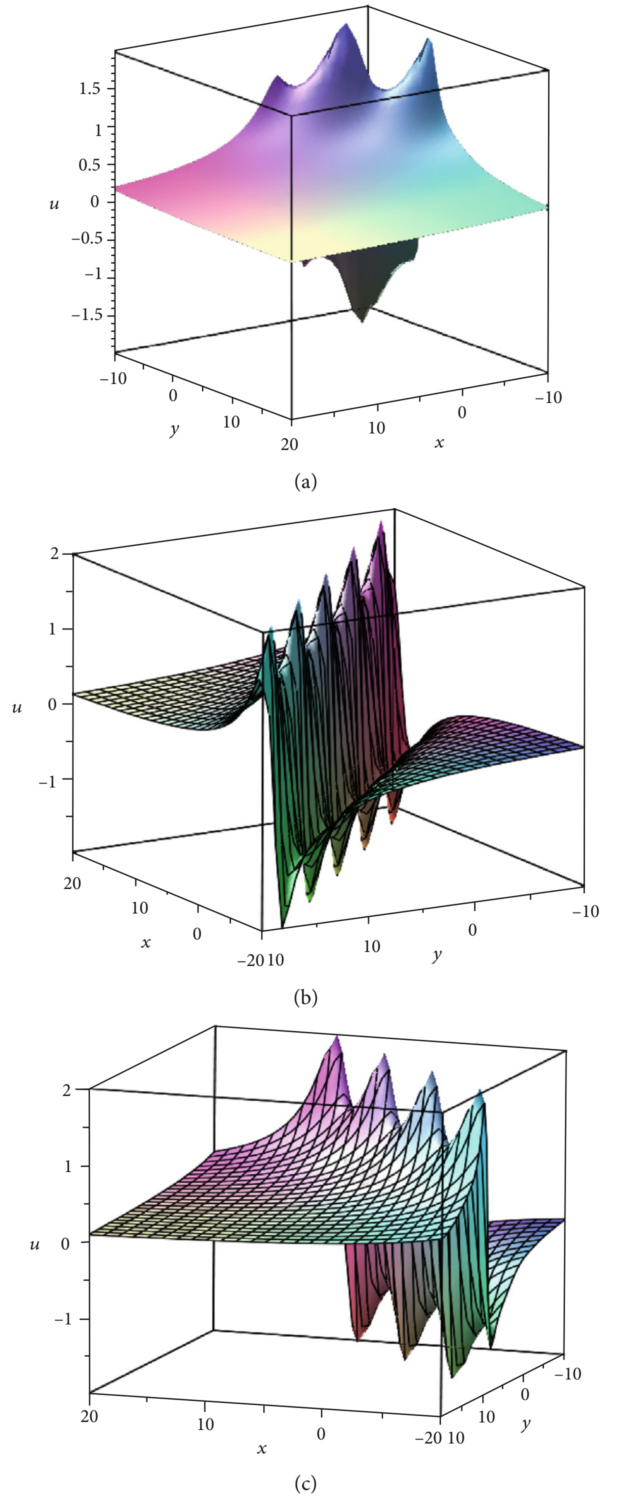

FIGURE 10: Spatiotemporal structure of solution (47) with the parameter selections: (a) $t = 0$, $a_5 = 1$, $a_6 = 1$, $a_7 = 1$, $p_1 = 1$, $b_1 = 1$, $\delta_1 = 1$, and $\delta_6 = 1$; (b) $t = 10$, $a_5 = 1$, $a_6 = 1$, $a_7 = 1$, $p_1 = 1$, $b_1 = 1$, $\delta_1 = 1$, and $\delta_6 = 1$; (c) $a_5 = 1$, $a_6 = 1$, $a_7 = 1$, $p_1 = 1$, $b_1 = 1$, $\delta_1 = 1$, and $\delta_6 = 1$.

With the help of Maple, the three-dimensional dynamic graphs are drawn as Figure 10. We can find that lump waves and periodic waves interact with each other and keep moving in the opposite direction.

5. Conclusions

In this paper, based on a bilinear differential equation, we study the breather wave solutions and the interaction solutions of the mixed Calogero-Bogoyavlenskii-Schiff and Bogoyavlensky-Konopelchenko equations. Compared with the existing results in the literature, our results are new. It will be extensively used to report many attractive physical phenomena in the fields of acoustics, heat transfer, fluid dynamics, classical mechanics, and so on. It is demonstrated that the Hirota operators are very simple and powerful in constructing new nonlinear differential equations, which possess nice math properties. It is interesting to study the interaction solutions between soliton solutions and period solution by making f as a combination of exponential function and trigonometry function. However, this method can be applied to those equations which have Hirota bilinear forms. Furthermore, we also can study the quardrilinear forms and even polylinearity forms of this equation in the future. These questions may also be interesting and worth studying.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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