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Research Article

A Potential Constraints Method of Finding Nonclassical Symmetry of PDEs Based on Wu's Method

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Solving nonclassical symmetry of partial differential equations (PDEs) is a challenging problem in applications of symmetry method. In this paper, an alternative method is proposed for computing the nonclassical symmetry of PDEs. The method consists of the following three steps: firstly, a relationship between the classical and nonclassical symmetries of PDEs is established; then based on the link, we give three principles to obtain additional equations (constraints) to extend the system of the determining equations of the nonclassical symmetry. The extended system is more easily solved than the original one; thirdly, we use Wu's method to solve the extended system. Consequently, the nonclassical symmetries are determined. Due to the fact that some constraints may produce trivial results, we name the candidate constraints as "potential" ones. The method gives a new way to determine a nonclassical symmetry. Several illustrative examples are given to show the efficiency of the presented method.

1. Introduction

The classical Lie symmetry (CLS) method, proposed by Sophus Lie in 1870s, has been widely used to solve nonlinear PDEs in mathematics, physics, and mechanics [1, 2]. When solving PDEs, it is not sufficient to use CLS method. There already have been several generalizations of the methods such as nonclassical symmetry, Q-conditional symmetry, conditional symmetry, reduction operators, weak symmetry, and hierarchy conditional symmetry [3–8]. In this article, we consider the standard nonclassical symmetry method proposed in [3, 4]. The nonclassical symmetr yields exact solutions to PDEs, which cannot be derived from CLS of the PDEs. The nonclassical symmetry method is also related to Bäcklund transformations, functionally invariant solutions, direct method, and so forth [9–13].

Closely related to our article, the very interesting references [14–16] have to be mentioned. In paper [14], the authors clarified the possibility of some bogus transformation in mechanics of continua that may be computed which yield trivial results. Particularly, they discuss why the invariance with respect to some well-known transformations must be used with care and explain why some of these universal transformations are useless to obtain invariant solutions of physical significance. In paper [15], the authors gave a deep insight into the relationship between potential symmetry and potential nonclassical symmetry of PDEs. They showed the link by potential symmetries and reduction of order two for PDEs. In paper [16], the authors discussed the problem of compatibility of a PDE of second order with several invariance surface conditions. Particularly, they revealed the relationships of some reduction methods for evolution equations based on invariant surface conditions related to functional separation of variables with nonclassical and weak point symmetries. These three papers made significant progress on applications of symmetry methods.

Likely, in this paper, we discuss a relationship between CLS and nonclassical symmetry of PDEs and some of its applications are given.

As well known, to use symmetry methods in the analysis of PDEs, the exact expressions of the symmetries are the prerequisites. While determining a symmetry, there is an inevitable step in which one solves the system of determining equations exactly. Nowadays, in using the symbolic computation programs [17–19], the calculations of the CLS become routine tasks. For the nonclassical symmetry, many algorithms and packages [20–22] can produce the system of determining equations. However, it is a difficult and challenging problem to solve the system, since it is nonlinear, unlike the situation of the CLS. It is difficult to deal with such system directly, even if one has been aided by computer. Up to now, the nonclassical symmetries of many equations have not been solved yet. Hence, it is necessary to overcome the difficult problem to extend the applications of the nonclassical symmetry method. More practically, it is useful to get some particular nonclassical symmetry when we cannot get whole set of the nonclassical symmetry due to the complexity of the determining equations. Some ansatz are usually used to get the nonclassical symmetry [23] of specific form. However, there are no systematic principles to follow.

It is worth mentioning that a system of polynomial formed differential equations can be viewed as one of differential polynomials, and the zero set of the polynomial system is equivalent to the solution of the system of the differential equations [19–22, 24, 25]. There are two main methods to deal with the set of zero points of a polynomial system. One is Groebner basis method used in [20]; another is Wu's method [21, 22, 24, 26]. In this paper, Wu's method is used as main computational tool to deal with the system of differential polynomials corresponding to the system of determining equations of nonclassical symmetries of PDEs.

Wu's method, proposed by Chinese mathematician Wentsun Wu in 1970s [26, 27], is one of the fundamental algorithmic methods in geometry algebra and computer algebra fields. Wu's method is designed to deal with the set of zero points of a system of polynomials. The main idea of the method is to characterize the set of zero points of a polynomial system by series of sets of zero points of so-called characteristic sets of the polynomial system. The zero set of characteristic set is easily determined due to its good order and passive structure. The basic results of Wu's method are the well-ordering principle and zero decomposition theorem (see Theorem 5). Wu gave an algorithm, called Wu's algorithm, to realize this procedure. In our paper, we take the left-hand side PS of the determining equations PS=0 of symmetries of PDEs as a differential polynomial system. Then the problem of solving the determining equations can be equivalently transformed to the one of determining the set of zero points of the differential polynomial system. Then Wu's method is used to determine the set of zero points of the polynomial system. Consequently, one obtains the corresponding symmetries of the PDEs.

In this article, we explore a constraints method for solving the system of determining equations through finding a relationship between CLS and nonclassical symmetries of PDEs. Based on the link, we get three principles to obtain additional equations (constraints) to the system of determining equations. By attaching the "constraints" on the system of determining equations, we get an extended system of the determining equations. Here, it is emphasized that the "constraints" are added on the system of determining equations not on the original PDEs. Then, Wu's method is used in dealing with the extended system. In the extended system, the original determining equations are reduced by the added additional equations, so the system becomes simpler. Consequently, a nonclassical symmetry is easily determined by solving this reduced system. Hence, the proposed method in this article differs from the existing "differential constraint" methods in [8, 14–16] and references therein.

Hopefully, the proposed method would be a complement to other existing methods. Particularly, we intend to present the following results in this article.

(R1) A set of identities showing the intrinsic relationship between the CLS and nonclassical symmetry of PDEs are derived.

(R2) A concept of the potential constraints is put forward. It serves additional auxiliary equations to the system of determining equations of the nonclassical symmetry.

(R3) We suggest three principles of getting the constraints from the identities given in (R1).

(R4) Assembling (R1)–(R3) and combining Wu's method of differential form, we propose a method of solving the system of determining equations that lead to determining nonclassical symmetries of PDEs.

(R5) Examples of finding nonclassical symmetries of several mathematical physics equations are given to show the efficiency of the presented method.

The given method can be used not only in nonclassical symmetry computation but also in symmetry classification problems.

2. Preliminary

In the following part, we give some preliminaries used in the article.

2.1. Notations and Basic Results. Suppose $X = (x_1, x_2, \dots, x_p)$ and $U = (u_1, u_2, \dots, u_q)$ are independent and dependent variables, respectively. A multi-index $\alpha = \{\alpha_1, \dots, \alpha_p\} \in \mathbb{N}_0^p$ ($\mathbb{N}_0$ is the set of nonnegative integers) denotes the derivative operator $D_\alpha = \partial^{|\alpha|}/\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_p}^{\alpha_p}$ with order $|\alpha| = \sum_{i=1}^{p}\alpha_i$. Let $U^\alpha = \{D_\alpha u_i,\ i = 1, 2, \dots, q\}$ for an index $\alpha \in \mathbb{N}_0^p$. Use notation $\partial U = \{U^\alpha, \alpha \in \mathbb{N}_0^p\}$ to denote the set of all derivatives of U with respect to X. Let $\mathscr{K}_X$ be a characteristic zero field of differential functions of X. Let $\mathscr{K}_X[U]$ be differential polynomial ring in the indeterminates ∂U over $\mathscr{K}_X$. In a differential polynomial system $\mathscr{D}$, $\mathrm{Z}(\mathscr{D})$ is the set of zero points of $\mathscr{D}$ in some extended field of $\mathscr{K}_X$.

2.2. Symmetry of PDEs. We consider kth order PDEs system

$$F[U] = F(X, U^\alpha) = 0, \quad |\alpha| \le k, \tag{1}$$

with independent variables X and dependent variables U. We suppose that F consisted of polynomials in its arguments. In this paper, we only consider PDEs (1) that are nondegenerate in the sense of Olver [2]; that is, the system has maximal rank.

The associated Lie algebra element of Lie symmetry is the infinitesimal generator

$$\mathcal{X} = \xi \cdot \partial_X + \eta \cdot \partial_U, \tag{2}$$

where $\partial_X = (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_p})^T$ and $\partial_U = (\partial_{u_1}, \partial_{u_2}, \ldots, \partial_{u_q})^T$ are derivative operator vectors with respect to X and U and $\xi = (\xi_1, \xi_2, \ldots, \xi_p)$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_q)$ are the vectors of infinitesimal functions of the generator (2), in which $\xi_i = \xi(z)$ and $\eta_j = \eta_j(z)$ are functions of $z = X \cup U$. The dot is the inner product of the involved vectors.

In the CLS method, the system of determining equations of generator is produced from the invariance criterion: $\mathrm{Pr}\mathscr{X}(F) = 0$ when $F = 0$. Here $\mathrm{Pr}\mathscr{X}$ is the prolongation of $\mathscr{X}$ on the space of U. Solving the infinitesimals functions $\xi(z)$ and $\eta(z)$ explicitly from the determining equations, one obtains generator (2).

Remark 1. In Lie's theory, the generator (2) corresponds to one parameter transformation group (symmetry for short). Hence, in this connection, the generator (2) is equivalently called symmetry.

In the standard nonclassical symmetry method, nonclassical symmetry (2) is found by requiring the PDEs (1) and the invariant surface conditions,

$$\psi = \xi \cdot U_X - \eta = 0, \tag{3}$$

are simultaneously invariant under symmetry transformations (2), where $U_X = (u_{i,x_j})^T$ is Jacobian matrix for U. In this case, the invariance criterion is given from $\mathrm{Pr}\mathscr{X}(\mathrm{F}) = 0$, which holds on the manifold defined by (1) and the differential consequences of (3). From this criterion, one derives the system of determining equations for the nonclassical symmetry. By solving the determining equations, one finds the generator (2), that is, the nonclassical symmetry of system (1). However, this is a hard task in general. To this end, we still need to continue to explore effective methods. This paper is one of the attempts.

2.3. Basic Concepts and Results of Wu's Method. In the following, we give some concepts and basic results of Wu's method used in this article. More details on the method can be found in [21, 22, 24–28].

Definition 2. A finite differential polynomial system,

$$\mathscr{ASC}\colon \mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_s, \tag{4}$$

is called a differential ascending chain in a differential polynomial rank $\prec$ if it satisfies the following two conditions:

(a) $\mathscr{A}_1 \prec \mathscr{A}_2 \prec \cdots \prec \mathscr{A}_s$;

(b) $\mathscr{A}_j$ is reduced with respect to $\mathscr{A}_i$ (the terms in $\mathscr{A}_j$ cannot be eliminated by $\mathscr{A}_i$) for $i = 1, 2, \ldots, j-1$.

In Wu's method, the basic reduction algorithm yields the pseudoreduction formula for a differential polynomial f with respect to a differential ascending chain $\mathscr{ASC}$. That is, there exist differential polynomials Q_α such that

$$\mathrm{IS} \cdot f = \sum_{g_\alpha \in \mathrm{ASC}} Q_\alpha D^\alpha g_\alpha + r, \tag{5}$$

where the differential polynomial r is reduced with respect to $\mathscr{ASC}$ (r cannot be reduced further by $\mathscr{ASC}$), and it is called the pseudoremainder of f with respect to the $\mathscr{ASC}$, denoted by $r = \mathrm{Prem}(f/\mathscr{ASC})$. The reduction formula is derived from eliminating the terms in f by each element in $\mathscr{ASC}$ until the procedure cannot be continuous. The IS or IS($\mathscr{ASC}$) (called IS product of $\mathscr{ASC}$) is a product of initials (the coefficients of the leading derivatives of polynomials) and separants (partial derivatives of differential polynomials with respect to leading derivatives) of the differential polynomials in $\mathscr{ASC}$.

In a differential polynomial system PS, we define

$$\mathrm{Prem}\left(\frac{\mathrm{PS}}{\mathscr{ASC}}\right) = \left\{\mathrm{Prem}(\frac{f}{\mathscr{ASC}} \text{ for } f \in \mathrm{PS}\right\}. \tag{6}$$

Definition 3. Under a differential ascending chain rank (order) [26, 29], a lowest-order differential ascending chain contained in a differential polynomial system PS is called a base set of the PS.

The concept of differential characteristic set of a differential polynomial system is the core of Wu's method. Its definition is given below.

Definition 4. For a differential polynomial system PS, if there exists a differential ascending chain CS satisfying the properties (a_1), (a_2), and (a_3),

(a_1) Z(*PS*) ⊂ Z(*CS*),

(a_2) Prem(PS/CS) = 0,

(a_3) Prem(IP/CS)=0,

where IP is nonzero integrability polynomial of CS, then we call the differential ascending chain CS the differential characteristic set of the PS.

The differential characteristic set of a differential polynomial system has many well-ordering algebraic properties, such as admitting triangular structure and containing all integrability conditions (passive). It reveals the essential properties of the zero set of a differential polynomial system.

In [26], Wu provided differential characteristic set algorithm (also called Wu's algorithm) to find a differential characteristic set for any given finite differential polynomial system. Based on the algorithm, we have the fundamental results of Wu's method shown in the following theorems.

Theorem 5. *Let PS be a finite differential polynomial system; then the well-ordering principle,*

$$\begin{gathered} \mathrm{Z}\left(\frac{\mathrm{CS}}{\mathrm{IS}}\right) \subset \mathrm{Z}(\mathrm{PS}) \subset \mathrm{Z}(\mathrm{CS}), \\ \mathrm{Z}(\mathrm{PS}) = \mathrm{Z}\left(\frac{\mathrm{CS}}{\mathrm{IS}}\right) \cup \mathrm{Z}(\mathrm{PS}, \mathrm{IS}), \end{gathered} \tag{7}$$

and zero decomposition,

$$\mathrm{Z}(\mathrm{PS}) = \bigcup_k \mathrm{Z}\left(\frac{\mathrm{CS}_k}{\mathrm{IS}_k}\right) = \bigcup_j \mathrm{Z}\left(\frac{\mathrm{ICS}_j}{\mathrm{IS}_j}\right), \tag{8}$$

hold true, where CS *and* $CS_k(ICS_j)$ *are the differential characteristic sets (the irreducible differential characteristic sets)* *of PS and its extensions obtained by adding some differential polynomials to PS.*

Remark 6. Theorem 5 gives not only relationship between zero sets of original PS and its differential characteristic set CS but also a judgment on the compatibility of the PS. If we have Z(CS) = ∅, then Z(PS) = ∅ by (7), which implies that PS = 0 has no solution; equivalently PS is no consistence. This property is used in the present article to find nontrivial constraints in the system of determining equations (see examples in the following sections).

In the following we give examples to show the efficiency of Wu's method and the reason why we use it.

Example 7. First we give an application of Wu's method for solving a linear system of variable coefficient overdetermined equations.

To compute the potential symmetries of the nonlinear wave equation $u_{tt} = ((1/u^2)u_x)_x + (1/u)_x$ with the potential system given by $v_t = (1/u^2)u_x + 1/u$, $v_x = u_t$, and infinitesimal generator $X = \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t + \eta(x,t,u,v)\partial_u + \phi(x,t,u,v)\partial_v$, we have to exactly solve the system of determining system PS = 0. Here the left-hand side differential polynomial system is

$$\text{PS} = \left\{\begin{array}{c} \xi_v - \tau_u, \eta_u - \phi_v + \xi_x - \tau_t, \eta_v + u(\eta_t - \phi_x) + \tau_x, u^2\xi_u - \tau_v, \\ u\xi_v + u^2\xi_t - \tau_x, u(\eta_u - \phi_v - \xi_x + \tau_t) + 2(\tau_v - \eta), \\ u(\phi_v - \tau_t) - (\tau_v + \eta_x - \eta) + u^2\phi_t, u^2\phi_u - u\tau_u - \eta_v, \end{array}\right\} \quad (9)$$

in $\mathcal{K}_X[\partial U]$ with $\mathbf{X} = (x,t,u)$ and $U = (\xi,\phi,\eta,\tau)$. Under the basic rank $x \prec t \prec u \prec v \prec \xi \prec \phi \prec \eta \prec \tau$, executing Wu's algorithm, we obtain the differential characteristic set of the PS as follows:

$$\text{CS} = \left\{\begin{array}{ccc} \xi_{tv}, \xi_{tt}, \xi_{xt}, & \phi_v, \phi_t, \phi_u + 2\xi_t, & \\ \xi_t + u\xi_{tu}, & \phi_x + 2u\xi_t; & \tau_x - u\xi_v - u^2\xi_t, \\ \xi_v + u\xi_{uv} + u\xi_t - \xi_{xv}, & & \tau_t + u\xi_u - \xi_x - u^{-1}\eta, \\ \xi_{vv} + \xi_x - \xi_{xx}, & \eta_x + u\xi_x, \eta_t + u\xi_t, & \tau_u - \xi_v, \tau_v - u^2\xi_u, \\ \xi_x + u^2\xi_{uu} + 2u\xi_u - \xi_{xx}, & u\eta_u - \phi + u^2\xi_u, & \\ \xi_x + u\xi_{xu} - \xi_{xx}; & \eta_v + u\xi_v + 2u^2\xi_t; & \end{array}\right\} \quad (10)$$

with IS = $u \not\equiv 0$. Hence one has

$$\text{Z}(\text{PS}) = \text{Z}(\text{CS}), \quad (11)$$

by Theorem 5 (see (7)). This shows the equivalence between solving PS = 0 and CS = 0. Compared with original system PS = 0, the well-ordering (triangular form) structure of the differential characteristic set CS makes the determination of Z(PS) easier through solving Z(CS). The zero set of ξ is obtained from the first part (the first eight equations) of CS; the zero sets of ϕ, η, and τ are obtained from the following parts of the CS sequentially by using the previously determined zero sets step by step. Hence, we easily have infinitesimal functions

$$\text{Z}(\text{PS}) = \left\{\begin{array}{c} \xi = \dfrac{c_1 t}{u} + c_2 - c_1 v + e^x A(v, e^x u), \\ \tau = (c_3 - c_1 v)t - c_1 u + c_4 + B(v, ue^x), \\ \eta = c_3 u - c_1(t + uv) - ue^x A(v, e^x u), \phi = -2c_1(\ln u + x) \end{array}\right\} \quad (12)$$

with $A_V - B_U = 0$, $B_V - U^2 A_U = 0$, $V = v$, and $U = ue^x$. Hence the the nonlinear wave equation admits four parameters finite symmetries and an infinite dimensional symmetry.

Example 8. In the example, we show the application of Wu's method for solving the system of nonlinear overdetermined equations.

To compute the nonclassical symmetry for the Burgers equation $u_t + uu_x + u_{xx} = 0$ with generator (as example just consider the case $\tau \equiv 1$) $\mathcal{X} = \partial_t + \xi(x,t,u)\partial_x + \eta(x,t,u)\partial_u$, we have to solve the nonlinear system PS=0; here

$$\begin{aligned} \text{PS} = \{&\xi_{uu}, \eta_t + u\eta_x + \eta_{xx} + 2\eta\xi_x, \eta_{uu} + 2u\xi_u - 2\xi\xi_u \\ &- 2\xi_{xu}, 2\eta_{xu} + 2\eta\xi_u - \xi_t + u\xi_x - 2\xi\xi_x - \xi_{xx} + \eta\}. \end{aligned} \quad (13)$$

After using Wu's algorithm on the polynomial system, we obtain its zero decomposition (see (8)):

$$\text{Z}(\text{PS}) = \text{Z}(\text{CS}_1) \cup \text{Z}(\text{CS}_2) \cup \text{Z}(\text{CS}_3). \quad (14)$$

Here

$$\begin{aligned} \text{CS}_1 &= \{\eta_{xx}, \xi_{xx}, \xi_u, \eta_u + \xi_x, \eta_t + u\eta_x + 2\xi_x\eta, u\xi_x - \xi_t \\ &\quad - 2\xi_x\xi + \eta\}, \\ \text{CS}_2 &= \{\eta, u - \xi\}, \\ \text{CS}_3 &= \{\eta_{uu} - u + \xi, 1 + 2\xi_u, \eta_t + u\eta_x + \eta_{xx} \\ &\quad + 2\eta\xi_x, 2\eta_{xu} - \xi_t + u\xi_x - 2\xi\xi_x - \xi_{xx}, 2\eta_{tu} + 2\eta_x \\ &\quad + u\xi_t + 4\eta_u\xi_x - u^2\xi_x + 2u\xi\xi_x + 2\xi_x^2 + \xi_{xt} + 2\xi\xi_{xx} \\ &\quad + \xi_{xxx}\}. \end{aligned} \quad (15)$$

Since CS_1 represents the CLS of the equation and $Z(PS)\backslash Z(CS_1) \neq \emptyset$, one knows that Burgers equation has the nontrivial nonclassical symmetries that correspond to solutions to the reduced systems $CS_1 = 0$ and $CS_2 = 0$. They are easily solved even by hand. Particularly,

$$\begin{aligned} Z(CS_1) &= \left\{\xi = \frac{c_1 tx + c_2 x + c_4 t + c_5}{c_1 t^2 + 2c_2 t + c_3}; \eta \right. \\ &\left. = \frac{c_1 (x - tu) - c_2 u + c_4}{c_1 t^2 + 2c_2 t + c_3}\right\}; \\ Z(CS_2) &= \{\xi = u, \eta = 0\}; \\ Z(CS_3) &= \left\{\xi = -\frac{1}{2}u + \alpha(x,t), \eta = \frac{1}{4}u^3 \right. \\ &\left. - \frac{1}{2}\alpha(x,t)u^2 - \beta(x,t)u + \gamma(x,t)\right\}, \end{aligned} \tag{16}$$

where $\alpha(x,t), \beta(x,t), \gamma(x,t)$ satisfy the PDE system

$$\begin{aligned} \alpha_t + \alpha_{xx} + 2\beta_x + 2\alpha\alpha_x &= 0, \\ \beta_t + \beta_{xx} - \gamma_x + 2\beta\alpha_x &= 0, \\ \gamma_t + \gamma_{xx} + 2\gamma\alpha_x &= 0. \end{aligned} \tag{17}$$

Remark 9. The above two examples are just to show the fundamental effect of Wu's algorithm. In practices, due to more complexity and heavy symbolic computations, the solving of the system of determining equations of nonclassical symmetries of PDEs cannot be always realized directly as in examples. We need some additional reasonable constraints to the system so that it is solved. As a result, the symmetries are derived. To this end, in the next section, we establish a relationship between CLS and nonclassical symmetry of PDEs from which one obtains some principles to get the additional constraints on the system of determining system of the nonclassical symmetry.

3. A Potential Constraints Method

3.1. An Identity. Let

$$\begin{aligned} \mathscr{X}' &= \xi'_i \partial_{x_i} + \eta'_j \partial_{u_j} \\ \mathscr{X} &= \xi_i \partial_{x_i} + \eta_j \partial_{u_j}, \end{aligned} \tag{18}$$

be the generators of the CLS and nonclassical symmetry of PDEs, respectively. $\xi' = (\xi'_1, \xi'_2, \ldots, \xi'_p)$ and $\eta' = (\eta'_1, \eta'_2, \ldots, \eta'_q)$; $\xi = (\xi_1, \xi_2, \ldots, \xi_p)$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_q)$. Let $\mathscr{D}' = 0$ and $\mathscr{D} = 0$ be the systems of the determining equations for $\mathscr{X}'$ and $\mathscr{X}$ in the sense of the definition in [3, 4, 20]. From the viewpoints in [20, 25], we call the left-hand sides $\mathscr{D}' \subset \mathscr{K}_z[\xi', \eta']$ and $\mathscr{D} \subset \mathscr{K}_z[\xi, \eta]$ of the two systems the determining polynomial systems of the CLS and nonclassical symmetry, respectively. Consequently, the zero sets $Z(\mathscr{D}')$ and $Z(\mathscr{D})$ represent the solution sets of the two systems of the determining equations, that is, the sets of the CLS and nonclassical symmetry of the PDEs.

Due to the fact that the CLS is a subset of the nonclassical symmetry, on the CLS, the infinitesimal functions of $\mathscr{X}'$ and $\mathscr{X}$ have relations

$$\begin{aligned} \xi'_i &= \xi_i, \\ \eta'_j &= \eta_j, \\ i(j) &= 1, 2, \ldots, p(q). \end{aligned} \tag{19}$$

Remark 10. Since only the regular case of the nonclassical symmetries is investigated in this article, without loss of generality, suppose that $\xi_1 \neq 0$. Then, according to the fact that if $\mathscr{X}$ is a symmetry of PDEs then $f\mathscr{X}$ so is, for any differential function f, set $\xi_1 = 1$. In this case, (19) becomes

$$\begin{aligned} \xi'_i &= a\xi_i, \\ \eta'_j &= a\eta_j, \\ i(j) &= 2, \ldots, p(q), \quad \text{where } a = \xi'_1. \end{aligned} \tag{20}$$

In evolutionary equations, ξ_1 is the coefficient of ∂_t in generator $\mathscr{X}$ corresponding to the time variable; it is usually notated by τ. In regular case of the nonclassical symmetries, we set $\tau = 1$.

We take a differential polynomial rank $\prec$ on $K_z[\xi', \eta']$ with $z = X \cup U$ and ξ'_1 with the highest rank. This rank induces a corresponding differential polynomial rank on $K_z[\xi, \eta, \xi'_1]$ by matching $\xi'_i \longrightarrow \xi_i, \eta'_j \longrightarrow \eta_j$. Let C' be the differential characteristic set of $\mathscr{D}'$ in the differential polynomial rank $\prec$ on $K_z[\xi', \eta']$. In [25], it is proven that $Z(C') = Z(\mathscr{D}')$ in PDEs; that is, $Z(C')$ represents the CLS of the PDEs.

Because of the linearity of C', the initials of the differential polynomials in C' are polynomials in z. Hence the IS product of C' is not zero; that is,

$$\text{IS}\left(C'\right) \not\equiv 0. \tag{21}$$

Suppose that $\mathscr{D}''$ is the subset of C' consisting of differential polynomials whose leading derivatives are the ones of ξ'_1. Substituting (20) into C' and $\mathscr{D}''$, one obtains differential polynomial systems belonging to $\mathscr{K}_z[\xi, \eta, \xi'_1]$ (still denote them as C' and D''). Since the linearity of both C' and transformations (20) in (ξ, η), the rank and initials of each differential polynomial in differential ascending chain C' are not changed after transformations (20) are applied to C'. Hence, the complement set $C' \setminus \mathscr{D}''$ and set $\mathscr{D}''$ keep the differential ascending chain structure and property (21) under transformations (20). Then we do the reduction $R = \text{Prem}((C' \setminus \mathscr{D}'')/\mathscr{D}'')$ by formula (5). After removing the factors on $\xi'_1 (\neq 0)$ from R (the resulting differential polynomial system is denoted again as R), we have the projection

$$C = R \cap \mathscr{K}_z[\xi, \eta]. \tag{22}$$

In the reduction for obtaining R, the operation on (ξ, η) is not involved; C inherits the differential ascending chain property of the $C' \setminus \mathscr{D}''$. In addition, through transformations (20), the set Z(C) represents the CLS expressed by the infinitesimal functions ξ and η of the nonclassical symmetry. Therefore, we have

$$\mathrm{Z}(\mathrm{C}) \subseteq \mathrm{Z}(\mathscr{D}). \tag{23}$$

For the sake of the following statement, we denote

$$C = \{q_1, q_2, \ldots, q_m\}, \tag{24}$$

as well as

$$\mathscr{D} = \{p_1, p_2, \ldots, p_n\}, \tag{25}$$

for the determining polynomial systems of the CLS and nonclassical symmetry of the PDEs.

One of main results of this paper is given below.

Theorem 11. *There exist differential polynomials coefficients differential operators $\mathfrak{D}_\nu^i$ such that the identities,*

$$\mathrm{IS}_i * p_i = \sum_{\nu=1}^{m} \mathfrak{D}_\nu^i q_\nu, \quad i = 1, 2, \ldots, n, \tag{26}$$

hold for all differential functions ξ and η and IS products IS_i of C for p_i.

Proof. By formula (5), we have the remainders $r_i = \mathrm{Prem}(p_i/C)$ reduced to C for each $p_i \in \mathscr{D}$. From (23), we have $r_i = 0$ on Z(C). Thus the irreducibility of C implies that $r_i \equiv 0$. The theorem is proven. □

This theorem shows the intrinsic connection between the CLS and nonclassical symmetry of PDEs. Thus we have the following corollary.

Corollary 12. *PDEs admit nontrivial nonclassical symmetry if and only if* Z(C) *is a proper subset of* Z($\mathscr{D}$)*; that is,* Z(C) ⊂ Z($\mathscr{D}$).

Summarizing the above procedure, we have an algorithm to obtain identities (26) in the following steps.

Step 1. Calculate the systems of determining equations $\mathscr{D}' = 0$ and $\mathscr{D} = 0$ of the CLS and nonclassical symmetry for the given PDEs.

Step 2. Calculate differential characteristic set C' of $\mathscr{D}'$ by Wu's algorithm and get $\mathscr{D}''$.

Step 3. Substitute (20) into C' and $\mathscr{D}''$ and calculate C as in (22).

Step 4. Establish (26) by reduction formula (5) in Wu's method.

Remark 13. In Step 2, since $\mathscr{D}'$ is linear (the system of determining polynomials of the CLS), its differential characteristic set C' is calculated easily by Wu's algorithm [25, 26].

The following is an illustrative example to show the above procedure of obtaining the relations (26).

Example 14. We establish relation (26) for the nonlinear heat equation

$$u_t = u_{xx} - u^3. \tag{27}$$

Step 1. The system of determining equations of the nonclassical symmetry $\mathscr{X} = \partial_t + \xi\partial_x + \eta\partial_u$ (see Remark 10) and CLS $\mathscr{X}' = \tau'\partial_t + \xi'\partial_x + \eta'\partial_u$ of the equation is given by $\mathscr{D} = \{p_1, p_2, p_3, p_4\} = 0$, where

$$\begin{aligned} p_1 &= \xi_{uu}, \\ p_2 &= \eta_{xx} - \eta_t + u^3\eta_u - 2\left(u^3 + \eta\right)\xi_x - 3u^2\eta, \\ p_3 &= 2\eta_{xu} - \xi_{xx} + \xi_t - \left(3u^3 + 2\eta\right)\xi_u + 2\xi\xi_x, \\ p_4 &= \eta_{uu} - 2\xi_{xu} + 2\xi\xi_u, \end{aligned} \tag{28}$$

and $\mathscr{D}' = 0$, in which

$$\begin{aligned} \mathscr{D}' = \Big\{ & 2\xi'_x - \tau'_t, \tau'_u, \tau'_x, \eta'_{uu}, \eta'_t - \eta'_{xx} + u^3\left(\tau'_t - \eta'_u\right) \\ & + 3u^2\eta', \xi'_{xx} - 2\eta'_{xu} - \xi'_t, \xi'_u \Big\}. \end{aligned} \tag{29}$$

Step 2. The differential characteristic set of $\mathscr{D}'$ under rank $\xi' \prec \eta' \prec \tau'$ is

$$\begin{aligned} & C' \\ & = \left\{\tau'_u, \tau'_x, u\tau'_t + 2\eta', \xi'_u, u\xi'_x + \eta', \xi'_t, \eta'_x, \eta'_t, \eta' - u\eta'_u\right\}. \end{aligned} \tag{30}$$

Hence

$$\mathscr{D}'' = \left\{\tau'_u, \tau'_x, u\tau'_t + 2\eta'\right\}. \tag{31}$$

And IS$(C') = u \not\equiv 0$.

Step 3. After substituting transformations (20) with $a = \tau'$, the sets C' and $\mathscr{D}''$ become

$$\begin{aligned} C' = \Big\{ & \tau'\xi_u, \tau'\left(\eta + u\xi_x\right), \xi_t\tau' + \xi\tau'_t, \tau'\eta_x, \eta_t\tau' \\ & + \eta\tau'_t, \tau'\left(\eta - u\eta_u\right) \Big\} \cup \mathscr{D}'', \end{aligned} \tag{32}$$

and

$$\mathscr{D}'' = \left\{\tau'_u, \tau'_x, u\tau'_t + 2\tau'\eta\right\}. \tag{33}$$

Then computing reduction $\mathrm{Prem}((C' \backslash \mathscr{D}'')/\mathscr{D}'')$ and deleting common factor $\tau' \not\equiv 0$, we have irreducible passive differential chain $C = \{q_1, \ldots, q_6\}$ under the corresponding rank $\xi \prec \eta$ with

$$\begin{aligned} q_1 &= u\xi_t - 2\xi\eta, \\ q_2 &= u\xi_x + \eta, \\ q_3 &= \xi_u, \\ q_4 &= u\eta_t - 2\eta^2, \\ q_5 &= u\eta_u - \eta, \\ q_6 &= \eta_x. \end{aligned} \tag{34}$$

Step 4. Using Wu's reduction formula (5), one obtains relations (26) as follows:

$$\begin{aligned}P_1 &= I_1 p_1 = D_u q_3,\\ P_2 &= I_2 p_2 = u D_x q_6 + u^3 q_5 - q_4 - 2\left(u^3+\eta\right) q_2,\\ P_3 &= I_3 p_3\\ &= 2D_x q_5 - D_x q_2 + 3q_6 - u\left(3u^3 + 2\eta\right) q_3 + 2\xi q_2\\ &\quad + q_1,\\ P_4 &= I_4 p_4 = D_u q_5 - 2uD_x q_3 + 2u\xi q_3,\end{aligned} \tag{35}$$

with initials $I_1 = 1, I_2 = I_3 = I_4 = u \neq 0$ of C.

Remark 15. Since $Z(C)$ and $Z(\mathscr{D})$ represent the CLS and nonclassical symmetry, the identities (26) (also see the identities (35) for (27)) reveal the relationship between the determining polynomial systems of the CLS and nonclassical symmetry of PDEs. This connection between the CLS and nonclassical symmetry provides us a way to get additional equations that can be used to reduce the system of the nonlinear determining equations of the nonclassical symmetry (see examples given in next section).

In practical computations, we use the following necessary and sufficient conditions for a generator (18) being a nonclassical symmetry of PDEs.

Theorem 16. *The operator $\mathscr{X}$ in 8 is a nontrivial nonclassical symmetry of the PDEs (1) if and only if the infinitesimal functions $(\xi,\eta) = (\xi_2,\ldots,\xi_p,\eta_2,\ldots,\eta_q) \in Z(\mathscr{D}/C)$ or, equivalently, at (ξ,η) all $p_i \in \mathscr{D}$ are zero and at least one of $q_j \in C$ is not zero.*

The theorem is easily proven by Corollary 12.

For example, for (27), $\xi = (3/\sqrt{2})u, \eta = -(3/2)u^3$ make all $p_i = 0 (i = 1, 2, 3, 4)$ in (35), but it is obvious that $q_3 \neq 0$. Hence $\mathscr{X} = \partial_t + 3u/\sqrt{2}\partial_x - (3/2)u^3\partial_u$ is a nontrivial nonclassical symmetry of the equation.

3.2. Potential Constraints. The determining equations of the nonclassical symmetry are solved relatively easily through extending some additional auxiliary equations, which are named potential constraint (PC).

Definition 17. An additional differential equation that is compatible with the system of the determining equations of the nonclassical symmetry of *PDEs* is called potential constraint (*PC*) for the system. The *PC* is called nontrivial if it provides a nontrivial nonclassical symmetry. An idea for obtaining suitable PCs comes from the following zero decomposition of the system $\mathscr{D}$; that is,

$$\begin{aligned}Z(\mathscr{D}) &= Z(\mathscr{D}\cup\mathscr{J}) \cup Z\left(\frac{\mathscr{D}}{\mathscr{J}}\right)\\ &= Z(\mathscr{D}\cup\mathscr{J}) \cup Z\left(\mathscr{D}\cup\frac{Q}{\mathscr{J}}\right)\\ &\quad \cup Z(\mathscr{D}\cup/\mathscr{J} * Q)\end{aligned} \tag{36}$$

for an arbitrary differential polynomial $\mathscr{J}$ and differential polynomial set Q. $\mathscr{D}\cup\mathscr{J}$ and $\mathscr{D}\cup Q$ are extensions of the system $\mathscr{D}$. Then, it is relatively easy to find the nonclassical symmetry of the considered PDEs from the subsets $Z(\mathscr{D}\cup\mathscr{J})$ and $Z(\mathscr{D}\cup Q/\mathscr{J})$ of $Z(\mathscr{D})$ by properly selecting some specific differential polynomial $\mathscr{J}$ and Q. Consequently, we have PCs $\mathscr{J} = 0$ or $Q = 0$. For this reason, it is obvious that Theorems 11 and 16 and identities in (26) provide feasible basis.

Particularly, the first subset $Z(\mathscr{D}\cup\mathscr{J})$ in the above decomposition results in the following two principles for selecting a PC.

Principle 1. Take the coefficient of ξ or η in some identities of (26) whose right-hand sides are collected in terms of ξ and η as $\mathscr{J}$.

Principle 2. Take the coefficient that is polynomial in ξ_i or η_j of some $\mathfrak{D}_\nu^i q_i$ in some identities of (26) as $\mathscr{J}$.

Example 18. Using (35), we give some examples to show how to use the principles.

Case 1 (use Principle 1). Let $\mathscr{J} = q_2$, which is the coefficient of ξ in the third identity P_3. Adding $\mathscr{J}$ to $\mathscr{D}$, we get the extended system $\mathscr{D}\cup\mathscr{J} = \{p_1, p_2, p_3, p_4, q_2\}$. Using Wu's method to the extended system, we get its nontrivial differential characteristic set $CS = \{3\eta+u\xi^2, \xi_u, \xi^2-3\xi_x, \xi_t\}$, which implies that $\mathscr{J} = 0$ is a PC for the system $\mathscr{D}$. By calculating CS, we get $\xi = -3/(3c_1 + x)$ and $\eta = -3u/(3c_1 + x)^2$. By $q_1 \neq 0$ and $p_i = 0$ $(i = 1, 2, 3, 4)$, we deduce that $\mathscr{J} = 0$ is a nontrivial PC for the system $\mathscr{D}$ by Theorem 16.

Case 2 (use Principle 2). Take the coefficient of q_2 in the second identity P_2 as $\mathscr{J} = u^3 + \eta$. After executing Wu's method for the system $\mathscr{D}\cup\mathscr{J}$, we have an empty differential characteristic set. This shows that differential equation $\mathscr{J} = 0$ is not compatible with $\mathscr{D}$. So it is not a PC for the system $\mathscr{D}$.

Case 3 (use Principle 2). Let $\mathscr{J} = 3u^3 + 2\eta$ be the coefficient of q_3 in the third identity P_3. The differential characteristic set of the extended system $\mathscr{D}\cup\mathscr{J}$ is $\{3u^3 + 2\eta, 9u^2 - 2\xi^2\}$ which directly yields $\xi = \pm 3u/\sqrt{2}, \eta = -3u^3/2$. Since $q_3 \neq 0$ and all $p_i = 0$ at current (ξ,η), the equation $\mathscr{J} = 0$ is an additional nontrivial PC.

One may choose other cases of $\mathscr{J} = 0$ as candidate PCs, but no new symmetry is found. Actually, one finds by directly solving the determining equations $\mathscr{D} = 0$ that the nonclassical symmetries in Cases 1 and 3 are complete set of the nonclassical symmetries of (27). Here, it was seen that the auxiliary constrains $\mathscr{J} = 0$ make the solving of the system of determining equations $\mathscr{D} = 0$ be easier and lead to the complete nonclassical symmetry set of (27).

The example shows that identity (26) combining Principles 1 and 2 efficiently yields the nontrivial PCs. It also indicates that a nonclassical symmetry of PDEs may be zero point of some partial terms of a polynomial in identities (26). This is the reason why we introduce Principles 1 and 2.

For the second subset $Z(\mathscr{D} \cup Q/\mathscr{J})$ in decomposition of $Z(\mathscr{D})$, if taking Q as a differential ascending chain and $\mathscr{J}$ is its IS product, then we get the reduction of the system $\mathscr{D}$ with respect to Q. In fact, let $\overline{\mathscr{D}} = \{\overline{p}_k = \text{Prem}(p_k/Q) : p_k \in \mathscr{D}\}$ be the set of pseudoremainders of $p_k \in \mathscr{D}$ with respect to Q in the sense of Wu's reduction (5); that is, for each $p_k \in \mathscr{D}$, there exist operators $\mathfrak{D}_i^k$ such that

$$\text{IS}_k * p_k = \sum_{q_i^k \in Q} \mathfrak{D}_i^k q_i^k + \overline{p}_k. \tag{37}$$

Thus, we have

$$Z\left(\mathscr{D} \cup \frac{Q}{\mathscr{J}}\right) = Z\left(\overline{\mathscr{D}} \cup \frac{Q}{\mathscr{J}}\right) \tag{38}$$

with $\mathscr{J} = \prod \text{IS}_k$. The right-hand side of (38) is the zero set of the reduced differential polynomial system $\overline{\mathscr{D}}$, which is more easily solved. Practically, we take the subset $Q \subset C$. Hence, more practically, we have the third principle to obtain PCs.

Principle 3. Take Q as a differential ascending chain or some proper subset of C.

In the next section, application of Principle 3 will be seen in Example 18.

4. More Examples

Because the determining equations involve arbitrary symbolic parameters [13], the symmetry classification problem for PDEs with arbitrary parameters is a difficult job for symmetry analysis. Finding classifying equations is the key step to successfully solve the problem. In the following, we give an example to show that our method can be used to solve the problem efficiently.

Example 19. We consider the symmetry classification problem for the nonlinear heat equation

$$u_t = u_{xx} + f(u), \tag{39}$$

with source $f(u) \neq 0$ [11]. Now we give the solution of the problem by using our method as compared with the method used in [11].

For the nonclassical symmetry $\mathscr{X} = \partial_t + \xi\partial_x + \eta\partial_u$ of (39), we have relations (26) as follows:

$$\begin{aligned} p_1 &= D_u q_2, \\ p_2 &= D_{uu} q_1 - 2D_x q_2 + 2\xi q_2, \\ p_3 &= 2D_{xu} q_1 + (3f(u) - 2\eta) q_2 - D_x q_3 + 2\xi q_3 + q_4, \\ p_4 &= D_{xx} q_1 - f(u) D_u q_1 - D_t q_1 + f'(u) q_1 \\ &\quad + 2(f(u) - \eta) q_3, \end{aligned} \tag{40}$$

where

$$\begin{aligned} q_1 &= \eta, \\ q_2 &= \xi_u, \\ q_3 &= \xi_x, \\ q_4 &= \xi_t \end{aligned} \tag{41}$$

is the differential characteristic set of the CLS of (39) for arbitrary function $f(u)$.

From the third identity in above identities, selecting the partial term $\mathscr{J} = (3f(u) - 2\eta)q_2$, we have two cases for the candidate PC $\mathscr{J} = 0$.

Case 1. Let $p_5 = 2\eta - 3f(u)$ and $q_2 = \xi_u \neq 0$. We execute Wu's algorithm on the extended system $\mathscr{D} \cup \{p_5\}$ and obtain a necessary condition on parameter f as follows:

$$3f^{(4)2} - 2f''' f^{(5)} = 0, \tag{42}$$

for the system having a nontrivial differential characteristic set. This is a classifying equation for symmetry classification of (39). It has general solutions,

$$\begin{aligned} f(u) &= (au + b)\ln(au + b) + c(au + b)^2 \\ &\quad + d(au + b) + e, \quad a \neq 0; \\ f(u) &= au^3 + bu^2 + cu + d, \end{aligned} \tag{43}$$

for arbitrary constants a, b, c, d, and e.

Case 2. Let $p_6 = q_2 = \xi_u$. In this case, we execute Wu's algorithm again on the extended system $\mathscr{D} \cup \{p_6\}$ and obtain an additional classifying equation:

$$f'''^2 f^{(4)} - 2f'' f^{(4)2} + f'' f''' f^{(5)} = 0. \tag{44}$$

It has general solutions,

$$\begin{aligned} f(u) &= A(au + b)^{\mu} + c(au + b) + d; \\ f(u) &= A(au + b)\ln(au + b) + c(au + b) + d; \\ f(u) &= A\ln(au + b) + c(au + b) + d; \\ f(u) &= Ae^{(au+b)} + c(au + b) + d, \end{aligned} \tag{45}$$

for arbitrary constants A, a, b, c, and d with $a \neq 0$.

Case 3. In other cases, of course, one can similarly select other partial terms from any of the identities as PCs. For example, we take coefficient $\mathscr{J} = f(u) - \eta$ of q_3 in the last identity. However, we are not able to get additional classifying equation.

Surprisingly, we have seen that functions (43)-(45) recovered all the cases given in [11]. This shows that the complete nonclassical symmetry classifications of the equation are determined by the selected PCs $p_5 = 0$ and $p_6 = 0$. Compared with the directly solving method used in [11], the presented potential constraint method is much more efficient.

Example 20. We consider the nonclassical symmetry and potential nonclassical symmetry of the nonlinear Boussinesq equation given by

$$u_{tt} + \beta\left(u^2\right)_{xx} + \gamma u_{xxxx} = 0 \quad (\beta\gamma\neq 0). \tag{46}$$

The nonclassical symmetry and nonclassical reductions of the equation are studied in [9, 30, 31]. To the authors' knowledge, the potential nonclassical symmetry of the equation has not been considered yet in literatures.

First, we consider the nonclassical symmetry of (46) by applying our method.

It is easily calculated that the CLS $\mathcal{X} = \tau'(t,x,u)\partial_t + \xi'(t,x,u)\partial_x + \eta'(t,x,u)\partial_u$ of the equation has determining polynomial system

$$\mathscr{D}' = \left\{\eta'_x, \eta'_t, \eta' - u\eta'_u, \tau'_u, \tau'_x, \eta' + u\tau'_t, \xi'_u, \eta' + 2u\xi'_x, \xi'_t\right\}, \tag{47}$$

with zero points (solutions to the system of determining equations),

$$\begin{aligned}\tau' &= c_1 t + c_2,\\ \xi' &= \left(\frac{1}{2}\right)c_1 x + c_3,\\ \eta' &= -c_1 u,\end{aligned} \tag{48}$$

for arbitrary constants c_1, c_2, and c_3. The differential characteristic set $C = \{q_1, \ldots, q_6\}$ of $\mathscr{D}'$ through transformations (20) is obtained by Wu's algorithm, where

$$\begin{aligned}q_1 &= \eta_x,\\ q_2 &= \xi_u,\\ q_3 &= \eta - u\eta_u,\\ q_4 &= \eta^2 - u\eta_t,\\ q_5 &= 2u\xi_x + \eta,\\ q_6 &= \eta\xi - u\xi_t\end{aligned} \tag{49}$$

are expressed by the infinitesimal functions ξ and η of the nonclassical symmetry $\mathcal{X} = \partial_t + \xi\partial_x + \eta\partial_u$. The determining polynomial system of the nonclassical symmetry for the equation is given by $\mathscr{D} = \{p_1, \ldots, p_7\}$ as follows:

$$\begin{aligned}p_1 &= \xi_u,\\ p_2 &= \eta_{uu},\\ p_3 &= \eta_u + 2\xi_x,\\ p_4 &= 2\eta_{xu} - 3\xi_{xx},\\ p_5 &= 12\gamma\eta_{xxu} - 8\gamma\xi_{xxx} + \xi\xi_t + 2\left(\beta u + \xi^2\right)\xi_x + \beta\eta,\\ p_6 &= 4\gamma\eta_{xxxx} + \eta_{tt} + 2\eta\eta_{tu} + 2\beta u\eta_{xx} - 2\xi_t\eta_x\\ &\quad + 4\left(\eta_t + \eta\eta_u - \xi\eta_x\right)\xi_x,\\ p_7 &= 4\gamma\left(4\eta_{xxxu} - \xi_{xxxx}\right) - \xi_{tt} - 2\xi\eta_{tu} + 4\beta u\eta_{xu}\\ &\quad - 2\beta u\xi_{xx} - 2\xi_t\eta_u + 4\xi\xi_x^2 - 2\xi_t\xi_x - 8\xi\eta_u\xi_x\\ &\quad + 4\beta\eta_x.\end{aligned} \tag{50}$$

For system $\mathscr{D}$, one easily obtains the following two identities in (26):

$$\begin{aligned}u * p_3 &= q_5 - q_3,\\ 2u * p_4 &= 7q_1 - 4D_x q_3 - 3D_x q_5.\end{aligned} \tag{51}$$

Since $p_3 = 0$ for the nonclassical symmetry, the first identity in the above yields $q_3 = q_5$. It is easy to check by Wu's method that when $q_3 = q_5 = 0$ only the CLS is obtained. Hence, for the nontrivial nonclassical symmetry, it has to be $q_3 = q_5\neq 0$. With second identity above and $p_4 = 0$ for the nonclassical symmetry, we have $q_1 = D_x q_3 = D_x q_5$. It leads to $\eta_{xu} = \xi_{xx} = 0$. Adding $p_8 = \eta_{xu}$ and $p_9 = \xi_{xx}$ to the system $\mathscr{D}$ and executing Wu's method on the extended system $\mathscr{D}\cup\{p_8, p_9\}$ under $\xi\neq 0$ (exclude the trivial case $\xi = 0$), we get the differential characteristic set of the extended system given by

$$\begin{aligned}\mathrm{CS} = \Big\{&\xi_{\mathrm{u}}, \beta\eta + \xi\xi_t + 2\left(\xi^2 + \beta \mathrm{u}\right)\xi_{\mathrm{x}}, \beta\eta + \xi\xi_{\mathrm{t}} - \eta_{\mathrm{u}}\left(\xi^2\right.\\ &\left.+ \beta\mathrm{u}\right), \eta\xi_t\left(2\xi^4 - \beta^2u^2 + 3\beta u\xi^2\right)\\ &- \xi\left(\eta_t\left(\xi^2 + \beta u\right)^2 + u\xi_t^2\left(\xi^2 + 2\beta u\right)\right) + \beta\eta^2\xi\left(\xi^2\right.\\ &\left.+ 2\beta u\right) + 2\eta_x\left(\xi^2 + \beta u\right)^3, \beta\eta^2\xi\xi_t\left(4\xi^4 - \beta^2u^2\right.\\ &\left.+ 9\beta u\xi^2\right) - \eta_{tt}\left(\xi^2 + \beta u\right)^4 + \beta^2\eta^3\left(\xi^2 + 2\beta u\right)^2\\ &+ \xi\xi_t\left(\beta u^2\xi_t^2\left(\xi^2 + 3\beta u\right)\right.\\ &\left.+ 2\eta_t\left(\xi^2 + 2\beta u\right)\left(\xi^2 + \beta u\right)^2\right)\\ &+ \beta\eta\left(\eta_t\left(\beta u - \xi^2\right)\left(\xi^2 + \beta u\right)^2\right.\\ &\left.- 2u\xi^2\xi_t^2\left(2\xi^2 + 5\beta u\right)\right), \beta^2\eta^2\xi + \left(2\xi^3\xi_t\right.\\ &\left.+ \beta\eta\left(3\xi^2 + \beta u\right) + \beta u\xi\xi_t\right)\xi_t - \left(\xi^2 + \beta u\right)^2\xi_{tt}\Big\},\end{aligned} \tag{52}$$

with initial $\mathrm{I} = \xi^2 + \beta u\neq 0$. Consequently, from the first two polynomials in CS, we easily get their zero points (solutions to the system of determining system of the nonclassical symmetry),

$$\begin{aligned}\xi &= f(t)\,x + g(t),\\ \eta &= -\frac{\left(\xi\xi_t + 2\left(\beta u + \xi^2\right)\xi_x\right)}{\beta},\end{aligned} \tag{53}$$

and the remaining polynomials in CS yield the conditions

$$f''(t) + 2f(t)f'(t) - 4f(t)^3 = 0,$$
$$g''(t) + 2f(t)g'(t) - 4f(t)^2 g(t) = 0, \tag{54}$$

for the functions f and g. This rediscovered the nonclassical symmetries given in [9].

In the above procedure of solving determining equations, PCs $p_8 = 0$ and $p_9 = 0$ are obtained just from two identities. This shows that even the parts of (26) provide nontrivial PCs.

Now, we consider the potential nonclassical symmetry of (46) through its potential system

$$u_t + v_x = 0,$$
$$v_t - \beta u_x^2 - \gamma u_{xxx} = 0. \tag{55}$$

It is easily calculated that the nonclassical symmetry $\mathscr{X} = \partial_t + \xi(t,x,u,v)\partial_x + \eta(t,x,u,v)\partial_u + \phi(t,x,u,v)\partial_v$ of the system has the determining polynomial system $\mathscr{D}$ consisting of eleven strong nonlinear differential polynomials and one of them admitting 35 terms. One hardly deals with this system directly. Hence, some PCs are necessary for getting a nontrivial nonclassical symmetry.

By Principle 3, we consider two cases of PCs as taking $Q = \{\eta, \phi\}$ and $Q = \{p_{12} = \xi_u, p_{13} = \xi_v\}$.

Case 1. $Q = \{\eta, \phi\}$. Under this PC, the system of determining equations is reduced to equation

$$\xi\xi_v + \xi_u = 0, \tag{56}$$

with $\xi_t = \xi_x = 0$. Its general solution is implicitly given by $F(\xi, v - u\xi) = 0$ for arbitrary functions $F(U, V)$. Hence, for any solution $\xi(u, v)$ of (56), (46) admits the nonclassical symmetry $\mathscr{X} = \partial_t + \xi\partial_x$. For example, we have specific potential nonclassical symmetries,

$$\mathscr{X} = \partial_t + \frac{(v+b)}{(u+a)\partial_x};$$
$$\mathscr{X} = \partial_t + \left(u + \sqrt{u^2 - 2v^2}\right)\partial_x, \tag{57}$$

for arbitrary constants a and b by selecting the functions $F(U,V) = aU - V - b$ and $F(U,V) = (1/2)U^2 - V$, respectively.

Case 2 ($Q = \{\xi_u, \xi_v\}$). Executing Wu's method on the extended system $\mathscr{D} \cup Q$, we get the differential characteristic sets C_1 and C_2 (see Appendix). Thus, the sets $Z(C_1)$ and $Z(C_2)$ are easily determined because of their characteristic set structure.

$Z(C_1)$ is given by

$$\xi = c_1 t + c_2,$$
$$\eta = -\left(\frac{c_1}{\beta}\right)(c_1 t + c_2), \tag{58}$$

and $\phi = \phi(t,x,u,v)$, which satisfies the system

$$\phi_t = \frac{\phi(\phi_u - c_1)}{\xi} - 2c_1\eta,$$
$$\phi_x = \left(\frac{c_1}{\beta}\right)\phi_u, \tag{59}$$
$$\phi_v = \frac{(c_1 - \phi_u)}{\xi}.$$

Since system (59) is an involution (standard) form, it has infinite number of solutions. This has been proven in [15]. Particularly, for any initial conditions $\phi(z_0) = c_1$, $\phi_1(z_0) = c_2$, $\phi_x(z_0) = c_3$, $\phi_v(z_0) = c_4$, and $\phi_u(t_0, x_0, u, v_0) = h(u)$ at $z_0 = (t_0, x_0, u_0, v_0) \in \mathbb{R}^n$, system (59) has unique Taylor series solution for any differential function $h(u)$. For example, $\phi = c_1(u + c_1 x/\beta) + (c_1^2/\beta)(c_1 t^2 + 2c_2 t) + c_3$ is a solution to system (59).

$Z(C_2)$ is given by (53) for ξ and η and

$$\phi = \left(\psi(t,x) + ug'(t)\right) + \left(\frac{f'(t)}{f(t)}\right)(v - ug(t)) - vf(t), \tag{60}$$

where

$$\psi(t,x) = \frac{\left[2f(t)\left(12x^2 f(t)^3 g(t) + 3\beta h(t) + 4x^3 f(t)^4\right) + 3x\left(xf'(t) + 2g'(t)\right)\left(f(t)g'(t) - g(t)f'(t) + 2f(t)^2 g(t)\right) + 2xf(t)^3\left(\left(2xf'(t) + 3g'(t)\right)x + 12g(t)^2\right)\right]}{(6\beta f(t))}. \tag{61}$$

Here $f(t)(\neq 0)$ and $g(t)$ are solutions to (54) and $h(t)$ satisfies

$$h'(t) + \left(\frac{f'(t)}{f(t)} + 4f(t)\right)h(t) - \left(\frac{2}{\beta}\right)g(t)\left(g'(t) + 2f(t)g(t)\right)^2 = 0. \tag{62}$$

The nonclassical symmetries (58)-(59) and (60) are two groups of potential nonclassical symmetries of (46)-(53).

Hence it is clear that under the current constraints $Q = \{\xi_u, \xi_v\}$ the nonlinear Boussinesq equation does not admit potential nonclassical symmetry in the sense of $\xi_v^2 + \eta_u^2 \neq 0$.

Particularly, using special ansatz, we have the following two classes of special solutions to (54) (as well as (62)) given by

$$f(t) = \frac{1}{(b + 2t)},$$

$$g(t) = \frac{c_1}{(b+2t)} + c_2(b+2t);$$
$$f(t) = \frac{1}{(b-t)},$$
$$g(t) = \frac{c_1}{(b-t)} + c_2(b-t)^4, \quad (63)$$

for arbitrary constants b, c_1, and c_2. c_2 of each case corresponds to nonclassical symmetries with

$$\xi = (2t+b) + \frac{x}{(2t+b)},$$
$$\eta = -\left(\frac{2}{\beta}\right)\left(2\xi + \frac{\beta u}{(b+2t)}\right); \quad (64)$$
$$\xi = (t-b)^4 - \frac{x}{(t-b)},$$
$$\eta = \frac{2u}{(t-b)} - \frac{\xi\left(3x + 2(t-b)^5\right)}{\left(\beta(t-b)^2\right)}. \quad (65)$$

Further reductions of (54) are given as

$$f(t) = \frac{p'(t)}{2p(t)},$$
$$p'(t)^2 = ap^3(t) + c, \quad (66)$$
$$f'(t) = f^2(t) + d\exp\left(\int f(t)\,dt\right),$$

where a, c, and d are arbitrary constants. Any solution to the reduced equations yields nonclassical symmetry of nonlinear Boussinesq equation (46) through solving $g(t)$ and $h(t)$ from (54) and (62). Although the specific nonclassical symmetries (64) and (65) are covered by the general reductions of (54) given in [9, 30, 31], they are not solved explicitly in these materials. In the next section, we show that the nonclassical symmetries yield a kind of rational function solutions with blow-up property to the nonlinear Boussinesq equation.

5. Invariant Solutions

We use symmetries (64) and (65) in Section 4 to construct invariant solutions to nonlinear Boussinesq equation (46). To end this, solving the characteristic equations $dt = dx/\xi = du/\eta$, we have invariants

$$I_1 = \frac{\left(3x - 4t^2 - 4bt - b^2\right)}{\left(3\sqrt{b+2t}\right)},$$
$$I_2 = \left(\frac{4}{9}\beta\right)\left(3x(b+2t) + 8t^3 + 12bt^2 + 6b^2t - b^3\right) + (b+2t)u, \quad (67)$$

in symmetry (64), and

$$I_3 = (t-b)\left(\frac{1}{6}(t-b)^5 - x\right),$$
$$I_4 = \frac{\left(6I_3^2/(t-b)^6 + (25/6)(t-b)^6\right)}{(12\beta)} + \frac{u}{(t-b)^2}, \quad (68)$$

in symmetry (65). Here, b is an arbitrary constant.

Corresponding to the two groups of the invariants, let $I_2 = F(I_1)$ and $I_4 = G(I_3)$ for two functions F and G be determined. Hence, we have

$$u = \frac{\left(F(I_1) + (4/9\beta)\left(b^3 - 6b^2t - 12bt^2 - 8t^3 - 3x(2t+b)\right)\right)}{(2t+b)};$$
$$u = \frac{1}{12\beta}\left(12\beta(t-b)^2 G(I_3) - 6\left(\frac{f(t,x)}{(t-b)}\right)^2 - \frac{25}{6}(t-b)^8\right). \quad (69)$$

Substituting these into (46), we get the reduced equations

$$36\gamma F^{(4)}(I_1) + \left(18\beta F(I_1) + 9I_1^2 + 16b^3\right)F''(I_1) + 18\beta F'(I_1)^2 + 63I_1F'(I_1) + 72F(I_1) + \frac{64b^3}{\beta} = 0;$$
$$36\gamma G^{(4)}(I_3) + 3\left(6\beta G(I_3) + 5I_3\right)G''(I_3) + 18\beta G'(I_3)^2 + 75G'(I_3) - \frac{175}{\beta} = 0, \quad (70)$$

respectively. Since the general solutions of the equations are difficult to solve, we get some special solutions to the equations as

$$F(t) = e - \frac{2}{\beta}t^2,$$
$$G(t) = \frac{5\epsilon}{3\beta}t + c, \quad (71)$$

where $\epsilon = -7/2$ or $\epsilon = 1$ (e and c are arbitrary constants). Hence, we obtain exact solutions

$$u(t,x) = \frac{2\left(b^3 - 18b^2t - 36bt^2 - 24t^3\right) + 9\beta e}{9\beta(b+2t)} - \frac{2x^2}{\beta(b+2t)^2}; \quad (72)$$

and

$$u(t,x) = \frac{1}{3\beta}(t-b)^2\left(3\beta c - \left(5\epsilon(t-b)f(t,x) + \frac{3f(t,x)^2}{2(t-b)^4} + \frac{25}{24}(t-b)^6\right)\right), \quad (73)$$

to (46), where $f(t,x) = x - (1/6)(t-b)^5$.

Since $u(t,x) \longrightarrow \infty$ when $t \longrightarrow -b/2$ or $t \longrightarrow b$, these solutions are blow-up solutions to (46) and the blow-ups occur on $t = -b/2$ and $t = b$. Also these solutions are rational solutions that are interested in the study of lump-type soliton solutions to mathematical physics equations (see [32] and the references therein).

6. Conclusions

We establish a set of intrinsical link identities (26) between the determining equations of the CLS and nonclassical symmetry through the corresponding determining polynomial system. From practical examples, we observed that the infinitesimal functions of a nonclassical symmetry of PDEs are zero points of some partial terms of the differential polynomials involved in the identities. By setting the partial terms to zero, one obtains an additional equation to extend the original system. As a result, we get auxiliary constraints to reduce the system of determining equations. Being encouraged by the observation, we introduce a concept called potential constraint to the system of determining equations of nonclassical symmetry of PDEs. Correspondingly, the three principles are suggested to obtain the PCs from the identities in (26). The PCs can be purposefully obtained from the relationship (26) from the given principles. Consequently, we obtain an extended system of the determining equations system by attaching the obtained PCs to the system. This system is relatively easy to solve than the original system, since it has more equations. By calculating the differential characteristic set of the extended system by Wu's algorithm, we reduce the system to differential characteristic set form. By determining the zero set of the differential characteristic set, we obtain the nonclassical symmetry of the PDEs.

The method provides a new way and idea to solve a nonclassical symmetry of PDEs. In principle, our method yields a part of the nonclassical symmetries of PDEs. However, the examples show that sometimes the method yields whole set of nonclassical symmetries of PDEs. The method can also be used to solve symmetry classification problem of PDEs.

Appendix

Differential Characteristic Sets in Example 20

Differential characteristic set $C_1 = \{q_1, q_2, \ldots, q_{11}\}$, where

$$
\begin{aligned}
q_1 &= \eta_v, \\
q_2 &= \eta_u, \\
q_3 &= \eta_x, \\
q_4 &= \xi_v, \\
q_5 &= \xi_u, \\
q_6 &= \xi_x, \\
q_7 &= \beta\eta + \xi\xi_t, \\
q_8 &= 2\beta\eta^2 - \xi\phi_t - \xi\phi_v\phi, \\
q_9 &= \beta\eta^2 + \xi^2\eta_t, \\
q_{10} &= 2\beta\eta^2\xi + \beta\eta\phi - \xi^2\phi_t + \xi\phi_u\phi, \\
q_{11} &= 2\beta\eta^3\xi + \beta\eta^2\phi - \eta\xi^2\phi_t - \xi^2\phi_x\phi.
\end{aligned} \tag{A.1}
$$

Differential characteristic set $C_2 = \{q_1, q_2, \ldots, q_{13}\}$, where

$$
\begin{aligned}
q_1 &= \eta_v, \\
q_2 &= \xi_v, \\
q_3 &= \xi_u, \\
q_4 &= \eta_u + 2\xi_x, \\
q_5 &= \beta\eta + \xi\xi_t - \beta u\eta_u - \xi^2\eta_u, \\
q_6 &= \beta\eta^2 + \xi^2\eta_t + 2\beta u^2\eta_u^2 - 3\beta u\eta\eta_u + u\xi^2\eta_u^2 \\
&\quad - 2\eta\xi^2\eta_u - 2\beta u\xi\eta_x - 2\xi^3\eta_x, \\
q_7 &= 2\beta^2\eta^2 + 2\beta\xi^2\eta_t - 2\beta^2 u\eta\eta_u + 2\beta u\xi\eta_u\phi_u \\
&\quad - 2\beta\eta\xi^2\eta_u + \xi^3\eta_u\phi_u - 2\beta\xi^3\eta_x, \\
q_8 &= 2\beta u\xi^2\eta_t - 3\beta\eta^2\xi^2 - 3\xi^4\eta_t - 2\beta^2 u^2\eta\eta_u \\
&\quad - 4\beta u^2\xi^2\eta_u\phi_v + 4\beta^2 u^2\xi\eta_x + 2\beta^2 u\eta^2 \\
&\quad + 7\beta u\eta\xi^2\eta_u + 6\eta\xi^4\eta_u - 2u\xi^4\eta_u\phi_v \\
&\quad + 6\beta u\xi^3\eta_x + 6\xi^5\eta_x, \\
q_9 &= \beta u\eta\xi^2\eta_t - \beta\eta^3\xi^2 - \eta\xi^4\eta_t - 2\beta u^2\xi^2\eta_t\eta_u \\
&\quad - u\xi^4\eta_t\eta_u - \beta^2 u^2\eta^2\eta_u + 2\beta^2 u^2\eta\xi\eta_x \\
&\quad - 2\beta u^2\xi^2\eta_u\phi_x + \beta^2 u\eta^3 + 2\beta u\eta^2\xi^2\eta_u \\
&\quad + 2\eta^2\xi^4\eta_u + 2\beta u\eta\xi^3\eta_x - u\xi^4\eta_u\phi_x \\
&\quad + 2\eta\xi^5\eta_x, \\
q_{10} &= \xi^6\eta_{tt} + 4\beta^2 u^2\xi^2\eta_{tt} + 4\beta u\xi^4\eta_{tt} + \beta^2\eta^3\xi^2 \\
&\quad + \beta\eta\xi^4\eta_t - 10\beta^2 u^2\xi^2\eta_t\eta_u + 3\beta^2 u\eta\xi^2\eta_t \\
&\quad - 9\beta u\xi^4\eta_t\eta_u - 2\xi^6\eta_t\eta_u - 12\beta^3 u^3\xi\eta_u\eta_x \\
&\quad - 3\beta^3 u^2\eta^2\eta_u + 18\beta^3 u^2\eta\xi\eta_x \\
&\quad - 10\beta^2 u^2\xi^3\eta_u\eta_x + 3\beta^3 u\eta^3 - 2\beta^2 u\eta^2\xi^2\eta_u \\
&\quad + 16\beta^2 u\eta\xi^3\eta_x - 2\beta u\xi^5\eta_u\eta_x + 4\beta\eta\xi^5\eta_x, \\
q_{11} &= 8\beta^2 u^3\xi^2\eta_{xx} + 8\beta u^2\xi^4\eta_{xx} + 2u\xi^6\eta_{xx} + \beta\eta^3\xi^2 \\
&\quad + \eta\xi^4\eta_t + 2\beta u^2\xi^2\eta_t\eta_u + \beta u\eta\xi^2\eta_t + u\xi^4\eta_t\eta_u \\
&\quad + 4\beta^2 u^3\xi\eta_u\eta_x - \beta^2 u^2\eta^2\eta_u - 2\beta^2 u^2\eta\xi\eta_x
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\beta u^2\xi^3\eta_u\eta_x + \beta^2 u\eta^3 - 4\beta u\eta^2\xi^2\eta_u \\
&- 2\eta^2\xi^4\eta_u - 4\beta u\eta\xi^3\eta_x - 2\eta\xi^5\eta_x, \\
q_{12} = {} & 4\beta^2u^3\xi^3\eta_{tx} + 4\beta u^2\xi^5\eta_{tx} + u\xi^7\eta_{tx} + \beta\eta^3\xi^4 \\
&+ \eta\xi^6\eta_t - 2\beta^2u^3\xi^2\eta_t\eta_u + \beta^2u^2\eta\xi^2\eta_t \\
&+ \beta u^2\xi^4\eta_t\eta_u + \beta u\eta\xi^4\eta_t + u\xi^6\eta_t\eta_u \\
&- 4\beta^3u^4\xi\eta_u\eta_x - \beta^3u^3\eta^2\eta_u + 6\beta^3u^3\eta\xi\eta_x \\
&- 2\beta^2u^3\xi^3\eta_u\eta_x + \beta^3u^2\eta^3 + 4\beta^2u^2\eta\xi^3\eta_x \\
&- 2\beta u^2\xi^5\eta_u\eta_x + \beta^2u\eta^3\xi^2 - 3\beta u\eta^2\xi^4\eta_u \\
&- 2\eta^2\xi^6\eta_u - 2\beta u\eta\xi^5\eta_x - u\xi^7\eta_u\eta_x \\
&- 2\eta\xi^7\eta_x, \\
q_{13} = {} & 4u\eta_x\phi_t\xi^7 - 8u^2\beta\eta_x^2\xi^7 - 4\eta\eta_x\phi\xi^7 - 4u\eta_u\eta_x\phi\xi^7 \\
&+ 4u^2\beta\eta_t\eta_x\xi^6 + 4u^2\beta\eta\eta_u\eta_x\xi^6 + 2\eta\eta_t\phi\xi^6 \\
&- 4\eta^2\eta_u\phi\xi^6 + 2u\eta_t\eta_u\phi\xi^6 - 2u\eta_t\phi_t\xi^6 \\
&+ 2u\eta\eta_u\phi_t\xi^6 - 24u^3\beta^2\eta_x^2\xi^5 + 4u\beta\eta^3\eta_u\xi^5 \\
&- 4u^2\beta\eta\eta_t\eta_u\xi^5 - 14u\beta\eta\eta_x\phi\xi^5 \\
&- 12u^2\beta\eta_u\eta_x\phi\xi^5 + 12u^2\beta\eta_x\phi_t\xi^5 \\
&+ 8u^3\beta^2\eta_t\eta_x\xi^4 + 10u^3\beta^2\eta\eta_u\eta_x\xi^4 \\
&+ 2\beta\eta^3\phi\xi^4 + 3u\beta\eta\eta_t\phi\xi^4 - 12u\beta\eta^2\eta_u\phi\xi^4 \\
&+ 6u^2\beta\eta_t\eta_u\phi\xi^4 - 2u\beta\eta^2\phi_t\xi^4 - 4u^2\beta\eta_t\phi_t\xi^4 \\
&+ 7u^2\beta\eta\eta_u\phi_t\xi^4 - 16u^4\beta^3\eta_x^2\xi^3 \\
&+ 2u^2\beta^2\eta^2\eta_t\xi^3 + 6u^2\beta^2\eta^3\eta_u\xi^3 \\
&- 8u^3\beta^2\eta\eta_t\eta_u\xi^3 - 18u^2\beta^2\eta\eta_x\phi\xi^3 \\
&+ u^3\beta^3\eta^2\eta_u\phi - 4u^3\beta^2\eta_u\eta_x\phi\xi^3 \\
&+ 8u^3\beta^2\eta_x\phi_t\xi^3 + 4u^3\beta^3\eta^2\eta_x\xi^2 \\
&+ 4u^4\beta^3\eta\eta_u\eta_x\xi^2 + 3u\beta^2\eta^3\phi\xi^2 \\
&- u^2\beta^2\eta\eta_t\phi\xi^2 - 8u^2\beta^2\eta^2\eta_u\phi\xi^2 \\
&+ 4u^3\beta^2\eta_t\eta_u\phi\xi^2 - 4u^2\beta^2\eta^2\phi_t\xi^2 \\
&+ 6u^3\beta^2\eta\eta_u\phi_t\xi^2 + 2u^2\beta^3\eta^4\xi - 2u^3\beta^3\eta^3\eta_u\xi \\
&- 10u^3\beta^3\eta\eta_x\phi\xi + 8u^4\beta^3\eta_u\eta_x\phi\xi \\
&- u^2\beta^3\eta^3\phi.
\end{aligned} \tag{A.2}
$$

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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