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Research Article

Infinitely Many Solutions of Schrödinger-Poisson Equations with Critical and Sublinear Terms

Xianzhong Yao, [1,2] Xia Li, [3] Fuchen Zhang, [4] and Chunlai Mu [2]

[1] Faculty of Applied Mathematics, Shanxi University of Finance and Economics, Taiyuan, Shanxi 030006, China
[2] College of Mathematics and Statistics, Chongqing University, Chongqing 401331, China
[3] Department of Mathematics, North University of China, Taiyuan, Shanxi 030051, China
[4] Chongqing Key Laboratory of Social Economy and Applied Statistics, College of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, China

Correspondence should be addressed to Xianzhong Yao; yaoxz416@163.com

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In this paper, we study the following Schrödinger-Poisson equations $\begin{cases} -\Delta u + u + \phi u = u^5 + \lambda a(x)|u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases}$ where the parameter $\lambda > 0$ and $p \in (0,1)$. When the parameter λ is small and the weight function $a(x)$ fulfills some appropriate conditions, we admit the Schrödinger-Poisson equations possess infinitely many negative energy solutions by using a truncation technology and applying the usual Krasnoselskii genus theory. In addition, a byproduct is that the set of solutions is compact.

1. Introduction and Main Results

In the present paper, we are interested in the existence of infinitely many negative energy solutions of the following Schrödinger-Poisson equations:

$$\begin{cases} -\Delta u + u + \phi u = u^5 + \lambda a(x)|u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1}$$

where the parameter $\lambda > 0$, $p \in (0,1)$, and $a(x)$ is a positive continuous weight function satisfying $a \in L^{2/(1-p)}(\mathbb{R}^3)$.

In recent decades, the Schrödinger-Poisson system has been studied widely by many authors, because it has strong physical background and interesting meaning. It arises in nonlinear quantum mechanics models and semiconductor theory. From a physical viewpoint, the system describes the interaction between identical charged particles, when the magnetic effects could be ignored in the interaction with each other and its solution is a standing wave for such a stationary system. The nonlinearity models the mutual interaction between many charged particles. The system consists of a Schrödinger equation coupled with a Poisson equation, which implies that the potential is determined by the charge of the wave function. The nonlocal term ϕu means that the particles interact with its own electric field. For more information about the mathematical and physical background of the system, we refer the readers to see papers [1–4] and the references therein.

The studies of the Schrödinger-Poisson system have been focused on the existence of positive solutions, ground state solutions, multiplicity of solutions, radial solutions and the semiclassical limit solutions, concentration behavior of solutions, and sign-changing solutions. See references [5–17] and the references therein.

When the nonlinear term is presented as a subcritical growth, there are many results in the literature. Ruiz [18] studied the following system:

$$\begin{cases} -\Delta u + u + \lambda\phi u = u^p, & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{2}$$

where the parameter $\lambda > 0$ and $p \in (1,5)$. When λ is small, the author showed that there exists at least one positive radial solutions for $p \in [2,5)$, and at least two positive radial solutions for $p \in (1,2)$. In particular, if $\lambda \geq 1/4$, the author proved that $p = 2$ is a threshold of existence and nonexistence of positive radial solutions. When $\lambda = 1$ in system (2), Azzollini and Pomponio [19] established the existence of ground state solution for $p \in (2,5)$. For related system and more results, please refer readers to see [20–28].

In the paper, we are concerned with a critical growth of nonlinearity term and perturbation of low order terms. In this case, there are some results in the references. As regards the following relevant system,

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = u^5 + \mu u^p, & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{3}$$

where the parameter $\lambda, \mu > 0$ and $p \in (1,5)$, under some suitable conditions, existence of a nontrivial solution was proved in [19] for $3 < p < 5$ and in [29] for $1 < p < 5$. Here, we would like to mention some other papers [30–34] for related results. We note that the existence of solutions is very seriously depending on the range of the p. As far as we know, there is no result of Schrödinger-Poisson system involving the combination with a critical nonlinearity and sublinear terms.

The compactness of the imbedding $H^1(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$ $(2 \leq p \leq 6)$ does not hold, and the nonlocal term $\phi_u u$ and the critical nonlinear term u^5 appear in the system, which cause many difficulties for us using the variational methods in a standard way to solve the Schrödinger-Poisson system.

Motivated by works mentioned above, particularly, by the results in [16, 24, 29, 35], we overcome these difficulties mentioned above and obtain the existence of infinitely many negative energy solutions to system (1) for $p \in (0,1)$ and small $\lambda > 0$.

We denote by S the best constant for the Sobolev space $\mathscr{D}^{1,2}(\mathbb{R}^3)$ imbedding into the Lebesgue space $L^6(\mathbb{R}^3)$, namely, $\mathscr{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$,

$$S := \inf_{u \in \mathscr{D}^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^6 dx\right)^{1/3}}. \tag{4}$$

Now, we give our main result as follows.

Theorem 1. *Assume $p \in (0,1)$. Then, there exists a positive constant Λ such that system (1) possesses infinitely many negative energy solutions for any $\lambda \in (0,\Lambda)$. Moreover, the set of solutions obtained above is compact.*

Remark 2. When $\lambda \leq 0$, the system (1) has no solution, which follows from Pohožaev's identity (see [36]). To some extent, we extend the results in [16, 24, 29, 35].

Remark 3. The key ingredient in the proof of Theorem 1 is the genus theory, which plays an important role in obtaining infinitely many solutions of Schrödinger-Poisson equations (1). We followed the methods of Yao and Mu in [37], where the authors studied nonlocal problem of Kirchhoff-type in high dimension $(N \geq 4)$.

The remainder of this paper is organized as follows. In Section 2, we present the abstract framework of the problem as well as some preliminary results. Theorem 1 shall be proved in Section 3.

2. Preliminaries and Functional Setting

In this section, we will define some notations and establish the variational setting for Schrödinger-Poisson equations (1) and list some fundamental results.

(i) Let $H^1(\mathbb{R}^3)$ be the usual Sobolev space endowed with the standard inner product and induced norm

$$\langle u, v\rangle = \int_{\mathbb{R}^3} \nabla u \cdot \nabla v + uv\,dx, \qquad \|u\| = \left(\int_{\mathbb{R}^3} |\nabla u|^2 + u^2 dx\right)^{1/2} \tag{5}$$

(ii) $L^p(\mathbb{R}^3)$ is the usual Lebesgue space equipped with the norm

$$|u|_p = \left(\int_{\mathbb{R}^3} |u|^p dx\right)^{1/p}, \text{for } 1 \leq p < \infty, \qquad |u|_\infty = \operatorname*{ess\,sup}_{x \in \mathbb{R}^3} |u(x)| \tag{6}$$

(iii) $\mathscr{D}^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^{1,2}(\mathbb{R}^3)} := \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{1/2} \tag{7}$$

(iv) The letters C and $C_i (i = 1, 2, \cdots)$ denote various positive constants which may vary from line to line and whose exact values are irrelevant

(v) The notations $\to$ and $\rightharpoonup$ mean strong convergence and weak convergence in corresponding to functional setting, respectively

(vi) We use $o(1)$ to denote any infinitely small quantity that tends to zero as $n \to \infty$

For any $u \in H^1(\mathbb{R}^3)$, the Lax-Milgram theorem implies that there exists a unique $\phi_u \in \mathscr{D}^{1,2}(\mathbb{R}^3)$ such that, for any $v \in \mathscr{D}^{1,2}(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \nabla\phi_u \cdot \nabla v\,dx = \int_{\mathbb{R}^3} u^2 v\,dx, \tag{8}$$

that is, ϕ_u is the weak solution of $-\Delta\phi = u^2$. Furthermore,

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}dy. \tag{9}$$

We then can rewrite Schrödinger-Poisson equations (1) as the following:

$$-\Delta u + u + \phi_u u = u^5 + \lambda a(x)u^p, \quad x \in \mathbb{R}^3, \tag{10}$$

and energy functional associated with equation (10) is

$$\begin{aligned} I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + u^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx \\ - \frac{1}{6}\int_{\mathbb{R}^3} u^6 dx - \frac{\lambda}{p+1}\int_{\mathbb{R}^3} a(x)|u|^{p+1}dx. \end{aligned} \tag{11}$$

It is readily seen that the energy functional I belongs to $\mathscr{C}^1(H^1(\mathbb{R}^3), \mathbb{R})$ and that

$$\begin{aligned} \left\langle I'(u), v\right\rangle = \int_{\mathbb{R}^3} \nabla u \cdot \nabla v + uvdx + \int_{\mathbb{R}^3} \phi_u uvdx \\ - \int_{\mathbb{R}^3} |u|^4 uvdx - \lambda\int_{\mathbb{R}^3} a(x)|u|^{p-1}uvdx, \end{aligned} \tag{12}$$

for any $v \in H^1(\mathbb{R}^3)$. Hence, if $u \in H^1(\mathbb{R}^3)$ is a critical point of functional I, then u is a solution of equation (10) and $(u, \phi_u) \in H^1(\mathbb{R}^3) \times \mathscr{D}^{1,2}(\mathbb{R}^3)$ is a solution of system (1). We denote $\Phi(u) := \phi_u$ for simple expressions.

In what follows, we start to state our preliminary results.

Lemma 4. *Φ satisfies the following results:*

(1) Φ is continuous in $H^1(\mathbb{R}^3)$ and $\Phi \geq 0$

(2) If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $\Phi(u_n) \rightharpoonup \Phi(u)$ in $\mathscr{D}^{1,2}(\mathbb{R}^3)$

(3) $\Phi(tu) = t^2\Phi(u)$ for any $t \in \mathbb{R}$

(4) $\|\phi_u\|_{\mathscr{D}^{1,2}(\mathbb{R}^3)} \leq C\|u\|^2$. Furthermore, $\int_{\mathbb{R}^3}\phi_u u^2 dx \leq \tilde{C}\|u\|^4$

The proof is omitted here and refers to [18, 29].

Lemma 5. *Assume $\{u_n\}$ is a $(PS)_c$ for functional I in $H^1(\mathbb{R}^3)$. Then, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$.*

Proof. Arguing by contradiction, assume $\|u_n\| \to \infty$ as $n \to \infty$. According to the Sobolev and Hölder inequality and $p \in (0, 1)$, we find that

$$\begin{aligned} c + 0(1) &= I(u_n) - \frac{1}{4}\left\langle I'(u_n), un\right\rangle \\ &= \frac{1}{4}\|u_n\|^2 - \lambda\left(\frac{1}{p+1} - \frac{1}{4}\right)\int_{\mathbb{R}^3} a(x)|u_n|^{p+1}dx \\ &\quad + \frac{1}{12}\int_{\mathbb{R}^3} u_n^6 dx \geq \frac{1}{4}\|u_n\|^2 \\ &\quad - \lambda\left(\frac{1}{p+1} - \frac{1}{4}\right)|a|_{2/(1-p)}|u_n|_2^{p+1} \\ &\geq \frac{1}{4}\|u_n\|^2 - \lambda C\|u_n\|^{p+1} \xrightarrow{n\to\infty} \infty. \end{aligned} \tag{13}$$

This is a contradiction, and $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$.

Lemma 6. *Suppose $\{u_n\}$ is a $(PS)_c$ for functional I in $H^1(\mathbb{R}^3)$. Then, $\{u_n\}$ has a convergent subsequence in $H^1(\mathbb{R}^3)$ provided that $c < (1/3)S^{3/2}$.*

Proof. By Lemma 5, we know that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, and up to a subsequence, there exists a $u \in H^1(\mathbb{R}^3)$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H^1(\mathbb{R}^3), \\ u_n &\to u \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^3) \text{ for } p \in (1, 6), \\ u_n &\to u \quad \text{a.e in } \mathbb{R}^3. \end{aligned} \tag{14}$$

In light of Lions' second concentration compactness lemma [38], there exist an at most countable index set J, a sequence of points $\{x_j\}_{j\in J} \subset \mathbb{R}^3$, and values $\{\mu_j\}_{j\in J}$, $\{\nu_j\}_{j\in J} \subset \mathbb{R}^+$ such that

$$\begin{aligned} |\nabla u_n|^2 &\rightharpoonup d\mu \geq |\nabla u|^2 + \sum_{j\in J}\mu_j\delta_{xj}, \\ u_n^6 &\rightharpoonup d\nu = u^6 + \sum_{j\in J}\nu_j\delta_{xj}, \\ S\nu_j^{1/3} &\leq \mu_j, \end{aligned} \tag{15}$$

in the measure sense, where δ_{x_j} is the Dirac mass at x_j. We next shall prove that the index set J is empty. By the reduction to absurdity, let us suppose that there exists a j_0 such that $\nu_{j_0} \neq 0$. Consider that some cut-off function $\phi \in C_0^\infty(\mathbb{R}^3)$: $\mathbb{R}^3 \to [0, 1]$ such that

$$\begin{aligned} \phi &\equiv 1 \text{ on } B\left(x_{j_0}, \varepsilon\right), \\ \phi &= 0 \text{ on } B\left(x_{j_0}, 2\varepsilon\right)^c, \\ |\nabla\phi| &\leq \frac{2}{\varepsilon}. \end{aligned} \tag{16}$$

Obviously, the sequence $\{\phi u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, and because $\{u_n\}$ is a $(PS)_c$ sequence of functional I, then $\lim_{n\to\infty}\langle I'(u_n), \phi u_n\rangle = 0$, *i.e.*, for large n, there is

$$\int_{\mathbb{R}^3} u_n \nabla u_n \nabla \phi dx + \int_{\mathbb{R}^3} |\nabla u_n|^2 \phi dx + \int_{\mathbb{R}^3} \phi_{u_n} \nabla u_n^2 \phi dx = \lambda \int_{\mathbb{R}^3} a(x) |u_n|^{p+1} \phi dx + \int_{\mathbb{R}^3} u_n^6 \phi dx + o(1). \quad (17)$$

Let us start to estimate each terms in the equation above. Using the Hölder inequality and (15), we compute simply

$$\begin{aligned} 0 &\le \int_{\mathbb{R}^3} |u_n \nabla u_n \nabla \phi| dx \\ &\le \left(\int_{B_{2\varepsilon}(x_{j_0})} |u_n|^2 dx \right)^{1/2} \left(\int_{B_{2\varepsilon}(x_{j_0})} |u_n|^2 |\nabla \phi|^2 dx \right)^{1/2} \\ &\le C_1 \left(\int_{B_{2\varepsilon}(x_{j_0})} |u_n|^2 |\nabla \phi|^2 dx \right)^{1/2} \\ &\le C_1 \left(\int_{B_{2\varepsilon}(x_{j_0})} |u_n|^6 dx \right)^{1/6} \left(\int_{B_{2\varepsilon}(x_{j_0})} |\nabla \phi|^3 dx \right)^{1/3} \\ &\le C_2 \left(\int_{B_{2\varepsilon}(x_{j_0})} |u_n|^6 dx \right)^{1/6} \xrightarrow{\varepsilon \to 0} 0, \end{aligned}$$
$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 \phi dx \ge \mu_{j_0}, \quad (18)$$

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^3} u_n^6 \phi dx = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} u^6 \phi dx + \nu_{j_0} = \nu_{j_0}, \quad (19)$$

$$\begin{aligned} &\lim_{\varepsilon \to 0} \lim_{n \to \infty} \lambda \int_{\mathbb{R}^3} a(x) |u_n|^{p+1} \phi dx \\ &= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \lambda \int_{B_{2\varepsilon}(x_{j_0})} a(x) |u_n|^{p+1} \phi dx \\ &= \lambda \lim_{\varepsilon \to 0} \int_{B_{2\varepsilon}(x_{j_0})} a(x) |u|^{p+1} \phi dx = 0. \end{aligned} \quad (20)$$

Together with formulas above, we note

$$\nu_{j_0} \ge \mu_{j_0}. \quad (21)$$

By combining this inequality with the third formula in (15), we show that

$$\nu_{j_0} \ge S^{3/2}. \quad (22)$$

Take a cut-off function $\psi \in C_0^\infty(\mathbb{R}^3)$: $\mathbb{R}^3 \to [0,1]$ such that

$$\begin{aligned} &\psi \equiv 1 \text{ on } B(0,R), \\ &\psi = 0 \text{ on } B(0,2R)^c, \\ &|\nabla \psi| \le \frac{3}{R}. \end{aligned} \quad (23)$$

For simplicity of computation, denote

$$\alpha := \frac{1}{4} \int_{\mathbb{R}^3} u^2 dx - \lambda \left(\frac{1}{p+1} - \frac{1}{4} \right) \int_{\mathbb{R}^3} a(x) |u|^{p+1} dx. \quad (24)$$

By Hölder's inequality, we easily know that

$$\alpha \ge \frac{1}{4} |u|_2^2 - \lambda \left(\frac{1}{p+1} - \frac{1}{4} \right) |a|_{2/(1-p)} |u|_2^{p+1}, \quad (25)$$

since $a(x) \in L^{2/(1-p)}(\mathbb{R}^3)$ and $p \in (0,1)$, there exists some positive constant Λ such that $\alpha > 0$ for $\lambda \in (0, \Lambda)$. Using (15), (21), and (22), we obtain

$$\begin{aligned} c &= \lim_{n \to \infty} I(u_n) - \frac{1}{4} \left\langle I'(u_n), u_n \right\rangle \\ &= \lim_{n \to \infty} \left[\frac{1}{4} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} u_n^2 dx \right. \\ &\quad \left. - \lambda \left(\frac{1}{p+1} - \frac{1}{4} \right) \int_{\mathbb{R}^3} a(x) |u_n|^{p+1} dx + \frac{1}{12} \int_{\mathbb{R}^3} u_n^6 dx \right] \\ &> \lim_{R \to \infty} \lim_{n \to \infty} \left(\frac{1}{4} \int_{\mathbb{R}^3} |\nabla u_n|^2 \psi dx + \frac{1}{4} \int_{\mathbb{R}^3} u_n^2 \psi dx \right. \\ &\quad \left. - \frac{\lambda(3-p)}{4(p+1)} \int_{\mathbb{R}^3} a(x) |u_n|^{p+1} \psi dx + \frac{1}{12} \int_{\mathbb{R}^3} u_n^6 \psi dx \right) \\ &> \lim_{R \to \infty} \left(\frac{1}{4} \int_{\mathbb{R}^3} |\nabla u|^2 \psi dx + \frac{1}{4} \mu_{j_0} + \frac{1}{12} \int_{\mathbb{R}^3} u^6 \psi dx + \frac{1}{12} \nu_{j_0} \right) \\ &\quad + \alpha > \frac{1}{4} \mu_{j_0} + \frac{1}{12} \nu_{j_0} > \frac{1}{4} S \nu_{j_0} + \frac{1}{12} \nu_{j_0} \\ &> \frac{1}{4} S^{3/2} + \frac{1}{12} S^{3/2} > \frac{1}{3} S^{3/2}. \end{aligned} \quad (26)$$

This is a contradiction with the hypothesis, so the index set J is empty.

Take $T > 0$ and define

$$\begin{aligned} \mu_\infty &:= \lim_{T \to \infty} \limsup_{n \to \infty} \int_{|x|>T} |\nabla u_n|^2 dx, \\ \nu_\infty &:= \lim_{T \to \infty} \limsup_{n \to \infty} \int_{|x|>T} u_n^6 dx. \end{aligned} \quad (27)$$

In line with lemma 1.40 in [39], μ_∞ and ν_∞ fulfill the following formulas

$$\begin{aligned} \limsup_{n \to \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx &= \int_{\mathbb{R}^3} d\mu + \mu_\infty, \\ \limsup_{n \to \infty} \int_{\mathbb{R}^3} u_n^6 dx &= \int_{\mathbb{R}^3} d\nu + \nu_\infty, \\ S \nu_\infty^{1/3} &\le \mu_\infty. \end{aligned} \quad (28)$$

Take a cut-off function $\varphi_R \in C_0^\infty(\mathbb{R}^3)$: $\mathbb{R}^3 \to [0,1]$ such that

$$\begin{aligned} &\varphi_R \equiv 0 \text{ on } B(0,R), \\ &\varphi_R = 1 \text{ on } B(0,2R)^c, \\ &|\nabla\varphi_R| \le \frac{3}{R}. \end{aligned} \tag{29}$$

As above, the sequence $\{\varphi_R u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and $\lim_{n\to\infty}\langle I'(u_n), \varphi_R u_n\rangle = 0$, namely,

$$\begin{aligned} &\int_{\mathbb{R}^3} u_n\nabla u_n\nabla\varphi_R dx + \int_{\mathbb{R}^3}|\nabla u_n|^2\varphi_R dx + \int_{\mathbb{R}^3}\phi_{u_n}u_n^2\varphi_R dx \\ &\quad = \lambda\int_{\mathbb{R}^3} a(x)|u_n|^{p+1}\varphi_R dx + \int_{\mathbb{R}^3} u_n^6\varphi_R dx + o(1). \end{aligned} \tag{30}$$

We need again to estimate every terms in the equation above. By the Hölder inequality and the definition of φ_R, it follows that

$$\begin{aligned} 0 &\le \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^3}|u_n\nabla u_n\nabla\varphi_R| dx \\ &\le \lim_{R\to\infty}\limsup_{n\to\infty}\left(\int_{B(0,2R)/B(0,R)}|\nabla u_n|^2 dx\right)^{1/2} \\ &\quad\cdot\left(\int_{B(0,2R)/B(0,R)}|\nabla\varphi_R|^2 u_n^2 dx\right)^{1/2} \\ &\le C\lim_{R\to\infty}\left(\int_{B(0,2R)/B(0,R)}|\nabla\varphi_R|^2 u^2 dx\right)^{1/2} \\ &\le C\lim_{R\to\infty}\left(\int_{B(0,2R)/B(0,R)}|\nabla\varphi_R|^3 dx\right)^{1/3} \\ &\quad\cdot\left(\int_{B(0,2R)/B(0,R)} u^6 dx\right)^{1/6} \\ &\le C_1\lim_{R\to\infty}\left(\int_{B(0,2R)/B(0,R)} u^6 dx\right)^{1/6} = 0, \end{aligned} \tag{31}$$

$$\begin{aligned} &\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2\varphi_R dx \ge \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|\nabla u_n|^2 dx \\ &\quad = \mu_\infty, \end{aligned} \tag{32}$$

$$\begin{aligned} \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^3} u_n^6\varphi_R{}^{dx} &= \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R} u_n^6\varphi_R{}^{dx} \\ &\le \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R} u_n^6 dx = \nu_\infty. \end{aligned} \tag{33}$$

According to the estimates above, we easily show that $\mu_\infty \le \nu_\infty$. Combination with (28), we know that $\nu_\infty = 0$ or $\nu_\infty \ge S^{3/2}$. If $\nu_\infty \ge S^{3/2}$ holds, then we have

$$\begin{aligned} C &= \lim_{n\to\infty}\left(I(u_n) - \frac{1}{4}\left\langle I'(u_n), u_n\right\rangle\right) \\ &> \lim_{n\to\infty}\left(\frac{1}{4}\int_{\mathbb{R}^3}|\nabla u_n|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}|u_n|^2 dx\right. \\ &\quad\left. - \frac{\lambda}{4}\int_{\mathbb{R}^3} a(x)|u_n|^{p+1}dx + \frac{1}{12}\int_{\mathbb{R}^3} u_n^6 dx\right) \\ &> \frac{1}{4}\int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{1}{4}\mu_\infty + \frac{1}{12}\int_{\mathbb{R}^3} u^6 dx + \frac{1}{12}\nu_\infty + \alpha \\ &> \frac{1}{4}\mu_\infty\frac{1}{12}\nu_\infty > \frac{1}{4}S\nu_\infty^{1/3}\frac{1}{12}\nu_\infty > \frac{1}{4}S^{3/2}\frac{1}{12}S^{3/2} > \frac{1}{3}S^{3/2}. \end{aligned} \tag{34}$$

This is a contradiction with the assumption, thus $\nu_\infty = 0$ holds. Together (15) and (28) with the empty index set J, we obtain that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3} u_n^6 dx = \int_{\mathbb{R}^3} u^6 dx. \tag{35}$$

By Fatou's lemma, we prove that

$$\int_{\mathbb{R}^3} u^6 dx \le \liminf_{n\to\infty}\int_{\mathbb{R}^3} u_n^6 dx \le \limsup_{n\to\infty}\int_{\mathbb{R}^3} u_n^6 dx = \int_{\mathbb{R}^3} u^6 dx. \tag{36}$$

Therefore, $u_n \to u$ in $L^6(\mathbb{R}^3)$. Set $w_n := u_n - u$, and $w_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. Suppose $\lim_{n\to\infty}\|w_n\| = w$. Since I' is sequentially weakly continuous in $H^1(\mathbb{R}^3)$, we have that $\lim_{n\to\infty}I'(w_n) \to 0$ and

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left\langle I'(w_n), w_n\right\rangle \\ &= \lim_{n\to\infty}\left(\int_{\mathbb{R}^3}|\nabla w_n|^2 + w_n^2 dx + \int_{\mathbb{R}^3}\phi_{w_n}w_n^2 dx\right. \\ &\quad\left. - \lambda\int_{\mathbb{R}^3} a(x)|w_n|^{p+1}dx - \int_{\mathbb{R}^3} w_n^2 dx\right) \\ &\ge \lim_{n\to\infty}\left(\int_{\mathbb{R}^3}|\nabla w_n|^2 + w_n^2 dx - \lambda\int_{\mathbb{R}^3} a(x)|w_n|^{p+1}dx\right) \\ &\ge \lim_{n\to\infty}\left(\|w_n\|^2 - C\lambda\|w_n\|^{p+1}\right) \ge w^2 - C\lambda w^{p+1}. \end{aligned} \tag{37}$$

Thus for small $\lambda > 0$, this formula forces $w = 0$ and $\|w_n\| \to 0$ as $n \to \infty$, and $u_n \to u$ in $H^1(\mathbb{R}^3)$. The proof is finished.

3. The Proof of Theorem 1

In this section, we shall utilize the Krasnoselskii genus theory to establish a minimax class of critical points for proving that Schrödinger-Poisson equations (1) possess infinitely many negative energy solutions. And we next need to introduce the classical concept and some properties of the genus.

Let E be a real Banach space and Γ denote the family of sets $A \subset E \setminus \{0\}$ such that A is closed in E and symmetric with respect to 0, namely, $x \in A$ implies $-x \in A$. For any $A \in \Gamma$, we define the genus of A as follows:

$$\gamma(A) := \min \Big\{ k \in \mathbb{N} : \exists \varphi \in C\left(A, \mathbb{R}^k \setminus \{0\}\right) \text{ such that } \varphi(x) \text{ is odd} \Big\}. \tag{38}$$

If there is no finite k, then define $\gamma(A) = \infty$, and $\gamma(\varnothing) = 0$ by definition of the genus. For $A \in \Gamma$ and $\delta > 0$, we denote by $N_\delta(A)$ a uniform δ-neighborhood of A, that is,

$$N_\delta(A) = \{x \in E : \operatorname{dist}(x, A) \le \delta\}. \tag{39}$$

In what follows, we shall list some properties of the genus that prepare for showing our results. More detail content about the genus may be seen in the references ([40], Propositions 7.5-7.8).

Proposition 7. *Assume $A, B \in \Gamma$. Then, following several results hold.*

(1) *Normalization: if $x \ne 0$, then $\gamma(\{x\} \cup \{-x\}) = 1$*

(2) *Mapping property: if there is an odd map $f \in C(A, B)$, then $\gamma(A) \le \gamma(B)$*

(3) *Monotonicity property: if $A \subset B$, then $\gamma(A) \le \gamma(B)$*

(4) *Subadditivity: $\gamma(A \cup B) \le \gamma(A) + \gamma(B)$*

(5) *Continuity property: if A is compact, then $\gamma(A) < \infty$ and there exists a $\delta > 0$ such that $N_\delta(A) \in \Gamma$ and $\gamma(N_\delta(A)) = \gamma(A)$*

(6) *If there is an odd homeomorphism between A and B, then $\gamma(A) = \gamma(B)$*

(7) *If S^{N-1} is the sphere in $\mathbb{R}^N$, then $\gamma(S^{N-1}) = N$*

(8) *If $\gamma(B) < \infty$, then $\gamma(\overline{A - B}) \ge \gamma(A) - \gamma(B)$*

(9) *If Y is a subspace of X with codimension k and $\gamma(A) \ge k$, then $A \cap Y \ne \varnothing$.*

Proposition 8. *Assume $A \in \Gamma$. If $\gamma(A) \ge 2$, then set A includes infinitely many points.*

We define presently an auxiliary function, which essentially follows the idea and method in [41]. Denote

$$f(t) := \frac{1}{2}t^2 - \lambda\beta t^{p+1} - \frac{1}{6S^3}t^6, \tag{40}$$

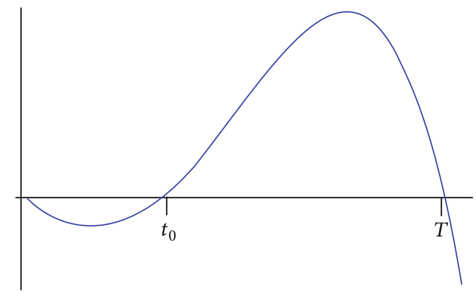

Figure 1

where β is a positive constant and determined below. By the Hölder and Sobolev inequality, it follows that

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + u^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx \\ &\quad - \frac{\lambda}{p+1}\int_{\mathbb{R}^3} a(x)|u|^{p+1} dx - \frac{1}{6}\int_{\mathbb{R}^3} u^6 dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{\lambda}{p+1}|a|_{2/(1-p)}|u|_2^{p+1} - \frac{1}{6S^3}\|u\|^6 \\ &\ge \frac{1}{2}\|u\|^2 - \lambda\beta\|u\|^{p+1} - \frac{1}{6S^3}\|u\|^6 = f(\|u\|). \end{aligned} \tag{41}$$

For $p \in (0,1)$, we observe that f gets its positive unique maximum (see Figure 1).

To find the critical points of the energy functional I, we truncate the functional I as the following:

$$\begin{aligned} \tilde{I}(u) &:= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_u|^2 + u^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx \\ &\quad - \frac{\lambda}{p+1}\int_{\mathbb{R}^3} a(x)|u|^{p+1} dx - \frac{\omega(u)}{6}\int_{\mathbb{R}^3} u^6 dx, \end{aligned} \tag{42}$$

where $\omega(u) = \tau(\|u\|)$ and $\tau \in C^\infty : \mathbb{R}^+ \to [0,1]$ is a decreasing function and satisfies

$$\begin{aligned} \tau(t) &= 1, \quad t \le t_0, \\ \tau(t) &= 0, \quad t \ge T, \end{aligned} \tag{43}$$

where t_0 and T are two roots of the function f (see Figure 1) and $t_0 < T$. As above, we have that $\tilde{I}(u) \ge \tilde{f}(\|u\|)$ with

$$\tilde{f}(t) := \frac{1}{2}t^2 - \lambda\beta t^{p+1} - \frac{\tau(t)}{6S^3}t^6, \tag{44}$$

and see Figure 2. Obviously, $\tilde{I} \in C^1$ and if $\tilde{I}(u) \le 0$, then we observe that $\|u\| \le t_0$ and $I(u) = \tilde{I}(u)$.

Therefore, we only need to find some negative critical values for the truncating functional $\tilde{I}$. To do it, we next shall give some lemmas for constructing the minimax sequence of negative critical values of the truncating functional $\tilde{I}$.

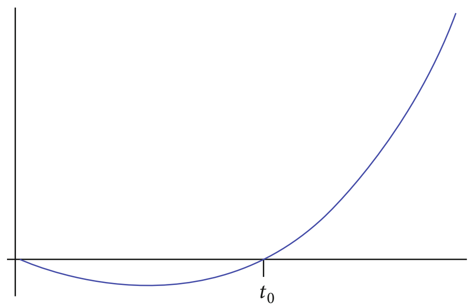

FIGURE 2

Before to state the following results, we give some definitions. Denote

$$\Gamma_k := \{A \in \Gamma : \gamma(A) \geq k\},$$
$$K_c := \left\{u \in H^1(\mathbb{R}^3) : \tilde{I}(u) = c, \tilde{I}'(u) = 0\right\}, \quad (45)$$

$$\tilde{I}^d := \left\{u \in H^1(\mathbb{R}^3) : \tilde{I}(u) \leq d\right\}, \quad (46)$$

and define

$$c_k := \inf_{A \in \Gamma_k} \sup_{u \in A} \tilde{I}(u). \quad (47)$$

Lemma 9. *For any positive integer $n \in \mathbb{N}$, there is a $\varepsilon = \varepsilon(n) > 0$ satisfying $\gamma(\tilde{I}^{-\varepsilon}) \geq n$.*

Proof. To show this lemma, we adopt to the argument used in [41], which handles with the local problem. Fix any n and suppose E_n is an n-dimensional subspace of $H^1(\mathbb{R}^3)$. Taking $v \in E_n$ with $\|v\| = 1$ and $t \in (0, t_0)$, by Figures 1 and 2, we observe

$$\tilde{I}(tv) = I(tv) < 0, \quad (48)$$

then, we can choose $\varepsilon > 0$ and $t < t_0$ such that $\tilde{I}(tu) \leq -\varepsilon$ for $u \in E_n$ and $\|u\| = 1$. Denote

$$\mathbb{S} := \{u \in E_n : \|u\| = t\}, \quad (49)$$

for $t < t_0$, it is easy to know $\gamma(\mathbb{S}) = n$ and note

$$\mathbb{S} \subset \left\{u \in H^1(\mathbb{R}^3) : \tilde{I}(u) \leq -\varepsilon\right\}. \quad (50)$$

Because E_n and $\mathbb{R}^n$ are isomorphic and $\mathbb{S}$ and S^{n-1} are homeomorphic, it follows from Proposition 7 that

$$\gamma\left(u \in H^1(\mathbb{R}^3) : \tilde{I}(u) \leq -\varepsilon\right\}\right) \geq \gamma(\mathbb{S}) = n. \quad (51)$$

Lemma 10. *Assume $\lambda \in (0, \Lambda)$. If $c = c_k = c_{k+1} = \cdots = c_{k+r}$ for some integer $r \in \mathbb{N}$, then $\gamma(K_c) \geq r + 1$.*

Proof. We first claim that each c_k is negative. Indeed, by Lemma 4, for every integer $k \in \mathbb{N}$, there exists $\varepsilon > 0$ such that $\gamma(\tilde{I}^{-\varepsilon}) \geq k$. Because functional $\tilde{I}$ is continuous in $H^1(\mathbb{R}^3)$ and even $\tilde{I}^{-\varepsilon} \in \Gamma_k$. In virtue of the definition of c_k and low boundedness of $\tilde{I}$, we know

$$-\infty < c_k = \inf_{A \in \Gamma_k} \sup_{u \in A} \tilde{I}(u) \leq \sup_{u \in \tilde{I}^{-\varepsilon}} \tilde{I}(u) \leq -\varepsilon < 0. \quad (52)$$

Then, we easily show that K_c is compact by Lemma 6. In the following argument, we will prove the expected result by contradiction. Assume $\gamma(K_c) \leq r$, according to the fifth in Proposition 7, then there exists a closed and symmetric set U with $K_c \subset U$ such that $\gamma(U) = \gamma(K_c) \leq r$. Due to $c < 0$, we can choose U such that $U \subset \tilde{I}^0$. Owing to the deformation lemma [42], there is an odd homeomorphism

$$\eta : H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3), \quad (53)$$

such that $\eta(\tilde{I}^{c+\delta} - U) \subset \tilde{I}^{c-\delta}$ for some δ with $0 < \delta < -c$. So $\tilde{I}^{c+\delta} \subset \tilde{I}^0$ and then by the definition of $c = c_k = c_{c+k}$, there exists $A \in \Gamma_{k+r}$ such that $\sup_{u \in A} \tilde{I} < c + \delta$, that is, $A \subset \tilde{I}^{c+\delta}$ and

$$\eta(A - U) \subset \eta\left(\tilde{I}^{c+\delta} - U\right) \subset \tilde{I}^{c-\delta}. \quad (54)$$

It means

$$\tilde{I}(\eta(A - U)) \leq c - \delta. \quad (55)$$

On the other hand, because of $\gamma(\overline{AU}) \geq \gamma(A) - \gamma(U) \geq k$ and $\gamma(\eta(\overline{AU})) \geq \gamma(\overline{AU}) \geq k$, then $\eta(\overline{AU}) \in \Gamma_k$, which implies

$$\sup_{u \in \eta(\overline{AU})} \tilde{I}(u) \geq c_k = c. \quad (56)$$

This is a contradiction with (54). Therefore, the proof is finished.

Proof of Theorem 1. By the analysis above, if $c_k < 0$, then we know that $\tilde{I}(u) = I(u)$. Thus, it is sufficient to show that the truncating functional $\tilde{I}$ possesses infinitely many negative critical values. Noting that c_k is nondecreasing with respect to k, there are only the following two cases:

Case I. There are $1 \leq k_1 < \cdots < k_i < \cdots$ satisfying $-\infty < c_{k_1} < c_{k_2} < \cdots < c_{k_i} < \cdots < 0$. In this case, obviously, the functional $\tilde{I}$ has infinitely many negative critical values $\{c_k\}$. It is done.

Case II. There is a positive l such that $c = c_k < 0$ for all $k \geq l$. In the case, by Lemma 10, we know $\gamma(K_c) = \infty$, which implies that the set K_c has infinitely many points, which means that the functional $\tilde{I}$ possesses infinitely many negative critical values. Finally, we shall see that the set of solutions is compact.

Let $\{u_k\}$ be a sequence of solutions to the equations (1). In Case I, we know that $0 \geq c_k = \tilde{I}(u_k)$ and $\tilde{I}'(u_k) = 0$; by Lemma 5, we then easily get that there is a $u \in H^1(\mathbb{R}^3)$ satisfying $u_k \to u$ as $k \to \infty$. Then, it follows conclusion. In Case

II, it is trivial to see that the set K_c is compact. Thereupon, the proof is completed.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

There are no competing interests regarding this research work.

Authors' Contributions

All authors equally have made contributions. All authors read and approved the final manuscript.

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