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Research Article

Fractional Hardy–Sobolev Inequalities with Magnetic Fields

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A fractional Hardy–Sobolev inequality with a magnetic field is studied in the present paper. Under appropriate conditions, the achievement of the best constant of the fractional magnetic Hardy–Sobolev inequality is established.

1. Introduction

During the last decades, researchers paid more and more attention to the study of Sobolev inequalities and Hardy–Sobolev inequalities, including the fractional case and magnetic case, see e.g. [1–9].

It is well known that the sharp constant of the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is attained (see [10]), where $2^* := 2N/(N-2)$ is the Sobolev critical exponent and $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}$. That is,

$$\mu := \inf_{\substack{u \in D^{1,2}(\mathbb{R}^N)\\ u \neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x\right)^{2/2^*}} \quad (1)$$

is achieved by the so-called Aubin–Talenti instanton (cf. [11, 12]) $U_{\varepsilon,x_0} \in D^{1,2}(\mathbb{R}^N)$ defined by

$$U_{\varepsilon,x_0}(x) := (N(N-2))^{(N-2)/4} \left(\frac{\varepsilon}{\varepsilon^2 + |x-x_0|^2}\right)^{(N-2)/2}, \quad (2)$$

where $\varepsilon > 0$, $x_0 \in \mathbb{R}^N$. Moreover, U_{ε,x_0} is a positive solution of $-\Delta u = |u|^{2^*-2}u$ in $\mathbb{R}^N$,

$$\int_{\mathbb{R}^N} \left|\nabla U_{\varepsilon,x_0}\right|^2 \mathrm{d}x = \int_{\mathbb{R}^N} \left|U_{\varepsilon,x_0}\right|^{2^*} \mathrm{d}x = \mu^{N/2}, \quad (3)$$

and $\{U_{\varepsilon,x_0} : \varepsilon > 0, x_0 \in \mathbb{R}^N\}$ contains all positive solutions of $-\Delta u = |u|^{2^*-2}u$ in $\mathbb{R}^N$.

For the best constant of the Hardy–Sobolev inequality

$$\mu_t := \inf_{\substack{u \in D^{1,2}(\mathbb{R}^N)\\ u \neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^N} (|u|^{2^*(t)}/|x|^t) \mathrm{d}x\right)^{2/2^*(t)}}, \quad (4)$$

where $0 < t < 2$ and $2^*(t) := 2(N-t)/(N-2)$ is the Hardy–Sobolev critical exponent, it follows from [13] that μ_t is achieved by functions of the form

$$U_\varepsilon(x) := ((N-t)(N-2))^{(N-2)/2(2-t)} \varepsilon^{(N-2)/2} \left(\varepsilon^{2-t} + |x|^{2-t}\right)^{-(N-2)/(2-t)}, \quad (5)$$

where $\varepsilon > 0$. The function U_ε is a positive solution of $-\Delta u = |u|^{2^*(t)-2} u/|x|^t$ in $\mathbb{R}^N$, and moreover,

$$\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 \mathrm{d}x = \int_{\mathbb{R}^N} \frac{|U_\varepsilon|^{2*(t)}}{|x|^t} \mathrm{d}x = (\mu_t)^{(N-t)/(2-t)}. \quad (6)$$

For the fractional Sobolev inequality, consider the Hilbert space $D^s(\mathbb{R}^N)$ defined as Gagliardo seminorm

$$D^s(\mathbb{R}^N) := \left\{u \in L^{2_s^*}(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{(N/2)+s}} \in L^2(\mathbb{R}^{2N})\right\}, \quad (7)$$

where $0 < s < 1$, $2_s^* := 2N/(N-2s)$ is the fractional Sobolev critical exponent and the norm

$$\|u\|_{D^s}^2 := \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \quad (8)$$

is induced by the scalar product

$$\langle u, v\rangle_{D^s} := \frac{c_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y. \quad (9)$$

Here $c_{N,s}$ is a dimensional constant precisely given by

$$c_{N,s} = \left(\int_{\mathbb{R}^N} \frac{1-\cos\zeta_1}{|\zeta|^{N+2s}}d\zeta\right)^{-1}. \quad (10)$$

Define the best constant for the fractional Sobolev inequality as

$$\mu_s := \inf_{\substack{u\in D^s(\mathbb{R}^N)\\ u\neq 0}} \frac{\|u\|_{D^s}^2}{\left(\int_{\mathbb{R}^N}|u|^{2_s^*}\mathrm{d}x\right)^{2/2_s^*}}. \quad (11)$$

From [14–16], we see that μ_s is achieved by $\tilde{u}(x) = \kappa\left(\varepsilon^2 + |x-x_0|\right)^{-(N-2s)/2}$, that is, $\mu_s = \|\tilde{u}\|_{D^s}^2/|\tilde{u}|_{2_s^*}^2$. Normalizing $\tilde{u}$ by $|\tilde{u}|_{2_s^*}$, we get that $\overline{u} = \tilde{u}/|\tilde{u}|_{2_s^*}$ fulfills

$$\mu_s = \inf_{\substack{u\in D^s(\mathbb{R}^N)\\ (u)_{2_s^*}=1}} \|u\|_{D^s}^2 = \|\overline{u}\|_{D^s}^2 \quad (12)$$

and $\overline{u}$ is a positive ground state solution of $(-\Delta)^s u = \mu_s|u|^{2_s^*-2}u$ in $\mathbb{R}^N$ (see Lemma 2.12 in [17]). Denote

$$U_{s,\varepsilon}(x) = \varepsilon^{-(N-2s)/2}\hat{u}\left(\frac{x}{\varepsilon}\right), \quad (13)$$

where $\hat{u} = \mu_s^{1/(2_s^*-2)}\overline{u}$ is a positive ground state solution of $(-\Delta)^s u = |u|^{2_s^*-2}u$ in $\mathbb{R}^N$ and

$$\|\hat{u}\|_{D^s}^2 = \int_{\mathbb{R}^N}|\hat{u}|^{2_s^*}\mathrm{d}x = (\mu_s)^{N/2s}. \quad (14)$$

Then, Lemma 2.12 in [17] yields that $U_{s,\varepsilon}$ solves $(-\Delta)^s u = |u|^{2_s^*-2}u$ in $\mathbb{R}^N$. The fractional Hardy–Sobolev inequality

$$\mu_{s,t} := \inf_{\substack{u\in D^s(\mathbb{R}^N)\\ u\neq 0}} \frac{\|u\|_{D^s}^2}{\left(\int_{\mathbb{R}^N}\left(|u|^{2_s^*(t)}/|x|^t\right)\mathrm{d}x\right)^{2/2_s^*(t)}}, \quad (15)$$

was considered in [18, 19], where $2_s^*(t) := 2(N-t)/(N-2s)$ is the fractional Hardy–Sobolev critical exponent. Marano and Mosconi [18] proved that $\mu_{s,t}$ is achieved by a optimizer $U\in D^s(\mathbb{R}^N)$, whose asymptotic behavior is

$$U(x) \simeq |x|^{2s-N} \quad \text{as } |x| \to +\infty. \quad (16)$$

For the magnetic Hardy–Sobolev inequality, we regard the range of function as $\mathbb{C}$, that is, $u:\mathbb{R}^N\to\mathbb{C}, N\geq 3$, $A=(A_1,\ldots,A_N):\mathbb{R}^N\to\mathbb{R}^N$ is a magnetic vector potential. Setting $-\Delta_A := (-i\nabla + A)^2, \nabla_A := \nabla + iA$, and

$$D_A^{1,2}(\mathbb{R}^N,\mathbb{C}) := \left\{u\in L^{2^*}(\mathbb{R}^N,\mathbb{C}) : |\nabla_A u|\in L^2(\mathbb{R}^N)\right\}, \quad (17)$$

then $-\Delta_A u = -\Delta u - iu\,\mathrm{div}A - iA\cdot\nabla u + |A|^2 u$ and $D_A^{1,2}(\mathbb{R}^N,\mathbb{C})$ is the Hilbert space obtained as the closures of $C_c^\infty(\mathbb{R}^N,\mathbb{C})$ with respect to scaler product

$$\mathrm{Re}\left(\int_{\mathbb{R}^N}\nabla_A u\cdot\overline{\nabla_A v}\mathrm{d}x\right), \quad (18)$$

where the bar denotes complex conjugation. Define

$$\mu_{t,A} := \inf_{\substack{u\in D_A^{1,2}(\mathbb{R}^N)\\ u\neq 0}} \frac{\int_{\mathbb{R}^N}|\nabla_A u|^2\mathrm{d}x}{\left(\int_{\mathbb{R}^N}\left(|u|^{2^*(t)}/|x|^t\right)\mathrm{d}x\right)^{2/2^*(t)}}, \quad (19)$$

where $2^*(t) := 2(N-t)/(N-2)$ is the Hardy–Sobolev critical exponent. Then, by Theorem 1.1 in [20], we see that if $A\in L^N_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N)$, then $\mu_{t,A}$ is attained by some $u\in D_A^{1,2}(\mathbb{R}^N,\mathbb{C})\backslash\{0\}$ if and only if curl $A\equiv 0$, where curl A is the usual curl operator for $N=3$ and the $N\times N$ skew-symmetric matrix with entries $a_{jk} = \partial_j A_k - \partial_k A_j$ for $N\geq 4$.

In our paper, we consider a fractional Hardy–Sobolev inequality with a magnetic field. To show our question, we first introduce the fractional magnetic Sobolev space $D_A^s(\mathbb{R}^N,\mathbb{C})$, which is the completion of $C_c^\infty(\mathbb{R}^N,\mathbb{C})$ with respect to the so-called fractional magnetic Gagliardo seminorm $[\cdot]_{D_A^s}$ given by

$$[u]_{D_A^s}^2 := \frac{c_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u(x)-u(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y, \quad (20)$$

where $c_{N,s}$ is given by (10), $u:\mathbb{R}^N\to\mathbb{C}$, $0<s<1$, $N\geq 3$, and $A=(A_1,\ldots,A_N):\mathbb{R}^N\to\mathbb{R}^N$ is a magnetic vector potential. The scalar product in $D_A^s(\mathbb{R}^N,\mathbb{C})$ is

$$\langle u, v\rangle_{D_A^s} := \frac{c_{N,s}}{2}\mathrm{Re}\int_{\mathbb{R}^{2N}} \frac{\left(e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u(x)-u(y)\right)}{|x-y|^{N+2s}} \cdot\overline{\left(e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}v(x)-v(y)\right)}\mathrm{d}x\mathrm{d}y. \quad (21)$$

Although $[\cdot]_{D_A^s}$ is a seminorm, by fractional magnetic Sobolev embeddings (see Lemma 3.5 in [21]), we can view $[\cdot]_{D_A^s}$ as a norm $\|\cdot\|_{D_A^s} := [\cdot]_{D_A^s}$ in the space $D_A^s(\mathbb{R}^N,\mathbb{C})$. As in Propositions 2.1 and 2.2 in [21], we can verify that $D_A^s(\mathbb{R}^N,\mathbb{C})$ is a Hilbert space. $L^p(\mathbb{R}^N,\mathrm{d}x/|x|^t)$ denotes the space of L^p-integrable functions with respect to the measure $\mathrm{d}x/|x|^t$, endowed with norm

$$|u|_{p,t} := \left(\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^t}\mathrm{d}x\right)^{1/p}. \quad (22)$$

The aim of the present paper is to investigate the following fractional magnetic Hardy–Sobolev inequality

$$\mu_{s,t,A} := \inf_{\substack{u\in D_A^s(\mathbb{R}^N)\\ u\neq 0}} \frac{\|u\|_{D_A^s}^2}{|u|_{2_s^*(t),t}^2}, \quad (23)$$

where $0<s<1$, $0<t<2s$ and $2_s^*(t) := 2(N-t)/(N-2s)$ is the fractional Hardy–Sobolev critical exponent. Problem (23) relates to the fractional magnetic Laplacian defined by

$$(-\Delta)_A^s u(x) = c_{N,s}\lim_{\varepsilon\to 0^+}\int_{B_\varepsilon^c(x)} \frac{u(x)-e^{i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u(y)}{|x-y|^{N+2s}}\mathrm{d}y, \quad x\in\mathbb{R}^N. \quad (24)$$

For $N = 3$ and with *mid-point prescription*, the fractional magnetic Laplacian was studied in [21]. In particular, d'Avenia and Squassina [21] considered the operator

$$(-\Delta)_A^s u(x) = c_{N,s} \lim_{\varepsilon\to 0^+} \int_{B_\varepsilon^c(x)} \frac{u(x) - e^{i(x-y)\cdot A((x+y)/2)} u(y)}{|x-y|^{N+2s}} \mathrm{d}y, \quad x \in \mathbb{R}^3. \tag{25}$$

Obviously, (24) can be regarded as an extension of the above-mentioned operator involving mid-point prescription. The fractional magnetic Laplacian $(-\Delta)_A^s : D_A^s(\mathbb{R}^N, \mathbb{C}) \to D_A^{-s}(\mathbb{R}^N, \mathbb{C})$ can also be defined by duality as

$$\begin{aligned}\langle (-\Delta)_A^s u, v \rangle &:= \frac{c_{N,s}}{2} \mathrm{Re} \int_{\mathbb{R}^{2N}} \frac{\left(e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} u(x) - u(y)\right)}{|x-y|^{N+2s}} \\ &\quad \cdot \overline{\left(e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} v(x) - v(y)\right)} \mathrm{d}x\mathrm{d}y \\ &= \frac{c_{N,s}}{2} \mathrm{Re} \int_{\mathbb{R}^{2N}} \frac{\left(u(x) - e^{i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} u(y)\right)}{|x-y|^{N+2s}} \\ &\quad \cdot \overline{\left(v(x) - e^{i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} v(y)\right)} \mathrm{d}x\mathrm{d}y.\end{aligned} \tag{26}$$

Denote

$$\mathcal{J} = \left\{u \in D_A^s(\mathbb{R}^N, \mathbb{C}) : |u|_{2_s^*(t),t} = 1\right\}. \tag{27}$$

Then (23) can be characterized as:

$$\mu_{s,t,A} = \inf_{u\in\mathcal{J}} \|u\|_{D_A^s}^2. \tag{28}$$

Our main result is:

Theorem 1. *If $A : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function with locally bounded gradient, then $\mu_{s,t,A}$ is achieved by a nonzero element $U_{s,t,A} \in D_A^s(\mathbb{R}^N, \mathbb{C})$.*

2. Proof of the Main Result

To prove Theorem 1, we need the following Lemma, which is obtained in [22].

Lemma 1 (Diamagnetic inequality). *For any $u \in D_A^s(\mathbb{R}^N, \mathbb{C})$, we have*

$$\begin{aligned}&||u(x)| - |u(x)|| \\ &\leq \left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} u(x) - u(y)\right| \quad \textit{for a.e. } x, y \in \mathbb{R}^N\end{aligned} \tag{29}$$

and

$$\||u|\|_{D^s} \leq \|u\|_{D_A^s}, \tag{30}$$

which means $|u| \in D^s(\mathbb{R}^N, \mathbb{R})$.

By the fractional Hardy–Sobolev inequality (15) and Lemma 1, we have the following lemma.

Lemma 2 (Fractional magnetic Sobolev embeddings). *The embedding*

$$D_A^s(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^{2_s^*(t)}(\mathbb{R}^N, \mathbb{C}) \tag{31}$$

is continuous.

Proof of Theorem 1. Since the best constant $\mu_{s,t}$ of fractional Hardy–Sobolev inequality (15) is achievable, we only need to show that

$$\mu_{s,t,A} = \mu_{s,t}. \tag{32}$$

In fact, for any $\varepsilon > 0$, there exists $u \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$ such that

$$\|u\|_{D^s}^2 \leq \mu_{s,t} + \varepsilon \quad \text{and} \quad |u|_{2_s^*(t),t} = 1. \tag{33}$$

Similarly to Lemma 4.6 in [21], for any $\epsilon > 0$, consider the scaling

$$u_\epsilon(x) := \epsilon^{(2s-N)/2} u\left(\frac{x}{\epsilon}\right), \quad x \in \mathbb{R}^N. \tag{34}$$

Substituting $x = \epsilon X$ and $y = \epsilon Y$, we derive that

$$\begin{aligned}\|u_\epsilon\|_{D_A^s}^2 &= \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} u_\epsilon(x) - u_\epsilon(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &= \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{\epsilon^{2s-N}\left|e^{-i\epsilon(X-Y)\cdot\int_0^1 A[\epsilon((1-\theta)X+\theta Y)]\mathrm{d}\theta} u(X) - u(Y)\right|^2}{\epsilon^{N+2s}|X-Y|^{N+2s}} \epsilon^{2N} \mathrm{d}X\mathrm{d}Y \\ &= \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{\left|e^{-i\epsilon(x-y)\cdot\int_0^1 A[\epsilon((1-\theta)x+\theta y)]\mathrm{d}\theta} u(x) - u(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y\end{aligned} \tag{35}$$

and the following invariance of scaling:

$$\begin{aligned}\|u_\epsilon\|_{D^s}^2 &= \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{|u_\epsilon(x) - u_\epsilon(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &= \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{\epsilon^{2s-N}|u_\epsilon(X) - u_\epsilon(Y)|^2}{\epsilon^{N+2s}|X-Y|^{N+2s}} \epsilon^{2N} \mathrm{d}X\mathrm{d}Y = \|u\|_{D^s}^2,\end{aligned} \tag{36}$$

$$\begin{aligned}|u_\epsilon|_{2_s^*(t),t}^{2_s^*(t)} &= \int_{\mathbb{R}^N} \frac{\epsilon^{((2s-N)/2)\cdot(2(N-t)/(N-2s))}|u(x/\epsilon)|^{2_s^*(t)}}{|x|^t} \mathrm{d}x \\ &= \int_{\mathbb{R}^N} \frac{\epsilon^{t-N}|u(X)|^{2_s^*(t)}}{\epsilon^t|x|^t} \epsilon^N \mathrm{d}x = |u|_{2_s^*(t),t}^{2_s^*(t)}.\end{aligned} \tag{37}$$

Noticing that $|u - v|^2 = |u|^2 + |v|^2 - 2\mathrm{Re}(u\overline{v})$ for $u, v \in \mathbb{C}$, we have

$$\begin{aligned}&\|u_\epsilon\|_{D_A^s}^2 - \|u\|_{D^s}^2 \\ &= \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{\left|e^{-i\epsilon(x-y)\cdot\int_0^1 A[\epsilon((1-\theta)x+\theta y)]\mathrm{d}\theta} u(x) - u(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &\quad - \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &= \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{2\mathrm{Re}\left(\left(1 - e^{-i\epsilon(x-y)\cdot\int_0^1 A[\epsilon((1-\theta)x+\theta y)]\mathrm{d}\theta}\right) u(x)u(y)\right)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &= c_{N,s} \int_{\mathbb{R}^{2N}} \frac{\left(1 - \cos\left(\epsilon(x-y)\cdot\int_0^1 A[\epsilon((1-\theta)x+\theta y)]\mathrm{d}\theta\right)\right)}{|x-y|^{N+2s}} u(x)u(y)\mathrm{d}x\mathrm{d}y \\ &=: c_{N,s} \int_{\mathbb{R}^{2N}} \Upsilon_\epsilon(x,y)\mathrm{d}x\mathrm{d}y = c_{N,s} \int_{K\times K} \Upsilon_\epsilon(x,y)\mathrm{d}x\mathrm{d}y,\end{aligned} \tag{38}$$

where K is compact support of u. Obviously, $\Upsilon_\epsilon(x,y)\to 0$ a.e. in $\mathbb{R}^{2N}$ as $\epsilon\to 0$. Since A is locally bounded, for $x,y\in K$, we have

$$1-\cos\left(\epsilon(x-y)\cdot\int_0^1 A[\epsilon((1-\theta)x+\theta y)]\mathrm{d}\theta\right)\le C|x-y|^2. \tag{39}$$

Then, there exists $C>0$ such that for $x,y\in K$,

$$|\Upsilon_\epsilon(x,y)|\le\begin{cases}\frac{C}{|x-y|^{N-2+2s}}, & \text{if } |x-y|<1,\\ \frac{C}{|x-y|^{N+2s}}, & \text{if } |x-y|\ge 1.\end{cases} \tag{40}$$

Define

$$m(x,y):=\begin{cases}\frac{C}{|x-y|^{N-2+2s}}, & \text{if } |x-y|<1,\\ \frac{C}{|x-y|^{N+2s}}, & \text{if } |x-y|\ge 1.\end{cases} \tag{41}$$

Then, $|\Upsilon_\epsilon(x,y)|\le m(x,y)$ for $x,y\in K$. Since

$$\begin{aligned}\int_{K\times K} m(x,y)\mathrm{d}x\mathrm{d}y &= \int_{(K\times K)\cap\{|x-y|<1\}} m(x,y)\mathrm{d}x\mathrm{d}y\\ &\quad+\int_{(K\times K)\cap\{|x-y|\ge1\}} m(x,y)\mathrm{d}x\mathrm{d}y\\ &\le C\int_{\{|z|<1\}}\frac{1}{|z|^{N-2+2s}}\mathrm{d}z + C\int_{\{|z|\ge1\}}\frac{1}{|z|^{N+2s}}\mathrm{d}z\\ &= C\int_0^1\frac{1}{r^{N-2+2s}}r^{N-1}\mathrm{d}r + C\int_1^{+\infty}\frac{1}{r^{N+2s}}r^{N-1}\mathrm{d}r\\ &<+\infty,\end{aligned} \tag{42}$$

where C may be different from line to line, we get that $m\in L^1(K\times K)$. By Lebesgue's Dominated Convergence Theorem, we see that $\lim_{\epsilon\to0}\|u_\epsilon\|^2_{D^s_A}=\|u\|^2_{D^s_0}$. Then, it follows from (33) that

$$\mu_{s,t,A}\le\lim_{\epsilon\to0}\|u_\epsilon\|^2_{D^s_A}=\|u\|^2_{D^s}\le\mu_{s,t}+\varepsilon, \tag{43}$$

which means that $\mu_{s,t,A}\le\mu_{s,t}$. The opposite inequality holds also because of Lemma 1. Thus, (32) holds, which completes the proof of Theorem 1. □

Data Availability

The [inequality] data used to support the findings of this study have been deposited in the [references 21 and 22] repository.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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